Is probability *the* measure of uncertainty?

Tommaso Flaminio and Hykel Hosni

Logics for Social Behaviour,
Lorenz Center
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Outline

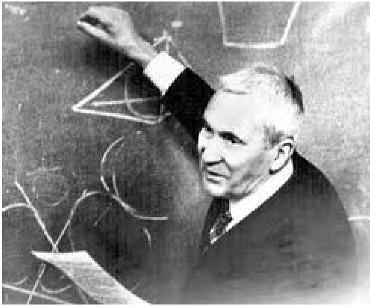
Uncertainty and Probability

Fact

For the past 300 years probability has been taken as a measure of uncertainty.

Justifications

- urn models (long run, Bernoulli's theorem)
- decision theoretic (Dutch Book)
- axiomatic (Cox's theorem)



Probabilistic models of uncertainty are largely successful in casinos, insurance, and even wether forecasts. However...

Uncertainty in the social sciences

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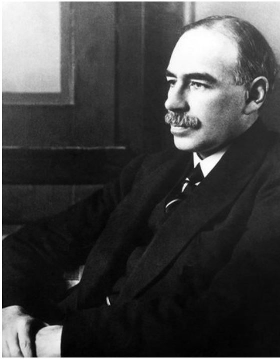

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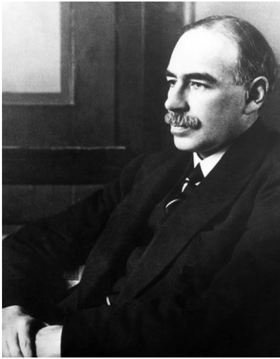

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Suppose θ is an event expressed in the classical propositional calculus (CPC)

Is it reasonable to have a normative theory of rational belief mandating that $P(\theta \vee \neg\theta) = 1$?

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- probability isn't the tool for sensible uncertainty quantification
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Fact

Relatively little attention (in economic theory) has been given to the **logical** analysis of the problem

Knightian Uncertainty (AKA Ambiguity)

Consider the popular rendering of *Knightian uncertainty*

A problem/situation in which a single probability distribution *cannot* be assigned to a given state-space

Intuitively appealing yet it conflates three logically distinct problems:

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2. practical intractability
   - the probability distribution cannot be defined with the cognitive/computational resources available to the agent
3. descriptive inadequacy
   - when presented with a given state space, the agent fails to define a probability on it

The adequacy of probability

Back to the initial problem. Is it reasonable to be normatively forced to assign

$$P(\theta \vee \neg\theta) = 1?$$

A standard argument in AI and Stats

I may have *unreliable information* (or no info at all!) about θ, so there is something odd about giving the highest degree of belief to an event mentioning θ.

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Knowledge vs Ignorance

> *The probability attached to an uncertain event does not reflect the heuristic amount of information that led to the assignment of that probability (Schmeidler 1989)*

The role of $\models$

A probability function over $\mathcal{L}$ is a map $P : \mathcal{SL} \to [0,1]$ satisfying

(P1) If $\models \theta$ then $P(\theta) = 1$

(P2) If $\models \theta \to \neg\phi$ then $P(\theta \vee \phi) = P(\theta) + P(\phi)$.

where

- $\mathcal{L} = \{p_1, \ldots, p_n\}$
- $\mathcal{SL} = \{\theta, \phi, \ldots\}$ is the set of sentences built recursively via the propositional connectives $\neg, \wedge, \vee, \to$
- $\models \subseteq 2^{\mathcal{SL}} \times \mathcal{SL}$ is the classical (Tarskian) consequence relation:

$$\Gamma \models \theta \Leftrightarrow \text{ every model of } \Gamma \text{ is a model of } \theta$$

The epistemic symmetry of $\models$

Facts

- $\models \theta \vee \neg\theta$, $\theta \equiv \neg\neg\theta$, etc hold because knowledge and ignorance are interpreted symmetrically in classical logic
- $P(\theta \vee \neg\theta) = 1$ holds "because of $\models$" and P1

The epistemic symmetry in probability logic

1. $P(\neg\theta) = 1 - P(\theta)$
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And it gets worse ...

The intractability of $\models$

A fact from complexity Even if the problem of deciding $\models \theta$ is decidable it is *intractable*, i.e. there exists no polynomial answer to it (in general).

Idea

It might well be that deciding $\models \theta$ (hence $\Gamma \models \theta, \theta \equiv \phi$, etc.) is **too hard** for practical purposes, i.e. practically unfeasible.

> *Even in the field of tautology [...] we always find ourselves in a state of uncertainty. In fact, even a single verification of a tautological truth [...] can turn out to be, at a given moment, to a greater or lesser extent accessible or affected with error, or to be just a doubtful memory. (de Finetti 1974, p.24)*

Summing up

Probability as a measure of rational belief depends heavily on the classical notion of consequence

Unhappiness with probability largely caused by

1. the epistemic symmetry of $\models$
2. the intractability of $\models$

Outline

1. The Problem

2. A proposal for an answer

Prescriptive approach

Grounding probability on Depth-bounded logics we put forward a theory of rational belief with *prescriptive* force, i.e. a normative theory which takes into account the cognitive resources of the agent

Outline of the project

1. replace $\models$ with tractable approximations (a hierarchy of logics of increasing complexity)
2. investigate the properties of the rational belief measures based on such logics

Depth-bounded logics

D' Agostino, M., Finger, M., & Gabbay, D. (2013). Semantics and proof-theory of depth bounded Boolean logics. Theoretical Computer Science, 480, 43-68.

Define an infinite sequence

$$L_0, L_1, \dots, L_n, \dots$$

of logical systems, such that:

- $\models_i \subset \models_{i+1} \subset \models_{\mathrm{CPL}}$ for all i;
- $\models_{\mathrm{CPL}}$ is the limit of $\{\models_i\}$ for $i \to \infty$;
- $\models_i$ is **tractable** for all i.

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- L_0 is *trivial* (i.e. it has no tautologies). For each propositional variable p, $p \models_0 p$, $p \models_0 \neg\neg p$.

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A hierarchy of agents

The depth k of virtual information which characterises L_k is a measure of

1. computational resources in determining validity
2. degree of idealisation of the agent

Depth-Bounded Probability Logic

Key challenges

- failure of deduction theorem in DBL implies lots of non equivalent axiomatisations of probability functions
- normalisation

Some interesting properties

- we can define a hierarchy of probability functions (mirroring $\models_k$)
- we can define $k-$coherence and prove that it is equivalent to de Finetti's coherence in the limit
- if a book is incoherent, there exists a minimal k for which it is $k-$incoherent