Finding a Forest in a Tree

Giorgio Bacci[1], Marino Miculan[2], and Romeo Rizzi[3]

[1] Department of Computer Science, Aalborg University, Denmark
grbacci@cs.aau.dk
[2] Dept. of Mathematics and Computer Science, University of Udine, Italy
marino.miculan@uniud.it
[3] Dept. of Computer Science, University of Verona, Italy. romeo.rizzi@univr.it

Abstract. In many models for mobile and distributed computations, agents and processes can be nested inside ambients, domains, networks, etc. Thus, the global state is a tree-like structure, which evolves according to some (sub)tree rewriting rules. In order to applying a subtree rewriting rule, we have to find a matching of the rule redex within the global state. In this paper we address this problem, namely: how to find a forest inside an unordered tree, with no overlaps? We show that the problem is NP-complete in general but, using the theory of Fixed Parameter Tractability, we prove that the exponential explosion depends only on the width of the forest to be found, and not on the size of the global tree (i.e., the system state). In most practical cases, the forest width is constant and small (e.g. ≤ 3), hence our results show that the problem is feasible in many situations of interest.

1 Introduction

Hierarchical structures are always been used for representing computational aspects like scoping, containment, security, locations, mobility, semi-structured data, etc. Specific languages have been developed to this end, the paradigmatic examples being Mobile Ambients and its derivations [5]. In these calculi, systems are composed by agents, possibly nested to form an *unordered tree* (possibly with other links). System evolution is governed by *rewriting rules* of the form $L(\boldsymbol{X}) \Rightarrow R(\boldsymbol{X})$, where L and R are unordered forests with "holes" in $\boldsymbol{X}$. Applying such a rule to a system G means finding a *context* $C(\_)$ and *parameters* $\boldsymbol{D}$ such that $G = C(L(\boldsymbol{D}))$; then L (the *redex*) is replaced by R (the *reactum*), transforming the system G into $G' = C(R(\boldsymbol{D}))$. A good example is given by the following rule from CaSPiS [4], a session-centered calculus with nesting:

$$C(s.P, \bar{s}.Q) \Rightarrow C(s.P|r \triangleright P, r \triangleright Q) \qquad (r \text{ fresh}) \qquad \text{(ServiceSync)}$$

where $C(\_,\_)$ is a suitable context with two holes, and P, Q are two generic processes. Actually, C, P, Q are not modified by the rule; hence the *actual* reduction rule is $\langle s.\_1, \bar{s}.\_2\rangle \Rightarrow \langle s.\_1|r \triangleright \_1, r \triangleright \_2\rangle$ whilst C is the context where the redex $\langle s.\_1, \bar{s}.\_2\rangle$ is found and P, Q are the *parameters* the redex is instantiated to. In fact, the redex and the reactum are two forests, composed by two trees each, and the rule is rendered graphically as in Figure 1(a).

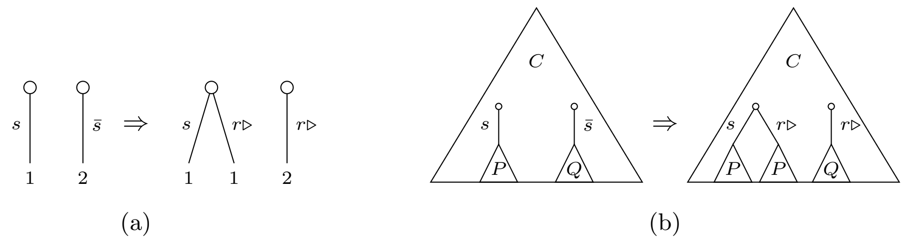


Fig. 1. A parametric rule (a), and its application as forest pattern matching (b).

Thus, in order to apply a parametric rule $\langle L_1, \ldots, L_n \rangle \Rightarrow \langle R_1, \ldots, R_n \rangle$ to a state G, we need to find in the tree of G, an occurrence of each tree L_i, possibly completed by some "grafted" subtrees of G, as in Figure 1(b). Notice that these occurrences cannot overlap (i.e., L_i cannot occur within any L_j nor its parameters). Moreover the trees are unordered, hence some rearrangements in G are possible to accomodate the redex forest.

In this paper, we address precisely this problem, i.e., the *forest pattern matching*: given a forest (the *pattern*) and an unordered tree (the *target*), how to match each of the trees of the pattern within the target (with no overlaps), singling out the subtrees that form the parameters for the pattern? This problem, which we will define formally in Section 2, arises whenever we rewrite unordered hierarchical structures; e.g., implementing an abstract machine for many calculi and languages for distributed systems requires an algorithm for this problem. The problem is at the core of the implementation of general graph-based metalanguages dealing explicitly with notions of containment, like Milner's Bigraphical Reactive Systems [14]. These graph-based formalisms are often used for modeling service-oriented architectures, autonomic systems, and cloud computing (e.g., bigraphs have been recently used for the design and prototyping of multi-agent systems [12]). Another application is semi-structured data transformation, *á la* XSLT on XML; e.g., some XPath queries can be reduced to the forest matching problem.

Our first result is that the forest pattern matching is NP-complete; we will prove it in Section 3 by means of a reduction from 3-Sat. However, this reduction points out the real source of time-complexity: the request that pattern trees are not overlapping in the target. Luckily, this aspect can be approached using Downey and Fellows' parameterized complexity theory [7]: in Section 4 we show that this combinatorial explosion does not depend on the size of the target tree, but only on the pattern width (i.e., the number of trees). As a consequence, the complexity of applying a set of parametric rules to a system is exponential in the maximum width of the rules (which is fixed, usually) and not on the size of the system (which varies during its evolution). Remarkably, in most real cases, rule width is small: e.g., for Ambients, CaSPiS, etc, it is no more than 3. As a side result, we introduce the new *rainbow antichain* problem, which is NP-complete but fixed/parameter tractable; we think that this problem can be a useful tool also for other complexity analysis and reductions of problems about trees.

Concluding remarks and some directions for future work are in Section 5.

2 Labeled Trees, Forest Patterns, and Matches

In this section we define the forest pattern matching problem with no overlaps. As a first step, we define edge-labeled unordered trees, adopting the syntax of ambient calculus without actions [5], and extending it to (linear) context trees.

Let m, n range over an enumerable set Λ of labels, and x, y, z over an enumerable set Ξ of variables. Finite sets of variables are ranged over by X, Y, Z. The set of terms is the set of labeled context trees, finitely branching and of finite depth, where variables are interpreted as leafs where other trees can be grafted. We denote by $T(X), S(X)$ trees whose variables are in X. The syntax of these trees is defined by the following grammar.

Syntax of context trees

$T(X) ::=$	$\mathbf{0}$	empty tree
	x	leaf, $x \in X$
	$m[\,T(X)\,]$	labeled tree
	$T(Y) \mid T'(Z)$	siblings, where $X = Y \uplus Z$

We often abbreviate $m[\,\mathbf{0}\,]$ as $m[\,]$, and $T(X)$ as T. We assume that "$|$" associates to the right, i.e. $T \mid T' \mid T''$ is read $T \mid (T' \mid T'')$. Let $lab(T) \subset \Lambda$ be the set of node labels in T, and $vars(T) \subset \Xi$ be the set of the variables occurring in T (obviously, $vars(T(X)) \subseteq X$). If $vars(T) = \emptyset$ we say that T is *ground*, otherwise it is not.

The intuitive interpretation of terms T as *unordered* trees induces an equivalence $T \equiv T'$ which is the minimal congruence that includes the commutative monoidal laws for $|$ and $\mathbf{0}$. This relation, similar to ambient calculus congruence, can be axiomatized as follows.

Structural congruence on context trees

$$\frac{}{T \equiv T}\ \textsf{(refl)} \qquad \frac{T \equiv T'}{T' \equiv T}\ \textsf{(symm)} \qquad \frac{T \equiv T' \qquad T' \equiv T''}{T \equiv T''}\ \textsf{(trans)}$$

$$\frac{T \equiv T'}{T \mid T'' \equiv T' \mid T''}\ \textsf{(sibl)} \qquad \frac{T \equiv T'}{m[\,T\,] \equiv m[\,T'\,]}\ \textsf{(rooting)}$$

$$\frac{}{T \mid T' \equiv T' \mid T}\ \textsf{(comm)} \qquad \frac{}{T \mid (T' \mid T'') \equiv (T \mid T') \mid T''}\ \textsf{(assoc)} \qquad \frac{}{T \mid \mathbf{0} \equiv T}\ \textsf{(nil)}$$

The axiomatization of structural congruence is adequate with respect to the semantic for unordered trees: $T \equiv T'$ iff T and T' represent the same tree structure (obviously, where siblings are not ordered). Moreover, if $T \equiv T'$ then $lab(T) = lab(T')$ and $vars(T) = vars(T')$.

Given two tree terms $T(X)$, $S(Y)$ with X, Y disjoint, we define term substitution, written $T\{S/x\}$, as usual: the occurrence x in T is replaced by the term S. For $x \in vars(T)$, $vars(T\{S/x\}) = (vars(T) \setminus \{x\}) \cup vars(S)$. Simultaneous substitution $T\{S_1/x_1, \ldots, S_k/x_k\}$ is defined by the substitution composition

$T\{S_1/x_1\}\cdots\{S_k/x_k\}$, where $x_1,\ldots,x_n$ are supposed to be pairwise distinct; we denote it by $T\{\boldsymbol{S}/\boldsymbol{x}\}$.

Lemma 1. *If $S_i \equiv S'_i$ for $i \in \{1,\ldots,k\}$, then $T\{\boldsymbol{S}/\boldsymbol{x}\} \equiv T\{\boldsymbol{S'}/\boldsymbol{x}\}$.*

Intuitively, given a tree list $S = S_1,\ldots,S_n$, called a "pattern", searching for a (sub-)match of S in a tree T means to find an occurrence of each $S_1,\ldots,S_n$ within T, without overlaps and possibly by instantiating variables in S_i. This means that we have to decompose T in a subtree C where all S_i can be grafted, and a list of subtrees to be grafted to the leaves of S_i. More formally:

Definition 2. *A* forest matching instance*, denoted by $T \succeq \boldsymbol{S}$, is given by a tree $T(Y)$ (*target*), and a list of trees $\boldsymbol{S}(X) = S_1(X_1),\ldots,S_n(X_n)$ (*pattern*) where X_i are all disjoint and $X = \cup_{i=1}^n X_i$. We say that $\boldsymbol{S}(X)$* matches *in $T(Y)$ if for some* context *$C(Z)$ and* parameters *$\boldsymbol{D} = D_1,\ldots,D_{|X|}$*

$$T \equiv \big(C\{\boldsymbol{S}/\boldsymbol{z'}\}\big)\{\boldsymbol{D}/\boldsymbol{x}\} \qquad \textit{where } \boldsymbol{z'} \subseteq Z$$

A match for $T \succeq \boldsymbol{S}$ is denoted by $C,\boldsymbol{D} \models T \succeq \boldsymbol{S}$, and we write $\models T \succeq \boldsymbol{S}$ if $C,\boldsymbol{D} \models T \succeq \boldsymbol{S}$ for some C, $\boldsymbol{D}$.

Proposition 3. *If $\models T \succeq \boldsymbol{S}$ and $\models S_i \succeq \boldsymbol{Q}$, for some $1 \le i \le n$ and $n = |\boldsymbol{S}|$, then $\models T \succeq \boldsymbol{Q}$; and in particular $\models T \succeq S_1,\ldots,S_{i-1},\boldsymbol{Q},S_{i+1},\ldots,S_n$.*

Proof. We have to prove that if $\models T \succeq \boldsymbol{S}$ and $\models S_i \succeq \boldsymbol{Q}$, for some $1 \le i \le n$ and $n = |\boldsymbol{S}|$, then $\models T \succeq S_1,\ldots,S_{i-1},\boldsymbol{Q},S_{i+1},\ldots,S_n$.

Since $\models T \succeq \boldsymbol{S}$ and $\models S_i \succeq \boldsymbol{Q}$ there exist contexts C, C' and parameters $\boldsymbol{D}$, and $\boldsymbol{D'}$ such that

$$\begin{aligned} T &\equiv (C\{S_1/x_1,\ldots,S_n/x_n\})\{\boldsymbol{D}/\boldsymbol{Z}\} && \text{for some } x_1,\ldots,x_n \in \mathit{vars}(C)\\ S_i &\equiv (C'\{\boldsymbol{Q}/\boldsymbol{X'}\})\{\boldsymbol{D'}/\boldsymbol{Z'}\} && \text{for some } X' \subseteq \mathit{vars}(C') \end{aligned}$$

Without loss of generality, suppose X' disjoint from $\{x_1,\ldots,x_n\}$, otherwise a variable renaming can be applied. Now, by an easy replacement of S_i and some rearrangements on the context and parameters, we obtain

$$\begin{aligned} T &\equiv (C\{S_1/x_1,\ldots,S_i/x_i,\ldots,S_n/x_n\})\{\boldsymbol{D}/\boldsymbol{Z}\}\\ &\equiv (C\{S_1/x_1,\ldots,(C'\{\boldsymbol{Q}/\boldsymbol{X'}\})\{\boldsymbol{D'}/\boldsymbol{Z'}\}/x_i,\ldots,S_n/x_n\})\{\boldsymbol{D}/\boldsymbol{Z}\}\\ &\equiv ((C\{C'/x_i\})\{S_1/x_1,\ldots,\boldsymbol{Q}/\boldsymbol{X'},\ldots,S_n/x_n\})\{\boldsymbol{D},\boldsymbol{D'}/\boldsymbol{Z},\boldsymbol{Z'}\} \end{aligned}$$

This states that $(C\{C'/x_i\},(\boldsymbol{D},\boldsymbol{D'}))$ is a match for $S_1,\ldots,S_{i-1},\boldsymbol{Q},S_{i+1},\ldots,S_n$, in T hence, $\models T \succeq S_1,\ldots,S_{i-1},\boldsymbol{Q},S_{i+1},\ldots,S_n$.

Similarly, we can prove that $(C\{S_i/x_1,\ldots,C'/x_i,\ldots,S_n/x_n\},\boldsymbol{D'})$ is a match for $\boldsymbol{Q}$ in T, hence $\models T \succeq \boldsymbol{Q}$ holds too. □

However, not all possible trees are interesting in patterns. First, the empty tree $\mathbf{0}$ matches all possible targets, since $T \equiv (T \mid x)\{\mathbf{0}/x\}$. Also, a tree composed by a sole variable trivially matches all subtrees; in fact, x has as many matches in T as nodes in T. A subtler situation happens to patterns with "unguarded" variables, e.g. of the form $x \mid R$. Intuitively, this pattern matches an occurrence of R "beside anything, possibly nothing". Thus, the unguarded variable allows to "move" subtrees between context and parameters in a match, yielding many redundant variants of the same. As an example, let $T = m[\,\mathbf{0}\,] \mid n[\,k[\,\mathbf{0}\,]\,]$ be the target and $S = x \mid m[\,\mathbf{0}\,]$ the pattern; then, we have three different matches $((y, n[\,k[\,\mathbf{0}\,]\,]),\ (n[\,y\,], k[\,\mathbf{0}\,]),\ (n[\,k[\,\mathbf{0}\,]\,], \mathbf{0}))$, despite $m[\,\mathbf{0}\,]$ occurs only once in T. Finally, sibling variables $x|y$ in patterns can be replaced by a single one z, because $(x \mid y)\{D_1/x, D_2/y\} = (z)\{D_1 \mid D_2/z\}$.

In all the cases above, a single occurrence of a pattern in a target yields many matches which are all redundant variants of the same. In order to avoid this plethora of redundant matches, we restrict our attention to a class of patterns, which we call *solid* after [11].

Definition 4. *A pattern $\boldsymbol{S}(X) = S_1(X_1), \ldots, S_n(X_n)$ is* solid *if for $1 \leq i \leq n$: $S_i \not\equiv \mathbf{0}$, for no $x \in X$ and S' it is $S_i \equiv x|S'$, and no two variables $x, y \in X_i$ are siblings, that is, $x \mid y$ cannot occur in S_i (up to $\equiv$).*

We can prove that any matching instance can be reduced to a matching instance whose pattern is solid. Let us define a function *solid* over forest patterns (i.e., tree lists), which drops empty trees and unguarded variables, and collapses sibling variables in one:

Transformation into solid patterns

$$solid(\epsilon) = \epsilon \qquad\qquad solid(T, \boldsymbol{S}) = \begin{cases} solid(\boldsymbol{S}) & \text{if } T \equiv \mathbf{0} \\ solid(Q, \boldsymbol{S}) & \text{if } T \equiv x \mid Q \\ solid(sld(T), \boldsymbol{S}) & \text{otherwise} \end{cases}$$

$$\begin{array}{ll} sld(\mathbf{0}) = \mathbf{0} & del(\mathbf{0}) = \mathbf{0} \\ sld(x \mid T) = x \mid del(T) & del(x \mid T) = del(T) \\ sld(m[T] \mid S) = m[sld(T)] \mid sld(S) & del(m[T] \mid S) = m[sld(T)] \mid del(S) \end{array}$$

Solid patterns enjoy the following properties.

Proposition 5. *The following statements hold:*

(a) ***no empty trees:*** $\models T \succeq \mathbf{0}, \boldsymbol{S} \iff \models T \succeq \boldsymbol{S}$;
(b) ***no sibling variables:*** $\models T \succeq x \mid y \iff \models T \succeq x$;
(c) ***no unguarded variables:*** $\models T \succeq x \mid S \iff \models T \succeq S$.

Due to the above, solid patterns suffice for checking match existence.

Lemma 6. $\models T \succeq solid(T)$ *if and only if* $\models solid(T) \succeq T$.

Proof. It is an easy application of proposition 5 and proposition 3. In fact, proposition 3 ensures that it suffices to check $\models \boldsymbol{T} \succeq \boldsymbol{T'} \iff \models \boldsymbol{T'} \succeq \boldsymbol{T}$ for each equation $\boldsymbol{T} = \boldsymbol{T'}$ defining *solid*. This is just a straightforward application of (a), (b), (c) of Proposition 5. □

Theorem 7. $\models T \succeq solid(\boldsymbol{S})$ *if and only if* $\models T \succeq \boldsymbol{S}$.

Proof. It follows directly from lemma 6 and proposition 3. ☐

Actually, all matches against a pattern $\boldsymbol{S}$ can be obtained from matches against $solid(\boldsymbol{S})$.

3 NP-completeness of Forest Pattern Matching

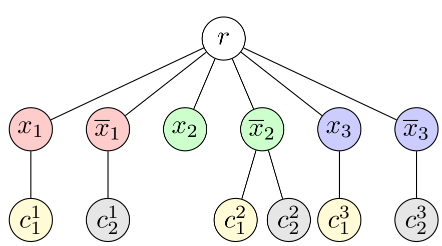
The main result in this section is that the problem of finding a sub-pattern matching of a tree list pattern $\boldsymbol{S} = S_1, \dots, S_n$ for a tree T is NP-complete. We show it by a reduction from 3-Sat [6]. Although the reduction can be done directly, we do it in two steps, introducing an intermediate problem which points out the actual source of time-complexity hardness.

Let us define the intermediate problem first, called RainbowAntichain. An instance of RainbowAntichain is a tree $\mathcal{T}(\mathcal{V}, \mathcal{E})$ with nodes $\mathcal{V}$ and edges $\mathcal{E}$, and a finite set $\mathcal{P}$ of colors, said palette. Some of the nodes in $\mathcal{T}$ have been colored with colors taken from the palette $\mathcal{P}$. Note that the same color can be associated with different nodes, and each node can be associated with more than one color. RainbowAntichain asks whether there exists a *rainbow antichain* $\mathcal{R} \subseteq \mathcal{V}$ in $\mathcal{T}$, i.e., a subset of nodes such that for no pair $u, v \in \mathcal{R}$ of distinct nodes u is an ancestor of v (hence, it is an *antichain*) and where each color $c \in \mathcal{P}$ has exactly one representative in $\mathcal{R}$ (hence, it is *colorful* w.r.t. $\mathcal{P}$).

Theorem 8. RainbowAntichain *is NP-complete.*

Proof. RainbowAntichain is in NP, since, given a set of nodes $\mathcal{R}$, checking whether $\mathcal{R}$ is a rainbow antichain for $\mathcal{T}$ can be done in polynomial time by a breadth-first visit of $\mathcal{T}$, and for each $v \in \mathcal{R}$ found, first increase the node counter nc, then the color counter $p[i]$ $(1 \leq i \leq |\mathcal{P}|)$ if v has color $c_i \in \mathcal{P}$. The check fails whether $nc > |\mathcal{P}|$ or $p[j] = 0$ for some $1 \leq j \leq |\mathcal{P}|$, otherwise $\mathcal{R}$ is a rainbow antichain for $\mathcal{T}$.

Let $C = \{c_1, \dots, c_m\}$ be an instance of 3-Sat on variables $\{x_1, \dots, x_n\}$. From C we define a colored tree $\mathcal{T}$ as follows. Let r be the root node which is left uncolored. For each variable x_i let x_i and $\overline{x}_i$ be child nodes of r, and color them with a fresh color c_{x_i}, distinct for each variable. For each clause $c_j \in C$, let c_j^1, c_j^2, c_j^3 be children nodes of l_i in T if c_j contains l_i as negated, and assign to each of them a fresh color c_{c_j}, distinct for each clause. An example of construction for $c_1 = (\overline{x}_1 \vee x_2 \vee \overline{x}_3)$, $c_2 = (x_1 \vee x_2 \vee x_3)$ is shown below.



Let φ be a truth assignment satisfying the formula C. By construction, selecting only literal nodes l_i which are satisfied by φ, we obtain a rainbow antichain $\mathcal{R}'$ in $\mathcal{T}$ for the palette $\{c_{x_i} : 1 \leq i \leq n\}$. Now, we extend $\mathcal{R}'$ to $\mathcal{R}$ adding all clause nodes which are not children of a element in $\mathcal{R}'$. Such $\mathcal{R}$ is clearly an antichain for $\mathcal{T}$, but we must ensure that is colorful and no more than one representative per color is taken. To do this, it suffices to prove that $\mathcal{R}$ is colorful, indeed if a color occurs more than once in $\mathcal{R}$ we remove the others. By hypothesis, each clause c_j is satisfied by φ, hence c_j has at least one literal l_i such that $\varphi(l_i) = \mathbf{T}$. By construction of $\mathcal{T}$, there exist a node c_j^k $(1 \leq k \leq 3)$ child of $\overline{l_i}$, hence already in $\mathcal{R}$. This holds for all clauses c_j, hence $\mathcal{R}$ is colorful.

Conversely, let $\mathcal{R}$ be a rainbow antichain for $\mathcal{T}$. Let $\varphi\colon \{x_1, \ldots, x_n\} \to \mathit{Bool}$ be defined by $\varphi(x_i) = \mathbf{T}$ if x_i is a node in $\mathcal{R}$, and $\varphi(x_i) = \mathbf{F}$ if x_i is a node not in $\mathcal{R}$. Since $\mathcal{R}$ has exactly one representative per color, no opposite literals are in $\mathcal{R}$, hence φ is a truth assignment for C. By colorfulness of $\mathcal{R}$, for all colors c_{c_j} $(1 \leq j \leq m)$ there exists a node $c_j^k \in \mathcal{R}$ $(1 \leq k \leq 3)$ such that c_j^k has color c_{c_j}. By construction of $\mathcal{T}$, each $c_j^k \in \mathcal{R}$ is a children of a literal node $l_i \notin \mathcal{R}$, and moreover the clause c_j contains $\overline{l_i}$. Since $l_i \notin \mathcal{R}$, by definition $\varphi(\overline{l_i}) = \mathbf{T}$, hence $\varphi(c_j) = \mathbf{T}$. This holds for all $1 \leq j \leq m$, hence φ satisfies C. ☐

It is easy to see that an instance $\mathcal{T}, \mathcal{P} = \{c_1, \ldots, c_n\}$ of RainbowAntichain can be reduced to a forest pattern matching problem, namely, the one that solves $\models T \succeq (c_1[\,x_1\,], \ldots, c_n[\,x_n\,])$, for a suitable tree term T defined upon $\mathcal{T}$. This states that the forest pattern matching problem is NP-complete. Formally,

Theorem 9. *The forest pattern matching problem is NP-complete.*

Proof. Given a match $(C, \boldsymbol{D})$ for $T \succeq \boldsymbol{S}$, checking that $T \equiv (C\{\boldsymbol{S}/\boldsymbol{X}\})\{\boldsymbol{D}/\boldsymbol{Z}\}$ corresponds to a tree isomorphism test, which is in P from [9, 10].

Let a colored tree $\mathcal{T}$ and a palette $\mathcal{P} = \{c_1, \ldots, c_n\}$ be and instance of RainbowAntichain. Let us transform $\mathcal{T}$ into a tree term T as follows. If $\mathcal{T}$ is a single node v (a leaf) T is $m[\,\mathbf{0}\,]$, where $m = c$ if v has color c, otherwise $m = *$, a fresh name not in $\mathcal{P}$ denoting an uncolored node. If $\mathcal{T}$ has root r and $\mathcal{T}_1, \ldots, \mathcal{T}_k$ are the (children) subtrees of r, T is $m[\,T_1 \mid \cdots \mid T_k\,]$, where m is as above for r, and $T_1, \ldots, T_k$ are transformed trees of $\mathcal{T}_1, \ldots, \mathcal{T}_k$.

Suppose $(C, \boldsymbol{D})$ be a match for $T \succeq (c_1[\,x_1\,], \ldots, c_k[\,x_n\,])$. In C, each $c_i[\,x_i\,]$ is grafted into a variable $z_i \in \mathit{vars}(C)$. Since variables can appear in terms only as leaves, in the transformation $\mathcal{T}$ of T, we have found a rainbow antichain for $\mathcal{P}$, since the matching pattern has all the colors in $\mathcal{P}$ exacty once.

Assume that $\mathcal{T}$ has a rainbow antichain $\mathcal{R}$. In order to recover context C and parameters $\boldsymbol{D}$, which are a match for $T \succeq (c_1[\,x_1\,], \ldots, c_k[\,x_n\,])$, it suffices to apply the construction explained above with some adjustments: we obtain C applying the transformation from the root of T, but if a node in $\mathcal{R}$ is reached it is transformed by a fresh variable z_i $(1 \leq i \leq n)$ one for each element in $\mathcal{R}$; D_j's are recovered applying the original transformation starting from the subtrees rooted at the children of nodes in $\mathcal{R}$. It is straightforward to prove that $T \equiv (C\{c_1[\,x_1\,]/z_1, \ldots, c_k[\,x_n\,]/z_n\})\{\boldsymbol{D}/\boldsymbol{X}\}$, for $X = \{x_1, \ldots, x_n\}$. ☐

The previous NP-reduction proves that the complexity hardness is merely due to finding a rainbow antichain in the given target, which corresponds to locate the list of trees of the pattern so that they are not in overlap in the target tree.

4 Tractability for Bounded Width

Despite the NP-completeness result from Theorem 9, in this section we give a tractability result for the forest pattern matching problem, when the number of trees in the matching pattern is bounded by a (relatively small) constant h and their roots have at most k children, for some (relatively small) constant k. We propose a parameterized algorithm whose running time is $f(h,k) + O(n_s \cdot n_t^{3/2})$, for n_t and n_s the number of nodes in the target and pattern, respectively. This proves that the forest pattern matching is a fixed-parameter tractable problem (FPT) (we refer to [7] for the formal definition of this complexity class).

In presenting the algorithm we switch from edge-labelled tree terms to a more convenient node-labelled tree representations of them. This translation eases the description of the proposed algorithm and provides a closer connection between the concept of (labelled) subtree isomorphism and tree pattern matching. Formally, a (rooted) node-labelled tree $\mathcal{T}(\mathcal{V}, \mathcal{E}, \mathit{label})$ is a triple, where $\mathcal{V}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of (oriented) edges, and $\mathit{label} \colon \mathcal{V} \to \Lambda^+ \times \{\mathit{op}, \mathit{cl}\}$ is a function associating to each node a label $m \in \Lambda^+ = \Lambda \uplus \{*\}$, and a flag *op* or *cl*. In the following we often abbreviate $\mathcal{T}(\mathcal{V}, \mathcal{E}, \mathit{label})$ with $\mathcal{T}$ and if $\mathit{label}(v) = (m, t)$ we say that v is m-labelled and *open* (resp. *closed*) if $t = \mathit{op}$ (resp. $t = \mathit{cl}$); $\mathit{root}(\mathcal{T})$ denotes the root node; $Ch(v)$ denotes the set of children of v; and $\mathcal{T} \restriction v$ denotes the subtree of $\mathcal{T}$ rooted at a node $v \in \mathcal{V}$.

Definition 10. *Given an edge-labelled tree term* $T \equiv m_1[T_1] \mid \cdots \mid m_n[T_n] \mid x_1 \mid \cdots \mid x_k$*, for* $n, k \geq 0$*, a node-labelled tree* $\mathcal{T}(\mathcal{V}, \mathcal{E}, \mathit{label})$ *is said a* graphical representation of T *if the following conditions hold:*

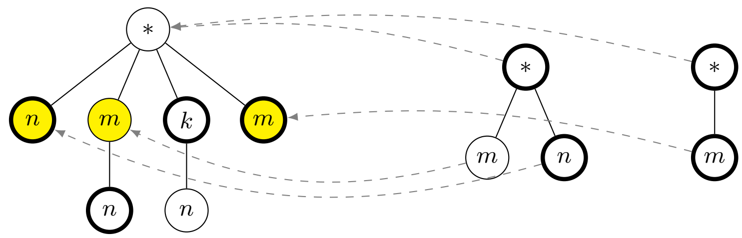
1. *if* $n = 0$ *then* $\mathcal{T}$ *is the empty tree (i.e.* $\mathcal{V} = \emptyset$*);*
2. *if* $n > 0$*, then* $\mathcal{V} = \{r, v_1, \ldots, v_n\} \cup \mathcal{V}'$*,* $\mathcal{E} = \bigcup_i (\{(r, v_i)\} \cup \{(v_i, w) \mid w \in Ch(\mathit{root}(\mathcal{T}_i))\}) \cup \mathcal{E}'$*, and for* $v \in \mathcal{V}$

$$\mathit{label}(v) = \begin{cases} (*, \mathit{op}) & \text{if } v = r \text{ and } k = 0 \\ (*, \mathit{cl}) & \text{if } v = r \text{ and } k > 0 \\ (m_i, t) & \text{if } v = v_i \text{ and } \mathit{label}(\mathit{root}(\mathcal{T}_i)) = (m, t) \\ \mathit{label}_i(v) & \text{if } v \in \mathcal{V}_i \end{cases}$$

where $\mathcal{T}_i(\mathcal{V}_i, \mathcal{E}_i, \mathit{label}_i)$ *be graphical tree representation of* T_i ($1 \leq i \leq m$) *with pairwise disjoint node sets not containing* r *and* v_i *for* $1 \leq i \leq n$, $\mathcal{V}' = \bigcup_i (\mathcal{V}_i \setminus \{\mathit{root}(\mathcal{T}_i)\})$*, and* $\mathcal{E}' = (\bigcup_i \mathcal{E}_i) \cap (\mathcal{V}' \times \mathcal{V}')$*.*

The graphical representation of a tree term is always rooted on a $*$-labelled node and converts m-labelled edges into m-labelled nodes, discarding variables. Note, however, that nodes in $\mathcal{T}$ are open iff they have a variable as a child in its tree term representation. In Figure 2 it is shown an example of translation into the graphical representation.

The following proposition relate the sub-isomorphism on trees with the notion of tree pattern matching on terms, when the pattern is supposed to be solid.



$T = n[\,\mathbf{0}\,] \mid m[\,y_1 \mid n[\,\mathbf{0}\,]\,] \mid k[\,n[\,y_2\,]\,] \mid m[\,\mathbf{0}\,] \mid y_3 \qquad S_1 = m[\,x\,] \mid n[\,\mathbf{0}\,] \qquad S_2 = m[\,\mathbf{0}\,]$

Fig. 2. The forest pattern $\boldsymbol{S} = S_1, S_2$ has a match in T: for $C = z_1 \mid k[\,n[\,y_2\,]\,] \mid z_2 \mid y_3$ and $D = y_1 \mid n[\,\mathbf{0}\,]$, $T \equiv (C\{S_1/z_1, S_2/z_2\})\{D/x\}$. Bold-circled nodes are closed.

Proposition 11. *For a term T and a solid one T', where $\mathcal{T}(\mathcal{V}, \mathcal{E}, label)$ and $\mathcal{T}'(\mathcal{V}', \mathcal{E}', label')$ are their tree representations, respectively, then $\models T \succeq T'$ if and only if there exists $\mathcal{V}'' \subseteq \mathcal{V}$, where $|\mathcal{V}'| = |\mathcal{V}''|$, and $\rho\colon \mathcal{V}' \to \mathcal{V}''$ a one-to-one function such that*

1. *$(u, v) \in \mathcal{E}'$ iff $(\rho(u), \rho(v)) \in \mathcal{E}$;*
2. *if v is m-labelled then $\rho(v)$ is m-labelled, for $m \in \Lambda$;*
3. *if $v \in \mathcal{V}' \setminus \{root(\mathcal{T}')\}$ is closed then $\rho(v)$ is closed and $|Ch(v)| = |Ch(\rho(v))|$.*

Apart (3), the conditions listed in Proposition 11 correspond exactly to require that there exists a subtree isomorphism between $\mathcal{T}$ and $\mathcal{T}'$ on Λ^+-labelled trees (when $*$ acts as a wildcard label). The last condition is required in situations like the following one: choose $T = m[\,n[\,\mathbf{0}\,]\,]$ and $T' = m[\,\mathbf{0}\,]$; T' has no match in T even though, considering their graphical tree representations, there exists a function ρ satisfying conditions (1) and (2).

Proposition 11 induces the definition of the following relation: $\rho \models \mathcal{T} \succeq \mathcal{T}'$ iff there exists ρ satisfying conditions (1–3). Obviously $\models T \succeq T'$ iff $\rho \models \mathcal{T} \succeq \mathcal{T}'$ and $\mathcal{T}$, $\mathcal{T}'$ are graphical representations for T, T', respectively.

Now, let us consider the forest pattern matching problem, that is, when the pattern is a list of arbitrary length $h \geq 0$.

Proposition 12. *Given a term T and a solid (forest) pattern $\boldsymbol{S} = S_1, \ldots, S_h$, where $\mathcal{T}$ and $\boldsymbol{\mathcal{S}} = \mathcal{S}_1, \ldots, \mathcal{S}_h$ are their tree representations (with disjoint node sets), then $\models T \succeq \boldsymbol{S}$ if and only if*

1. *$\rho_i \models \mathcal{T} \succeq \mathcal{S}_i$, for $1 \leq i \leq h$;*
2. *$\mathcal{R} = \{\rho_i(v) \mid v \in Ch(root(\mathcal{S}_i)), 1 \leq i \leq h\}$ is an antichain in $\mathcal{T}$.*

Condition (1) is obvious, and it is due to Proposition 11. Condition (2) states that the children of each $\mathcal{S}_i$-root must be mapped by ρ_i to form an antichain in $\mathcal{T}$; this ensures that the mapping of trees in the pattern are not overlapping in $\mathcal{T}$. Note that, different roots of the pattern can be mapped to the same target node, and that the antichain condition must be satisfied by the roots children nodes only (see Figure 2 for an example).

4.1 A Parameterized Algorithm for Forest Pattern Matching

Proposition 12 offers an alternative characterization for the forest pattern matching problem through which it is easier to provide a parameterized algorithm that solves it, when $h = |\boldsymbol{\mathcal{S}}|$ and $k = \max_i |Ch(root(\mathcal{S}_i))|$ are the chosen parameters. The key idea is to find all possible matches of each $\mathcal{S}_i$ separately, identifying them by coloring nodes in $\mathcal{T}$, and finally search for a rainbow antichain. The proposed algorithm uses the *reduction to kernel size* technique. Formally, the parameterized algorithm solving $\models T \succeq \boldsymbol{S}$ acts in three steps:

1. for each S_i in the pattern, we identify all possible mappings ρ_i satisfying $\rho_i \models \mathcal{T} \succeq \mathcal{S}_i$. These mappings corresponds to tree matches and we identify them by means of colors: each $\mathcal{S}_i$ is associated with a color $f \in \mathcal{F}$, and nodes in $Ch(root(\mathcal{S}_i))$ with colors from the palette $\mathcal{P}_i$ (a color for each node). Palettes are supposed to be disjoint.
2. we bound the size of the returned colored target tree, yielding a *kernel* of size which depends only on the parameters h and k.
3. we perform an exhaustive search for a rainbow antichain on palette $\bigcup_i \mathcal{P}_i$.

Coloring the target tree: By Proposition 11 we know that this corresponds to solving the subtree isomorphism problem for each $\mathcal{S}_i$ in the pattern and ensuring that the closedness property holds (that is, condition (3) in Proposition 11). It is not hard to see that the Matula's algorithm [13] for the subtree isomorphism can be adapted to our aims. Let M be a Boolean matrix of size $n_s \times n_t$, where n_t and n_s are respectively the number of nodes of the target tree and of the pattern (the summation of each node set of the whole tree list). By dynamic programming on $\mathcal{T}$ and $\boldsymbol{\mathcal{S}}$ we can fill M as follows: for each node u in $\boldsymbol{\mathcal{S}}$ and node v in $\mathcal{T}$, $M[u, v] = \mathbf{T}$ if there exists an embedding (respecting node labeling and the closedness property) of $\boldsymbol{\mathcal{S}}\restriction u$ in $\mathcal{T}$ rooted at v, otherwise $M[u, v] = \mathbf{F}$ (see Matula [13] for details on how the matrix M is obtained).

From the matrix M we can define the coloring functions for $\mathcal{T}$. Let $\mathcal{F}$ and $\mathcal{P}_i$, for $1 \leq i \leq h$, be disjoint palettes such that $|\mathcal{F}| = h$ and $|\mathcal{P}_i| = |Ch(root(\mathcal{S}_i))| \leq k$, and $\alpha \colon \bigcup_i \{\mathcal{S}_i\} \to \mathcal{F}$ and $\beta_i \colon Ch(root(\mathcal{S}_i)) \to \mathcal{P}_i$ be bijections associating a color $f \in \mathcal{F}$ with each $\mathcal{S}_i$ in the pattern, and a color $p \in \mathcal{P}_i$ with each children of $root(\mathcal{S}_i)$. We define $\mathcal{V}$-indexed family color sets $color_R(v) \subseteq \mathcal{F}$ and $color_i(v) \subseteq \mathcal{P}_i$, for $1 \leq i \leq h$ as follows:

$$\alpha(\mathcal{S}_i) \in color_R(v) \iff M[root(\mathcal{S}_i), v] \qquad\qquad \beta_i(u) \in color_i(v) \iff M[u, v]$$

Note that nodes may take color from different palettes, indeed a subtree of the target may have a match with more than one tree in the pattern. The family of color sets $color_R$ and $color_i$ enjoy the following property:

Proposition 13. *If $\alpha(\mathcal{S}_i) \in color_R(v)$ then $\bigcup_{u \in Ch(v)} color_i(u) = \mathcal{P}_i$.*

The above proposition says that if a node v in the target has color $\alpha(\mathcal{S}_i)$, then $\mathcal{S}_i$ has a match in $\mathcal{T}$ rooted at v, hence there must exists $C \subseteq Ch(v)$ such that $|C| = |Ch(root(\mathcal{S}_i))|$ and for each $u \in Ch(root(\mathcal{S}_i))$, $\mathcal{S}_i\restriction u$ has a match rooted at a node in C.

Reduction to kernel size: The reduction of $\mathcal{T}$ to kernel size consists in a decoloring procedure that aims at leaving as much nodes as possible completely uncolored in order to remove them from $\mathcal{T}$. Indeed, uncolored nodes have no influence in the detection of a possible rainbow antichain in $\mathcal{T}$.

Before starting with the description of the reduction, we need some technical definitions and notations. We say that a node is c-decolored if we remove c from all its color sets (note that $color_R$ and $color_i$ are disjoint, hence the set deletion of c influences only the right color set). By $\mathcal{T} \setminus v$ we denote the tree obtained from $\mathcal{T}$ removing the node v and such that the children of u are adopted by its parent (if u is the root node we just decolor it).

Definition 14. *Let $\mathcal{T}$ be a tree and u a node. We denote by $\mathsf{fout}(u)$ the* fan-out *of u, defined as $\mathsf{fout}(u) = \sum_{v \in an(u)} |Ch(v)| - 1$, where $an(v)$ is the set of all ancestors of v; and by $\mathsf{fout}(\mathcal{T}) = \max_{v \in \mathcal{V}} \mathsf{fout}(v)$ the maximal fan-out in $\mathcal{T}$.*

Intuitively, $\mathsf{fout}(u)$ is the out-degree of the whole path from u to the root of $\mathcal{T}$.

Lemma 15. *If v is uncolored and $\mathcal{T}$ admits a $\mathcal{P}$-rainbow antichain, then also $\mathcal{T} \setminus v$ has $\mathcal{P}$-rainbow antichain.*

Lemma 16. *If $\mathcal{T}$ has a $\mathcal{P}$-rainbow antichain, then it has one also when u is c-decolored, for color $c \in \mathcal{P}$, if one of the following conditions hold:*
(a) u is an ancestor of v, and both u, v are c-colored;
(b) $\mathcal{T}$ has only c-colored leaves and u is a leaf such that $\mathsf{fout}(u) \geq |\mathcal{P}|$.

Proof. (a) Let $\mathcal{T}$ be a colored tree on palette $\mathcal{P}$, where there exist two nodes u and v, such that u is an ancestor of v and $c \in \mathcal{P}$ is assigned both to u and v. We want to prove that if $\mathcal{T}$ has a rainbow antichain, it continues to have one also if we c-decolor node u. Let $\mathcal{R}$ be a rainbow antichain for $\mathcal{T}$ such that $u \in \mathcal{R}$. Since u belongs to $\mathcal{R}$, for some color $c_\mathcal{R} \in \mathcal{P}$ assigned to u, $\mathcal{R}$ must be rainbow on the palette $\mathcal{P}$. If we decolor u by c, there are two cases. If $c \neq c_\mathcal{R}$, $\mathcal{R}$ continues to be a rainbow antichain for $\mathcal{T}$, conversely, if $c = c_\mathcal{R}$, $\mathcal{R}$ is no more colorful on $\mathcal{P}$, since one of the representative of $\mathcal{P}$ lacks (i.e. c). By hypothesis, u has a c-colored descendant v. It is easy to see that $\mathcal{R}' = (\mathcal{R} \setminus \{u\}) \cup \{v\}$ is still an antichain and moreover it is colorful for $\mathcal{P}$.

(b) Let $\mathcal{T}$ be a colored tree on palette $\mathcal{P}$ such that, all its leaves are colored by $c \in \mathcal{P}$, and v is a leaf in $\mathcal{T}$ for which $\mathsf{fout}(v) \geq |\mathcal{P}|$. We want to prove that if $\mathcal{T}$ has a rainbow antichain, it continues to have one also if we c-decolor v. Let P be the path from the leaf v to the root of $\mathcal{T}$. To each outer-neighbour n_i ($1 \leq i \leq \mathsf{fout}(v)$) of P corresponds a subtree $\mathcal{T}{\restriction}n_i$ with all leaves colored by c, since $\mathcal{T}$ has only c-colored leaves. It is worth noting that all $\mathcal{T}{\restriction}n_i$ are not overlapping with each other, since $\bigcup_i \{n_i\}$ is an antichain for $\mathcal{T}$.

Suppose $\mathcal{R}$ be a rainbow antichain for $\mathcal{T}$ such that $v \in \mathcal{R}$. Since $v \in \mathcal{R}$, for some color $c_\mathcal{R} \in \mathcal{P}$ assigned to v, $\mathcal{R}$ must be rainbow on the palette $\mathcal{P}$. If we c-decolor v, there are two cases. If $c \neq c_\mathcal{R}$, $\mathcal{R}$ continues to be a rainbow antichain for $\mathcal{T}$, conversely, if $c = c_\mathcal{R}$, $\mathcal{R}$ is no more rainbow on $\mathcal{P}$, since one of the representative of $\mathcal{P}$ lacks. Note that $\mathcal{R}$, apart v, must reside in $\bigcup_i \mathcal{T}{\restriction}n_i$.

Since $\mathsf{fout}(v) \geq |\mathcal{P}|$, there are more than $|\mathcal{P}|$ subtrees $\mathcal{T}\restriction n_i$ $(1 \leq i \leq \mathsf{fout}(v))$, hence there is no way to choose $|\mathcal{P}|$ distinct nodes from $\bigcup_i \mathcal{T}\restriction n_i$ such that each $\mathcal{T}\restriction n_i$ as at lest one of these nodes. Therefore, since each $\mathcal{T}\restriction n_i$ contains at least one node colored by c (all leaves are c-colored!), we can substitute the node $v \in R$ with one of the leaf node in the "untouched" $\mathcal{T}\restriction n_i$, thus obtaining a new antichain where v is not choosen (hence v can be safely decolored). □

Applying (a) we c-decolor all nodes that have a c-colored descendant, and by Lemma 15 we remove all the nodes left uncolored. Note that this procedure can be applied both on palette $\mathcal{F}$ and on palette $\mathcal{P}_i$, for $1 \leq i \leq h$. This reduction returns a tree where all paths do not have color repetitions, hence, by Proposition 13 its height is at most $2h$. Condition (b) induces another decoloring procedure. In fact, once the previous reduction is applied, node colored the same must form an antichain and, in particular for each $f \in \mathcal{F}$ we can apply (b) just ignoring paths from a leaf up to a f-colored node. Note that this time we do not apply the reduction on palettes $\mathcal{P}_i$'s.

Proposition 17. *If* $\mathsf{fout}(\mathcal{T}) \leq m$*, then* $\mathcal{T}$ *has at most* 2^m *leaves.*

Proof. The proof is by induction on $m \geq 0$. If $m = 0$, then $\mathsf{fout}(\mathcal{T}) = 0$, hence $\mathcal{T}$ must be a single path, hence it has exactly one leaf. Let $m > 0$, and $\mathcal{T}$ be a tree with $t > 0$ children under its root (the case when $t = 0$ is trivial). By inductive hypothesis, each subtree rooted at a child of the root have at most 2^{k-t+1} leaves, since their fan-out is at most $k - (t-1)$. Since there are t of those subtrees, the number of the leaves in $\mathcal{T}$ is at most $t \cdot 2^{k-t+1}$. We have $t \cdot 2^{k-t+1} = 2 \cdot \frac{t}{2^t} \cdot 2^k \leq 2^k$, since, for all $t > 0$, $\frac{t}{2^t} \leq \frac{1}{2}$. □

By Proposition 17, the reduced target tree have at most $2^{|\mathcal{F}|}$ (hence, 2^h) f-colored nodes, for each $f \in \mathcal{F}$. Note, however, that we do not have a bound on the total number of nodes in the reduced tree, indeed the reduction (b) is not applied on c-colored nodes, for $c \in \bigcup_i \mathcal{P}_i$. This problem is overcome just checking that for each color $f \in \mathcal{F}$, all f-colored nodes have no more than $|\bigcup_i \mathcal{P}_i|$ c-colored children, for $c \in \bigcup_i \mathcal{P}_i$. Since $|\bigcup_i \mathcal{P}_i| \leq h \cdot k$, we obtain a reduced tree $\mathcal{T}_{red}$ with at most $h\,(k+1) \cdot 2^h$ nodes.

Look for rainbow antichains: What we actually need is the following for each node v in the reduced target tree: for each $X \subseteq \mathcal{F}$, determine whether the pattern trees corresponding to color in X can be mapped *simultaneously* in the subtree $\mathcal{T}_{red}\restriction v$. To calculate this, we determine all the possible tuples $t = (c_1, \ldots, c_{|Ch(v)|})$ of colors associated to each child of v, then we check that for each $\alpha(S_i) \in X$, the tuple t contains $\mathcal{P}_i$. Since both $Ch(v)$ and $\bigcup_i \mathcal{P}_i$ have at most $h \cdot k$ elements, for each node v and subset X we need to check at most $(h \cdot k)^2$ tuples at a cost of $h \cdot k$ per tuple. We denote this by the predicate $N(v, X)$.

In order to determine whether there exists a rainbow antichain in the reduced target tree $\mathcal{T}$, we need to check that $A(\mathcal{T}_{red}, \mathcal{F})$ hold, where the predicate $A(\mathcal{T}, X)$, for $\mathcal{T}$, subtree of $\mathcal{T}_{red}$, and $X \subseteq \mathcal{F}$, is defined as follows:

$$A(\mathcal{T}, X) = N(v, X) \vee \bigvee_{Y \subseteq X} \Big(A(\mathcal{T}', Y) \wedge A(\mathcal{T}'', Y \setminus X) \Big),$$

where, $v = root(\mathcal{T})$, $\mathcal{T}' = \mathcal{T}\upharpoonright u_1$ and $\mathcal{T}''$ is the tree obtained by collecting all $\mathcal{T}\upharpoonright u_j$ under a fresh copy of the node v, for $Ch(v) = \{u_1, \ldots, u_m\}$ and $2 \leq j \leq m$.

Saying that the predicate $A(\mathcal{T}, X)$ holds means that $\mathcal{T}$ admits a rainbow antichain $\mathcal{R}$ for the palette $\bigcup_{\alpha(\mathcal{S}_i)\in X} \mathcal{P}_i$. Indeed, the antichain is either a subset of the immediate children of root (in this case $N(root(\mathcal{T}), X)$ holds), or it is split in the subtrees of $\mathcal{T}$ (in this case the right part of the formula holds). A formal argument for this intuition can be provided by a straightforward induction on the height of $\mathcal{T}$.

In order to calculate $A(\mathcal{T}, X)$ we must solve a *subset convolution* problem for each node in the reduced target tree. Each subset convolution can be calculated in time $O(h^2 \cdot 2^h)$, by means of the fast subset convolution algorithm of [2], hence we can check $A(\mathcal{T}_{red}, \mathcal{F})$ in time $O(h \cdot k)^3 + O(h^3(k+1) \cdot 2^{2h})$ using a dynamic programming algorithm working bottom-up on the structure of $\mathcal{T}_{red}$.

Complexity analysis of the algorithm: The coloring phase costs $O(n_s \cdot n_t^{3/2})$ where n_t and n_s are the number of nodes in the target and pattern, respectively, [13]. Note that while coloring the nodes from leaves up to the root, it can be easily performed the first decoloring step, just do not coloring nodes by colors already assigned to some descendant.

The second decoloring phase must be performed after the previous decoloring. This is both necessary for the correctness of the reduction, and useful to increase the node fan-outs. The decoloring, for each $f \in \mathcal{F}$, first calculates the fan-out of each f-colored node just performing a simple depth-first visit of the tree, then it decolors the nodes by other h depth-first visits, one for each color in $\mathcal{F}$. The overall cost of the reduction is linear in n_t.

The cost for checking the existence of rainbow antichains in $\mathcal{T}_{red}$ has been already shown to be in $O(h \cdot k)^3 + O(h^3(k+1) \cdot 2^{2h})$.

Concluding, the overall cost of the algorithm is $O(h^3(k+1) \cdot 2^{2h}) + O(n_s \cdot n_t^{3/2})$.

Notice that the proposed algorithm proves also that the forest pattern matching problem is fixed-parameter tractable also if we choose as parameter simply $K = |\bigcup_i Ch(\mathcal{S}_i)|$; indeed, in this case the upper bound would be $O(K^3 \cdot 2^{2K}) + O(n_s \cdot n_t^{3/2})$. We have preferred to consider the two parameters h, k, instead of the single K, because our approach leads to a lower and more precise upper-bound for the problem.

5 Conclusions

In this paper we have considered the problem of finding a forest within an unordered tree, with no overlaps. This problem arises often with languages using unordered hierarchical structures, e.g. to represent scoping, containment, etc. Although the problem is NP-complete in general, we have shown that the combinatorial explosion depends only on the forest width. This parameter is usually fixed (i.e., reduction rules do not change, for a given calculus) and often it is small (i.e. ≤ 3), thus the problem is feasible. We have given an algorithm for computing the solutions for this problem, respecting these complexity bounds.

As a side result of our proof techniques, we have singled out the new *rainbow antichain* problem, which is NP-complete but fixed-parameter tractable; we think that this problem can be a useful tool also for other complexity analysis and reductions of problems about trees.

Future work. First, we plan to apply the results and algorithm presented in this paper to real calculi and frameworks. The cases of Bigraphical Reactive Systems [14], BioBigraphs [1] and Synchronized Hyperedge Replacement [8] are of particular interest. In these cases, we have to integrate forest pattern matching with sub(hyper)graph isomorphisms (needed to match e.g. the link part of bigraphs). Subgraph isomorphism is a notoriously hard problem; we hope that the tractability results given in this paper will help to tame its hardness.

An important question is whether there are other possible reductions to be applied in the target tree in order to yield a smaller kernel instance. A positive result in this direction would provide a significant improvement of both time and space complexity upper bounds. At the moment, we know only that our problem does not fulfill the criteria in [3] that would imply the nonexistence of a polynomial-bounded kernel, so there is still hope.

Another interesting situation is when we consider rules with *reaction rates*. These cases are of great interest in quantitative models of networks, biological systems, etc. Here, we are interested to pick out a single match among many possible matches of many different rules, but still respecting rates and stochastic distributions. We plan to adapt our results accordingly, with a suitable *counting* algorithm from the one presented in this paper.

References

1. G. Bacci, D. Grohmann, and M. Miculan. Bigraphical models for protein and membrane interactions. In G. Ciobanu, editor, *Proc. MeCBIC*, volume 11 of *Electronic Proceedings in Theoretical Computer Science*, pages 3–18, 2009.
2. A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto. Fourier meets möbius: fast subset convolution. In *STOC '07: Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 67–74, New York, NY, USA, 2007. ACM.
3. H. L. Bodlaender, R. G. Downey, M. R. Fellows, and D. Hermelin. On problems without polynomial kernels (extended abstract). In L. Aceto, I. Damgård, L. A. Goldberg, M. M. Halldórsson, A. Ingólfsdóttir, and I. Walukiewicz, editors, *Proc. ICALP*, volume 5125 of *Lecture Notes in Computer Science*, pages 563–574. Springer, 2008.
4. M. Boreale, R. Bruni, R. De Nicola, and M. Loreti. Sessions and pipelines for structured service programming. In G. Barthe and F. S. de Boer, editors, *Proc. FMOODS*, volume 5051 of *Lecture Notes in Computer Science*, pages 19–38. Springer, 2008.
5. L. Cardelli and A. D. Gordon. Mobile ambients. In *Foundations of Software Science and Computation Structures: First International Conference, FOSSACS '98*, pages 140–155. Springer-Verlag, Berlin Germany, 1998.

6. S. A. Cook. The complexity of theorem-proving procedures. In *STOC '71: Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158, New York, NY, USA, 1971. ACM.
7. R. G. Downey and M. R. Fellows. Fixed-parameter tractability and completeness i: Basic results. *SIAM J. Comput.*, 24(4):873–921, 1995.
8. G. L. Ferrari, D. Hirsch, I. Lanese, U. Montanari, and E. Tuosto. Synchronised hyperedge replacement as a model for service oriented computing. In F. S. de Boer, M. M. Bonsangue, S. Graf, and W. P. de Roever, editors, *Proc. FMCO*, volume 4111 of *Lecture Notes in Computer Science*, pages 22–43. Springer, 2005.
9. J. E. Hopcroft and R. E. Tarjan. A v^2 algorithm for determining isomorphism of planar graphs. *Inf. Process. Lett.*, 1(1):32–34, 1971.
10. J. E. Hopcroft and J. K. Wong. Linear time algorithm for isomorphism of planar graphs (preliminary report). In *Proceedings of Conference Record of 6th Annual ACM Symposium on Theory of Computing, (STOC '74)*, pages 172–184, 1974.
11. J. Krivine, R. Milner, and A. Troina. Stochastic bigraphs. In *Proc. MFPS*, volume 218 of *Electronic Notes in Theoretical Computer Science*, pages 73–96, 2008.
12. A. Mansutti, M. Miculan, and M. Peressotti. Multi-agent system desing and prototyping with bigraphical reactive systems. In E. Kalyvianaki, K. Magoutis, and P. Pietzuch, editors, *Distributed Applications and Interoperable Systems*, Lecture Notes in Computer Science. Springer, 2014. To appear.
13. D. W. Matula. Substree isomorphism in $O(n^{5/2})$. *Annals of Discrete Mathematics*, 2:91–106, 1978.
14. R. Milner. *The Space and Motion of Communicating Agents*. Cambridge University Press, 2009.

A Proofs of technical lemmata

Proof (of Proposition 5). We prove each point separately.

(a, $\Longrightarrow$) Since $\models T \succeq \mathbf{0}, \boldsymbol{S}$, there exist a context C and parameters $\boldsymbol{D}$ such that $T \equiv (C\{\mathbf{0}/z, \boldsymbol{S}/\boldsymbol{Z}\})\{\boldsymbol{D}/\boldsymbol{X}\}$ for some $\{z\} \uplus Z \subseteq \mathit{vars}(C)$. It is simple to prove that $C\{\mathbf{0}/z, \boldsymbol{S}/\boldsymbol{Z}\} \equiv C\{\mathbf{0}/z\}\{\boldsymbol{S}/\boldsymbol{Z}\}$, hence, by associativity of substitution composition, $T \equiv ((C\{\mathbf{0}/z\})\{\boldsymbol{S}/\boldsymbol{Z}\})\{\boldsymbol{D}/\boldsymbol{X}\}$, that is, $\models T \succeq \boldsymbol{S}$.

(a, $\Longleftarrow$) Since $\models T \succeq \boldsymbol{S}$, there exist a context C and parameters $\boldsymbol{D}$ such that $T \equiv (C\{\boldsymbol{S}/\boldsymbol{Z}\})\{\boldsymbol{D}/\boldsymbol{X}\}$ for some $Z \subseteq \mathit{vars}(C)$. Now, observing that $C \equiv C \mid \mathbf{0} \equiv (C \mid z)\{\mathbf{0}/z\}$ for some $z \notin Z$, we obtain $T \equiv ((C \mid z)\{z/\mathbf{0}, \boldsymbol{S}/\boldsymbol{Z}\})\{\boldsymbol{D}/\boldsymbol{X}\}$, that is, $\models T \succeq \mathbf{0}, \boldsymbol{S}$.

(b, $\Longrightarrow$) Since $\models T \succeq x \mid y$, there exist a context C and parameters $\boldsymbol{D}$ such that $T \equiv (C\{x \mid y/z\})\{\boldsymbol{D}/\boldsymbol{X}\}$ for some $z \in \mathit{vars}(C)$. Observing that $C\{x \mid y/z\} \equiv C\{y \mid w/z\}\{w/x\}$ (for w fresh), by associativity of substitution composition, we obtain $T \equiv ((C\{y \mid w/z\})\{x/w\})\{\boldsymbol{D}/\boldsymbol{X}\}$, that is, $\models T \succeq x$.

(b, $\Longleftarrow$) Since $\models T \succeq x$, there exist a context C and parameters $\boldsymbol{D}$ such that $T \equiv (C\{x/z\})\{\boldsymbol{D}/\boldsymbol{X}\}$ for some $z \in \mathit{vars}(C)$. It is easy to prove that $C\{x/z\} \equiv C\{x \mid \mathbf{0}/z\} \equiv C\{x \mid y/z\}\{\mathbf{0}/y\}$ for y fresh. Now by associativity and from the freshness of y, we obtain $T \equiv (C\{x \mid y/z\})\{\mathbf{0}/y, \boldsymbol{D}/\boldsymbol{X}\}$, that is, $\models T \succeq x \mid y$.

(c) has the same proof of (b), just replace x in (b) with S. $\square$