The problem of judgment aggregation in the framework of boolean-valued models

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Abstract. A framework for boolean-valued judgment aggregation is described. The simple (im)possibility results in this paper highlight the role of the set of truth values and its algebraic structure. In particular, it is shown that central properties of aggregation rules can be formulated as homomorphy or order-preservation conditions on the mapping between the power-set algebra over the set of individuals and the algebra of truth values. This is further evidence that the problems in aggregation theory are driven by information loss, which in our framework is given by a coarsening of the algebra of truth values.

1 Introduction and motivation

One of the most elementary problems in multiagent systems is the problem of aggregating the distributed information coming from different sources. For collective decision making by autonomous software agents, the canonical version of this problem (which will be used in the following for the illustration of the more general aggregation framework) is, of course, the aggregation of the preferences that the individual agents express over a given set of alternatives (see e.g. [20], Chapter 12). In its almost ubiquitous form, this problem is given by a set A of alternatives (e.g. candidates) which has to be ranked by a set I of agents, based on the individual orderings of these alternatives. A preference is then a binary relation $P \subset A \times A$, which is typically assumed to be a linear order, i.e. an anti-symmetric, transitive and complete binary relation on the set of alternatives. For all alternatives $x, y \in A$, $(x, y) \in P$ then denotes the strict preference of x over y. Denoting by $L(A)$ the set of all linear orders on A, the problem of preference aggregation consists in finding a rule that assigns to each product, or profile, of individual preferences $\langle P_i \rangle_{i \in I} \in L(A)^I$ a collective preference $P \in L(A)$. As preference aggregation is the core problem of social choice theory, namely the classical Arrovian aggregation problem of the (im)possibility of constructing a social welfare function which assigns to each profile of individual preferences a collective preference relation and satisfies a set of normatively desirable properties, the significance of social choice theory as a fundamental tool for the study of multiagent systems has always been recognized [18], — especially so since the incorporation of computational issues in the new field of computational social choice [4]. This significance of social choice theory has greatly been increased by the recent generalization of the classical Arrovian aggregation problem, culminating in the new field of judgment aggregation (for a survey see [14]). An essential feature of this generalization is the extension of the problem of aggregation from the aggregation of preferences to the aggregation of arbitrary information represented by individual "judgments" on a set of logically interconnected propositions (the agenda) expressed in some formal language (typically propositional logic), the truth values of which are to be collectively determined.

Especially, in order to also exploit the expressive power of first-order logic, it seems natural to use the potential of model theory which, broadly speaking, studies the relation between abstract structures and statements about them (for an introduction to model theory see [3]) and to analyse the problem of of aggregating judgments as the problem of aggregating the models that satisfy these judgments (see [11], following [13]). In a model theoretic perspective, the aggregation problem as it underlies Arrovian impossibility results can be related to the well known fact (see [1], p. 174) that a (direct) product of individual models (e.g. a profile of individual preference relations) may not share the first-order properties of its factor models (e.g. transitivity). For this reason the direct product construction is often modified by using another boolean algebra than $\mathbf{2} = \{0, 1\}$ and in particular the power-set algebra over the index set as an algebra of truth values (see e.g. [2]). This approach was first applied to social welfare functions in [19] as one of the many attempts to overcome Arrow's dictatorship result and is here extended to the problem of aggregating judgments in first-order logic. While the major body of the literature on judgment aggregation studies the (in)consistency between properties of the aggregation rule and properties of the agenda (for a survey see [6]), the significance of our simple (im)possibility results consists in stressing the importance of the set of truth values and its algebraic structure.[4] This significance is closely related to a property of order preservation of mappings between the power-set algebra over the set of individuals and the algebra of truth values.

The theory of boolean algebras can be seen as the natural method to analyse axiom systems in first-order predicate logic. The reason is that axiom systems under first-order predicate logic induce an algebraic structure on the set of well-formed formulae: The axiom system combined with the deduction rules of first-order logic induces a notion of provability, and the quotient of the set of well-formed formulae with respect to the equivalence relation of provable equivalence turns out to be a boolean algebra, called the Lindenbaum algebra.

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[4] Among the relatively few many-valued extensions of judgment aggregation [17], [7], and [10] deserve to be noted. Closest in spirit to our (im)possibility results is, however, [8] which establishes a characterization of the possibility/impossibility boundary in the framework of t-norms.

2 Formal framework and results

Fix an arbitrary set A, and let $\mathcal{L}$ be a language consisting of constant symbols for all elements a of A as well as (at most countably many) predicate symbols P_n, $n \in \mathbf{N}$. We shall denote the arity of P_n by $\delta(n)$ (for all $n \in \mathbf{N}$).

In the case of preference aggregation, A is interpreted as the set of alternatives and the (unique) binary predicate symbol P denotes strict preference.

Let $\mathcal{S}$ be the set of atomic formulae in $\mathcal{L}$, and let $\mathcal{T}$ be the *boolean closure* of $\mathcal{S}$, i.e. the closure of $\mathcal{S}$ under the logical connectives $\neg$, $\wedge$, $\vee$.

Obviously, in the case of preference aggregation, $\mathcal{S} = \{P(x, y) : x, y \in A\}$.

The relational structure $\mathfrak{A} = \langle A, \langle R_n \; : \; n \in \mathbf{N}\rangle\rangle$ is called a *realisation of* $\mathcal{L}$ *with domain* A or an $\mathcal{L}$*-structure with domain* A if and only if the arities of the relations R_n correspond to the arities of the predicate symbols P_n and the relations are evaluated in A, that is if $R_n \subseteq A^{\delta(n)}$ for each n.

An $\mathcal{L}$-structure $\mathfrak{A}$ is a *model* of the theory T if $\mathfrak{A} \models \varphi$ for all $\varphi \in T$, i.e. if all sentences of the theory hold true in $\mathfrak{A}$ (with the usual Tarski definition of truth).

In the case of preference aggregation with linear preferences, T is the set of $\mathcal{L}$-sentences which axiomatize the class of linear orders, i.e.

$\forall x \neg P(x, x)$ (irreflexivity),

$\forall x \forall y \forall z[(P(x, y) \wedge P(y, z)) \rightarrow P(x, z)]$ (transitivity),

$\forall x \forall y(P(x, y) \vee P(y, x) \vee x = y)$ (completeness).

A boolean-valued model for $\mathcal{L}$ is a mapping which assigns to each $\mathcal{L}$-formula λ a truth value $\|\lambda\|$ in some arbitrary complete boolean algebra $\mathbf{B} = \langle B, \sqcup, \sqcap, ^*, 0_B, 1_B\rangle$ in such a way that boolean connectives and logical connectives commute:

$\|\neg\lambda\| = \|\lambda\|^*$; $\|\phi \vee \varphi\| = \|\phi\| \sqcup \|\varphi\|$; $\|\phi \wedge \varphi\| = \|\phi\| \sqcap \|\varphi\|$ (see [12]).

Boolean-valued models stand in a natural relation to products of models, like they play a role in aggregation theory. Indeed, in a model theoretic framework, a profile of individual judgments is nothing else than the direct product of the individual (factor) models, and this makes the power-set algebra over the index set of individuals a natural choice for a modification of the direct product construction and an alternative boolean valuation (see [1], p. 174f.)

Let Ω be the collection of models of T with domain A.

Let I be a (finite or infinite) set. Elements of I will be called *individuals*, elements of Ω^I will be called *profiles* and will be denoted by $\underline{\mathfrak{A}} := \langle \mathfrak{A}\rangle_{i\in I}$.

Thus, in the case of preference aggregation, Ω^I represents the set of all logically possible profiles of preferences.

For simplicity, let us assume for our preference aggregation example that $I = \{1, 2, 3\}$, $A = \{a, b, c\}$, and that the preferences of the individuals are given by the classical configuration of the Condorcet paradox, respectively

$\mathfrak{A}_1 \models P(a, b) \wedge P(b, c) \wedge P(a, c)$

$\mathfrak{A}_2 \models P(b, c) \wedge P(c, a) \wedge P(b, a)$

$\mathfrak{A}_3 \models P(c, a) \wedge P(a, b) \wedge P(c, b)$.

Remark 1. Observe that any such profile $\underline{\mathfrak{A}} \in \Omega^I$ as a mapping $I \to \Omega$ induces a map from the set of $\mathcal{L}$-formulae to the power-set algebra $P(I) = \langle 2^I, \cup, \cap, \complement, \varnothing, I\rangle$[5], which maps every $\mathcal{L}$-formula λ to the coalition of all individuals whose models satisfy λ, i.e. $\{i \in I : \mathfrak{A}_i \models \lambda\}$.

Thus, e.g. in our simple preference aggregation example $\{i \in I : \mathfrak{A}_i \models P(a,c)\} = \{1\}$ and $\{i \in I : \mathfrak{A}_i \models P(a,b)\} = \{1,3\}$.

We now call a boolean-valued map f which assigns to each profile $\underline{\mathfrak{A}} \in \Omega^I$ and each formula λ a truth value $\|\lambda\|_f^{\underline{\mathfrak{A}}}$ in some arbitrary complete boolean algebra $\mathbf{B} = \langle B, \sqcup, \sqcap, ^*, 0_B, 1_B\rangle$ a **boolean-valued aggregation rule** (BVAR) if and only if $\|\neg\lambda\|_f^{\underline{\mathfrak{A}}} = \left(\|\lambda\|_f^{\underline{\mathfrak{A}}}\right)^*$; $\|\phi \vee \varphi\|_f^{\underline{\mathfrak{A}}} = \|\phi\|_f^{\underline{\mathfrak{A}}} \sqcup \|\varphi\|_f^{\underline{\mathfrak{A}}}$; $\|\phi \wedge \varphi\|_f^{\underline{\mathfrak{A}}} = \|\phi\|_f^{\underline{\mathfrak{A}}} \sqcap \|\varphi\|_f^{\underline{\mathfrak{A}}}$ (see [12]).

If we now take for our preference aggregation example the power-set algebra $P(I)$ as an algebra of truth valuations,[6] we obtain a boolean-valued map F which assigns to each atomic formula the set of individuals in the models of which it holds true. Thus, e.g. $\|P(a,b)\|_F^{\underline{\mathfrak{A}}} = \|\neg P(b,a)\|_F^{\underline{\mathfrak{A}}} = \{1,3\}$, whereas $\|P(a,b) \wedge P(b,c)\|_F^{\underline{\mathfrak{A}}} = \|P(a,c)\|_F^{\underline{\mathfrak{A}}} = \{1\}$ and $\|P(a,b) \vee P(b,c)\|_F^{\underline{\mathfrak{A}}} = \{1,2,3\}$.

The following properties are reformulations of standard conditions for judgment aggregation rules in the framework of BVARs.

In particular, the non-dictatorship condition can be expressed in the following way:

Definition 1. *A BVAR f is* ***non-dictatorial*** *if there exists no individual $i \in I$ such that for every $\mathcal{L}$-formula λ and every profile $\underline{\mathfrak{A}} \in \Omega^I$*
$\mathfrak{A}_i \models \lambda \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} = 1_B$ *(where 1_B the top element of the set of truth values).*

Obviously, non-dictatorship is only relevant if the set I consists of at least two individuals, which will be assumed throughout.

Intuitively, non-dictatorship in the framework of BVARs guarantees that there exists no individual who can ensure for her judgments the highest truth

[5] Wherein $\complement D = I \setminus D$ for all $D \subseteq I$.

[6] For another simple example which does not involve the power-set boolean algebra, consider a set of three agents $I = \{1,2,3\}$ facing a set of four different alternatives $A = \{a,b,c,d\}$. Suppose each of them linearly ranks the alternatives according to their own subjective preferences.

Let $\mathcal{L}$ be the first-order language consisting of four constants a,b,c,d and one relation symbol P, and let T be the theory of linear orders. Let Ω be the set of models of T with domain A. A profile is then simply a triple of linear orders on the set $\{a,b,c,d\}$, i.e. an element of Ω^I.

A particularly simple aggregation function is a map $f : \Omega^I \times \mathcal{L} \to 2$ which maps to each pair $\langle\underline{\mathfrak{A}}, \lambda\rangle$ of a profile $\underline{\mathfrak{A}} \in \Omega^I$ and an $\mathcal{L}$-formula λ the truth value which a majority of the agents assigns. In other words, for all $\underline{\mathfrak{A}} \in \Omega^I$ and all $\mathcal{L}$-formulae λ,

$$f(\underline{\mathfrak{A}}, \lambda) = \begin{cases} 1, & \#\{i \in I \ : \ \mathfrak{A}_i \models \lambda\} \geq 2, \\ 0, & \text{otherwise} \end{cases}$$

Verifying that this is a paretian, systematic and non-dictatorial boolean-valued aggregation function with values in $\{0,1\}$ is left as an exercise to the reader.

degree. On the other hand, the intuitively appealing Pareto principle requires that unanimous agreement be respected by a judgment aggregation rule:

Definition 2. *A BVAR f is **paretian** if for every $\mathcal{L}$-formula λ and every profile $\underline{\mathfrak{A}} \in \Omega^I$*
$\{i \in I : \mathfrak{A}_i \models \lambda\} = I \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} = 1_B$.

Central to aggregation problems are independence conditions of various strength:

Definition 3. *A BVAR f is **independent** if for every $\mathcal{L}$-formula λ and every pair of profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$*
$\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}'_i \models \lambda\} \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} = \|\lambda\|_f^{\underline{\mathfrak{A}}'}$.

Definition 4. *A BVAR f is **neutral** if for every $\mathcal{L}$-formulae λ, λ' and every profile $\underline{\mathfrak{A}} \in \Omega^I$*
$\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}_i \models \lambda'\} \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} = \|\lambda'\|_f^{\underline{\mathfrak{A}}}$.

Definition 5. *A BVAR f is **systematic** if it is independent and neutral, i.e. if for every pair of $\mathcal{L}$-formulae λ, λ' and every pair of profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$*
$\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}'_i \models \lambda'\} \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} = \|\lambda'\|_f^{\underline{\mathfrak{A}}'}$.

The property of systematicity might appear strong at first sight but it is well-known in the literature on judgment aggregation that it is implied by the independence property and a condition of logical richness known as total blockedness, i.e. if every formula is related to every other one by a sequence of conditional entailments.

The framework of BVARs allows to use the partial order structure $\langle P(I), \subseteq \rangle$ of the power-set algebra $P(I)$ over the set of individuals (the "coalition algebra"), respectively of the algebra of truth values $\langle \mathbf{B}, \sqsubseteq \rangle$ for the formulation of conditions on aggregation rules.[7] In particular, the monotonicity property can be formulated in a natural way as such an order preservation property:

Definition 6. *A BVAR f is **monotonic** if for every $\mathcal{L}$-formula λ and every pair of profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$*
$\{i \in I : \mathfrak{A}_i \models \lambda\} \subsetneq \{i \in I : \mathfrak{A}'_i \models \lambda\} \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} \sqsubseteq \|\lambda\|_f^{\underline{\mathfrak{A}}'}$.

Monotonicity is known to be an important property of aggregation rules because it guarantees non-manipulability, i.e. the impossibility for any individual to increase the collectively assigned truth value of a formula by signalling its negation.

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[7] Herein, $\sqsubseteq$ is the canonical partial order on the boolean algebra; it can be defined algebraically, for all $x, y \in B$, by

$$x \sqsubseteq y \Leftrightarrow x \sqcap y^* = 0_B$$

(or equivalently $x \sqsubseteq y \Leftrightarrow x \sqcap y = x$).

The conjunction of monotonicity and independence (known in the judgment aggregation literature as monotone independence, see [16]) can now be formulated as an order preservation property of the aggregation rule with respect to the partial orders of the coalition algebra and the algebra of truth values.

Proposition 1. *A BVAR f satisfies* ***monotone independence*** *(i.e. is monotonic and independent) if and only if for every pair of profiles $\underline{\mathfrak{A}}, \underline{\mathfrak{A}}' \in \Omega^I$ and every formula $\lambda \in \mathcal{T}$*

$$\{i \in I : \mathfrak{A}_i \models \lambda\} \subseteq \{i \in I : \mathfrak{A}'_i \models \lambda\} \Rightarrow \|\lambda\|_f^{\underline{\mathfrak{A}}} \sqsubseteq \|\lambda\|_f^{\underline{\mathfrak{A}}'} . \tag{1}$$

A natural BVAR F can now be defined by assigning to every $\mathcal{L}$-formula λ and every profile $\underline{\mathfrak{A}} \in \Omega^I$ precisely the subset of individuals in whose models it holds true, i.e. $\|\lambda\|_F^{\underline{\mathfrak{A}}} = \{i \in I : \mathfrak{A}_i \models \lambda\}$. Thus, the algebra of truth values is simply identified with the coalition algebra.

This construction immediately leads to the following possibility result:

Theorem 1. *The BVAR F is a neutral, paretian and non-dictatorial judgment aggregation rule which satisfies monotone independence.*

For a proof, see the Appendix; the easy verification for the case of our simple preference aggregation example being left to the reader.

The main interest of this simple boolean-valued construction consists in highlighting the implications for the aggregation problem of the structure of the set of truth values and the significance of the condition of order preservation with respect to the power-set algebra over the set of individuals and the algebra of truth values (for a deeper exploration of the relation between judgment aggregation rules and boolean algebra homomorphisms see [9]).

This significance is closely related to a property of homomorphisms of boolean algebras. [8] Note that systematicity (i.e. the conjunction of independence and neutrality) permits a decomposition of every BVAR as $h \circ F$. One can show that this h is a homomorphism and thus order-preserving, whence neutrality and independence already entail monotonicity.

By the **agenda richness condition** we mean that there are $\lambda, \mu \in \mathcal{S}$ such that T is consistent with each of $\lambda \wedge \mu$, $\neg\lambda \wedge \mu$ and $\lambda \wedge \neg\mu$.

In preference aggregation, this condition is satisfied if there are at least three alternatives $x, y, z \in A$ such that $\lambda = P(x, y)$ and $\mu = P(y, z)$.

Theorem 2. *Let the agenda richness condition be satisfied. A neutral and independent BVAR induces a homomorphism h_f of the coalition algebra $P(I) = \langle 2^I, \cup, \cap, \complement, \varnothing, I\rangle$ to its co-domain, the boolean algebra of truth values $\mathbf{B} = \langle B, \sqcup, \sqcap, ^*, 0_B, 1_B\rangle$*

[8] A homomorphism of a boolean algebra B into a boolean algebra B' is a map $h : B \to B'$ which preserves the algebraic operations, i.e. such that for all $x, y \in B$, $h(x \sqcap y) = h(x) \sqcap h(y)$, $h(x \sqcup y) = h(x) \sqcup h(y)$, $h(x^*) = h(x)^*$. A homomorphism is always order-preserving with respect to the canonical partial orders of the corresponding boolean algebras.

As we shall see presently, using the notion of the Lindenbaum algebra, Theorem 2 can be reformulated as an algebraic factorization result. Let $\vdash$ be the provability relation of classical first-order logic, let $T \subseteq \mathcal{L}$ be consistent (possibly empty), and let $\equiv$ denote provable equivalence given T (i.e., $\phi \equiv \psi$ if and only if both $T \cup \{\phi\} \vdash \psi$ and $T \cup \{\psi\} \vdash \phi$). The set of equivalence classes of $\mathcal{L}$-formulae under $\equiv$ is known as the *Lindenbaum algebra* and will be denoted $\mathcal{L}/_{\equiv}$. It is obvious that for every BVAR f, the map

$$H_f : \mathcal{L}/_{\equiv} \times \Omega^I \to B, \quad \langle [\lambda]_{\equiv}, \underline{\mathfrak{A}} \rangle \mapsto \|\lambda\|_f^{\mathfrak{A}}$$

is well-defined. It is also clear that for every $\underline{\mathfrak{A}} \in \Omega^I$, $H_f(\cdot, \underline{\mathfrak{A}})$ is a homomorphism. Given any profile $\underline{\mathfrak{A}} \in \Omega^I$, we then have the following commutative diagram or factorization:

$$\begin{array}{ccc} \mathcal{L}/_{\equiv} & \overset{H_F(\cdot,\underline{\mathfrak{A}})}{\longrightarrow} & P(I) \\ {\scriptstyle H_f(\cdot,\underline{\mathfrak{A}})} \downarrow & \swarrow {\scriptstyle h_f} & \\ B & & \end{array}$$

Hence, every boolean-valued aggregation rule can be, for an arbitrary fixed profile, decomposed into a (a) a structure-preserving map from the set of $\mathcal{L}$-formulae (modulo provable equivalence) to the coalition algebra and (b) another structure-preserving map from the coalition algebra to the actual algebra of truth values. This latter step can be seen as a coarsening of the set of the algebra of truth values at the social level compared to the richness of "social valuations" of the formulae by the coalition algebra. The extreme case is the classical situation where the truth values at the social level are just binary.

Now there is a connection between the homomorphy among boolean algebras and the source of dictatorship in this classical case of binary social truth values, viz. the existence of an ultrafilter on the set of individuals.

Recall that a non-empty subset $F \subsetneq B$ of a boolean algebra B is a **(proper) filter** if and only if for all $x, y \in F$ and any $z \sqsupseteq x$, both $x \sqcap y \in F$ and $z \in F$ (*meet closure* and *successor closure*). A filter $U \subsetneq B$ is an **ultrafilter** if and only if it is *maximal* in the sense that there exists no filter F with $U \subsetneq F \subsetneq B$.[9] In the case of the power-set boolean algebra 2^I, a proper filter is a proper non-empty subset of 2^I which is closed under the intersection operation $\cap$ and the superset relation $\supseteq$; a proper filter U in 2^I is an ultrafilter if and only for every set $C \in 2^I$ either C or its complement $\complement C = I \setminus C$ is an element of U. It is well known that every ultrafilter on a finite set is the collection of all supersets of a singleton — the dictator —, and 2-valued homomorphisms have an ultrafilter as its shell (see e.g. [3]):

Lemma 1. *Let $g : B' \to B$ be a homomorphism between boolean algebras. Then the shell of g, i.e. the set $\{x \in B' : g(x) = 1_B\}$ is a filter. If B is the two-valued algebra $\mathbf{2} = \{0, 1\}$ of truth values, then the shell $g^{-1}\{1_B\}$ of g is an ultrafilter.*

[9] The maximality condition is equivalent to the so-called *ultrafilter property*: A filter U is an ultrafilter if and only if for all $x \in B$, either $x \in U$ or $x^* \in U$.

With the help of such a purely algebraic result, we obtain in the BVAR framework a typical Arrow-style dictatorship result, as a simple corollary of the previous theorem:

Corollary 1. *Let f be a neutral BVAR which satisfies monotone independence and has co-domain $\mathbf{2} = \{0,1\}$. If the set I of individuals is finite, then f is a dictatorship.*

It is thus the rigidity of the truth-value algebra at the social level which forces dictatorship results. This finding confirms the intuition behind the recent unification of probabilistic opinion pooling and judgment aggregation through the overarching concept of propositional attitudes [5]. In the former case, there is a continuum of possible propositional attitudes at the social level, allowing for a beautiful possibility result in terms of linear opinion pools [15], and in the latter case, only binary propositional attitudes are admissible, leading to dictatorial impossibility results [17].

3 Conclusion

We have thus described a framework for boolean-valued judgment aggregation. While the major body of the literature on judgment aggregation draws attention to inconsistencies between properties of the agenda and properties of the aggregation rule, the simple (im)possibility results in this paper highlight the role of the set of truth values and its algebraic structure. In particular, it is shown that central properties of aggregation rules can be formulated as homomorphy or order-preservation conditions on the mapping between the power-set algebra over the set of individuals and the algebra of truth values. This is further evidence that the problems in aggregation theory are driven by information loss, which in our framework is given by a coarsening of the algebra of truth values.

Appendix: Proofs

Proof (Proof of Proposition 1). (if part) a) Monotonicity of f can easily be seen from the fact that the antecedent of the property in formula (1) is just a weakening of the antecedent of the monotonicity property. b) Independence of f follows from the fact that in case $\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}'_i \models \lambda\}$, formula (1) requires both $\|\lambda\|^{\mathfrak{A}}_f \sqsubseteq \|\lambda\|^{\mathfrak{A}'}_f$ and $\|\lambda\|^{\mathfrak{A}'}_f \sqsubseteq \|\lambda\|^{\mathfrak{A}}_f$, and thus by the antisymmetry of the partial order $\sqsubseteq$ on B, $\|\lambda\|^{\mathfrak{A}}_f = \|\lambda\|^{\mathfrak{A}'}_f$. (only if part) Suppose f is monotonic and independent. If the antecedent in formula (1) is satisfied, then either $\{i \in I : \mathfrak{A}_i \models \lambda\} \subsetneq \{i \in I : \mathfrak{A}'_i \models \lambda\}$ or $\{i \in I : \mathfrak{A}_i \models \lambda\} = \{i \in I : \mathfrak{A}'_i \models \lambda\}$. In the former case, the monotonicity yields $\|\lambda\|^{\mathfrak{A}}_f \sqsubseteq \|\lambda\|^{\mathfrak{A}'}_f$, and in the latter case, so does the independence of f.

Proof (Proof of Theorem 1). By construction, F is neutral. Also, F satisfies monotone independence, since the antecedent and the consequent in formula

(1) become identical if F is inserted for f. That F is both non-dictatorial and paretian can be verified easily by noting that $\|\lambda\|_F^{\mathfrak{A}} = 1_{P(I)}$ $(= I)$ is tantamount to $\{i \in I : \mathfrak{A}_i \models \lambda\} = I$ (for every profile $\underline{\mathfrak{A}} \in \Omega^I$ and every formula λ).

Proof (Proof of Theorem 2). By the agenda richness, it is easy to see that

$$\begin{aligned}&\text{for every } D, E \subseteq I \text{ there is a profile } \underline{\mathfrak{D}} \text{ such that}\\ &D = \{i \in I : \mathfrak{D}_i \models \lambda\} \text{ and } E = \{i \in I : \mathfrak{D}_i \models \mu\}.\end{aligned} \tag{2}$$

Let $h_f(D) = \|\lambda\|_f^{\mathfrak{D}}$. Then, h_f (henceforth h for brevity) is well-defined — in the sense of being independent of the choice of $\underline{\mathfrak{D}}$ and λ — because whenever $\{i \in I : \mathfrak{D}_i \models \lambda\} = D = \{i \in I : \mathfrak{D}'_i \models \lambda'\}$, the independence and neutrality (i.e., systematicity) of h ensures that $\|\lambda\|_f^{\mathfrak{D}} = \|\lambda\|_f^{\mathfrak{D}'} = \|\lambda'\|_f^{\mathfrak{D}'}$. Now, for every $D \subseteq I$, one can find (by our above observation (2), applied to $\complement D$ instead of D) a profile $\underline{\mathfrak{C}}$ such that $\complement D = \{i \in I : \mathfrak{C}_i \models \lambda\}$ whence $h(\complement D) = \|\lambda\|_f^{\mathfrak{C}}$. Now by Tarski's definition of truth, $\{i \in I : \mathfrak{D}_i \models \neg\lambda\} = \complement D = \{i \in I : \mathfrak{C}_i \models \lambda\}$. Since f is both neutral and independent (hence systematic), this entails $\|\neg\lambda\|_f^{\mathfrak{D}} = \|\lambda\|_f^{\mathfrak{C}}$. By our definition of a BVAR, this amounts to $\left(\|\lambda\|_f^{\mathfrak{D}}\right)^* = \|\lambda\|_f^{\mathfrak{C}}$, whence $h(D)^* = h(\complement D)$ for arbitrary $D \in 2^I$. In a similar vein, one can establish $h(D) \sqcap h(E) = h(D \cap E)$ for all $D, E \subseteq I$. Indeed, let $D, E \subseteq I$. Then there will (by (2)) be a profile $\underline{\mathfrak{D}}$ such that

$$\begin{aligned} D &= \{i \in I : \mathfrak{D}_i \models \lambda\}\\ E &= \{i \in I : \mathfrak{D}_i \models \mu\}\end{aligned}$$

and (by the consistency of $T \cup \{\lambda \wedge \mu\}$) another profile $\underline{\mathfrak{C}}$ such that

$$D \cap E = \{i \in I : \mathfrak{C}_i \models \lambda \wedge \mu\},$$

whence

$$h(D \cap E) = \|\lambda \wedge \mu\|_f^{\mathfrak{C}}, \quad h(D) = \|\lambda\|_f^{\mathfrak{D}}, \quad h(E) = \|\mu\|_f^{\mathfrak{D}}.$$

Now by Tarski's definition of truth, $\{i \in I : \mathfrak{D}_i \models \lambda \wedge \mu\} = D \cap E = \{i \in I : \mathfrak{C}_i \models \lambda \wedge \mu\}$. Since f is independent, this entails $\|\lambda \wedge \mu\|_f^{\mathfrak{D}} = \|\lambda \wedge \mu\|_f^{\mathfrak{C}}$. By our definition of a BVAR, this amounts to $\|\lambda\|_f^{\mathfrak{D}} \sqcap \|\mu\|_f^{\mathfrak{D}} = \|\lambda \wedge \mu\|_f^{\mathfrak{C}}$, whence $h(D) \sqcap h(E) = h(D \cap E)$ for arbitrary $D, E \in 2^I$. By De Morgan's formulae in the boolean algebras B and 2^I as well as iterated application of the preservation of meets and complements by h one can now deduce that $h(D) \sqcup h(E) = h(D \cup E)$ for arbitrary $D, E \in 2^I$.[10] So, h preserves joins, too, and thus is a homomorphism.

[10]

$$\begin{aligned} h(D) \sqcup h(E) &= (h(D)^* \sqcap h(E)^*)^* = \left(h(\complement D) \sqcap h(\complement E)\right)^* = \left(h\left((\complement D) \cap (\complement E)\right)\right)^*\\ &= h\left(\complement\left((\complement D) \cap (\complement E)\right)\right) = h(D \cup E).\end{aligned}$$

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