LOGICS FOR SOCIAL BEHAVIOR

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Tutorial Lecture 1.1

Arithmetic Probability Pooling

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DEDICATED TO

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26 December 1924 –
in the year of his 90\textsuperscript{th} birthday
1. Probability Mass Functions

- If $\Omega$ is a finite or denumerably infinite set of possible states of the world, a function $p: \Omega \rightarrow [0,1]$ is a *probability mass function* (pmf) if
  \[ \sum_{\omega \in \Omega} p(\omega) = 1. \]

- When $\Omega$ is finite, such functions can also represent *(normalized) allocations* of a fixed sum of money, or other quantifiable resource, among various enterprises, for example, a cost or benefit sharing arrangement among participants in a cooperative game.

Every pmf $p$ on $\Omega$ induces a (discrete) probability measure (pm) (which, abusing notation, we also denote by $p$) on $2^\Omega$, defined for every subset $A$ of $\Omega$, by
  \[ p(A) := \sum_{\omega \in A} p(\omega). \]
2. Pooling Operators

*The Probability Pooling Problem:*

What is a reasonable way to aggregate the possibly differing *pmfs* (resp., *pms*) $p_1, \ldots, p_n$ of $n$ individuals into a single *pmf* (resp, *pm*) $p$? We’ll treat the case of *pmfs*.

- If $\mathcal{P} =$ the set of all *pmfs* on $\Omega$ and $n > 1$, a *pooling operator* is any function

$$T : \mathcal{P}^n \rightarrow \mathcal{P}.$$ 

In specifying $\mathcal{P}^n$ as the domain of $T$, we are implicitly adopting as a pooling axiom the *universal domain condition* (UD). Each $n$-tuple $(p_1, \ldots, p_n) \in \mathcal{P}^n$ is called a *profile* (of individual probability assessments). UD demands that $T$ produce a *pmf* $p := T(p_1, \ldots, p_n)$ for every logically possible *profile* $(p_1, \ldots, p_n)$. 

• The justification of additional axiomatic restrictions on T depends on the intended interpretation of \( p: = T(p_1, \ldots, p_n) \). Is \( p \) conceived of as ....

(i) a rough summary of the current pmfs \( p_1, \ldots, p_n \) of \( n \) individuals?

(ii) a compromise adopted by these individuals in order to complete an exercise in group decision making?

(iii) a “rational” consensus to which all individuals have freely revised their initial pmfs after extensive discussion?

(iv) the pmf of a decision maker external to the group (who may or may not have assessed his own prior before consulting the group) upon being apprised of the pmfs \( p_1, \ldots, p_n \) of \( n \) “experts”?
(v) the revision of the pmf $p_i$ of a particular individual $i$ in the group, upon being apprised of the pmfs of the remaining $n - 1$ individuals in the group, each of whom he considers to be his epistemic peer (much discussed recently for $n = 2$)?

Put the issue of interpretation aside, and consider a natural, and familiar, pooling operator, the \textit{weighted arithmetic mean}

\begin{equation}
    p: = T(p_1, \ldots, p_n) = w_1 p_1 + \cdots + w_n p_n,
\end{equation}

where the weights $w_i$ are nonnegative, and sum to 1. Formula (2.1) is interpreted \textit{pointwise}, as asserting for all $\omega \in \Omega$ that

\begin{equation}
    p(\omega) = w_1 p_1(\omega) + \cdots + w_n p_n(\omega).
\end{equation}
3. The Charms of Arithmetic Pooling

Note that

(3.1) \[ \sum_{\omega \in \Omega} [w_1 p_1(\omega) + \cdots + w_n p_n(\omega)] = 1, \]

*with no need to normalize the summands.*

- Other pooling functions based, say on the geometric mean, the harmonic mean, the root-mean-square, etc. lack this property.


- The solution of Cauchy’s functional equation \( f(x + y) = f(x) + f(y) \) on the restricted domain \( 0 \leq x, y, x+y \leq 1 \) lies at the core of the proof of this result, and the proofs of many subsequent theorems on probability pooling.
4. A Characterization of Weighted Arithmetic Pooling


Consider the following axiomatic restrictions on a pooling operator $T : \mathcal{P}^n \rightarrow \mathcal{P}$:

**State-wise Pooling** (SP). For each $\omega \in \Omega$, there exists a function $f_\omega : [0,1]^n \rightarrow [0,1]$ such that, for all $\omega \in \Omega$,

$$ p(\omega) : = T(p_1,\ldots,p_n)(\omega) = f_\omega(p_1(\omega),\ldots,p_n(\omega)). $$

(Since $p \in \mathcal{P}$, we must have $\sum_{\omega \in \Omega} p(\omega) = 1.$)

**Zero Preservation** (ZP). For all $\omega \in \Omega$, if $p_1(\omega) = \cdots = p_n(\omega) = 0$, then $T(p_1,\ldots,p_n)(\omega) = 0$.

**Remark 1.** In the presence of SP, ZP reduces to the assertion that $f_\omega(0,\ldots,0) = 0$ for all $\omega \in \Omega$. 
Remark 2. SP alone allows for different pooling methods for different states of the world. But, as we’ll see, ZP forces such pooling to be uniform across all $\omega \in \Omega$.

Theorem 4.1. If $|\Omega| \geq 3$, a pooling operator $T : \mathcal{P}^n \rightarrow \mathcal{P}$ satisfies SP and ZP if and only if there exists a sequence $w_1, \ldots, w_n$ of nonnegative numbers, summing to 1, such that, for all $\omega \in \Omega$,

$$T(p_1, \ldots, p_n)(\omega) = w_1 p_1(\omega) + \cdots + w_n p_n(\omega).$$

Proof. Sufficiency: obvious.

Necessity: Denote by $X, Y, \text{ etc.}$ elements of $[0,1]^n$, and by $c$ the n-dimensional vector $(c, \ldots, c)$. 
Let $\omega_1$, $\omega_2$, and $\omega_3$ denote arbitrary distinct elements of $\Omega$.

Suppose that $0 \leq X, Y, X+Y \leq 1$.

Define $P = (p_1, \ldots, p_n) \in \mathcal{P}^n$ by

\begin{align*}
(4.1) & \quad (p_1(\omega_1), \ldots, p_n(\omega_1)) = X, \\
(4.2) & \quad (p_1(\omega_2), \ldots, p_n(\omega_2)) = Y, \\
(4.3) & \quad (p_1(\omega_3), \ldots, p_n(\omega_3)) = 1 - X - Y, \text{ and} \\
(4.4) & \quad (p_1(\omega), \ldots, p_n(\omega)) = 0 \text{ otherwise.}
\end{align*}

By SP and ZP, (4.1) – (4.4) imply

\begin{equation}
(4.5) \quad f_{\omega_1}(X) + f_{\omega_2}(Y) + f_{\omega_3}(1 - X - Y) = 1.
\end{equation}
Now define $Q = (q_1, \ldots, q_n) \in \mathcal{P}^n$ by

(4.6) \hspace{1em} (q_1(\omega_1), \ldots, q_n(\omega_1)) = 0,

(4.7) \hspace{1em} (q_1(\omega_2), \ldots, q_n(\omega_2)) = X + Y,

(4.8) \hspace{1em} (q_1(\omega_3), \ldots, q_n(\omega_3)) = 1 - X - Y, \text{ and}

(4.9) \hspace{1em} (q_1(\omega), \ldots, q_n(\omega)) = 0 \text{ otherwise.}

By SP and ZP, (4.6) – (4.9) yield

(4.10) \hspace{1em} f_{\omega_2}(X+Y) + f_{\omega_3}(1 - X - Y) = 1.

Recalling (4.5) that

\[ f_{\omega_1}(X) + f_{\omega_2}(Y) + f_{\omega_3}(1 - X - Y) = 1, \]

we get

(4.11) \hspace{1em} f_{\omega_2}(X+Y) = f_{\omega_1}(X) + f_{\omega_2}(Y)

Setting $Y = 0$ in (4.11) yields $f_{\omega_1}(X) = f_{\omega_2}(X)$ for all $X \in [0,1]^n$. But since $\omega_1$ and $\omega_2$ were arbitrary, the functions $f_\omega$ are identically equal to some function $f: [0,1]^n \rightarrow [0,1]$. 
So, for \(0 \leq X,Y, X+Y \leq 1\), (4.11) becomes

\[
(4.12) \quad f(X+Y) = f(X) + f(Y),
\]

which is Cauchy’s equation for vectors, but on a restricted domain.

- From \(f\) we extract \(n\) scalar functions \(f^{<i>}: [0,1] \rightarrow [0,1], \ i = 1,\ldots,n\), by

\[
(4.13) \quad f^{<i>}(x) := f(0,\ldots,0,x,0,\ldots,0),
\]

where \(x\) denotes the \(i\)th coordinate.

- From the obvious extension of (4.12) to \(n\) summands, we get

\[
(4.14) \quad f(x_1,\ldots,x_n) = f^{<1>}(x_1) + \cdots + f^{<n>}(x_n),
\]

and, if \(0 \leq x, y, x+y \leq 1\), then

\[
(4.15) \quad f^{<i>}(x + y) = f^{<i>}(x) + f^{<i>}(y).
\]

It is crucial to be able to extend \(f^{<i>}\) to a nonnegative function on the domain \([0,\infty)\) so that (4.15) continues to hold for all \(x, y \geq 0\)....

- Since the additivity of $f^{<i>}$ also extends to finitely many summands, it follows immediately (with $\phi$ denoting $f^{<i>}$ for the sake of simplicity) that

\[(4.16) \quad \phi(mx) = m \phi(x), \; m = 1,2,\ldots\]

- If $x = (k/m)t$, then $mx = kt$, and so $m\phi(x) = \phi(mx) = \phi(kt) = k\phi(t)$, whence

\[(4.17) \quad \phi[(k/m)t] = (k/m)\phi(t).\]

- Suppose that $\phi(1) = w$. Setting $t = 1$ above shows that

\[(4.18) \quad \phi(r) = wr \text{ for every rational } r > 0.\]

Also, $\phi(x + y) = \phi(x) + \phi(y) \Rightarrow \phi(0) = 0$, so (4.18) holds for all rational $r \geq 0$. 
• Since \( \varphi(y) \geq 0 \) for all \( y \geq 0 \),
\[
\varphi(x + y) = \varphi(x) + \varphi(y) \geq \varphi(x)
\]
i.e., \( \varphi \) is weakly increasing, with, as noted above, \( \varphi(r) = wr \), for all rational \( r \geq 0 \).

• For each \( x \geq 0 \), let \((r_j)\) and \((R_j)\) be, respectively, increasing and decreasing rational sequences converging to \( x \). Then
\[
(4.19) \quad wr_j = \varphi(r_j) \leq \varphi(x) \leq \varphi(R_j) = wR_j,
\]
and so
\[
(4.20) \quad \varphi(x) = wx \quad \text{for all } x \geq 0.
\]

• From (4.20), with \( \varphi = f^{<i>}, \ i = 1,\ldots,n \),
along with \( f(x_1,\ldots,x_n) = f^{<1>}(x_1) + \cdots + f^{<n>}(x_n) \),
it follows that there exist nonnegative weights \( w_1,\ldots,w_n \) such that, for all
\( X = (x_1,\ldots,x_n) \in [0,1]^n \),
\[
(4.21) \quad f(x_1,\ldots,x_n) = w_1x_1 + \cdots + w_nx_n.
\]
Set \( x_i = p_i(\omega) \), where \( (p_1,\ldots,p_n) \in \mathcal{P}^n \). Then
(4.22) \[ p(\omega) : = T(p_1, \ldots, p_n)(\omega) = \]
\[ f(p_1(\omega), \ldots, p_n(\omega)) = w_1 p_1(\omega) + \cdots + w_n p_n(\omega), \]
and since \( \sum_{\omega \in \Omega} p(\omega) = 1, \)
(4.23) \[ w_1 + \cdots + w_n = 1. \quad \square \]

- See also the beautiful paper of Kevin McConway,

Marginalization and linear opinion pools, *J. Amer. Stat. Assoc.* 76 (1981) 410-414, conceptualized in terms of the pooling of all profiles of probability measures defined on every possible sigma algebra \( \mathcal{A} \) on \( \Omega \). In this context, McConway shows that the commutativity of the family \( \{ T_{\mathcal{A}} \} \) with marginalization (restriction to a sub-sigma algebra) is equivalent to postulating “event-wise pooling.”
5. Pooling Under SP Alone

**Theorem 5.1.** (Aczél, Ng, & Wagner, 1984) Suppose that $|\Omega| \geq 3$ and $T: \mathcal{P}^n \rightarrow \mathcal{P}$ satisfies SP, so that for each $\omega \in \Omega$, there exists a function $f_\omega : [0,1]^n \rightarrow [0,1]$ such that $p(\omega) = T(p_1(\omega),...,p_n(\omega)) = f_\omega(p_1(\omega),...,p_n(\omega))$.

- If $\Omega$ is finite, then, for all $\omega \in \Omega$, either
  (i) $p(\omega) = \sum_i w_i p_i(\omega)$
      with all $w_i \geq 0$ and $w_1 + \cdots + w_n = 1$, or
  (ii) $p(\omega) = \sum_i w_i p_i(\omega) + [1-\sum_i w_i]q(\omega)$,
      where all $w_i \in [-1,1]$, $\sum_i w_i < 1$, and
      $q$ is a pmf on $\Omega$.

- If $\Omega$ is denumerably infinite, then either (i) above holds, or (ii) holds, but with all $w_i \geq 0$.

Details of proof in …..

- **Example with negative weights** (|Ω| = m)
  
  \[ w_1 = \cdots = w_{n-1} = 0, \quad w_n = -1/(m-1), \text{ and} \]
  
  \[ q(\omega) = 1/m \quad \text{for all} \quad \omega \in \Omega, \text{ which yields} \]

  \[ p(\omega) = \left[-1/(m-1)\right] p_n(\omega) + 1/(m-1) \]

*Exercise.* Show that all weights \( w_i \) are nonnegative (though they need not sum to 1) if and only if, for all \( \omega \in \Omega \), and all \( X \) and \( Y \) in \([0,1]^n\),

\[ X \geq Y \quad \Rightarrow \quad f_\omega(X) \geq f_\omega(Y) \]

*(Weak Dominance)*
6. **SP is stringent**

SP looks innocuous at first glance, but it is really quite stringent, restricting pooling to weighted arithmetic averaging, with the same weights for each state, or to an affine version thereof. (More on the problems with such pooling in tutorial 1.2)

More strikingly, if probability values are restricted (as they always are in practice) to a *finite* subset of $[0,1]$, **SP alone rules out any reasonable pooling methods.**

See: Shattuck & Wagner, An impossibility theorem for allocation aggregation, *J. Philos. Logic*, published online 24 June 2014)

Details…..
• Let $V$ denote the set of values that may be assigned as probabilities. Suppose that $V$ satisfies the closure conditions

(6.1) $0 \in V,$

(6.2) $x \in V \Rightarrow 1 - x \in V,$ and

(6.3) $x, y \in V \& x + y \leq 1 \Rightarrow x + y \in V.$

• The subsets $V$ of $[0,1]$ satisfying the preceding conditions fall into just two radically distinct categories:

**Theorem 6.1**

A subset $V$ of $[0,1]$ satisfying (6.1) – (6.3) is either *dense in* $[0,1]$, or *finite*. In particular, every *discrete* subset of $[0,1]$ satisfying these closure conditions must be finite.
A pooling operator $T: \mathcal{P}_V^n \rightarrow \mathcal{P}_V$ is *dictatorial* if there exists $d \in \{1, \ldots, n\}$ such that, for all $(p_1, \ldots, p_n) \in \mathcal{P}_V^n$, $T(p_1, \ldots, p_n) = p_d$. $T$ is *imposed* if there exists $q \in \mathcal{P}_V$ such that, for all $(p_1, \ldots, p_n) \in \mathcal{P}_V^n$, $T(p_1, \ldots, p_n) = q$.

**Theorem 6.2**

Suppose that $\Omega$ is finite and $|\Omega| \geq 3$. Let $\mathcal{P}_V := \{ \text{pmfs } p \text{ on } \Omega : p(\omega) \in V \text{ for all } \omega \in \Omega \}$. A pooling operator $T: \mathcal{P}_V^n \rightarrow \mathcal{P}_V$ satisfies SP if and only if it is dictatorial, or imposed. If $T$ satisfies ZP as well, $T$ must be dictatorial.

- When $V = \{0,1\}$, this yields a corollary of Franz Dietrich on *judgment aggregation*.

(Judgment aggregation: (im)possibility theorems, *J. Economic Theory* **126** (2006), 286-298.)