Coalgebraic Logics for Knowledge Representation and Reactive Systems

Lutz Schröder

DFKI Bremen

University of Leicester, June 24, 2011
Introduction: Modal Logic in Computer Science

- Description logics
  - Core formalism of KR and the Semantic Web
  - Underlying logic of OWL-DL
- Temporal logics (CTL, LTL)
- (and many more: epistemic, deontic, . . .)
- Relational semantics
  - Binary relations between individuals
  - Guarded universal and existential quantification
Beyond Relational Semantics

Many modes of expression need more than relational semantics, e.g.

- Uncertainty (Probabilities)
- Vagueness (Fuzzy truth values)
- Defeasibility (Preference orderings)
- Causation and agency (Games)

Large variety of domain-specific logics

+ Suitable expressive means for every purpose
- Multiplied need for tools and algorithms
Enter Coalgebra

Coalgebra acts as a **unified framework** for real-life reasoning

- semantically

- logically

  - generic complete axiomatizations

- algorithmically

  - generic decidability results

  - generic algorithms and **complexity analysis**
Overview

- Real-life reasoning
- Review of relational semantics
- Coalgebraic logic
- One-step rules and generic algorithms
OWL in CAD Quality Control

▶ CATIA DMU Analyser:
  ▶ Overlaps of parts
OWL in CAD Quality Control

- CATIA DMU Analyser:
  - Overlaps of parts
  - Not every overlap is an error

- OWL Ontology:

  \[
  \text{part} \sqsubseteq \text{overlaps only gasket} \\
  \sqcap (\text{bolt} \sqcap \text{overlaps only nut}) \\
  \sqcap \ldots
  \]

(Franke/Klein/Schröder/Thoben CIRP Design 2010)
Conditional logic in CAD Quality Control

\[ a \Rightarrow b:\]
If \( a \) then normally \( b \).

- part \( \Rightarrow \) overlaps only nothing
- gasket \( \Rightarrow \) overlaps some part
- bolt \( \Rightarrow \) overlaps some nut
- bolt \( \sqcap \) hasExplicitPart some thread
- \( \Rightarrow \) overlaps only nothing

(Franke/Klein/Schröder/Thoben CIRP Design 2010)
From the German CPG for coronary heart disease:

**7-13** In presence of medium prior probability and inconclusive ergometry, an exercise test with imaging should be carried out.

Approximation in relational DL:

\[
\forall \text{hasPriorRiskCHG. Medium} \sqcap \\
\forall \text{hasDiagnostics.} \neg \text{Ergometry} \sqcup \text{Inconclusive} \\
\sqsubseteq \exists \text{hasRecommendedDiagnostics.} \\
(\text{ExerciseTest} \sqcap \exists \text{hasObservation. Imaging})
\]
OWL in the Representation of CPGs

[BMBF KMU Innovativ project SIMPLE
Semantically founded implementation of clinical practice guidelines]

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From the German CPG for coronary heart disease:

7-13 In presence of medium prior probability and inconclusive ergometry, an exercise test with imaging should be carried out.

Better approximation in coalgebraic description logic:

moderately(probably (∃.hasDisorder.CHD))\cap
∀hasDiagnostics.¬Ergometry ∪ Inconclusive
⇒∃hasRecommendedDiagnostics.
(ExerciseTest \cap ∃hasObservation.Imaging)
More examples from CPGs

Nested defeasible implication:

*Units normally seeing at least 100 new cases of cancer per annum should be able to maintain their expertise.*

Comparison of probabilities:

*Radiotherapy should be given following mastectomy or breast conserving surgery [...] where the benefit to the individual is likely to outweigh risks of radiation related morbidity.*

*(SIGN breast cancer CPG)*

Combined vague temporality, belief, and uncertainty:

*Aspirin should be given to all patients with a STEMI as soon as possible after the diagnosis is deemed probable.*

*(European CPG for acute ST-segment elevated myocardial infarction)*
Relational Semantics of DL

Concepts

\[ C ::= \bot \mid A \mid \neg C \mid C_1 \cap C_2 \mid \forall R. C \]

Interpretations \( \mathcal{I} \):

- \((\Delta^\mathcal{I}, (A^\mathcal{I}), (R^\mathcal{I}))\) where
  - \(A^\mathcal{I} \subseteq \Delta^\mathcal{I}\)
  - \(R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}\)

- Extension \(C^\mathcal{I} \subseteq \Delta^\mathcal{I}\) of concepts \(C\):

\[
(\forall R. C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I}. x R^\mathcal{I} y \Rightarrow y \in C^\mathcal{I} \}
\]

E.g.

\(\text{ChessFanatic} = \text{ChessPlayer} \cap \forall \text{hasFriend}. \text{ChessFanatic}\)
Incomplete Overview of Non-relational Logics

<table>
<thead>
<tr>
<th>Logic</th>
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<th>Reading</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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<td>$\geq nR. C$</td>
<td>$\geq n R$-successors satisfy $C$</td>
</tr>
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**KR**
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## Reactive systems

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### Multi-agent systems
... for such logics has been notoriously limited:

- CondLean: weak conditional logics
- Pronto: $P\mathcal{SHIQ}(D)$. 
A Reformulation of Relational Semantics

- Interpretations of role $R$ are $\mathcal{P}$-coalgebras

\[ \xi_R : \Delta^I \rightarrow \mathcal{P}(\Delta^I) \]

functor

- Extension of $\forall R. C$:

\[
(\forall R. C)^I = \{ x \in \Delta^I \mid \xi_R(x) \in \{ A \in \mathcal{P}(\Delta^I) \mid A \subseteq C^I \} \}
\]

\[=: \lbrack \forall R \rbrack_{\Delta^I}(C^I)\]

predicate lifting
Coalgebraic Logic

- General modal signatures $\Sigma$
  (sets of finitary modal operators)
- Abstraction of the type of interpretations:
  - Functor (parametrized data type) $T : \text{Set} \rightarrow \text{Set}$
  - Interpretations = $T$-coalgebras

$$\xi : \Delta^T \rightarrow T(\Delta^T)$$

- Abstraction of the semantics of operators $L \in \Sigma$:
  - predicate liftings $[L]_X : \mathcal{P}(X) \rightarrow \mathcal{P}(TX)$, natural in $X$
  - $(LC)^T = \xi^{-1}[[L]_{\Delta^T}(C^T)]$

(Pattinson 2003, Schröder 2005)
Nearly Everything is Coalgebraic

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<tr>
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<td>Relational models</td>
<td>( \forall R. C )</td>
<td>Powerset ( P(X) )</td>
</tr>
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<td>( L_p C )</td>
<td>Distributions ( D(X) )</td>
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<td>( \geq nR. C )</td>
<td>Multisets ( B(X) = X \rightarrow \mathbb{N}_\infty )</td>
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<td>Preference orders ( \exists (S, \preceq). S \rightarrow X )</td>
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<td>( [c]C )</td>
<td>Games ( \exists (S_i). (\prod S_i \rightarrow X) )</td>
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<td>( \langle \gamma \rangle C )</td>
<td>Upclosed nbhd. systems</td>
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(Schröder/Pattinson/Cirstea/Kurz/Venema et al. 2004–2010)
Example: Local Type-1 Probabilistic Logic

(Fagin/Halpern JACM 1994)

Functor $D(X) = \text{distributions on } X$

Interpretations $\Delta^\mathcal{I} \rightarrow D(\Delta^\mathcal{I}) = \text{Markov chains}$

Operators $L_p \ '\text{with probability } \geq p'$

$$[L_p]_X(A) = \{\mu \in D(X) \mid \mu(A) \geq p\}$$
Example: Alternating Time

(Alur et al. JACM 2002)

\[ N = \{1, \ldots, n\} \] set of agents, \( c \subseteq N \) coalition

Functor:

\[ F(X) = \left\{ (k_1, \ldots, k_n, f) \mid f : \left( \prod_{i \in N} \{1, \ldots, k_i\} \right) \to X \right\} \]

Interpretations \( \Delta^I \rightarrow F(\Delta^I) = \) concurrent game structures

Operators \([c] \) ‘c can force \ldots in the next step’

\[ \lbrack [c] \rbrack_X(A) = \{ f \in F(X) \mid \exists \sigma_c. \forall \sigma_{N-c}. f(\sigma_c, \sigma_{N-c}) \in A \} \]
Generic Deduction Systems

Parametrized Systems:

- Fixed propositional part

- Further fixed parts depending on orthogonal features (nominals, fixed points)

- Parameter: Axiomatization of the functor through (cutfree complete) one-step rules
  (Schröder/Pattinson LICS 06; see my Leicester seminar talk of March 2006)
One-Step Rules

One-step logic: $V$ set of prop. var.,

$$\Sigma V = \{ La \mid a \in V, L \in \Sigma \}.$$

Given $\tau : V \rightarrow \mathcal{P}(X)$, interpret

- propositional formulas $\varphi$ over $V$ as $\llbracket \varphi \rrbracket \tau \subseteq X$
- propositional formulas $\psi$ over $\Sigma V$ as $\llbracket \psi \rrbracket \tau \subseteq TX$ by

$$\llbracket La \rrbracket \tau = \llbracket L \rrbracket_{\chi} \tau(a)$$

One-step rules:

- $\varphi \quad$ propositional over $V$
- $\psi \quad$ clause over $\Sigma V$

$\varphi \quad$ one-step sound if $\llbracket \varphi \rrbracket \tau = X \implies \llbracket \psi \rrbracket \tau = TX$. 

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The Cut Rule

\[
\frac{A \rightarrow C \quad C \rightarrow B}{A \rightarrow B}
\]

Undesirable for proof search.
Set $\mathcal{R}$ of one-step rules one-step cut-free complete if for clauses $\chi$ over $\Sigma V$

\[
\begin{array}{c}
[\chi] \tau = TX \iff \exists \phi/\psi \in \mathcal{R}, \sigma : V \rightarrow V. \\
[\phi\sigma] \tau = X, \quad \psi\sigma \text{ contracted}, \quad \psi\sigma \subseteq \chi.
\end{array}
\]

One-step cut-free complete rule sets (OSCCR)

- induce tableau-based model constructions
- yield cut-free complete deduction systems for the full logic
  \rightarrow proof search
One-Step Rules: Examples

\[ ALC: \]
\[
\bigcup_{i=1}^{n} \neg a_i \sqcup b \\
\bigcup_{i=1}^{n} \neg \forall R. a_i \sqcup \forall R. b 
\]
\( (n \geq 0) \)

Local type-1 probabilistic logic:

Arithmetic of characteristic functions

\[
\bigcup_{0 \leq i \leq n} \text{sgn}(r_i) L_{p_i} a_i
\]

where \( n \geq 0, \ r_i \in \mathbb{Z} - \{0\}, \ \text{\geq} = \begin{cases} > & \text{if } r_i < 0 \text{ for all } i \\ \geq & \text{otherwise} \end{cases} \)
Generic Algorithms via OSCC Rule Sets

- PSPACE for next-step-logics
- PSPACE for coalgebraic hybrid logic
- EXPTIME for coalgebraic description logics (i.e. with TBoxes)
- Completeness and EXPTIME global caching for flat fixed point logics via $\mathcal{O}$-adjointness (Schröder/Venema 2010)
  - Alternating $\mu$-calculus (Alur et al. 2002)
  - Graded $\mu$-calculus (Kupferman et al. 2002)
Flat fixed point operators

$$\sharp_\gamma(\varphi) \equiv \mu x. \gamma(\varphi, x)$$

$$\flat_\gamma(\varphi) \equiv \nu x. \gamma(\varphi, x) \quad (\gamma \text{ modal})$$

→ fragments of single-variable coalgebraic $\mu$-calculus.

E.g.

- CTL: $AF\varphi = \#p \lor \square x \varphi$
- $\flat p \land \square \square x$ not in CTL*
- ATL: $\langle\langle C\rangle\rangle F\varphi = \#p \lor [C]x \varphi$
- Graded $\mu$-calculus (Kupferman et al. 2002):

$$\#p \lor \Diamond_2 x \varphi$$

‘the current state is the root of a binary tree whose leaves satisfy $\varphi$’.
The Kozen-Park Axioms

Briefly: ‘$\sharp\gamma(\varphi)$ is a least fixed point’, i.e.:

Unfolding:

$$\gamma(\varphi, \sharp\gamma \varphi) \to \sharp\gamma \varphi$$

Fixed-point induction:

$$\frac{\gamma(\varphi, \chi) \to \chi}{\sharp\gamma(\varphi) \to \chi}$$

Are these complete?

- Do imply that $\sharp\gamma(\varphi)$ is a least fixed point in the Lindenbaum algebra
Strategy for the Completeness Proof

- Show constructivity of the Lindenbaum algebra:

  \[ \#\gamma(\varphi) = \bigvee_{i<\omega} \gamma(\varphi)^i(\bot) \]

  via \(\mathcal{O}\)-adjointness of \(\gamma(\varphi)\): for all \(\psi\) there is a finite set \(G_{\gamma(\varphi)}(\psi)\) s.t.

  \[ \gamma(\varphi, \rho) \leq \psi \iff \rho \leq \chi \quad \text{for some } \chi \in G_{\gamma(\varphi)}(\psi) \]

- Constructivity implies

  \[ \#\gamma \varphi \land \psi \text{ consistent} \implies \gamma(\varphi)^i(\bot) \land \psi \text{ consistent for some } i < \omega. \]

- Tableau construction with time-outs
- Adjointness via OSCC Rule Sets

- Unfolding & guardedness:
  w.l.o.g. the top level of every formula is modal

- Rigidity lemma:
  w.l.o.g. proofs of modal clauses end in modal one-step rules

**Example**: Adjointness of □. Recall rule:

\[
\frac{\bigwedge_{i=1}^{n} a_i \rightarrow b}{\bigwedge_{i=1}^{n} \Box a_i \rightarrow \Box b}
\quad (n \geq 0)
\]

Calculate:

\[
\Box \rho \leq \psi = \bigwedge_{i=1}^{n} \Box \chi_i \rightarrow \bigvee_{j=1}^{m} \Box \theta_j
\]
O-Adjointness via OSCC Rule Sets

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$$
\begin{align*}
\bigwedge_{i=1}^n a_i & \rightarrow b \\
\bigwedge_{i=1}^n \Box a_i & \rightarrow \Box b
\end{align*}
$$

$(n \geq 0)$

Calculate:

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\iff \vdash \Box \rho \rightarrow \bigwedge_{i=1}^n \Box \chi_i \rightarrow \bigvee_{j=1}^m \Box \theta_j
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Rigidity

\[
\iff \vdash \rho \land \bigwedge_{i=1}^n \chi_i \rightarrow \theta_j \quad \text{für ein } j
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$$\iff \vdash \Box \rho \land \wedge_{i=1}^{n} \Box x_i \rightarrow \bigvee_{j=1}^{m} \Box \theta_j$$

Rigidity

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Rigidity

\[\iff \vdash \rho \rightarrow \bigwedge_{i=1}^{n} \chi_i \rightarrow \theta_j \quad \text{für ein } j\]

Thus put \(G_{\Box \chi}(\psi) = \{\bigwedge_{i=1}^{n} \chi_i \rightarrow \theta_j \mid j = 1, \ldots, m\}\)
Conclusions

- Coalgebra provides a **uniform framework** for modal and hybrid logics
  - Graded operators (knowledge representation, redundancy)
  - Probabilistic operators (quantitative uncertainty, reactive systems)
  - Conditional operators (nonmonotonic reasoning)
  - Alternating-time logics, game logic, logics of agency (multi-agent systems)

- Wide range of generic decision procedures and complexity bounds

- Modular

- Frequently new bounds and calculi for instance logics, in particular in presence of
  - nominals
  - fixed points

(Schröder/Pattinson ICALP 2007)
Ongoing and Future Work

- Manydimensional coalgebraic logics

- Fuzzy coalgebraic logics
  - E.g. the logic of probably

- Vision: generic, efficient modular reasoning tools
  - Ongoing optimization of CoLoSS (PhD thesis Hausmann)
  - Enable use in realistic applications, e.g. CPGs
Thanks for your attention!
CDL with Nominals

- Nominals $i, j, \ldots$ are atomic concepts to be interpreted as singletons

- Internalize ABoxes via satisfaction operators

\[ @_i C = \text{`$i$ satisfies $C$'} \]
Deduction over GCIs

▶ General concept inclusions $C \sqsubseteq D$

▶ Tableaux diverge without blocking:
  for gci $\top \sqsubseteq \exists R. A$,

\[
\begin{align*}
\top, \exists R. A \\
\hline
A, \exists R. A \\
\hline
A, \exists R. A \\
\hline
\ldots
\end{align*}
\]

▶ Tedious analysis even for $\mathcal{ALC}$ (Donini/Massacci 99)
A Global Caching Algorithm

Collect $@$-formulas along a winning strategy:

$\Diamond \Diamond @i p, \Diamond (\Diamond @i q \lor \Diamond A)$

$\rightarrow$

$@i p, @i q$

$\downarrow$

$@i p$

$\downarrow$

$\Diamond @i q \lor \Diamond A$

$\downarrow$

$\Diamond @i q$

$\downarrow$

@i q

- Decidability in EXPTIME
- Room for heuristic optimization
- Novel algorithm even for the relational case

(Goré/Kupke/Pattinson/Schröder IJCAR 10)