A Crash Course in Judgment Aggregation

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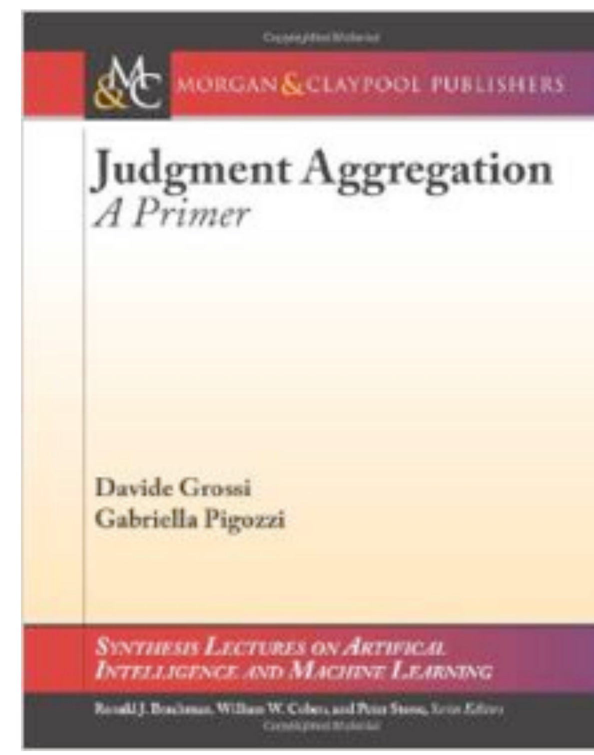
Outline

- PART I: voting rules & paradoxes
- PART II: why is (good) judgment aggregation (really) difficult
- PART III: possible judgment aggregation



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PART I

FINANCIAL TIMES MONDAY AUGUST 31 20

Companies | UK

Individual rationality can mean collective irrationality

Tony Jackson ON MONDAY

based on behaviour rather than rationality, do a better job.

But when it comes to the big stuff, our actions belie that. When we are grappling with the subprime debacle or Chinese economic policy, we ask ourselves what people are up to – not how they behave, but how they are reasoning.

In fact, the theory of rational behaviour

savings to institutions, who hand them on to specialist fund managers.

If those managers underperform the market, it is hard for the investor to know whether they are deliberately avoiding overvalued stocks, or simply messing up. If the situation persists, then investors infer the latter and switch their money.

The germ of the idea came to Dr

from the present one, they should move to that price immediately.

In reality, of course, prices overshoot over long periods, then go into reverse. The dotcom example illustrates why.

As investors were bailing out of value funds such as GMO, they were gradually switching more cash into the bubble stocks. Thus those stocks were pushed up

outcome. That might seem odd to mainstream economists, but not to the rest of us. Mutually destructive wars have been fought on the same basis.

What are we to do about this? Unstable and irrational markets can be socially harmful, besides wasting resources within the financial system.

Dr Woolley hopes the very fact of

The hiring committee

good researcher

good teacher

to be hired!

The hiring committee

REFable $\wedge$ good teacher $\leftrightarrow$ to be hired!

REFable	good teacher	to be hired!
✓	✓	✓
✗	✓	✗
✓	✗	✗

Hire or not?

- “Doctrinal paradox” (Vacca, 1921) (Kornhauser & Sager, 1993)

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The hiring committee

	p	q	$p \wedge q$
J_1	1	1	1
J_2	1	0	0
J_3	0	1	0
J	?	?	?

- "Doctrinal paradox" (Vacca, 1921) (Kornhauser & Sager, 1993)

Voting by propositionwise majority

Issues $I \subset \mathcal{L}$

Agenda $A = \{\varphi \mid \varphi \in I\} \cup \{\neg\varphi \mid \varphi \in I\}$

	p	q	$p \wedge q$
J_1	1	1	1
J_2	1	0	0
J_3	0	1	0
J	?	?	?

Majority rule:

$$f_{maj}(P) = \left\{\varphi \in A \mid |P_\varphi| \geq \left\lceil \frac{|N|+1}{2} \right\rceil \right\}$$

where, for $x \in \mathbb{Q}$, $\lceil x \rceil$ is the smallest integer greater or equal to x. I.e., φ is collectively accepted iff there is a majority of voters accepting it.

Aggregation function

$f : \mathbf{J}^{|N|} \longrightarrow \wp(A)$

Set of profiles

Profiles $P = \langle J_i \rangle_{i \in N}$

Judgment sets

$J_i \subseteq A$ s.t. consistent and complete

The two 'souls' of JA

- Logic-based JA

$$f : \langle J_1, \ldots, J_n \rangle \mapsto J$$

Dietrich & List

- Abstract aggregation & Opinion pooling (consensus formation)

$$f : \begin{pmatrix} J_1(p_1) & J_1(p_2) & \ldots & J_1(p_{|A|}) \\ J_2(p_1) & J_2(p_2) & \ldots & J_2(p_{|A|}) \\ \vdots & \vdots & \ddots & \vdots \\ J_{|N|}(p_1) & J_{|N|}(p_2) & \ldots & J_{|N|}(p_{|A|}) \end{pmatrix} \mapsto (\ J(p_1) J(p_2) \quad \ldots \quad J(p_{|A|})\ )$$

Wilson, Dokow & Holzman, Nehring & Puppe, Lehrer & Wagner

Threshold-based rules

Majority rule:

$$f_{maj}(P) \quad = \quad \left\{ \varphi \in A \mid |P_\varphi| \geq \left\lceil \frac{|N| + 1}{2} \right\rceil \right\}$$

where, for $x \in \mathbb{Q}$, $\lceil x \rceil$ is the smallest integer greater or equal to x. I.e., φ is collectively accepted iff there is a majority of voters accepting it.

Unanimity rule:

$$f_u(P) \quad = \quad \{ \varphi \in A \mid |P_\varphi| \geq |N| \}$$

I.e., φ is collectively accepted iff all voters accept it.

Quota rule:

$$f_t(P) \quad = \quad \{ \varphi \in A \mid |P_\varphi| \geq t_\varphi \}$$

where $t = \langle t_\varphi \rangle_{\varphi \in A}$ is a tuple of integer thresholds or quotas t_φ, one for each formula in the agenda. I.e., φ is collectively accepted iff there are at least t_φ voters accepting it.

Premise- and conclusion-based rules

$Prem \cup Conc$ is a partition of A

Premise-based rule:

$$f_{pb}(P) \quad = \quad f_{maj}(P^{Prem}) \cup \{\varphi \in Conc \mid f_{maj}(P^{Prem}) \models \varphi\}$$

I.e., φ is collectively accepted iff it is a premise and it has been voted by the majority of the individuals or it is a conclusion entailed by the premises accepted by the majority.

Conclusion-based rule:

$$f_{cb}(P) = f_{maj}(P^{Conc})$$

I.e., φ is collectively accepted iff it is a conclusion and it has been voted by the majority of the individuals.

Example

	p	$p \rightarrow q$	q
J_1	1	1	1
J_2	1	0	0
J_3	0	1	0
f_{maj}	1	1	0
f_u			
$f_{t'}$	1	1	0
$f_{t''}$	1	0	0
f_{pb}	1	1	1
f_{cb}			0

- Is this a problem of only these rules? Or a genuine difficulty?
- **NOTE**: these rules are "**nice**"! (they are anonymous, unbiased, monotonic, independent, ...)
- **BUT**: they do not preserve rationality

PART II

Oligarchs (Ultra)filters and Dictators

Arrow's devastating discovery is to mathematical politics what Kurt Goedel's 1931 impossibility-of-proving-consistency theorem is to mathematical logic

P. Samuelson

[Scientific American, October 1974, p. 120]

(Im)possibility

- Given a certain type of agenda, does an aggregation function exist, which satisfies some desirable aggregation conditions?
- **If** the agenda satisfies the agenda conditions C1, ... , Cn **then** the aggregation function satisfies the aggregation conditions C'1 ... C'm **if and only** if the aggregation function is a dictatorship (or an oligarchy)

Theorem (Dietrich & List, 2007). *Let $\mathcal{J} = \langle N, A \rangle$ be a judgment aggregation structure such that A satisfies NS and EN, and let f be an aggregation function: f satisfies* **U**, **RAT** *and* **SYS** *iff f satisfies* **D**.

Agenda conditions

Definition (Non-simple agendas). *An agenda A is* non-simple *(NS) iff it contains at least one set X s.t.:*

- $3 \leq |X|$;
- X *is* minimally inconsistent, *i.e.:*
  - X *is inconsistent;*
  - $\forall Y$ s.t. $Y \subset X$: Y *is consistent.*

An agenda is called simple *if it is not non-simple.*

- **NOTE**: an agenda is simple iff it contains only minimally inconsistent sets of size 2.
- **If** an agenda is simple **then** the majority rule works very well (it is actually the "best")

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Aggregation conditions

An aggregation function f is:

Collectively rational (**RAT**) iff $f(P)$ is consistent and complete.
I.e., the collective set is a judgment set.

Unanimous (**U**) iff $\forall \varphi \in A, \forall P \in \mathbf{P}$: IF $[\forall i \in N : P_i \models \varphi]$ THEN $f(P) \models \varphi$.
I.e., if all voters agree on accepting φ, so does also the collective set.

Systematic (**SYS**) iff $\forall \varphi, \psi \in A, \forall P, P' \in \mathbf{P}$: IF $[\forall i \in N : P_i \models \varphi$ IFF $P'_i \models \psi]$ THEN $[f(P) \models \varphi$ IFF $f(P') \models \psi]$.
I.e., if all voters in two different profiles agree on the acceptance or rejection pattern of two formulae (φ is accepted iff ψ is accepted), the aggregated judgments of the two profiles also do.

- **SYS**: "all that matters are the columns of a vote matrix"

	p	q	$p \wedge q$
J_1	1	1	1
J_2	1	0	0
J_3	0	1	0
J	?	?	?

$$\begin{array}{ccc} P_1(\varphi_1) & \dots & P_1(\varphi_m) \\ \dots & \dots & \dots \\ P_n(\varphi_1) & \dots & P_n(\varphi_m) \\ \hline f(P)(\varphi_1) & \dots & f(P)(\varphi_m) \end{array}$$

$$\begin{array}{ccc} P'_1(\varphi_1) & \dots & P'_1(\varphi_m) \\ \dots & \dots & \dots \\ P'_n(\varphi_1) & \dots & P'_n(\varphi_m) \\ \hline f(P')(\varphi_1) & \dots & f(P')(\varphi_m) \end{array}$$

(Ultra)filters of winning coalitions

Winning coalitions

$$\mathcal{W}_\varphi := \{C \subseteq N \mid \forall P \in \mathbf{P}: \text{ IF } C = P_\varphi \text{ THEN } f(P) \models \varphi\}$$

$$\forall \varphi, \psi \in A : \mathcal{W}_\varphi = \mathcal{W}_\psi$$

Lemma (Ultrafilter lemma). *Let $\mathcal{J} = \langle N, A \rangle$ be a judgment aggregation structure and f an aggregation function such that A satisfies NS and EN and f satisfies* **U**, **SYS** *and* **RAT**. *The set $\mathcal{W}$ is an* ultrafilter, *i.e.:*

i) $N \in \mathcal{W}$

~~*ii)* $C \in \mathcal{W}$ *iff* $-C \notin \mathcal{W}$;~~ $\emptyset \notin \mathcal{W}$ Proper filter

iii) $\mathcal{W}$ *is upward closed: if* $C \in \mathcal{W}$ *and* $C \subseteq C'$ *then* $C' \in \mathcal{W}$;

iv) $\mathcal{W}$ *is closed under finite meets: if* $C, C' \in \mathcal{W}$ *then* $C \cap C' \in \mathcal{W}$.

(Ultra)filters & (singleton)oligarchs

Lemma (Existence of dictators (resp. oligarchs)). *Let $\mathcal{W}$ be an ultrafilter (resp. a proper filter) on a finite set N. Then $\mathcal{W}$ is* principal, *i.e.,* $\exists i \in N$ S.T. $\{i\} \in \mathcal{W}$. *(resp.,* $\emptyset \neq \bigcap \mathcal{W} \in \mathcal{W}$*)*

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(Ultra)filters & (singleton)oligarchs

Theorem (Dietrich & List '07). *Let $\mathcal{J} = \langle N, A \rangle$ be a judgment aggregation problem such that A satisfies NS and EN, and let f be an aggregation function: f satisfies* **U**, **RAT** *and* **SYS** *iff f satisfies* **D**.

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(Ultra)filters & (singleton)oligarchs

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$$\begin{array}{rcr} \{NS\} & + & \{\mathbf{RAT}, \mathbf{SYS}, \mathbf{MON}\} \\ \{PC, EN\} & + & \{\mathbf{RAT}, \mathbf{U}, \mathbf{IND}\} \\ \{PC\} & + & \{\mathbf{RAT}, \mathbf{U}, \mathbf{IND}, \mathbf{MON}\} \\ \ldots & & \ldots \end{array}$$

Impossibility: the general pattern

- Show that the properties of the judgment aggregation problem at hands (i.e., agenda and aggregation conditions) force the set of winning coalitions to be an ultrafilter or proper filter
- If the set of voters is finite, then the set of winning coalitions is generated by a singleton or a smallest set
- First ultrafilter proof of Arrow's theorem due to [Fishburn, 1971]
- Ways around impossibility:

PART III

‘Escaping’ impossibility

by Lex Drewinski

PART III

'Escaping' impossibility

Infinite electorates

Irresolute rules

$$f : \mathbf{J}^{|N|} \longrightarrow \wp(A)$$

Restricted domains

Weaker agenda and/or aggregation conditions

Infinite electorates

Theorem (Non-dictatorial aggregation with infinite electorates). *Let $\mathcal{J} = \langle N, A \rangle$ be an aggregation problem where $|N|$ is infinite. There exists an aggregation function f which satisfies* **RAT**, **U**, **SYS** *and does not satisfy* **D**.

- Take the Frechet filter over N
- Complete it to an ultrafilter [Tarski, 30]
- Define the rule: "an issue is collectively accepted iff the set of voters supporting it belongs to that filter"

Restricted domains

Unidimensional alignment. A profile P is *unidimensionally aligned* if there exists a strict linear order $>$ such that, $\forall \varphi \in A$: it is *either* the case that $\forall i, j \in N$ if $i \in P_\varphi$ and $j \in P_{\neg\varphi}$ then $i > j$, *or* it is the case that $\forall i, j \in N$ if $i \in P_\varphi$ and $j \in P_{\neg\varphi}$ then $j > i$.

	Voter 3	Voter 2	Voter 5	Voter 4	Voter 1
p	0	0	0	1	1
q	1	1	0	0	0
r	1	0	0	0	0

Theorem (List'05). *Let $\mathcal{J} = \langle N, A \rangle$ be an aggregation problem and let the domain of the aggregation consist of only* unidimensionally aligned *profiles. Then propositionwise majority is the only rule that satisfies* **SYS** *and* **AN**.

The ground we covered ...

- JA as a general theory of "aggregation"
- Doctrinal paradox
- Agenda and aggregation conditions
- Ultrafilter proof technique
- 'Escape' routes for impossibility results

Thank you!