Functorial Coalgebraic Logic: The case of many-sorted varieties

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Abstract

Following earlier work, a modal logic for T-coalgebras is a functor L on a suitable variety. Syntax and proof system of the logic are given by presentations of the functor. This paper makes two contributions. First, a previous result characterizing those functors that have presentations is generalized from endofunctors on one-sorted varieties to functors between many-sorted varieties. This yields an equational logic for the presheaf semantics of higher-order abstract syntax. As another application, we show how the move to functors between many-sorted varieties allows to modularly combine syntax and proof systems of different logics. Second, we show how to associate to any set-functor T a complete (finitary) logic L consisting of modal operators and Boolean connectives.

Keywords: Coalgebra, Modal Logic, Stone Duality, Coalgebraic Logic, Sifted Colimits, Variety, Universal Algebra, Presentation by Operations and Equations

1 Introduction

Beginning with [6,17], it has been argued that logics for T-coalgebras (where T is an endofunctor on Set) are suitably described by endofunctors L on the category of Boolean algebras. Syntactically, L specifies an extension of Boolean propositional logic by modal operators and axioms. Semantically, L gives a logical description of the 'transition type' T of the coalgebras.

[17] showed that the modal logics for coalgebras usually considered [22,26,15] give rise to such functors L.[1] To describe the class of functors arising in such a way, [8] introduced the notion of a functor having a finitary presentation by operations and equations. This notion was investigated more systematically in [20] where it is shown that a functor L on a (finitary) variety $\mathcal{A}$ has a presentation by operations and equations if and only if L preserves sifted colimits. Although not as well known

[1] Although quite different from the above, Moss's coalgebraic logic [21] also fits into our framework.

as filtered colimits, sifted colimits are the right concept when working with varieties (as opposed to locally finitely presentable categories): Each variety $\mathcal{A}$ is the free cocompletion by sifted colimits of the dual of the Lawvere theory of $\mathcal{A}$. The reason is that algebras for a Lawvere theory are set-valued product-preserving functors and sifted colimits are precisely those colimits that commute in Set with finite products.

This paper continues this line of research. We start by generalizing the results of [20] on functors on varieties from the one-sorted to the many sorted case (Section 3). This generalization in itself is not difficult, but it has interesting applications. The first, maybe somewhat unexpected, is that it provides an equational logic for the binding algebras used in the work of [11] (Section 4). The second application shows how to modularly compose presentations of different functors (Section 5). Even if one was only interested in one-sorted coalgebras, only the move to many-sorted logics gives the desired modularity. Finally, whereas [20] shows how to prove strong completeness results by generalizing the Jónsson-Tarski representation theorem for Boolean algebras with operators, we show here how to prove completeness results for categories of coalgebras that do not satisfy the restrictions needed for the strong completeness result of [20]. In particular, we show that an arbitrary set-functor has a complete finitary logic (Section 6).

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2 Some preliminaries

For an endofunctor L on a category $\mathcal{A}$, we consider the category of L-*algebras*, denoted by $\mathsf{Alg}(L)$, whose objects are defined as pairs (A, α) such that $\alpha : LA \to A$ is a morphism in $\mathcal{A}$. A morphism of $L-$algebras $f : (A, \alpha) \to (A', \alpha')$ is a morphism $f : A \to A'$ of $\mathcal{A}$ such that $f \circ \alpha = \alpha' \circ Lf$. Dually, for an endofunctor $T : \mathcal{A} \to \mathcal{A}$ we consider the category of $T-$*coalgebras*, denoted by $\mathsf{Coalg}(T)$, whose objects are pairs (A, γ), such that $\gamma : A \to TA$. A morphism of $T-$coalgebras $f : (A, \gamma) \to (A', \gamma')$ is an arrow $f : A \to A'$ of $\mathcal{A}$ such that $Tf \circ \gamma = \gamma' \circ f$.

Let S be a set (of sorts). A *signature* Σ is a set of operation symbols together with an arity map $a : \Sigma \to S^* \times S$ which assigns to each element $\sigma \in \Sigma$ a pair $(s_1, \dots, s_n; s)$ consisting of a finite word in the alphabet S indicating the sort of the arguments of σ and an element of S indicating the sort of the result of σ. To each signature we can associate an endofunctor on Set^S, which will be denoted for simplicity with the same symbol Σ:

$$(\Sigma X)_s = (\coprod_{k \in \omega_f^S} \Sigma_{k,s} \times X^k)_s$$

Here, by ω_f^S we denote the set of functions from S to ω which have finite support (i.e. which vanish everywhere except for a finite set) and by X^k the set of presheaf morphisms $\mathsf{Set}^S(k, X)$. In detail, if $k \in \omega_f^S$ has support $\{s_1, ..., s_n\}$ then $\Sigma_{k,s}$ is a set of operations of arity $(s_1 ... s_n; s)$ and X^k is isomorphic in Set with the finite product $X_{s_1}^{k(s_1)} \times \cdots \times X_{s_n}^{k(s_n)}$. Conversely to each endofunctor on Set^S given as above corresponds a signature $\coprod_{k\in\omega_f^S} \Sigma_{k,s}$. Throughout this paper we will make no notational difference between the signature and the corresponding functor, and it will be clear from the context when we refer to the set of operation symbols or to a Set^S endofunctor. The algebras for a signature Σ are precisely the algebras for the corresponding endofunctor, and form the category denoted by $\mathsf{Alg}(\Sigma)$. The terms over an S-sorted set of variables X are defined in the standard manner and and form a set denoted by $\mathrm{Term}_\Sigma(X)$, and in fact this is the underlying set of the free Σ-algebra generated by X. An equation consists of a pair (τ_1, τ_2) of terms of the same sort, usually denoted $\tau_1 = \tau_2$. A Σ-algebra A satisfies this equation if and only if, for any interpretation of the variables of X, we obtain equality in A. A class $\mathcal{A}$ of Σ-algebras is called a *variety* or an *equational class* if there exists a set of equations E such that an algebra lies in $\mathcal{A}$ if and only if it satisfies all the equations of E. Such an equational class will be denoted by $\mathsf{Alg}(\Sigma, E)$. The forgetful functor $U : \mathcal{A} \to \mathsf{Set}^S$ has a left adjoint F and $\mathsf{Alg}(\Sigma, E)$ is monadic over Set^S, i.e it is isomorphic with the Eilenberg-Moore category $(\mathsf{Set}^S)^T$ for the monad T corresponding to the adjoint pair $F \dashv U$. Also notice that UF is a finitary, that is, UF preserves filtered colimits. In [3], varieties are described independently of signatures and equations, via finite product sketches, and it is proved that any variety is a locally finitely presentable category. Recall that a category is locally finitely presentable provided that it is cocomplete and it has a set K of finitely presentable objects such that any object is a filtered colimit of objects of K. We will say that a category is *lfp* if it is locally finitely presentable.

Endofunctors may appear via composition of functors between different varieties. Therefore, it is useful to consider a slight generalization of the notion of signature. If S_1 and S_2 are sets of sorts we will consider operations with arguments of sorts in S_1 and returning a result of a sort in S_2, encompassed in the signature functor $\Sigma : \mathsf{Set}^{S_1} \to \mathsf{Set}^{S_2}$:

$$\Sigma X = \big(\coprod_{k\in\omega_f^{S_1}} \Sigma_{k,s} \times X^k\big)_{s\in S_2} \tag{1}$$

An important example of a (finitary) variety of algebras is the functor category $\mathsf{Set}^{\mathcal{C}}$ for any small category $\mathcal{C}$. The sorts are the objects of $\mathcal{C}$, the operations symbols are the morphisms of $\mathcal{C}$ (all of them with arity 1), and the equations are given by the composition of morphisms in $\mathcal{C}$.

A *sifted category* $\mathcal{D}$ is a small category such that colimits over $\mathcal{D}$ commute in Set with finite products. A sifted colimit in a category $\mathcal{C}$ is a colimit over $\mathcal{D}$. The most important examples of sifted colimits are filtered colimits and reflexive coequalizers. An object in a category is called *strongly finitely presentable* if its hom-functor preserves sifted colimits. It is shown in [4] that any object in a variety is a sifted colimit of strongly finitely presentable algebras, which in a variety are the retracts of finitely generated free algebras. An important observation is that sifted colimit preserving functors on varieties are determined by their action on free algebras. Another useful observation is that for an S-sorted variety $\mathcal{A}$ the forgetful functor

$U : \mathcal{A} \to \mathsf{Set}^S$ and its left adjoint F are sifted colimit preserving functors. In fact F preserves all colimits.

3 Presenting algebras and functors

The purpose of this section is an algebraic investigation of logics for coalgebras. The general idea is as follows. Just as coalgebras are given wrt a functor T on, say, Set, so are logics for coalgebras given by a functor L on, say, Boolean algebras. The following example shows how logics for coalgebras given in a more conventional style give rise to a functor on the category BA of Boolean algebras.

Example 3.1 Let $T = \mathcal{P}$ be the covariant powerset functor. The modal logic **K** associated to $\mathcal{P}$-coalgebras (=Kripke frames) can be described by the functor L which maps a Boolean algebra A to the Boolean algebra LA freely generated by $\{\Box a \mid a \in A\}$ modulo the relations $\Box\top = \top$ and $\Box(a \wedge b) = \Box a \wedge \Box b$. We see that the modal operators appear as generators and the modal axioms as relations. Of course, from a logical point of view, we want the generators to be operations and the relations to be universally quantified equations. In other words, we need that the description of LA in terms of generators and relations is uniform in A. This is exactly captured by Definition 3.3 below.

It is not difficult to see that the category $\mathsf{Alg}(L)$ of algebras for the functor L is isomorphic to the category of Boolean algebras with operators, which constitute the standard algebraic semantics of **K** in modal logic. In particular, the initial L-algebra is the Lindenbaum-Tarski algebra of the modal logic **K**.

To simply replace a concrete modal logic by the corresponding functor is a powerful abstraction that makes a number of category theoretic methods available to modal logic. This section makes sure that the move from logics to functors is not an over-generalisation: Every suitable functor L will come from a modal logic in exactly the same way as in the example above. The reader who wants to know more about the relationship between T-coalgebras and L-algebras before reading this section might want to skip ahead to Section 6.

Definition 3.2 Let A be a many-sorted algebra in a variety $\mathcal{A}$. We say that (G, E) is a presentation for A if G is an S-sorted set of generators and $E = (E_s)_{s\in S},\ E_s \subset (UFG)_s \times (UFG)_s$ is an S sorted set of equations such that A is the coequalizer of the following diagram:

$$FE \underset{\pi_2^\sharp}{\overset{\pi_1^\sharp}{\rightrightarrows}} FG \xrightarrow{q_A} A \tag{2}$$

The maps $\pi_1^\sharp, \pi_2^\sharp$ are induced, via the adjunction, by the projections π_1, π_2 of E on UFG.

Next we want to define a presentation for a functor $L : \mathcal{A}_1 \to \mathcal{A}_2$ between many-sorted varieties. For $i \in \{1, 2\}$, denote by S_i the set of sorts for $\mathcal{A}_i$ respectively, by $U_i : \mathcal{A}_i \to \mathsf{Set}^{S^i}$ the corresponding forgetful functor, and by F_i its left adjoint. We will do this in the same fashion as in [20] and [8], keeping in mind that we need to extend (2) uniformly: this means that the generators and equations for each LA will depend functorialy on A. Suppose A is a many-sorted algebra in $\mathcal{A}_1$. The generators $\Sigma U_1 A$ for the algebra LA will be given by a signature functor $\Sigma : \mathsf{Set}^{S_1} \to \mathsf{Set}^{S_2}$ as in (1). The equations that we will consider are of rank 1, meaning that in the terms involved every variable is under the scope of precisely one operation symbol in Σ, and are given by an S_2-sorted set E. For each sort $s \in S_2$ and each S_1-sorted set of variables V with the property that $\bigcup_{t\in S_1} V_t$ is finite, we consider a set $E_{V,s}$ of equations over the set V of terms of sort s, which is defined as a subset of $(U_2F_2\Sigma U_1F_1V)^2_s$. Now take $E_V = (E_{V,s})_{s\in S_2}$ and $E = \bigcup_{V\in\omega_f^{S_1}} E_V$.

Definition 3.3 Let S_1, S_2 be sets of sorts, $\mathcal{A}_1$ be an S_1-sorted variety and $\mathcal{A}_2$ be an S_2-sorted variety. A presentation for a functor $L : \mathcal{A}_1 \to \mathcal{A}_2$ is a pair $\langle \Sigma, E\rangle$ defined as above. A functor $L : \mathcal{A}_1 \to \mathcal{A}_2$ is presented by $\langle \Sigma, E\rangle$, if

(i) for every algebra $A \in \mathcal{A}_1$ the algebra LA is the joint coequalizer:

$$\begin{tikzcd} F_2E_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2\Sigma U_1F_1V \arrow[r, "F_2\Sigma U_1 v^\sharp"] & F_2\Sigma U_1 A \arrow[r, "q_A"] & LA \end{tikzcd} \tag{3}$$

taken after all finite sets of S_1-sorted variables V and all valuations $v : V \to U_1A$. Here $v^\sharp$ denotes the adjoint transpose of a valuation v.

(ii) for all morphisms $f : A \to B$ the diagram commutes:

$$\begin{tikzcd} F_2\Sigma U_1 A \arrow[r, "q_A"] \arrow[d, "F_2\Sigma U_1 f"] & LA \arrow[d, "Lf"] \\ F_2\Sigma U_1 B \arrow[r, "q_B"] & LB \end{tikzcd} \tag{4}$$

The importance of this notion, emphasized in [8], resides in the fact that endofunctors having finitary presentations give rise to modal logics, where the modal operators are the operation symbols of Σ and the axioms are the equations of E.

If $\mathcal{A} = \mathsf{Alg}(\Sigma_\mathcal{A}, E_\mathcal{A})$ is an S-sorted variety and the endofunctor $L : \mathcal{A} \to \mathcal{A}$ has a finitary presentation $\langle \Sigma_L, E_L\rangle$, we can obtain an equational calculus for $\mathsf{Alg}(L)$, regarding the equations $E_\mathcal{A}$ and E_L as equations containing terms in $\mathrm{Term}_{\Sigma_\mathcal{A}+\Sigma_L}$. First remark, that formally, for an arbitrary set of variables V, $E_{L,V}$ is a subset of the $S-$sorted set $(UF\Sigma_L UFV)^2$. But for each set X, UFX is a quotient of $\mathrm{Term}_{\Sigma_\mathcal{A}}X$ modulo the equations. Thus, if we choose a representative for each equivalence class in $UF\Sigma_L UFV$, we can obtain a set of equations in $\mathrm{Term}_{\Sigma_\mathcal{A}}\Sigma_L\mathrm{Term}_{\Sigma_\mathcal{A}}$. Using the natural map from $\mathrm{Term}_{\Sigma_\mathcal{A}}\Sigma_L\mathrm{Term}_{\Sigma_\mathcal{A}}$ to $\mathrm{Term}_{\Sigma_\mathcal{A}+\Sigma_L}V$, we obtain a set of equations on terms $\mathrm{Term}_{\Sigma_\mathcal{A}+\Sigma_L}V$. By abuse of notation we will denote this set with E_L as well.

Theorem 3.4 *Let $\mathcal{A} = \mathsf{Alg}(\Sigma_\mathcal{A}, E_\mathcal{A})$ be an S-sorted variety and let $L : \mathcal{A} \to \mathcal{A}$ be a functor presented by operations Σ_L and equations E_L. Then $\mathsf{Alg}(L) \cong \mathsf{Alg}(\Sigma_\mathcal{A} + \Sigma_L, E_\mathcal{A} + E_L)$.*

Proof. We define a functor $H : \mathsf{Alg}(L) \to \mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$. Suppose $\alpha : LA \to A$ is an L-algebra. Then the underlying set of HA is defined to be UA. HA inherits the algebraic structure of A: the interpretation of the operation symbols of $\Sigma_{\mathcal{A}}$ is the same as in the algebra A and it satisfies the equations $E_{\mathcal{A}}$. As far as the operation symbols of Σ_L are concerned, their interpretation is given by the composition:

$$F\Sigma_L UA \xrightarrow{q_A} LA \xrightarrow{\alpha} A \tag{5}$$

Explicitly, the interpretation of an operation symbol σ of arity $(s_1 \dots s_n; s)$ is the morphism $\sigma^A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ defined by

$$\sigma^A(x_1, \dots, x_n) = \alpha(q_A((\sigma, x_1, \dots, x_n)))$$

Now it is clear that the equations E_L are satisfied in HA, because q_A is a coequalizer as in (3). If f is a morphism of $L-$algebras, we define $Hf = f$ and we only have to check that $f(\sigma(a_1, \dots, a_k)) = \sigma(f(a_1), \dots, f(a_k))$ for all $\sigma \in \Sigma_L$. But this follows from the fact the definition of the interpretation of the operations, the commutativity of diagram (4) and the fact that f is an L-algebra morphism.

Conversely, we define a functor $J : \mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L) \to \mathsf{Alg}(L)$. Suppose A is an algebra in $\mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$. The map $\rho_A : \Sigma_L UA \to UA$ defined by:

$$(\sigma_{(s_1 \dots s_n; s)}, x_{i_1}, \dots, x_{i_n}) \mapsto \sigma_{(s_1 \dots s_n; s)}(x_{i_1}, \dots, x_{i_n})$$

induces a map $\rho_A^\sharp : F\Sigma_L UA \to A$. The fact that equations E_L are satisfied implies that $\rho_A^\sharp \circ F\Sigma_L Uv^\sharp \circ \pi_1^\sharp = \rho_A^\sharp \circ F\Sigma_L Uv^\sharp \circ \pi_2^\sharp$ as depicted in (6). But LA is a coequalizer in $\mathsf{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$, therefore there exists a morphism $\alpha_A : LA \to A$ such that $\alpha_A \circ q_A = \rho_A^\sharp$. We define JA to be the L- algebra α_A. For any morphism $f : A \to B$ in $\mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L)$ we define $Jf = U_0 f$, where $U_0 : \mathsf{Alg}(\Sigma_{\mathcal{A}} + \Sigma_L, E_{\mathcal{A}} + E_L) \to \mathsf{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$ is the forgetful functor. This is well defined and we can check this easily by proving that the rightmost square of diagram (6) is commutative:

$$\begin{tikzcd}
 & & F\Sigma_L UA \arrow[r, "q_A"] \arrow[d, "F\Sigma U f"] \arrow[rr, bend left, "\rho_A^\sharp"] & LA \arrow[r, dashed, "\alpha_A"] \arrow[d, "Lf"] & A \arrow[d, "f"] \\
FE_L \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F\Sigma_L UFV \arrow[ur, "F\Sigma U v_1^\sharp"] \arrow[dr, "F\Sigma U v_2^\sharp"'] & & & \\
 & & F\Sigma_L UB \arrow[r, "q_B"] \arrow[rr, bend right, "\rho_B^\sharp"'] & LB \arrow[r, dashed, "\alpha_B"] & B
\end{tikzcd} \tag{6}$$

Now it is straightforward to check that $J \circ H$ and $H \circ J$ are the identities. $\square$

The characterization of endofunctors having finitary presentation was given in [20] for monadic categories over Set and it can be easily extended if we replace Set with the presheaf category Set^S. The result holds even if we work with functors between different varieties.

Theorem 3.5 *Let S_1, S_2 be sets of sorts, $\mathcal{A}_1$ be an S_1-sorted variety and $\mathcal{A}_2$ be an S_2-sorted variety. For a functor $L : \mathcal{A}_1 \to \mathcal{A}_2$ the following conditions are equivalent:*

(i) L has a finitary presentation by operations and equations;

(ii) L preserves sifted colimits.

Proof. $(i) \Rightarrow (ii)$. Assume L has a finitary presentation $\langle \Sigma, E \rangle$. Let D be a sifted category and $a_i : A_i \to A$ be a sifted colimit in $\mathcal{A}_1$. Let $d_i : LA_i \to B$ be an arbitrary cocone. As we have seen in the preliminaries, the corresponding forgetful functors and their left adjoints U_1, U_2, F_1, F_2 preserve sifted colimits. Σ shares the same property because sifted colimits are computed point-wise and commute with finite products. Therefore we obtain that $F_2\Sigma U_1 a_i : F_2\Sigma U_1 A_i \to F_2\Sigma U_1 A$ is a colimiting cocone in $\mathcal{A}_2$, hence there exists a map $d : F_2\Sigma U_1 A \to B$ such that $d \circ F_2\Sigma U_1 a_i = d_i \circ q_{A_i}$ for all i in D.

Choose an arbitrary S_1-sorted set of variables $V = (V_s)_{s\in S_1}$ such that $\bigcup\limits_{s\in S_1} V_s$ is finite and a morphism $v : V \to U_1 A$. Then V is strongly finitely presented in the category Set^{S_1}, and $U_1 a_i : U_1 A_i \to U_1 A$ is a sifted colimit, so there exists $v_i : V \to U_1 A_i$ such that $v = U_1 a_i \circ v_i$. From the fact that q_{A_i} is a joint coequalizer, it follows that d makes the bottom line of diagram (7) commutative.

$$
\begin{tikzcd}
 & & F_2\Sigma U_1 A_i \arrow[d, "F_2\Sigma U_1 a_i"] \arrow[dr, "d_i\circ q_{A_i}"] & \\
F_2 E_V \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & F_2\Sigma U_1 F_1 V \arrow[ur, "F_2\Sigma U_1 v_i"] \arrow[r, "F_2\Sigma U_1 v"] & F_2\Sigma U_1 A \arrow[r, "d"] & B
\end{tikzcd}
\tag{7}
$$

Using that LA is a joint coequalizer we obtain $b : LA \to B$ such that $b \circ q_A = d$. Now it is immediate to check that diagram (8) is commutative, and this shows that the cocone $La_i : LA_i \to LA$ is universal.

$$
\begin{tikzcd}
F_2\Sigma U_1 A_i \arrow[r, "q_{A_i}"] \arrow[d, "F_2\Sigma U_1 A_i"] & LA_i \arrow[d, "La_i"] \arrow[ddr, "d_i"] & \\
F_2\Sigma U_1 A \arrow[r, "q_A"] \arrow[drr, "d"] & LA \arrow[dr, "b"] & \\
 & & B
\end{tikzcd}
\tag{8}
$$

$(ii) \Rightarrow (i)$ Being a sifted colimit preserving functor, L is determined by its values on finitely generated free algebras. Given $k \in \omega_f^{S_1}$ with support $\{s_1, \ldots, s_n\}$ and given $s \in S_2$ we can view the elements of the set $(U_2 L F_1 k)_s$ as operations symbols which take $k(s_i)$ arguments of sort s_i for all $1 \le i \le n$ and return a result of sort s. More explicitly we can consider for all algebras A the map r_A given component-wise by:

$$\coprod_{k\in\omega_f^S}(U_2LF_1k)_s\times U_1A^k \xrightarrow{r_{A,s}} (U_2LA)_s \tag{9}$$
$$(\sigma,x)\mapsto (U_2L\epsilon_A\circ U_2LF_1x)_s(\sigma)$$

where $\epsilon_A : F_1U_1A\to A$ is the counit of the adjunction. In the definition of the map $r_{A,s}$ we have interpreted x as a morphism in $\mathsf{Set}^S(k,U_1A)$. Now the operations that we will consider are encompassed in the functor $\Sigma:\mathsf{Set}^{S_1}\to\mathsf{Set}^{S_2}$ defined by

$$\Sigma X=(\coprod_{k\in\omega_f^{S_1}}(U_2LF_1k)_s\times X^k)_{s\in S_2} \tag{10}$$

Note that r is a natural transformation from ΣU_1 to U_2L.

For an arbitrary S_1-sorted set of variables V, the equations are induced by the map $r_{F_1V}:\Sigma U_1F_1V\to U_2LF_1V$ as in (9), more precisely E_V is defined to be the kernel pair of the map $Ur^\sharp_{F_1V}:U_2F_2\Sigma U_1F_1V\to U_2LF_1V$. We will prove that L is presented by $\langle\Sigma,E\rangle$. For all $k\in\omega_f^{S_1}$ the following diagram is a split coequalizer because E_k is a kernel pair.

$$\begin{tikzcd} E_k \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & U_2F_2\Sigma U_1F_1k \arrow[l, bend left, "t"] \arrow[r, "U_2r^\sharp_{F_1k}"] & U_2LF_1k \arrow[l, bend left, "s"] \end{tikzcd} \tag{11}$$

One can check that it follows that

$$\begin{tikzcd} U_2F_2E_k \arrow[r, shift left, "U_2\pi_1^\sharp"] \arrow[r, shift right, "U_2\pi_2^\sharp"'] & U_2F_2\Sigma U_1F_1k \arrow[l, bend left, "U_2F_2t\,\circ\,\eta_{U_2F_2\Sigma U_1F_1k}"] \arrow[r, "U_2r^\sharp_{F_1k}"] & U_2LF_1k \arrow[l, bend left, "s"] \end{tikzcd} \tag{12}$$

is again a split coequalizer. U_2 is a monadic functor, hence it creates split coequalizers, and we obtain that

$$\begin{tikzcd} F_2E_k \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2\Sigma U_1F_1k \arrow[r, "r^\sharp_{F_1k}"] & LF_1k \end{tikzcd} \tag{13}$$

is a coequalizer. Now it is straightforward to show that

$$\begin{tikzcd} F_2E_V \arrow[r, shift left, "\pi_1^\sharp"] \arrow[r, shift right, "\pi_2^\sharp"'] & F_2\Sigma U_1F_1V \arrow[r, "F_2\Sigma U_1v^\sharp"] & F_2\Sigma U_1F_1k \arrow[r, "r^\sharp_{F_1k}"] & LF_1k \end{tikzcd} \tag{14}$$

is a joint coequalizer. This proves that L coincides on finitely generated algebras with the functor presented by the finitary presentation $\langle\Sigma,E\rangle$, and therefore it is presented by $\langle\Sigma,E\rangle$. $\square$

4 Equational logic for higher-order abstract syntax

Syntax with variable binders cannot be captured as an initial algebra in the usual way. But Fiore, Plotkin and Turi [11] (see also Hofmann [13] and Gabbay and

Pitts [12]) showed that this is possible if one moves from algebras for a functor on Set to algebras for a functor on a suitable presheaf category. In particular, they showed that λ-terms up to α-equivalence form an initial algebra for a functor. These functors generalize the notion of a signature, but a notion of equational theory for these algebras is missing in [11] (but see the more recent work [10]).

This section starts from the observation that a category of presheaves is a many-sorted variety. From Theorem 3.5 we know that a large class of functors on presheaf categories have a presentation. To illustrate an application of Theorem 3.4 we give an algebraic structure of the presheaf of λ-terms up to α-equivalence. Canonical representatives for λ-terms up to α-equivalence can be obtained in different ways, for example, using the method of De Bruijn levels or the method of De Bruijn indices. Using the method of De Bruijn levels, normal forms up to α-equivalence are obtained by specifying well-formedness rules for λ-terms within a context:

$$\frac{1 \leq i \leq n}{x_1, \ldots, x_n \vdash x_i}\ ,\ \frac{x_1, \ldots, x_n, x_{n+1} \vdash t}{x_1, \ldots, x_n \vdash \lambda x_{n+1}.t}\ ,\ \frac{x_1, \ldots, x_n \vdash t_1 \quad x_1, \ldots, x_n \vdash t_2}{x_1, \ldots, x_n \vdash t_1 t_2} \tag{15}$$

The appropriate notion to encompass contexts and the operations allowed on them is the full subcategory $\mathbb{F}$ of Set with objects $\underline{n} = \{1, \ldots, n\}$ and $\underline{0} = \emptyset$. The equivalence classes of λ-terms over a countable set of variables $V = \{x_1, x_2, \ldots\}$ form a presheaf in $\mathsf{Set}^{\mathbb{F}}$, which we will denote by ΛV_α. Explicitly $\Lambda V_\alpha(\underline{n})$ is defined as the set of equivalence classes of λ-terms with the free variables contained in the set $\{x_1, \ldots, x_n\}$. For any morphism $\rho : \underline{n} \to \underline{m}$, $\Lambda V_\alpha(\rho)$ acts on an equivalence class of a term by substituting the free variables x_i with $x_{\rho(i)}$. More generally we can work with an arbitrary presheaf of variables V and again we can see that the λ-terms over V form a presheaf in $\mathsf{Set}^{\mathbb{F}}$. Contexts, which correspond to natural numbers, stratify λ-terms up to α-equivalence, and we can capture this by regarding them as the set of sorts. As we have seen in Section 2, $\mathcal{A} = \mathsf{Set}^{\mathbb{F}}$ is a many-sorted unary variety, the sorts being the set of objects of $\mathbb{F}$, which is isomorphic to the set of non-negative integers $\mathbb{N}$. For this many-sorted variety we denote by $U : \mathcal{A} \to \mathsf{Set}^{\mathbb{N}}$ the forgetful functor and by $F : \mathsf{Set}^{\mathbb{N}} \to \mathcal{A}$ its left adjoint.

We endow $\mathbb{F}$ with the coproduct structure:

$$\begin{tikzcd} & \underline{1} \arrow[d, "new"] \\ \underline{n} \arrow[r, "i"] & \underline{n+1} \end{tikzcd} \tag{16}$$

where i is the inclusion and $new(1) = n + 1$. The type constructor for context extension can be defined as a functor $\delta : \mathcal{A} \to \mathcal{A}$ given by $\delta(A)(\rho) = A(\rho + id_1)$ for all $A \in \mathcal{A}$ and for all morphisms ρ in $\mathbb{F}$. Let $L : \mathcal{A} \to \mathcal{A}$ be the functor given by

$$LX = \delta X + X \times X \tag{17}$$

If V is a presheaf (of variables), then an immediate consequence of Theorem 2.1 of [11] states that ΛV_α is the the free L-algebra over V. We obtain the algebraic structure of ΛV_α by giving an equational presentation for $\mathsf{Alg}(L)$, arising from a finitary presentation of the functor L and an equational presentation of the variety $\mathcal{A}$.

In order to obtain **the equational presentation for $\mathcal{A}$** we consider the signature:

$$\Sigma_{\mathcal{A}} = \{\sigma_n^{(i)} \mid \ 1 < n,\ 1 \leq i < n\} \cup \{w_n \ \mid \ n \geq 0\} \cup \{c_n \ \mid \ n > 0\} \tag{18}$$

with the intended interpretation being the following: $\sigma_n^{(i)}$ can be interpreted as the transposition $(i, i+1)$ of the set $\underline{n}$, c_n as a contraction $c_n : \underline{n+1} \to \underline{n}$ defined by $c_n(i) = i$ for $i \leq n$ and $c_n(n+1) = n$, and w_n as the inclusion of $\underline{n}$ into $\underline{n+1}$.

Firstly, we consider the equations coming from the presentation of the symmetric group, see for example [24]:

$$\begin{array}{lll} (\sigma_n^{(i)})^2(x) = id_n(x) & 1 \leq i < n & \\ \sigma_n^{(i)}\sigma_n^{(j)}(x) = \sigma_n^{(j)}\sigma_n^{(i)}(x) & j \neq i \pm 1; 1 \leq i, j < n & \quad (\mathrm{E}_1) \\ (\sigma_n^{(i)}\sigma_n^{(i+1)})^3(x) = id_n(x) & 1 \leq i < n-1 & \end{array}$$

where x is a variable of sort n. Each permutation of the set $\underline{n}$ can be written as a composition of transpositions $\sigma_n^{(i)}$ and we choose for each permutation such a representation. In the next equations the permutations that will appear should be regarded as abbreviations of their representation in terms of $\sigma_n^{(i)}$.

Secondly, we consider the equations coming from the presentation of the monoid of functions from $\underline{n}$ to $\underline{n}$. [5] gives a presentation of this monoid for $n \geq 4$ in terms of generators of the symmetric group and an additional generator:

$$A = \begin{pmatrix} 1 \ 2 \ 3 \ \dots \ n \\ 1 \ 1 \ 3 \ \dots \ n \end{pmatrix}$$

Apart from the equations giving the presentation of the symmetric group, Aizenštat uses the following seven equations:

$$A\sigma_n^{(1)} = \sigma_n^{(3)}A\sigma_n^{(3)} = (3, 4, \dots, n)A(3, 4, \dots, n) = [(1, n)A]^2 = A$$

$$[\sigma_n^{(2)}A]^2 = A\sigma_n^{(2)}A = [A\sigma_n^{(2)}]^2$$

$$[\sigma_n^{(2)}(1, n)A]^2 = [A\sigma_n^{(2)}(1, n)]^2$$

If we replace A in the equations above by $(1, n-1)(2, n)w_{n-1}c_{n-1}(1, n-1)(2, n)$ and each permutation by its representation in terms of $\sigma_n^{(i)}$ we will obtain the equations which we will label by (E_2). For each function from $\underline{n}$ to $\underline{n}$ we can choose a canonical representation in terms of $\sigma_n^{(i)}$ and $w_{n-1}c_{n-1}$, and one can prove that any other representation can be reduced to this canonical one using (E_1) and (E_2).

Thirdly, we use the next set of equations:

$$c_n\sigma^{(n)}_{n+1}(y) = c_n(y) \tag{E$_3$}$$

$$c_n w_n(x) = id_n(x) \tag{E$_4$}$$

$$\sigma^{(i)}_{n+1} w_n(x) = w_n\sigma^{(i)}_n(x) \qquad 1 \leq i < n \tag{E$_5$}$$

$$\sigma^{(n+1)}_{n+2} w_{n+1} w_n(x) = w_{n+1} w_n(x) \tag{E$_6$}$$

$$\sigma^{(i)}_n c_n(y) = c_n\sigma^{(i)}_{n+1}(y) \qquad i < n-1 \tag{E$_7$}$$

$$c_n\sigma^{(n-1)}_{n+1}\sigma^{(n)}_{n+1} w_n(x) = \sigma^{(n-1)}_n w_{n-1} c_{n-1}(x) \tag{E$_8$}$$

$$c_1 c_2 \sigma^{(1)}_3 = c_1 c_2 \tag{E$_9$}$$

where x is a variable of sort n and y is a variable of sort $n+1$.

We can see that the equation of (E$_2$) corresponding to $A\sigma^{(1)}_n = A$ can be obtained from (E$_3$), so we can remove it.

For each positive integers n, k such that $n > k$, we call a k-partition of n a k-uple $p = (i_1, \ldots, i_k)$ such that $i_1 + \cdots + i_k = n$ and $1 \leq i_1 \leq \cdots \leq i_k$. For any k-partition p of n we denote by $C^p_{n,k} : \underline{n} \to \underline{k}$ the morphism which maps the first i_1 elements of $\underline{n}$ to 1, the next i_2 elements to 2 and so on, the last i_k elements to k. If $n \geq 4$, if we compose $C^p_{n,k}$ with the inclusion of $\underline{k}$ in $\underline{n}$ we obtain a function from $\underline{n}$ to $\underline{n}$, which has a canonical representation in terms of $\sigma^{(i)}_n$ and $w_{n-1}c_{n-1}$, abbreviated by $N^p_{n,k}$. Note that $C^p_{n,k} = c_k \ldots c_{n-1} N^p_{n,k}$.

Proposition 4.1 *Let $E_{\mathcal{A}}$ denote the set of equations of the form $(E_1)-(E_9)$. Then $\mathcal{A}$ is isomorphic to $\mathsf{Alg}(\Sigma_{\mathcal{A}}, E_{\mathcal{A}})$.*

Proof (Sketch) An exhaustive presentation of $\mathcal{A}$ can be obtained if we take an operation symbol for each morphism in $\mathbb{F}$ and if we consider all the equations induced by the composition of morphisms. Any morphisms of $\mathbb{F}$ can be generated using only morphisms of the form $\sigma^{(i)}_n, c_n, w_n$. In fact each morphism $f : \underline{n} \to \underline{m}$ determines a k-partition of n denoted by p_f, where k is the cardinal of the image of f. If $n \geq 4$ then there exist permutations π_n and π_m, such that:

$$f = \pi_m w_{m-1} \ldots w_k c_k \ldots c_{n-1} N^{p_f}_{n,k} \pi_n$$

If $n \leq 3$ then there exists permutations π_n and π_m, such that:

$$f = \pi_m w_{m-1} \ldots w_k c_k \ldots c_{n-1} \pi_n$$

To make the notation easier we can just make the convention to denote $N^{p_f}_{n,k} = id_n$ for $n \leq 3$. We prove that any other representation of f in terms of the generators can be reduced to this canonical form using the equations (E$_1$) $-$ (E$_9$). We will use implicitly the equations (E$_1$) when we manipulate the permutations. Let us consider another representation of f. The first step is to prove that we can transform it into a representation of the form $\pi'_m w_{m-1} \ldots w_k g$ where g is written only in terms of transpositions and contractions and π'_m is a permutation. This can be done using equations $(\mathrm{E}_3) - (\mathrm{E}_8)$. Next we show that there exists a permutation τ such that we have the following equalities in $\mathbb{F}$:

$$\pi'_m w_{m-1} \dots w_k \tau = \pi_m w_{m-1} \dots w_k$$

$$\tau^{-1} g = c_k \dots c_{n-1} N^{pf}_{n,k} \pi_n$$

Therefore it is enough to show that we can derive the above two relations from the equations $(\mathrm{E}_1) - (\mathrm{E}_9)$. The first can be obtained using $(\mathrm{E}_5), (\mathrm{E}_6)$. In order to prove the second, if $n \geq 4$, we can use (E_4) and obtain:

$$\tau^{-1} g \pi_n^{-1} = c_k \dots c_{n-1} w_{n-1} \dots w_k \tau^{-1} g \pi_n^{-1}$$

Now $w_{n-1} \dots w_k \tau^{-1} g \pi_n^{-1}$ can be reduced, via equations $(\mathrm{E}_5), (\mathrm{E}_8)$, to an expression in terms of $\sigma_n^{(i)}$ and $w_{n-1} c_{n-1}$. From Aizenštat's result it follows that we can further reduce this expression to $N^{pf}_{n,k}$ using (E_1) and (E_2). If $n \leq 3$ then we can just use (E_7) or (E_9). $\Box$

A presentation for L can be obtained considering for each $n \in \mathbb{N}$ the operation symbols lam_n, app_n which semantically correspond to λ-abstraction and application. The respective signature functor $\Sigma_L : \mathsf{Set}^{\mathbb{N}} \to \mathsf{Set}^{\mathbb{N}}$ is given by

$$(\Sigma_L X)_m = \{lam_{m+1}\} \times X_{m+1} + \{app_m\} \times X_m \times X_m \tag{19}$$

For any presheaf $V \in \mathcal{A}$ let $\rho_V : \Sigma UV \to ULV$ be the map defined by

$$(lam_{n+1}, t) \mapsto t \qquad \forall t \in V(n+1) = (\delta V)(n)$$

$$(app_n, t_1, t_2) \mapsto (t_1, t_2) \qquad \forall t_1, t_2 \in V(n)$$

The equations E_L should correspond to the kernel pair of the adjoint transpose $\rho_V^\sharp : F\Sigma UV \to LV$. We will consider the set of equations of the following form:

$$\begin{array}{lr}
\sigma_n^{(i)}(lam_{n+1}, t) = (lam_{n+1}, \sigma_{n+1}^{(i)} t) & [t] \\
w_n(lam_{n+1}, t) = (lam_{n+2}, \sigma_{n+2}^{(n+1)} w_{n+1} t) & [t] \\
c_n(lam_{n+2}, t') = (lam_{n+1}, \sigma_{n+1}^{(n)} c_{n+1} \sigma_{n+2}^{(n)} \sigma_{n+2}^{(n+1)} t') & [t'] \\
\sigma_n^{(i)}(app_n, t_1, t_2) = (app_n, \sigma_n^{(i)} t_1, \sigma_n^{(i)} t_2) & [t_1, t_2] \\
w_n(app_n, t_1, t_2) = (app_n, w_n t_1, w_n t_2) & [t_1, t_2] \\
c_n(app_n, t_1, t_2) = (app_n, c_n t_1, c_n t_2) & [t_1, t_2]
\end{array} \tag{20}$$

where t is a variable of sort $n+1$, t' is a variable of sort $n+2$ and t_1, t_2 are variables of sort n and n is an arbitrary positive integer.

Proposition 4.2 *L is presented by $\langle \Sigma_L, E_L \rangle$.*

Proof. Suppose $A \in \mathcal{A}$. We have to check that LA is a coequalizer as in diagram (3). Observe that $\rho^\sharp_A(f(lam_{n+1}, t)) = \delta(A)(f)(t) = A(f + id_1)(t)$. Then the equations (20) are satisfied by LA because we have:

$$\sigma^{(i)}_{n+1} = \sigma^{(i)}_n + id_1$$

$$\sigma^{(n+1)}_{n+2} w_{n+1} = w_n + id_1$$
$$\sigma^{(n)}_{n+1} c_{n+1} \sigma^{(n)}_{n+2} \sigma^{(n+1)}_{n+2} = c_n + id_1$$

Conversely, suppose that $(f(lam_{n+1}, t), f'(lam_{n'+1}, t'))$ is in the kernel pair of $\rho^\sharp_A$. We have to show that $f(lam_{n+1}, t)$ and $f'(lam_{n'+1}, t')$ can be identified in $F\Sigma UA$ by different valuations of the equations. The proof follows from the fact that $A(f + id_1)(t) = A(f' + id_1)(t')$ and that for any morphisms ρ, ρ' which can be composed in $\mathbb{F}$, we have that $\rho \circ \rho' + id_1 = (\rho + id_1) \circ (\rho' + id_1)$. Hence $f(lam_{n+1}, t)$ and $(lam_{m+1}, A(f + id_1)(t))$, can be identified in $F\Sigma UA$ for any morphism $f : \underline{n} \to \underline{m}$.□

Remark 4.3 The presentation of L depends on the operations $\Sigma_{\mathcal{A}}$ used to describe $\mathcal{A} = \mathsf{Set}^{\mathbb{F}}$ but is independent of the equations $E_{\mathcal{A}}$.

Representing different implementations of λ-terms If V is the presheaf defined by $V(\rho) = \rho$ for all morphisms ρ in $\mathbb{F}$, the free L-algebra over V gives an implementation of λ−terms by the De Bruijn levels method. In [11] it is suggested that different implementations of λ-terms can be obtained by equipping $\mathbb{F}$ with different coproduct structures. But this implies working with a different functor than L. Instead, we can use another approach, namely to consider the free L-algebra over different presheaves of variables. For example, if W is the presheaf of variables defined explicitly by

$$W(n) = n \quad W(c_n)(1) = 1 \quad W(c_n)(i) = i - 1; i > 1$$
$$W(w_n)(i) = i + 1$$
$$W(\sigma^{(i)}_n) = \sigma^{(n-i)}_n$$

we obtain the presheaf ΛW_α of λ−terms implemented by the De Bruijn indices method.

5 Modular coalgebraic logic

In this section we are interested in logics for $(T_2 \circ T_1)$-coalgebras. We assume that we have logics for the T_i described by functors L_i with presentations $\langle \Sigma_i, E_i \rangle$. We know that $L = L_2 \circ L_1$ gives a logic for coalgebras for the functor $T = T_2 \circ T_1$. We also know that properties like expressiveness and completeness extend from the components L_i to the composition L. Here we are interested to show how to combine the syntax and the proof systems of the two logics. In our framework, this amounts to showing how to obtain a presentation for the functor $L = L_2 \circ L_1$ from presentations $\langle \Sigma_2, E_2 \rangle$ and $\langle \Sigma_1, E_1 \rangle$.

We know that such a presentation exists, because sifted-colimits preserving functors are closed under composition. But this in itself does not give us a recipe to compute a presentation $\langle \Sigma, E\rangle$ from the presentations $\langle \Sigma_i, E_i\rangle$ *in a simple modular way.* For example, in the case that $L_i : \mathsf{BA} \to \mathsf{BA}$, even if the Σ_i contain only one unary operation symbol $\Box_i$, one may need an infinite set of operation symbols of arbitrary (finite) arities to present $L = L_2 \circ L_1 : \mathsf{BA} \to \mathsf{BA}$. The reason is that operation symbols for L are of the form $\Box_2\phi$ where ϕ can be *any Boolean combination* of terms of the kind $\Box_1\psi$, or, more formally, in the notation of Section 3, operation symbols for L are terms in G_1UFG_2UFV.

The solution is to replace L by a two-sorted functor $\bar{L} : \mathsf{BA}^S \to \mathsf{BA}^S$ where we write $S = \{\mathsf{s}, \mathsf{i}\}$. The intuition here is that a coalgebra $X \to T_2 \circ T_1(X)$ goes first to an intermediate state in $T_1(X)$ and then to a (proper) state in $T_2(T_1(X))$. This point of view introduces a 2-sorted semantics: (proper) states, of sort s, and intermediate states, of sort i. We could make this explicit using a two sorted functor $\bar{T} : \mathsf{Set}^S \to \mathsf{Set}^S$, but we do not need to do this here. On the other hand, on the dual side, to construct the logics, introducing a new sort for intermediate states allows us to compose presentations in a modular way. This has been used (implicitly) in our work on π-calculus [7] and goes back in coalgebraic logic to Rößiger [25] (see also Jacobs [14] and Cîrstea and Pattinson [9]). But the technique of reflecting the structure of the type constructors in the syntax of the logic appears already in Abramsky's use of a 'meta-language' in [1]. In our framework, the details can be formulated as follows. First, it is convenient to introduce the notion of a two-sorted composition.

Below we consider the more general case $L_1 : \mathcal{A}_s \to \mathcal{A}_i$ and $L_2 : \mathcal{A}_i \to \mathcal{A}_s$, which allows us to also treat in a modular way binary functors such as product and coproduct.

Definition 5.1 [two-sorted composition of functors] Given two functors $L_1 : \mathcal{A}_s \to \mathcal{A}_i$ and $L_2 : \mathcal{A}_i \to \mathcal{A}_s$ between any two categories, the two-sorted composition of L_1 with L_2 is the functor $\bar{L} : \mathcal{A}_i \times \mathcal{A}_s \to \mathcal{A}_i \times \mathcal{A}_s$ mapping $A = (A_\mathsf{i}, A_\mathsf{s})$ to $(\bar{L}A)_\mathsf{s} = L_2(A_\mathsf{i})$ and $(\bar{L}A)_\mathsf{i} = L_1 A_\mathsf{s}$.

This composition is symmetric: Swapping L_1 and L_2 just means that the indices i and s change role. It is therefore tempting to suppress the distinction between 1 and i and between 2 and s in our notation. We do not do this because we want to use the notation $(-)_1$ to refer to the functor L_1 and the notation $(-)_\mathsf{i}$ to refer to a projection onto sort i.

The next proposition ensures that we can extract the initial $L_2 \circ L_1$-algebra from the initial algebra of the two-sorted composition. (We continue to write again L_2L_1 instead of $L_2 \circ L_1$).

Proposition 5.2 *Consider categories $\mathcal{A}_\mathsf{i}, \mathcal{A}_\mathsf{s}$ which are lfp and two finitary functors $L_1 : \mathcal{A}_s \to \mathcal{A}_i$ and $L_2 : \mathcal{A}_i \to \mathcal{A}_s$. Let $\bar{L}$ be the two-sorted composition of L_1 with L_2. Then the s-component of the initial $\bar{L}$-algebra is the initial L_2L_1-algebra.*

Proof. $\mathcal{A}_i \times \mathcal{A}_\mathsf{s}$ is lfp and $\bar{L}$ is finitary. Therefore, the initial $\bar{L}$-algebra is the colimit of the initial algebra chain $\bar{L}^n 0$ where 0 denotes the initial object and n runs through finite ordinals. As colimits are calculated sort-wise, it is enough to show that the projected sequence $(\bar{L}^n 0)_\mathsf{s}$ has the same colimit as the initial sequence of L_2L_1, which is easy to see as the latter sequence is a subsequence of the former.□

As the relation between logics and coalgebras is provided by the initial algebras, the proposition tells us that we do not loose anything if we present $\bar{L}$ instead of L_2L_1. It is obvious how to do this.

Theorem 5.3 *Consider (many-sorted) varieties $\mathcal{A}_\mathsf{i} \times \mathcal{A}_\mathsf{s}$ and two functors $L_1 : \mathcal{A}_s \to \mathcal{A}_i$ and $L_2 : \mathcal{A}_i \to \mathcal{A}_s$ with presentations $\langle \Sigma_1, E_1 \rangle$ and $\langle \Sigma_2, E_2 \rangle$, respectively. Then $\langle \bar{\Sigma}, \bar{E} \rangle$ is a presentation of the two-sorted composition $\bar{L}$ of L_1 with L_2 as follows.*

$$(\bar{\Sigma}X)_\mathsf{s} = \Sigma_2 X_\mathsf{i}$$
$$(\bar{\Sigma}X)_\mathsf{i} = \Sigma_1 X_\mathsf{s}$$

where we use that the signatures Σ_1, Σ_2 are given by functors $\mathsf{Set}^{S_2} \to \mathsf{Set}^{S_1}, \mathsf{Set}^{S_1} \to \mathsf{Set}^{S_2}$ and $X = (X_\mathsf{i}, X_\mathsf{s})$ denotes and element of $\mathsf{Set}^{S_1} \times \mathsf{Set}^{S_2}$. Equations are given by $\bar{E}_\mathsf{s} = E_2$, $\bar{E}_\mathsf{i} = E_1$.

Example 5.4 Let us illustrate this theorem using more familiar notation. To be specific, we assume that $\mathcal{A}_i$ and $\mathcal{A}_s$ are both BA. We write $\vdash_\mathsf{i} \psi$ and $\vdash_\mathsf{s} \phi$ to assert that ψ, ϕ are formulas of sort i, s, respectively. The theorem then states that formulas of both sorts are closed under Boolean operations and, for all n-ary operation symbols σ_i in Σ_i, formulas are closed under

$$\frac{\vdash_\mathsf{i} \psi_1, \ldots, \vdash_\mathsf{i} \psi_n}{\vdash_\mathsf{s} \sigma_2(\psi_1, \ldots, \psi_n)} \qquad \frac{\vdash_\mathsf{s} \phi_1, \ldots, \vdash_\mathsf{s} \phi_n}{\vdash_\mathsf{i} \sigma_1(\phi_1, \ldots, \phi_n)}$$

The axioms are given by equations E_1, E_2, sortwise. The rules of the calculus are those of equational logic. The only rules that make the two sorts interact are the congruence rules:

$$\frac{\vdash_\mathsf{i} \psi_1 = \psi_1', \ldots, \vdash_\mathsf{i} \psi_n = \psi_2'}{\vdash_\mathsf{s} \sigma_2(\psi_1, \ldots, \psi_n) = \sigma_2(\psi_1', \ldots, \psi_n')} \qquad \frac{\vdash_\mathsf{s} \phi_1 = \phi_1', \ldots, \vdash_\mathsf{s} \phi_n = \phi_n'}{\vdash_\mathsf{i} \sigma_1(\phi_1, \ldots, \phi_n) = \sigma_1(\phi_1', \ldots, \phi_n')}$$

Here, we use $\vdash_\mathsf{i} \psi = \psi'$ and $\vdash_\mathsf{s} \phi = \phi'$ to denote derivability of equations of the respective sorts.

Example 5.5 In the literature, one often considers inductively defined classes of functors [25,14,9], such as, for example, the class of so-called Kripke polynomial functors $T : \mathsf{Set} \to \mathsf{Set}$ built according to

$$T ::= Id \mid K_C \mid T + T \mid T \times T \mid T \circ T \mid \mathcal{P}$$

where Id is the identity functor, K_C is the constant functor that maps all sets to a finite set C, $\mathcal{P}$ is covariant powerset.

Typically, these functors are not only built from basic ingredients (such as $Id, K_C, \mathcal{P}$) and composition, but also from binary operations (such as $+, \times$). To give an example, the functor $L_+ : \mathsf{BA} \times \mathsf{BA} \to \mathsf{BA}$ capturing the logic of $+ : \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$ has the following presentation.

There are two unary operation symbols $[\kappa_1]$ and $[\kappa_2]$. Equations specify that the $[\kappa_i]$

preserve finite joins and binary meets and that $[\kappa_1]a_1 \wedge [\kappa_2]a_2 = \bot$, $[\kappa_1]\top \vee [\kappa_2]\top = \top$, $\neg[\kappa_1]a_1 = [\kappa_2]\top \vee [\kappa_1]\neg a_1$, $\neg[\kappa_2]a_2 = [\kappa_1]\top \vee [\kappa_2]\neg a_2$.

If L is now the logic of T, then the logic of $T + T$ is given by the presentation of the two-sorted composition of $\mathsf{BA} \xrightarrow{\langle L,L\rangle} \mathsf{BA} \times \mathsf{BA} \xrightarrow{L_+} \mathsf{BA}$ according to Theorem 5.3.

6 Uniform completeness proofs

In this section we show how to associate to an arbitrary set-functor T a functor L on BA and a semantics $\delta : LP \to PT$ so that the resulting logic is complete. The definition of L from T is the same as in [20,19], but as we do not insist on strong completeness [2] here, *we don't need to put any assumptions on* T. Instead we use an induction along the final sequence as first done in Pattinson [23] and adapted to the setting of functorial logics over BA in [17].

Definition of L. First, let us recall from [20,19] the definition of L from T (see also Klin [16]). The essential ingredients are as follows. Two contravariant functors P and S that are adjoint on the right

$$\begin{tikzcd} \mathcal{A} \arrow[loop left, "L"] \arrow[r, bend right, "S"'] & \mathcal{X} \arrow[l, bend right, "P"'] \arrow[loop right, "T"] \\ \mathcal{A}_0 \arrow[u, "I"] \end{tikzcd} \tag{21}$$

where $\mathcal{A}$ is lfp with a small subcategory $\mathcal{A}_0$ of finitely presentable objects. We then define L on $\mathcal{A}_0$, eliding the inclusion I, as

$$LA = PTSA$$

and extend L continuously from $\mathcal{A}_0$ to $\mathcal{A}$. Note that L thus defined preserves filtered colimits, whereas PTS need not to do so.

Example 6.1 Take $\mathcal{A} = \mathsf{BA}$ and $\mathcal{X} = \mathsf{Set}$. Then P is contravariant powerset and S takes ultrafilters. On arrows, P and S map a function to its inverse image. The adjunction restricts to a dual equivalence between finite Boolean algebras and finite sets. The ultrafilters of a finite Boolean algebra A are the atoms of A, that is, those elements $a \in A$ such that there are no elements strictly between bottom and a. Thus, on finite Boolean algebras, the duality reduces to the well known fact that every finite Boolean algebra is isomorphic to the powerset of its atoms. We will also make use of the fact that the finitely presentable Boolean algebras coincide with the finite ones.

Definition of $\delta : LP \to PT$. The idea that the semantics of a logic for coalgebras should be described by a natural transformation $LP \to PT$ goes back to [17,6]. The

[2] A logic is strongly complete if, whenever ϕ holds in all models satisfying a possibly infinite set of formulas Γ, then one can also derive ϕ from Γ. Strong completeness is closely related to compactness. So, for example, the procedure below will not give rise to strongly complete logics if T is the probability distribution functor or if $TX = A \times X$ for an infinite set A.

following definition is again from [20].

$$\begin{tikzcd} PX & LPX \arrow[r, "\delta_X"] & PTX \\ A_i \arrow[u, "c_i"] & LA_i \arrow[u, "Lc_i"] \arrow[r, "\cong"] & PTSA_i \arrow[u, "PTc_i^\sharp"'] \end{tikzcd} \tag{22}$$

PX is a filtered colimit $c_i : A_i \to PX$. Under the adjunction, this cocone corresponds to a cone $c_i^\sharp : X \to SA_i$ which is turned into a cocone under PT (recall that P is contravariant). Now δ_X exists uniquely, since L preserves filtered colimits .

Intuitively, the logic L is complete if any two formulas identified in the semantics, are already identified in the syntax. This is the content of the following lemma.

Assumption: From now on we take $\mathcal{A} = \mathsf{BA}^{S'}$ and $\mathcal{X} = \mathsf{Set}^{S'}$ with the functors P and S sort-wise as in Example 6.1.

Lemma 6.2 *δ_X as defined above is injective.*

Proof. Consider two distinct $\phi_1, \phi_2 \in LPX$. By filteredness, we find some A_i and $\phi'_j \in A_i$ such that $c_i(\phi'_j) = \phi_j$. Moreover, since in BA the finitely presentable objects are closed under quotients, we can assume c_i to be injective. The following fact is easily proved.

Claim: Let A be finite. $c : A \to PX$ is injective iff the adjoint transpose $c^\sharp : X \to SA$ is surjective.

Indeed, by the laws of adjunction and A being finite, we have that c is $A \cong PSA \xrightarrow{Pc^\sharp} PX$; now $Pc^\sharp = (c^\sharp)^{-1}$ is injective iff $\sigma^\sharp$ is surjective, which proves the claim. Using that T, as any functor on $\mathsf{Set}^{S'}$, preserves surjective maps and that P maps surjective maps to injective $\mathsf{BA}^{S'}$-homomorphisms, we conclude that $PTc_i^\sharp$ is injective, hence $\delta_X(\phi_1) \neq \delta_X(\phi_2)$. □

Theorem 6.3 *The logic given by L as defined above is complete for T-coalgebras.*

Proof. (The proof is essentially the one from [17], where the reader can find the missing technical details.) Let $L : \mathsf{BA}^{S'} \to \mathsf{BA}^{S'}$ be the functor defined above. L preserves filtered colimits and therefore, using a special property of BA and following [20, Proposition 3.4], L preserves sifted colimits. It follows that L has a presentation, which induces an equational logic, which in turn can be written in the usual modal-logic style, using the correspondences between equations $\phi = \psi$ and formulas $\phi \leftrightarrow \psi$ and between formulas ϕ and equations $\phi = \top$.

The semantics of an L-formula wrt a coalgebra $\xi : X \to TX$ is determined by the arrow $[\![-]\!]_{(X,\xi)}$ from the initial L-algebra to the algebra $LPX \to PTX \to PX$. Because of the naturality of δ, the semantics wrt to all coalgebras is determined by the semantics wrt to the final coalgebra. Since we don't assume that the final coalgebra exists, we replace it by the corresponding final sequence $T^n\mathbb{1}$ which is defined as follows. We denote by $\mathbb{1} = T^0\mathbb{1}$ the final object in $\mathsf{Set}^{S'}$. $p_0 : T\mathbb{1} \to \mathbb{1}$ is given by finality and $p_{n+1} : T(T^n\mathbb{1}) \to T^n\mathbb{1}$ is defined to be Tp_n. We think of the $T^n\mathbb{1}$ as approximating the final coalgebra.[3] In the same way as any coalgebra $\xi : X \to TX$ has a unique arrow into the final coalgebra, there are canonical arrows $\xi_n : X \to T^n\mathbb{1}$ to the approximants of the final coalgebra, defined inductively by $\xi_{n+1} = T(\xi_n) \circ \xi$. The idea now is to interpret a formula ϕ 'of depth n' as a subset $[\![\phi]\!]_n$ of $T^n\mathbb{1}$. The semantics of ϕ in X is then $\xi_n^{-1}([\![\phi]\!]_n)$.[4] To say what it means for a formula to be of depth n we need the initial sequence of L, which we define next.

Since L is finitary the initial algebra is the colimit of the sequence $L^n\mathbb{2}$ defined as follows. We denote by $\mathbb{2} = L^0\mathbb{2}$ the initial object in $\mathsf{BA}^{S'}$. $e_0 : \mathbb{2} \to L\mathbb{2}$ is given by initiality and $e_{n+1} : L^n\mathbb{2} \to L(L^n\mathbb{2})$ is defined to be Le_n. Since L preserves sifted colimits and hence injective maps [20, Corollary 4.10], all maps in the sequence are injective. This means that we can consider the initial L-algebra as a union of its approximants $L^n\mathbb{2}$. We call the elements of $L^n\mathbb{2}$ formulas of depth n. The semantics of a formula of depth n is given by a $\mathsf{BA}^{S'}$-morphism $[\![-]\!]_n : L^n\mathbb{2} \to PT^n\mathbb{1}$ as follows.

$$\begin{tikzcd} P\mathbb{1} \arrow[r, "Pp_0"] & \cdots & PT^n\mathbb{1} \arrow[r, "Pp_n"] & PT^{n+1}\mathbb{1} & \cdots \\ \mathbb{2} \arrow[u, "{[\![-]\!]_0}"] \arrow[r, "e_0"'] & \cdots & L^n\mathbb{2} \arrow[u, "{[\![-]\!]_n}"] \arrow[r, "e_n"'] & L^{n+1}\mathbb{2} \arrow[u, "{[\![-]\!]_{n+1}}"] & \cdots \end{tikzcd} \tag{23}$$

$[\![-]\!]_0$ is given by initiality (and is actually the identity). $[\![-]\!]_{n+1}$ is defined to be $\delta_{T^n\mathbb{1}} \circ L([\![-]\!]_n)$. Observe that the semantics of a formula is independent of the particular approximant we choose (all squares in the diagram commute). Moreover, given a coalgebra $\xi : X \to TX$ and a formula of depth n, the semantics via the initial L-algebra and the semantics via the final sequence coincide: $[\![\phi]\!]_{(X,\xi)} = \xi_n^{-1}([\![\phi]\!]_n)$. Since δ is injective and L preserves injective maps, all $[\![-]\!]_n$, $n \in \mathbb{N}$, are injective.

To show completeness, suppose $\phi_1 \neq \phi_2$ in the initial L-algebra. We find an approximant $L^n\mathbb{2}$, in which ϕ_1 and ϕ_2 are different. Any one-sided inverse ξ of $p_n : TT^n\mathbb{1} \to T^n\mathbb{1}$ is a T-coalgebra with carrier $T^n\mathbb{1}$. We have $[\![\phi]\!]_{(T^n\mathbb{1},\xi)} = [\![\phi]\!]_n$. Now injectivity of $[\![-]\!]_n$ shows that $(T^n\mathbb{1}, \xi)$ is a counter-example for $\phi_1 = \phi_2$. □

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[3] Indeed, if we let run the final sequence through all ordinals, we obtain the final coalgebra as a limit if it exists, see Adamek and Koubek [2].

[4] This point of view has been elaborated in [18].

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