Lambda Calculus between Algebra and Topology

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Lambda Calculus between Algebra and Topology

\[ \lambda \text{-calculus} \]

- Algebra ← Lambda abstraction algebras
- Church algebras
- Boolean-like algebras
- \( n \)-subtractive algebras

- Topology

- From Lambda calculus to Universal Algebra:
  - Lambda abstraction algebras
  - Church algebras
  - Boolean-like algebras
  - \( n \)-subtractive algebras

- From Universal Algebra to Lambda calculus:
  - The structure of the lattice of the \( \lambda \)-theories
  - Boolean algebras, Stone representation theorem and the indecomposable semantics
  - The order-incompleteness problem
Lambda Calculus between Algebra and Topology

Topology refines partial orderings through the separation axioms:

A space \((X, \tau)\) is \(T_0\) iff the specialization preorder \(\leq_\tau\) is a partial order. Every partial order is the specialization order of a space.

- **From Lambda calculus to Topology:**
  (i) new axioms of separation
  (ii) topological algebras
  (iii) Visser spaces and Priestley spaces

- **From Topology to Lambda calculus:**
  (i) Topological incompleteness/completeness theorems
  (ii) Topological models
  (iii) The equational completeness problem for Scott semantics
Part 0

Lambda Calculus: Church, Curry, Scott
Scott

- Church (around 1930): Lambda calculus

- $\lambda$-theory $= \text{congruence w.r.t. application and } \lambda\text{-abstractions containing } \alpha\beta\text{-conversion}$

- **Scott:** First model and Continuous Semantics (1969) A *continuous model* $D$ is a reflexive object in the category $\text{CPO}$ of complete partial orderings.

- All known models of $\lambda$-calculus admits a compatible (w.r.t. application) partial order and are topological algebras (w.r.t. Scott topology).

- Each model $D$ defines an equational theory and an order theory:

  $$Eq(D) = \{(M, N) : |M|^D = |N|^D\}; \quad Ord(D) = \{(M, N) : |M|^D \leq |N|^D\}$$
Theory (In)Completeness Problem

- A class $\mathcal{C}$ of models of $\lambda$-calculus is theory complete if
  $$(\forall \text{ consistent } \lambda\text{-theory } T)(\exists M \in \mathcal{C}) \ Th=(M) = T.$$  
  Theory incomplete, otherwise.

**Theorem 1** (*Theory incompleteness*) All known semantics are theory incomplete.  
Honsell-Ronchi: Scott semantics; Bastonero-Gouy: stable semantics; Salibra: strongly
stable semantics and all pointed po-models.

- Selinger (1996) asked: Are partial orderings intrinsic to computations?

- A *po-model* is a pair $(M, \leq)$, where $M$ is a model and $\leq$ is a nontrivial partial ordering on $M$ making the application monotone.

- The order-completeness problem by Selinger asks whether the class $\text{PO}$ of po-models is theory complete or not.  
  ANSWER: Unknown.
  The best we know about order-incompleteness:

**Theorem 2** *(Carraro-S. 2013)* There exists a $\lambda$-theory $T$ such that, for every po-model $(M, \leq)$,

$$Th=(M) \supset T \Rightarrow (M, \leq) \text{ has infinite connected components}$$

and the connected component of the looping term $\Omega$ is a singleton set.
Two Other Open Problems

- **Equational Completeness Problem** asks whether there exists a Scott continuous model whose equational theory is the least $\lambda$-theory $\lambda\beta$:

  $$Eq(D) = \lambda\beta,$$

  for some Scott continuous model $D$

  ANSWER: Unknown

- **Equational Consistency Problem** (Honsell-Plotkin 2006) asks whether, for every finite set $E$ of equations between $\lambda$-terms consistent with the $\lambda$-calculus, there exists a Scott continuous model contemporaneously satisfying all equations of $E$.

  $$(\forall E \text{ finite set of identities})[(E \cup \lambda\beta \text{ consistent}) \rightarrow (\exists D \in \text{Scott}) D \models E]?$$

  ANSWER: No (Carraro-S. 2013).

- These problems and the order incompleteness problem are interconnected
Part I

Topology: Scott, Selinger, Visser, Priestley
The technique for the Equational Completeness Problem

Remember that the Equational Completeness Problem asks whether there exists a Scott continuous model whose equational theory is the least $\lambda$-theory $\lambda\beta$:

$$Eq(D) = \lambda\beta,$$

for some Scott continuous model $D$.

Given a class $\mathcal{C}$ of po-models, we sometimes are able to construct an “effective” po-model $\mathcal{E}$ (maybe not in the class $\mathcal{C}$) such that

$$\text{Ord}(\mathcal{E}) \subseteq \text{Ord}(D), \quad \text{for all } D \in \mathcal{C}.$$  

Lemma 1 If $\mathcal{E}$ is an effective po-model, then, after encoding,

1. $|M|^{\mathcal{E}}$ is an r.e. element of the model for every closed $\lambda$-term $M$;
2. $|M|^{\mathcal{E}}$ is a decidable element for every closed normal form $M$.
3. $\{N : \mathcal{E} \models N \leq \lambda x.x\}$ is co-r.e.
Theorem 3 (Berline-Manzonetto-S. 2007) Given a class $\mathbb{C}$ of models, if there exists an “effective” model $\mathcal{E}$ such that

$$(\forall D \in \mathbb{C}) \text{Ord}(\mathcal{E}) \subseteq \text{Ord}(D),$$

then, for every model $D \in \mathbb{C}$, we have:

(i) $\text{Ord}(D)$ is not r.e.
(ii) $\text{Eq}(D) \neq \lambda \beta$.

Proof. Define Visser topology over the set $\Lambda$ of $\lambda$-terms (modulo $\lambda \beta$):

$X \subseteq \Lambda$ is Visser base open if it is $\beta$-closed and co-r.e.

Theorem 4 (Visser 1980) Visser topology is hyperconnected on $\Lambda$.

(i) Assume $\text{Ord}(D)$ to be r.e. for some $D \in \mathbb{C}$.

$\{N : \mathcal{E} \models N \leq \lambda x.x\}$ co-r.e. $\subseteq \{N : D \models N \leq \lambda x.x\}$ r.e.

Visser open Visser closed

$\downarrow$

$\{N : D \models N \leq \lambda x.x\} = \Lambda.$

(ii) (Selinger 1996) If $\text{Eq}(D) = \lambda \beta$ for a po-model $D$, then the term denota-
tions are an antichain. Consequence: $\text{Eq}(D) = \text{Ord}(D) = \lambda \beta$ are r.e.
SOME RESULTS (Carraro-S. 2009):

- $\lambda_\beta\eta$ is not the theory of a model living in the category of Scott domains.
- $\lambda_\beta$ is not the theory of a filter model living in $\text{CPO}$. 
Leaving Lambda Calculus Towards Computability Theory
(Work in Progress)

• An enumerated set is a pair $X = (|X|, \phi_X)$, where $\phi_X : \omega \to |X|$ is an onto map (Mal’cev 1964)

• Recursive Functions: $\phi : \omega \to \text{RecFun}$ mapping the “program” $n$ into the function $\phi_n$ computed by the program “$n$”.

• Lambda Calculus: $\phi_\Lambda : \Lambda \to \Lambda/\lambda\beta$ mapping a $\lambda$-term $M$ into its equivalence class $[M]_\beta$.

• Given an enumerated set $X$, we define $Y \subseteq X$ r.e. (co-r.e., decidable) if $\phi_X^{-1}(Y)$ is r.e. (co-r.e., decidable).
The Visser topology

- The r.e. sets of $X$ are a ring $\mathcal{R}_X$ of sets generating a topology $\tau_E$ on $|X|$.

- The co-r.e. sets of $X$ are a ring co-$\mathcal{R}_X$ of sets generating a topology $\tau_V$ on $|X|$.

$$(|X|, \tau_E, \tau_V)$$ is a bitopological space

$\tau_E$ is the Ershov topology and $\tau_V$ is the Visser topology

**Lemma 2** $\tau_E$ is $T_0$ iff $\tau_V$ is $T_0$ ($a \leq_E b$ iff $b \leq_V a$).

- Recursive Functions: $\tau_E$ on RecFun is the Scott topology; while $\tau_V$ is $T_0$ with $f \downarrow$ $\tau_V$-open iff $\text{graph}(f)$ is decidable.

- Lambda Calculus: $\tau_E$ on $\Lambda/\lambda\beta$ is the discrete topology, while $\tau_V$ is non-trivial, hyperconnected and $T_1$. 

The Visser topology

**Definition 1** (Visser) An enumerated set \( X = (|X|, \phi) \) is pre-complete if, for every partial recursive function \( f \), there exists a total recursive function \( g \) such that

\[
f \downarrow n \Rightarrow \phi_f(n) = \phi_g(n).
\]

**Proposition 1**

(i) The set of computable functions is precomplete.

(ii) (Visser) \( \land/T \) (\( T \) a \( \lambda \)-theory) is precomplete.

Proof. 
(i) Define

\[
\phi_g(x)(y) = \begin{cases} 
\phi_f(x)(y), & \text{if } x \in \text{dom}(f) \\
\uparrow, & \text{otherwise}
\end{cases}
\]

(ii) Let \( \overline{M} \) be the Godel number of \( \lambda \)-term \( M \) and let \( n_\lambda \) be the \( \lambda \)-term denoted by the number \( n \). Barendregt has shown that there exists a \( \lambda \)-term \( E \) such that, for every \( M \), \( E \overline{M} =_{\lambda \beta} M \). Let \( F \) be a \( \lambda \)-term representing the computable function \( f \). Define

\[
g(n) = E(Fn),
\]

\[
g(n)_\lambda = E(Fn) =_{\lambda \beta} E(f(n)) =_{\lambda \beta} f(n)_\lambda.
\]
Proposition 2 (Visser) If the enumerated set \( X = (|X|, \phi) \) is pre-complete then

1. \( \tau_V \) is hyperconnected;

2. \( \tau_E \) is compact iff \( \leq_{\tau_E} \) has a bottom element.

Proof: (1) If \( V \cup U = \omega \), where \( V \) and \( U \) are r.e. and \( \phi \)-closed sets of natural numbers, then either \( V = \omega \) or \( U = \omega \).
By contraposition assume that neither \( V \) nor \( U \) is \( \omega \). Let \( a \in V \setminus U \) and \( b \in U \setminus V \). Let \( A \) and \( B \) be two recursively inseparable sets of natural numbers. Define the partial function

\[
 f(x) = \begin{cases} 
 a, & \text{if } x \in A \\
 b, & \text{if } x \in B \\
 \uparrow, & \text{otherwise}
\end{cases}
\]

Consider a total recursive function \( g \) completing \( f \) up to \( \phi \)-equivalence. We have \( g^{-1}(V) \cup g^{-1}(U) = \omega \), \( A \subseteq g^{-1}(V) \setminus g^{-1}(U) \) and \( B \subseteq g^{-1}(U) \setminus g^{-1}(V) \). Then \( A \) and \( B \) are recursively separable. Contradiction.

(2) By (1) every finite covering of \( |X| \) must contain \( |X| \).
The Priestley space of computability

Hereafter we always assume that \( \tau_E \) is \( T_0 \).

**Proposition 3** \( \tau_E \lor \tau_V \) is zero-dimensional (i.e., it has a base of clopens), Hausdorff and satisfies the Priestley separation axiom (w.r.t. \( \leq_E \)).

Proof: Let \( a, b \in X \). Since \( \tau_E \) is \( T_0 \), either \( a \not\leq_E b \) or \( b \not\leq_E a \). In the first case there is an r.e. open \( U \) such that \( a \in U \) but \( b \notin U \). \( X \setminus U \) is a co-r-e open such that \( b \in X \setminus U \) and \( a \notin X \setminus U \).

Let \( x \not\leq_E y \). Then there is an r.e. set \( U \) such that \( x \in U \) but \( y \notin U \). \( U \) is E-upper. The complement is E-down which contains \( y \) but not \( x \).

**Proposition 4** Let \( X \) be an enumerated set. If \( (|X|, \tau_E) \) is a \( T_0 \)-space, then the compactification of \( (|X|, \tau_E \lor \tau_V) \) is a Priestly space and \( (|X|, \tau_E \lor \tau_V) \) is a dense subspace of this compactification.

Proof: We consider the ring \( \mathcal{R}_X \) of r.e. subsets of \( |X| \) and consider the product topology on \( 2^{\mathcal{R}_X} \). Consider the closed subspace of lattice homomorphisms \( HOM(\mathcal{R}_X^E, 2) \). It is Priestley (because closed), and \( (|X|, \tau_E \lor \tau_V) \) embeds into \( HOM(\mathcal{R}_X, 2) \) as a dense subspace.
Consider $2 = \{0, 1\}$ with three topologies:

- The discrete top $\tau_d$; The top $\tau_0$ with $0 < 1$; The top $\tau_1$ with $1 < 0$.

We have $\tau_d = \tau_0 \lor \tau_1$. We consider the ring $\mathcal{R}_X^E$ of r.e. subsets of $|X|$ and consider the product topology on $2^{\mathcal{R}_X^E}$. We have:

- The topology $\prod \tau_0$ on $2^{\mathcal{R}_X^E}$ is the Scott topology w.r.t. $\subseteq$;
- The topology $\prod \tau_1$ on $2^{\mathcal{R}_X^E}$ is the Scott topology w.r.t. $\supseteq$;
- The topology $\prod \tau_d = \prod \tau_0 \lor \prod \tau_1$ on $2^{\mathcal{R}_X^E}$ is a Priestley space.

Consider the closed subspace of lattice homomorphisms $HOM(\mathcal{R}_X^E, 2)$. It is Priestley (because closed), and $(|X|, \tau_E \lor \tau_V)$ embeddes into $HOM(\mathcal{R}_X^E, 2)$ as a dense subspace.

We consider a map $e : X \rightarrow \text{Hom}(\mathcal{R}X, 2)$ defined as follows, for every r.e. set $Y$ and every $x \in X$: $e(x)(Y) = 1$ iff $x \in Y$. 
The map $e$ is bi-continuous because, for every r.e. set $Y$, $Y = e^{-1}(\{f : f(Y) = 1\})$ and $X \setminus Y = e^{-1}(\{f : f(Y) = 0\})$.

The codomain of $X$ is a dense subspace $Y$ of $\text{Hom}(RX, 2)$.

$X$ is homeomorphic to $Y$ iff the ring $RX$ distinguishes the points of $X$.

Remark: What is the compactification of lambda calculus? We extend the application operator and the lambda-abstractions to its compactification.
Part II
Algebras: Stone, Boole and Church
Theorem 5 *(Stone Representation Theorem)*

- Every Boolean algebra is isomorphic to a field of sets.
- Every Boolean algebra can be embedded into a Boolean product of indecomposable Boolean algebras (*2* is the unique indecomposable Boolean algebra!).

Then every Boolean algebra is isomorphic to a subalgebra of \(2^I = \mathcal{P}(I)\) for a suitable set \(I\).

**Generalisations to other classes of algebras by Pierce (rings with unit) Comer and Vaggione.**

Combinatory algebras (CA) and \(\lambda\)-abstraction algebras (LAA) satisfy an analogous theorem...
The untyped \(\lambda\)-calculus has truth values 0, 1 and “if-then-else” construct \(q(x, y, z)\) of programming:

- \(\lambda\)-calculus (LAA): \(1 \equiv \lambda xy.x\); \(0 \equiv \lambda xy.y\); \(q(e, x, y) = (ex)y\)
- Combinatory logic (CA): \(1 \equiv k\); \(0 \equiv sk\); \(q(e, x, y) = (ex)y\)
- Boolean algebras: \(q(e, x, y) = (e \land x) \lor (\neg e \land y)\)
- Rings with unit 1: \(q(e, x, y) \equiv ex + (1 - e)y\).

**Definition 2** (Manzonetto-Salibra 2008) An algebra \(A\) is a Church algebra if it admits two constants 0, 1 and a ternary term \(q(x, y, z)\) satisfying:

\[ q(1, x, y) = x; \quad q(0, x, y) = y. \]

There are equations which are contemporaneously satisfied by 0 and 1: for example,

\[ q(1, x, x) = x; \quad q(0, x, x) = x. \]
Central elements

An element $e$ of a Church algebra $A$ is central if

$$A \cong A/Cong(e = 1) \times A/Cong(e = 0).$$

**Lemma 3** Let $A$ be a Church algebra et $e \in A$. The following conditions are equivalent:

- $e$ is central;
- $e$ satisfies the following identities:
  1. $q(e, x, x) = x$.
  2. $q(e, q(e, x, y), z) = q(e, x, z) = q(e, x, q(e, y, z))$.
  3. $q(e, f(x), f(y)) = f(q(e, x_1, y_1), \ldots, q(e, x_n, y_n))$, ∀ operation $f$
  4. $e = q(e, 1, 0)$.

Central elements are the unique way to decompose the algebra as Cartesian product.

$A$ is indecomposable if the unique central elements are 0, 1.
Theorem 6  

- The central elements of a Church algebra $A$ constitute a Boolean algebra:

\[ e \lor d = q(e, 1, d); \quad e \land d = q(e, d, 0); \quad \neg e = q(e, 0, 1) \]

- Let $\mathcal{V}$ be a variety of Church algebras, $A \in \mathcal{V}$ and $\mathcal{F}$ be the Boolean space of maximal ideals of the Boolean algebra of central elements of $A$. Then the map

\[ f : A \to \prod_{I \in \mathcal{F}} (A/\theta_I), \]

defined by

\[ f(x) = (x/\theta_I : I \in \mathcal{F}), \]

gives a weak Boolean product representation of $A$. The quotient algebras $A/\theta_I$ are directly indecomposable if the indecomposable members of $\mathcal{V}$ constitute a universal class. (True for CA and LAA!)
Central elements at work in lambda calculus!

The indecomposable CAs (models of λ-calculus) are the building blocks of CA.

The **indecomposable semantics** is the class of models which are indecomposable as combinatory algebras.

**Theorem 7**  *Scott is always simple!*

Proof: Scott continuous semantics (and the other known semantics of λ-calculus) are included within the indecomposable semantics, because every Scott model is simple (i.e., it admits only trivial congruences) as a combinatory algebra.
Central elements at work in lambda calculus!

**Theorem 8** The algebraic incompleteness theorem: There exists a continuum of \( \lambda \)-theories which are not equational theories of indecomposable models.

Proof:
1. Decomposable CAs are closed under expansion.
2. \( \Omega \equiv (\lambda x.xx)(\lambda x.xx) \) is a non-trivial central element in the term algebra of a suitable \( \lambda \)-theory \( \phi \), because
   - the \( \lambda \)-theory \( \psi_1 \) generated by \( \Omega = \lambda xy.x \) is consistent;
   - the \( \lambda \)-theory \( \psi_2 \) generated by \( \Omega = \lambda xy.y \) is consistent;
   - \( \Omega \) is central in the term algebra of \( \phi = \psi_1 \cap \psi_2 \).
3. All models of \( \phi \) are decomposables!

The algebraic incompleteness theorem encompasses all known theory incompleteness theorems:

(Honsell-Ronchi 1992) Scott continuous semantics;
(Bastonero-Guy 1999) Stable semantics;
(Salibra 2001) Strongly stable semantics.
Central elements at work in universal algebra!

1. **Boolean-like algebras**: Church algebras (of any algebraic type), where all elements are central.

2. **Semi-Boolean-like algebras**: Church algebras (of any algebraic type), where all elements are semi-central.

**Theorem 9**  A double pointed variety is discriminator iff it is idempotent semi-Boolean-like and 0-regular.

3. Lattices of equational theories
Part III: The $\lambda$-calculus is algebraic
Lambda terms

• Algebraic similarity type Σ:
  – Nullary operators: \( x, y, z, \ldots \) (names = variables of \( \lambda \)-calculus)
  – Binary operator: \( \bullet \) (application)
  – Unary operators: \( \lambda x, \lambda y, \lambda z, \ldots \) (\( \lambda \)-abstractions)

• A \( \lambda \)-term is a ground \( \Sigma \)-term (no algebraic variable)

\[ \lambda x.x y \]

• A context is just a term of type \( \Sigma \); algebraic variables \( a, b, c, \ldots \) (holes in Barendregt's terminology) may be involved

\[ \lambda x.x a \]
Two substitutions

- Substitution for names (with $\alpha$-conversion)
  
  $$(\lambda x. xy)[y := x] = \lambda z. zx$$

  We care...

- Substitution for variables (without $\alpha$-conversion)
  
  $$(\lambda x. xa)[a := x] = \lambda x. xx$$

  We do not care...
The untyped λ-calculus

- Let

\[ \Lambda = (\Lambda, \cdot, \lambda x, x)_{x \in \text{Name}} \]

be the absolutely free Σ-algebra over an empty set of generators.

A λ-theory is any congruence on \( \Lambda \) (i.e., equivalence relation compatible w.r.t. application and λ-abstractions) including \( \alpha \beta \)-conversion.

It seems that Universal Algebra cannot be applied to λ-calculus because \( \alpha \beta \)-conversion does not involve algebraic variables!
Two starting points for the algebraic $\lambda$-calculus

- The lattice $\lambda T$ of $\lambda$-theories
  
  $\cong$

  The lattice of congruences of the term algebra $\Lambda/\lambda\beta$
  ($\lambda\beta$ is the least congruence on $\Lambda$ including $\alpha$- and $\beta$-conversion)

- The variety (equational class) generated by $\Lambda/\lambda\beta$
  
  =

  Class of $\Sigma$-algebras satisfying all identities between contexts satisfied by $\Lambda/\lambda\beta$

- $CA = \text{class of combinatory algebras (Curry-Schönfinkel 1920-30)}$
- $LAA = \text{class of } \lambda\text{-abstraction algebras (Pigozzi-Salibra 1993)}$

**Theorem 10** (*Salibra 2000*) $LAA = \text{Variety}(\Lambda/\lambda\beta)$
The algebraic lambda calculus

**Theorem 11** (S. 2000) The variety generated by $\Lambda/\lambda\beta$ is axiomatized by:

1. $(\beta_1)$ $(\lambda a.a)x = x$
2. $(\beta_2)$ $(\lambda a.b)x = b$ \hspace{1cm} $(b \neq a)$
3. $(\beta_3)$ $(\lambda a.x)a = x$
4. $(\beta_4)$ $(\lambda a.a.x)y = \lambda a.x$
5. $(\beta_5)$ $(\lambda a.x)y z = (\lambda a.x)y z \cdot (\lambda a.y)z$
6. $(\beta_6)$ $(\lambda b.y)c = y \Rightarrow (\lambda ab.x)y = \lambda b.(\lambda a.x)y$ \hspace{1cm} $(c \neq b, a \neq b)$
7. $(\alpha)$ $(\lambda b.x)c = x \Rightarrow \lambda a.x = \lambda b.(\lambda a.x)b$ \hspace{1cm} $(a \neq b)$

Algebras satisfying $(\beta_1)$-$(\beta_6)$ and $(\alpha)$ are called lambda abstraction algebras (LAAs, for brevity) and were introduced by Pigozzi-S. (1993)
The algebraic lambda calculus

• An element of a LAA may depend on all possible names in $Na = \{x, y, z, x_1, y_1, z_1, \ldots\}$

  – Cartesian product: $\langle x, y, z, x_1, y_1, z_1, \ldots \rangle \in (\Lambda/\lambda\beta)^{Na}$

  – Lambda theories of infinitary $\lambda$-calculus: $\lambda x.x(y(z(x_1(y_1(x_1(\ldots))))$.

• Examples of LAAs:

  – The term algebra $\Lambda/\phi$ of a (infinitary) $\lambda$-theory $\phi$

  – Algebras of functions obtained by the models of $\lambda$-calculus
LAA and Universal Algebra

Universal Algebra: A variety $\mathcal{V}$ is studied by means of the lattice identities contemporaneously satisfied by all congruence lattices of the algebras in $\mathcal{V}$.

A priori we can apply the last 30 years of Universal Algebra to the variety LAA:

**Theorem 12** (Lusin-Salibra 2004) *Every lattice identity holding in (all congruence lattices of algebras in) LAA is trivial.*

But Universal Algebra is at work!
Universal Algebra at work

Theorem 13 (Salibra 2000) The lattice $\lambda T$ of $\lambda$-theories is isomorphic to the lattice of equational theories of LAA’s.

Corollary 1 Every variety of LAA’s is generated by the term algebra $\Lambda/\phi$ of a suitable $\lambda$-theory $\phi$.

- $\lambda$-theory $\phi \iff$ Variety generated by term algebra $\Lambda/\phi$.
- $\lambda$-theory $\phi \iff$ Lattice interval $\{\psi : \psi \geq \phi\} \cong$ congruence lattice of $\Lambda/\phi$
- $\lambda$-calculus problem $= \text{Problem of existence of a subvariety of LAA}$.

Example: Order-incompleteness problem (by Selinger)
The lattice $\lambda T$ of $\lambda$-theories

Conjecture: Every nontrivial lattice identity fails in $\lambda T$

- (Visser 1980)
  - Every countable poset embeds into $\lambda T$ by an order-preserving map.
  - Every lattice interval $[\phi, \psi]$ ($\phi, \psi$ r.e.) $\lambda$-theories has a continuum of elements.

- (Lusin-Salibra 2004) $\lambda T$ satisfies the Zipper condition:
  \[
  \phi \lor \psi = 1 \text{ and } \delta \land \phi = \delta \land \psi \Rightarrow \delta \leq \phi \land \psi.
  \]

- (Salibra 2001) $\lambda T$ is not modular.

- (Berline-Salibra 2006) $\exists$ a finite axiomatisable $\lambda$-theory $\phi$ such that the lattice interval $[\phi) = \{\psi : \psi \geq \phi\}$ is distributive.

- (Statman 2001) The meet of all coatoms of $\lambda T$ is $\neq \lambda \beta$. (i.e., there exist equations $M = N$ such that $\phi \cup \{M = N\}$ is consistent for every consistent $\lambda$-theory $\phi$).

- (Manzonetto-Salibra 2008)
  (\forall\text{natural number } n)(\exists \lambda\text{-theory } \phi_n) such that the interval sublattice $[\phi_n) = \{\psi : \psi \geq \phi_n\}$ is isomorphic to the finite Boolean lattice $2^n$. 
Part IV
Separability: Selinger, Coleman, Kearnes, Sequeira
Theory (In)Completeness Problem

- A class $\mathcal{C}$ of models of $\lambda$-calculus is **theory complete** if

  $$(\forall \text{consistent } \lambda\text{-theory } T)(\exists D \in \mathcal{C}) \ E_q(D) = T.$$  

  Theory incomplete, otherwise.

**Theorem 14 (Theory incompleteness)** All known semantics are theory incomplete.

- *Honsell-Ronchi (1984):* Scott semantics;
- *Bastonero-Gouy (1996):* stable semantics;

- A **po-model** is a pair $(D, \leq)$, where $D$ is a model and $\leq$ is a nontrivial partial ordering on $D$ making monotone the application operator.

- Selinger (1996) The order-completeness problem asks whether the class $\text{PO}$ of po-models is **theory complete** or not.

  **ANSWER:** Unknown.

  The best we know about order-incompleteness:

**Theorem 15 (Carraro-S. 2013)** There exists a finitely axiomatizable $\lambda$-theory $T$ such that, for every po-model $(D, \leq)$,

$$E_q(D) \supseteq T \Rightarrow (D, \leq) \text{ has infinite connected components}$$

and the connected component of the looping term $\Omega$ is a singleton set.
The Order-Incompleteness Problem

**Theorem 16** (Hagemann 73, Selinger 96, Coleman 96-97) Let $\mathcal{V}$ be a variety of algebras. Then the following conditions are equivalent:

1. $\mathcal{V}$ is $n$-permutable for some $n \geq 2$ (i.e., $\theta \lor \phi = \theta \circ \phi \circ \theta \circ \cdots \circ \phi$ ($n$-times)).

2. There exist a natural number $n \geq 2$ and ternary terms $p_1, \ldots, p_{n-1}$ in the type of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the following Mal’cev identities:

   $$
   x = p_1(x, y, y);
   $$

   $$
   p_i(x, x, y) = p_{i+1}(x, y, y) \quad (i = 1, \ldots, n-2);
   $$

   $$
   p_{n-1}(x, x, y) = y.
   $$

3. Every $T_0$-topological algebra in $\mathcal{V}$ is $T_1$.

4. Every $T_0$-topological algebra in $\mathcal{V}$ is $T_1$ and sober.

5. Every algebra in $\mathcal{V}$ is unorderable.

6. Every compatible preorder on an algebra in $\mathcal{V}$ is symmetric (and thus a congruence).

The order incompleteness problem is equivalent to find an $n$-permutable variety of combinatory algebras.
The Order-Incompleteness Problem

In the case a variety $\mathcal{V}$ has two constants 0 and 1, the Mal’cev identities give:

\[
\begin{align*}
0 &= p_1(0, 1, 1); \\
p_i(0, 0, 1) &= p_{i+1}(0, 1, 1) \quad (i = 1, \ldots, n - 2); \\
p_{n-1}(0, 0, 1) &= 1.
\end{align*}
\]

If we define the unary term operations $f_i(x) = p_i(0, x, 1)$, then the above identities can be written as follows:

\[
\begin{align*}
0 &= f_1(1); \\
f_i(0) &= f_{i+1}(1) \quad (i = 1, \ldots, n - 2); \\
f_{n-1}(0) &= 1. \quad (1)
\end{align*}
\]

This suggests the following theorem:

**Theorem 17** Let $\mathcal{V}$ be a variety with two constants 0 and 1. Then the constants 0 and 1 are incomparable in all ordered algebras in $\mathcal{V}$ if, and only if, there exist a natural number $n \geq 2$ and unary terms $f_1, \ldots, f_{n-1}$ in the type of $\mathcal{V}$ such that the identities (1) hold in $\mathcal{V}$. 
The Order-Incompleteness Problem

In the case a variety $\mathcal{V}$ has a constant 0, then we can relativise the Mal’cev identities of Theorem ?? as follows:

$$0 = p_1(0, y, y);$$
$$p_i(0, 0, y) = p_{i+1}(0, y, y) \quad (i = 1, \ldots, n - 2);$$
$$p_{n-1}(0, 0, y) = y.$$

If we define the binary term operations $s_i(y, x) = p_i(0, x, y)$, then the above identities can be written as follows:

$$0 = s_1(x, x)$$
$$s_i(x, 0) = s_{i+1}(x, x) \quad (i = 1, \ldots, n - 2);$$
$$s_{n-1}(x, 0) = x.$$
Relaxing the Order-Incompleteness Problem

**Theorem 18** (Carraro-S. 2013) Let $\mathcal{V}$ be a variety of algebras with 0. Then the following conditions are equivalent:

1. $\mathcal{V}$ is $n$-subtractive for some $n \geq 2$, that is, there exist $n \geq 2$ and binary terms $s_1, \ldots, s_{n-1}$ such that $\mathcal{V}$ satisfies the Mal’cev identities:
   \[
   0 = s_1(y, y);
   s_i(y, 0) = s_{i+1}(y, y) \quad (i = 1, \ldots, n - 2);
   s_{n-1}(y, 0) = y.
   \]

2. Every $T_0$-topological algebra in $\mathcal{V}$ is $T_1$-separated in 0.
3. Every algebra in $\mathcal{V}$ is 0-unorderable.

**Theorem 19** 2-subtractivity is consistent with lambda calculus.

**Equational Consistency Problem** (Honsell-Plotkin 2006) asks whether

$(\forall E \text{ finite set of identities})[(E \cup \lambda \beta \text{ consistent}) \rightarrow (\exists D \in \text{Scott}) \ D \models E]?$

**ANSWER:** No
**Separability in $n$-permutable varieties: $n$-step Hausdorff**

**Theorem 20** Kearnes-Sequeira (2002) *Every* $n$-*permutable variety of algebras* *is* $\lfloor n/2 \rfloor$-*step Hausdorff.*

Let $X$ be a topological space. For every $a \in X$, we define:

1. $\Gamma^a_0 = \emptyset$;

2. $\Gamma^a_{i+1} = \{ b : \exists$ open $U, V$ with $a \in U, \ b \in V$ and $U \cap V \subseteq \Gamma^a_i \}$.

**Definition 3** Coleman(1997) $X$ *is* $n$-*step Hausdorff* *if* $\Gamma^a_n = A/\{a\}$ *for all* $a \in X$.

$n$-step Hausdorff implies $T_1$.

$n$-step Hausdorff implies $k$-step Hausdorff for every $k \geq n$.

1-step Hausdorff is equivalent to $T_2$.

A variety of algebras is $n$-step Hausdorff if every topological algebra in the variety is $n$-step Hausdorff.
Separability in $n$-subtractive varieties

Axioms of $n$-subtraction:

\[
0 = s_1(y, y); \\
\quad s_i(y, 0) = s_{i+1}(y, y) \quad (i = 1, \ldots, n-2); \\
\quad s_{n-1}(y, 0) = y.
\]

The rank $r(y) = \min\{k : s_k(y, 0) \neq 0\}$ \quad ($y \neq 0$).

The rank $r(y)$ exists and $1 \leq r(y) \leq n - 1$.

Let $A$ be an $n$-subtractive $T_0$-topological algebra. Define the opens

\[
R_i = \{a : \kappa(a) \leq i\} = \bigcup_{1 \leq j \leq i} \{a : s_j(a, 0) \neq 0\}
\]

Then

\[
R_0 = \emptyset; \quad R_i \subseteq R_{i+1}; \quad R_{n-1} = A/\{0\}.
\]
Define the opens
\[ \Sigma_i = \{ a : (\exists U, V \text{ opens}) \ a \in U, \ 0 \in V \text{ and } U \cap V \subseteq R_{i-1} \} \quad (1 \leq i \leq n) \]

Then we have:

- \( \Sigma_1 = \{ a : a \text{ and } 0 \text{ are } T_2\text{-separated}\} \);
- \( R_{i-1} \subseteq \Sigma_i \subseteq \Sigma_{i+1} \);
- \( \Sigma_n = A \setminus \{0\} \).
Theorem 21 Let $A$ be an $n$-subtractive $T_0$-topological algebra. Then we have:

$$R_i \subseteq \Sigma_i = \{a : (\exists U, V \text{ opens}) \ a \in U, \ 0 \in V \text{ and } U \cap V \subseteq R_{i-1}\}.$$

**Proof**: We show that $a \in \Sigma_{r(a)}$. Since $s_{r(a)}(a, 0) \neq 0$, then there exists an open neighbourhood $W$ of $s_{r(a)}(a, 0)$ such that $0 \notin W$. By the continuity of $s_{r(a)}$ there exist two open neighbourhoods $U$ and $V$ of $a$ and 0 respectively such that

$$s_{r(a)}(U, V) \subseteq W.$$

If $r(a) = 1$ and there exists $b \in U \cap V$, then $0 = s_1(b, b) \in W$, contradicting the hypothesis on $W$. Then $V \cap U = \emptyset$; thus $a$ and 0 are $T_2$-separated, and $a \in \Sigma_1$.

If $r(a) > 1$, for every $b \in U \cap V$ we have that $s_{r(a)}(b, b) \in W$, that implies

$$s_{r(a)-1}(b, 0) = s_{r(a)}(b, b) \neq 0.$$

This means that the rank of $b$ is less than the rank of $a$ for every $b \in U \cap V$. Then $U \cap V \subseteq R_{r(a)-1}$, so that $a \in \Sigma_{r(a)}$. 
Separability in $n$-subtractive varieties

**Proposition 5** Every $n$-subtractive $T_0$-topological algebra is $n-1$-step Hausdorff in 0.

*Proof:* We show by induction that

$$
\Sigma_i \subseteq \Gamma_i^0 = \{ b : \exists \text{ open } U, V \text{ with } b \in U, \ 0 \in V \text{ and } U \cap V \subseteq \Gamma_i^{0-1} \}
$$

for all $1 \leq i \leq n$. For $i = 0$ the result is trivial.

\[
\begin{align*}
\Sigma_{i+1} &= \{ a : \exists \text{ open } U, V \text{ with } a \in U, \ 0 \in V \text{ and } U \cap V \subseteq R_i \} \quad \text{by definition} \\
\subseteq &\quad \{ a : \exists \text{ open } U, V \text{ with } a \in U, \ 0 \in V \text{ and } U \cap V \subseteq \Sigma_i \} \quad \text{by } R_i \subseteq \Sigma_i \\
\subseteq &\quad \{ a : \exists \text{ open } U, V \text{ with } a \in U, \ 0 \in V \text{ and } U \cap V \subseteq \Gamma_i^0 \} \quad \text{by induction} \\
= &\quad \Gamma_{i+1} \quad \text{by definition}
\end{align*}
\]

The conclusion follows because $R_{n-1} \subseteq \Sigma_{n-1} \subseteq \Gamma_{n-1}^0$ and $R_{n-1} = A \setminus \{0\}$. 
Let \((X, \tau)\) be a topological space and \(a \in X\).

A sequence \(Y\) of length an ordinal \(\alpha\) of open sets is called an \(a\)-sequence if

\[ Y_0 = \emptyset; \ Y_i \subseteq Y_{i+1} \text{ and } a \notin Y_i \text{ for every } i < \alpha. \]

For every \(i \geq 1\) define the opens

\[ \Sigma_{i,Y}^a = \{ b : (\exists U, V \text{ opens}) \ a \in U, \ b \in V \text{ and } U \cap V \subseteq Y_{i-1} \} \]

\(X\) is \(\beta\)-step \(Y, a\)-Hausdorff if \(\beta\) is the least ordinal satisfying \(\Sigma_{\beta,Y}^a = X \setminus \{a\}\).

\(X\) is \(\alpha\)-step \(Y, a\)-Hausdorff if \(\bigcup_{\beta \geq 1} \Sigma_{\beta,Y}^a = X \setminus \{a\}\).

**Proposition 6** If \(Y_i \subseteq \Sigma_{i,Y}^a\) then \(\Sigma_{i,Y}^a \subseteq \Gamma_i^a\).

**Corollary 2** If \(Y_i \subseteq \Sigma_{i,Y}^a\) for every \(i\), and \(X\) is \(n\)-step \(Y\)-Hausdorff, then \(X\) is \(n\)-step Hausdorff.

What are the \(a\)-sequences \(Y\) satisfying \(Y_i \subseteq \Sigma_{i,Y}^a\) for every \(i\)?