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Enriched Logical Connections

Alexander Kurz · Jiří Velebil

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Abstract In the setting of enriched category theory, we describe dual adjunctions of the form $L \dashv R : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$ between the dual of the category Spa of "spaces" and the category Alg of "algebras" that arise from a schizophrenic object Ω, which is both an "algebra" and a "space". We call such adjunctions logical connections. We prove that the exact nature of Ω is that of a module that allows to lift optimally the structure of a "space" and an "algebra" to certain diagrams. Our approach allows to give a unified framework known from logical connections over the category of sets and analyzed, e.g., by Hans Porst and Walter Tholen, with future applications of logical connections in coalgebraic logic and elsewhere, where typically, both the category of "spaces" and the category of "algebras" consist of "structured presheaves".

Keywords Logical connection · Schizophrenic object · Module

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A. Kurz
Department of Computer Science, University of Leicester,
Leicester, UK
e-mail: kurz@mcs.le.ac.uk

J. Velebil (✉)
Faculty of Electrical Engineering, Czech Technical University in Prague,
Prague, Czech Republic
e-mail: velebil@math.feld.cvut.cz

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1 Introduction

Many adjunctions in category theory are induced by an object, living in both categories in question. For example, the well-known adjunction

$$\mathsf{pt} \dashv \Sigma : \mathsf{Top}^{op} \longrightarrow \mathsf{Frm}$$

between the category Top of topological spaces and continuous maps and the category Frm of frames and their morphisms, is induced by a two-element set $\Omega = 2$ viewed, on one side, as a topological space when endowed with Sierpiński topology, and, on the other side as a frame free on the empty set. Moreover, both functors Σ and pt are essentially given by homming into Ω, once as an object in Top, or as an object in Frm. See, for example, the book [10] by Peter Johnstone. In fact, as it turns out, one can completely recover the above adjunction knowing just that the two-element set "is both a topological space and a frame" and that the underlying functors of both Top and Frm to the category Set of sets "behave nicely". We refer to the paper by Hans Porst and Walter Tholen [18] for an excellent account of this phenomenon.

Another example of an adjunction that is induced by an object living in two categories is the following well-known adjunction of module theory. For unital rings S and A, to know an adjunction

$$L \dashv R : \left(\mathsf{Mod}\text{-}S\right)^{op} \longrightarrow A\text{-}\mathsf{Mod}$$

is the same thing as to give a left-A right-S *module* Ω, i.e., the same thing as to give an Abelian group equipped with actions of A and S, respectively. Both functors L and R are then given by "homming into" Ω that is, due to its schizophrenic nature, considered on the one hand as a left-A module and on the other hand as a right-S module. The above adjunction is best understood as an adjunction in the sense of category theory *enriched* over the category of Abelian groups: both L and R are functors that are "locally" Abelian group homomorphisms.

In fact, we prove below that the example of modules is quite typical: two-sided modules give rise to dual adjunctions. Moreover, it makes the reasoning not harder when we decide to work with dual adjunctions in the setting *enriched* over a base category $\mathscr{V}$. The example of frames and topological spaces above becomes then an example of our setting when $\mathscr{V}$ is the category of sets and the above example of modules becomes an instance of our theory when one takes for $\mathscr{V}$ the category of Abelian groups.

Thus, in full generality, we choose the category Spa of "spaces" and the category Alg of "algebras" that both are categories of "structured presheaves", i.e., they are equipped with faithful functors

$$V : \mathsf{Spa} \longrightarrow [\mathscr{S}, \mathscr{V}] \quad \text{and} \quad U : \mathsf{Alg} \longrightarrow [\mathscr{A}, \mathscr{V}]$$

and we study the conditions under which an adjunction of the form

$$L \dashv R : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$$

is fully described by a functor of the form

$$\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$$

With the examples of Section 5 in mind we adopt the terminology of [17] and call such an adjunction a *logical connection*.

The precise conditions that Ω should satisfy are, for the simplest case $\mathscr{V} = \mathsf{Set}$ and one-element categories $\mathscr{S}$ and $\mathscr{A}$, summed up in the definition of a *schizophrenic object*, see [18]. Definition 4.12 below only extends these ideas to the enriched and "many-sorted" setting.

Organization of the Paper In Section 2 we introduce the necessary terminology and concepts from enriched category theory. Section 3 introduces the main technical concept we will use, namely, an *initial lift* along a functor. Such a concept is well-established in the ordinary category theory, see, e.g., [2], we use its generalization due to Constantin Anghel [3, 4] to the enriched setting. Section 4 contains the main result: in Definition 4.12 we introduce the notion of a *schizophrenic module* and we prove in Theorem 4.16 that these modules give rise to logical connections. Finally, in Section 5 we give examples of logical connections.

2 Preliminaries

In this section we gather notions from enriched category theory that we will need. For the full account of the general theory we refer the reader to the monograph [13].

We work with enrichment over a *symmetric monoidal closed category* $\mathscr{V} = (\mathscr{V}_o, \otimes, I, [-,-])$, where the category $\mathscr{V}_o$ is complete and cocomplete.

Remark 2.1 Examples of such $\mathscr{V}$'s abound. For various examples to which our theory will apply, see, e.g., Example 2.4 below and [13].

When we say category, functor, natural, etc. below, we always mean $\mathscr{V}$-category, $\mathscr{V}$-functor, $\mathscr{V}$-natural, etc. Categories, functors, transformations, etc. that are enriched over Set will often be called *ordinary*.

Every category $\mathscr{X}$ has an *underlying ordinary category* $\mathscr{X}_o$ with the same objects as $\mathscr{X}$ and homsets

$$\mathscr{X}_o(X, Y) = \mathscr{V}_o(I, \mathscr{X}(X, Y))$$

One can use the monoidal structure on $\mathscr{V}$ to define identity morphisms and composition in $\mathscr{X}_o$ and to verify that $\mathscr{X}_o$ is indeed a category in the ordinary sense.

Whenever we say a morphism $f : X \longrightarrow Y$, we always mean a morphism in $\mathscr{X}_o$.

The *opposite* category $\mathscr{X}^{op}$ is defined in the usual way: it has the same objects as $\mathscr{X}$ and $\mathscr{X}^{op}(X, Y) = \mathscr{X}(Y, X)$.

We will need *tensors* and *cotensors* in what follows. The first notion generalizes copowers, the second generalizes powers.

Definition 2.2 A *tensor* of X in $\mathscr{X}$ by S in $\mathscr{V}$ is an object $S \bullet X$ in $\mathscr{X}$, together with an isomorphism

$$\mathscr{X}(S \bullet X, Y) \cong \mathscr{V}(S, \mathscr{X}(X, Y))$$

natural in Y.

A *cotensor* of X in $\mathscr{X}$ by S in $\mathscr{V}$ is the object $S \pitchfork X$ that is the corresponding tensor in $\mathscr{X}^{op}$, explicitly, there is an isomorphism

$$\mathscr{X}(Y, S \pitchfork X) \cong \mathscr{V}(S, \mathscr{X}(Y, X))$$

natural in Y.

Example 2.3 Suppose $\mathscr{V} = \mathsf{Set}$ and let $\mathscr{X}$ be an arbitrary category. Then $S \bullet X$ is the S-fold coproduct of X in $\mathscr{X}$, and $S \pitchfork X$ is the S-fold product of X in $\mathscr{X}$.

Observe that when $\mathscr{X} = \mathsf{Set}$, one has $S \bullet X \cong S \times X$ and $S \pitchfork X \cong [S, X]$. This has an obvious generalization to any $\mathscr{V}$: $S \bullet X \cong S \otimes X$ and $S \pitchfork X \cong [X, S]$ hold in any $\mathscr{V}$. The last observation (probably) explains the tensor-cotensor terminology.

Recall that, for every small category $\mathscr{S}$ we can form the category

$$[\mathscr{S}^{op}, \mathscr{V}]$$

having functors as objects and where the hom-objects are defined as follows:

$$[\mathscr{S}^{op}, \mathscr{V}](F, G) = \int_s [Fs, Gs]$$

More generally, one can define a $\mathscr{V}$-category having $\mathscr{V}$-functors from $\mathscr{A}$ to $\mathscr{B}$ as objects and

$$[\mathscr{A}, \mathscr{B}](F, G) = \int_a [Fa, Ga]$$

as hom-objects. In fact, this procedure provides us with a plethora of base categories that we can enrich in:

Example 2.4 Consider any $\mathscr{V}$ and denote by

$$\mathscr{W}_o = \mathscr{V}\text{-}\mathsf{Cat}$$

the *ordinary* category of *small* $\mathscr{V}$-categories and $\mathscr{V}$-functors. Then there exists a canonical[1] symmetric monoidal closed structure

$$\mathscr{W} = (\mathscr{W}_o, \mathscr{I}, \otimes, [-, -])$$

on $\mathscr{W}_o$ with:

1. $\mathscr{A} \otimes \mathscr{B}$ is the category having pairs (A, B) as objects and

$$\mathscr{A} \otimes \mathscr{B}((A, B), (A', B)) = \mathscr{A}(A, A') \otimes \mathscr{B}(B, B')$$

2. The unit $\mathscr{I}$ of $\otimes$ is the unit $\mathscr{V}$-category. The unit category $\mathscr{I}$ has just one object, say $*$, and $\mathscr{I}(*, *) = I$. Composition in $\mathscr{I}$ is defined in the obvious way.

[1]We denote the closed monoidal structures on $\mathscr{V}$ and $\mathscr{W}$ by the same symbols, there is no danger of confusion.

3. The internal hom $[\mathscr{A}, \mathscr{B}]$ is the $\mathscr{V}$-category having $\mathscr{V}$-functors from $\mathscr{A}$ to $\mathscr{B}$ as objects and

$$[\mathscr{A}, \mathscr{B}](F, G) = \int_a [Fa, Ga]$$

Hence $\mathscr{W}$ can serve as a basis for enriched category theory and we can study logical connections of $\mathscr{W}$-categories.

Examples are:

1. $\mathscr{V} = \mathsf{Set}$. Then $\mathscr{W}$ is the category of small categories and functors. $\mathscr{W}$-categories are commonly called *2-categories*. Every hom-object of a general $\mathscr{W}$-category is a small (ordinary) category. $\mathscr{W}$-functors are exactly the *2-functors*.
2. $\mathscr{V} = \mathbb{2}$ (the two-element chain). Then $\mathscr{W}$ is the category of preorders. $\mathscr{W}$-categories are exactly the ordinary categories such that every hom-set is a preorder and the composition is a monotone map. $\mathscr{W}$-functors are exactly locally monotone functors.
3. $\mathscr{V}$ is a complete lattice. Such base categories are called *commutative quantales*. A $\mathscr{V}$-category is then a *metric space* [16] in a rather broad sense. Examples:

   (a) $\mathscr{V}$ is the unit interval $[0, 1]$ with the opposite ordering, $a \otimes b$ is the maximum of a and b.
   It is easy to see that a $\mathscr{V}$-category can be identified with a set X and a mapping $d : X \times X \longrightarrow [0, 1]$ such that $\langle X, d\rangle$ is a *generalized ultrametric space*. The slight generalization of the usual notion lies in the fact that $d(x, x') = 0$ does not necessarily entail $x = x'$.
   A $\mathscr{V}$-functor $f : (X, d) \longrightarrow (Y, s)$ is then a exactly a *nonexpanding mapping*, i.e., one satisfying the inequality $s(fx, fx') \leq d(x, x')$ for every $x, x' \in X$.

   (b) $\mathscr{V}$ is the interval $[0, +\infty]$ with the opposite ordering, $a \otimes b$ is the sum of a and b (with $a + \infty = +\infty + a = +\infty$).
   Then a $\mathscr{V}$-category is a *generalized metric space*. Again, the slight generalization of the usual notion lies in the fact that $d(x, x') = 0$ does not necessarily entail $x = x'$.
   Again, $\mathscr{V}$-functors are nonexpanding maps.

   $\mathscr{W}$-categories are exactly the ordinary categories with homsets that are metric spaces in the above broad sense. $\mathscr{W}$-functors are functors that are locally nonexpanding.
4. $\mathscr{V}$ is the cartesian closed category $\omega\mathsf{CPO}$ of posets having suprema of ω-chains and the bottom element, and not necessarily strict continuous maps. A $\mathscr{V}$-category is an ordinary category such that every homset is in $\omega\mathsf{CPO}$ and composition is continuous in each variable. A $\mathscr{V}$-functor is a locally continuous one.

For every example above, our theory will apply.

Notation 2.5 For any functor $F : \mathscr{S} \longrightarrow \mathscr{A}$, where $\mathscr{S}$ is a small category, we denote by

$$\widetilde{F} : \mathscr{A} \longrightarrow [\mathscr{S}^{op}, \mathscr{V}] \tag{2.1}$$

the functor that sends A to the functor $\mathscr{A}(F-, A) : \mathscr{S}^{op} \longrightarrow \mathscr{V}$. The functor $\widetilde{F}$ is called the *tilde-conjugate* of F.

Dually, we define the *hat-conjugate*

$$\widehat{F} : \mathscr{A}^{op} \longrightarrow [\mathscr{S}, \mathscr{V}] \tag{2.2}$$

the functor that sends A to the functor $\mathscr{A}(A, F-) : \mathscr{S} \longrightarrow \mathscr{V}$.

It is useful to think of $\widetilde{F}$ and $\widehat{F}$ as of "generalized hom-functors", since if we choose $\mathscr{S}$ to be the unit category $\mathscr{I}$, then $F : \mathscr{I} \longrightarrow \mathscr{A}$ is just a choice of an object $F*$ in $\mathscr{A}$ and $\widetilde{F} : \mathscr{A} \longrightarrow [\mathscr{I}^{op}, \mathscr{V}]$ can be identified with the corresponding hom-functor $\mathscr{A}(F*, -) : \mathscr{A} \longrightarrow \mathscr{V}$ and $\widehat{F}$ with the contravariant representable functor $\mathscr{A}(-, F*) : \mathscr{A}^{op} \longrightarrow \mathscr{V}$.

Recall that in enriched category theory one needs to work with *weighted* (co)limit notions.

Definition 2.6 A *colimit* of $D : \mathscr{S} \longrightarrow \mathscr{X}$ weighted by $W : \mathscr{S}^{op} \longrightarrow \mathscr{V}$ is an object $W * D$ in $\mathscr{X}$ together with an isomorphism

$$\mathscr{X}(W * D, X) \cong [\mathscr{S}^{op}, \mathscr{V}](W, \widetilde{D}X))$$

natural in X.

A *limit* of $D : \mathscr{S} \longrightarrow \mathscr{X}$ weighted by $W : \mathscr{S} \longrightarrow \mathscr{V}$ is an object $\{W, D\}$ in $\mathscr{X}$ together with an isomorphism

$$\mathscr{X}(X, \{W, D\}) \cong [\mathscr{S}, \mathscr{V}](W, \widehat{D}X)$$

natural in X.

Remark 2.7 We will work later with *cylinders* for limit diagrams, see Example 3.3. Given a diagram $D : \mathscr{S} \longrightarrow \mathscr{X}$ weighted by $W : \mathscr{S} \longrightarrow \mathscr{V}$, then a *cylinder* is a natural transformation $\lambda : W \longrightarrow \mathscr{X}(X, D-)$ or, using the hat conjugate, $\lambda : W \longrightarrow \widehat{D}X$. Here, X is called the *vertex* and D the *base* of the cylinder.

A special cylinder is the *limit cylinder* that is the image of identity under the isomorphism

$$\mathscr{X}(\{W, D\}, \{W, D\}) \cong [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(\{W, D\}, D-))$$

Remark 2.8 Recall that a functor $U : \mathscr{A} \longrightarrow \mathscr{X}$ is *faithful* provided that every action

$$U_{A,B} : \mathscr{A}(A, B) \longrightarrow \mathscr{X}(UA, UB)$$

of U on hom-objects is a monomorphism in $\mathscr{V}_o$.

Remark 2.9 We will work with *adjunctions* in the enriched setting. For a detailed account of this notion, see, e.g., Section 1.11 of [13]. We just make the following brief comments.

That $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ means that we have an isomorphism

$$\mathscr{A}(FX, A) \cong \mathscr{X}(X, UA)$$

natural in A and X. Let us stress the fact that the above isomorphism takes place in $\mathscr{V}_o$.

As always, the adjunction can be described by a unit and a counit satisfying the triangle identities. For example, the unit η_X is defined as the composite

$$I \xrightarrow{id} \mathscr{A}(FX, FX) \xrightarrow{\cong} \mathscr{X}(X, UFX)$$

i.e., as the *transpose* of identity on FX under $F \dashv U$. Notice that

$$\eta_X : I \longrightarrow \mathscr{X}(X, UFX)$$

making it a morphism $\eta_X : X \longrightarrow UFX$ in the underlying *ordinary* category $\mathscr{X}_o$. However, the *naturality* of η is considered in the *enriched* sense, see Section 1.2 of [13].

Hence the triangle equalities that we have in mind are, of course, the equations

$$\varepsilon F \cdot F\eta = id_F, \quad U\varepsilon \cdot \eta U = id_U$$

but in the sense how *enriched* natural transformations compose.

3 Lifts Along a Functor

The crucial point of understanding dual adjunctions (see [18]) is the notion of a schizophrenic object that, in turn, requires the notion of an *initial lift*. Below we summarize the bare basics on initial lifts in the enriched setting due to Constantin Anghel [3] and [4] (see Definition 3.8 and Remark 3.9 below) that we will need later.

In case $\mathscr{V} = \mathsf{Set}$, general initial lifts form the essence of what it means to be "topological". For example, suppose that we want to make a mapping $f : Z \longrightarrow A$, where $\langle A, \sigma \rangle$ is a topological space, into a continuous map. There are numerous way how to do it, but there is a distinguished topology τ on Z that has the following universal property:

> Given any topological space $\langle X, \rho \rangle$, a mapping $h : X \longrightarrow Z$ is continuous (w.r.t. ρ and τ) if and only if the composite $f \cdot h : X \longrightarrow A$ is continuous (w.r.t. ρ and σ).

Observe that a single map as the above f can be written as a map

$$f : \mathbb{1} \longrightarrow \mathsf{Set}(Z, U\langle A, \sigma \rangle)$$

where $\mathbb{1}$ denotes the one-element set and $U : \mathsf{Top} \longrightarrow \mathsf{Set}$ denotes the forgetful functor from the category of topological spaces and continuous maps to the category of sets and mappings. The (necessarily unique) topology τ is called the U-initial U-lift of the above U-structured source f.

In fact, in topology one is interested in families of the form

$$f_d : Z \longrightarrow U\langle A_d, \sigma_d \rangle$$

indexed by a *set* D. The task is then to make *all* the mappings f_d continuous in an optimal way as described above. This means that an element $\langle A, \sigma \rangle$ of Top is replaced by a *generalized element* $A : D \longrightarrow \mathsf{Top}$ where the set D is regarded as a small discrete category.

The above family $\langle f_d \rangle$ can then be written as a *natural transformation*

$$f : \mathrm{const}_{\mathbb{1}} \longrightarrow \mathsf{Set}(Z, U \cdot A-)$$

where $\mathrm{const}_{\mathbb{1}} : D \longrightarrow \mathsf{Set}$ is constantly the one-element set $\mathbb{1}$. See [2] for more details.

We are going to generalize the above concept in several ways:

1. The functor $U : \mathsf{Top} \longrightarrow \mathsf{Set}$ is going to be replaced by an *arbitrary* functor $U : \mathscr{A} \longrightarrow \mathscr{X}$.
2. The discrete category D above is going to be replaced by an *arbitrary* small category $\mathscr{S}$. Accordingly, the mere family $A : D \longrightarrow \mathsf{Top}$ is going to be replaced by a functor $A : \mathscr{S} \longrightarrow \mathscr{A}$.
3. The "indexing functor" $\mathrm{const}_{\mathbb{1}}$ is going to be replaced by *any* functor $W : \mathscr{S} \longrightarrow \mathscr{V}$. Hence a "family" will be a natural transformation of the form

$$\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$$

and we will study its lift along the functor $U : \mathscr{A} \longrightarrow \mathscr{X}$.

Definition 3.1 Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ is a functor. Suppose $\mathscr{S}$ is a small category.

1. A *U-structured source* is a natural transformation

$$\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-) \tag{3.1}$$

where $W : \mathscr{S} \longrightarrow \mathscr{V}$ is the *weight* of λ and $A : \mathscr{S} \longrightarrow \mathscr{A}$ is the *base* of λ. The object Z is called the *vertex* of λ.
2. A *U-lift* of λ in Eq. 3.1 is a natural transformation

$$\overline{\lambda} : W \longrightarrow \mathscr{A}(\overline{Z}, A-) \tag{3.2}$$

such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\overline{\lambda}} & \mathscr{A}(\overline{Z}, A-) \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle U_{\overline{Z},A-}} \\ & & \mathscr{X}(Z, U \cdot A-) \end{array} \tag{3.3}$$

commutes.

Remark 3.2 Let us emphasize that a U-structured source $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$ is a *$\mathscr{V}$-natural* transformation. Hence every component $\lambda_s : Ws \longrightarrow \mathscr{X}(Z, U(As))$ is a morphism in the *ordinary* category $\mathscr{V}_o$.

Example 3.3 For $\mathscr{V} = \mathsf{Set}$ Definition 3.1 describes the usual notions. To see it, consider any ordinary functor $U : \mathscr{A} \longrightarrow \mathscr{X}$.

1. Suppose that the category $\mathscr{S}$ is discrete.

(a) Instead of a natural transformation $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$ one speaks of the collection

$$\langle \lambda_{s,i} : Z \longrightarrow U(As) \mid i \in Ws \rangle$$

of morphisms in $\mathscr{X}$.

(b) The lift of the above family $\langle \lambda_{s,i} \mid i \in Ws \rangle$ is a family

$$\langle \overline{\lambda}_{s,i} : \overline{Z} \longrightarrow As \mid i \in Ws \rangle$$

with the property that $U\overline{\lambda}_{s,i} = \lambda_{s,i}$ for every $i \in Ws$ and every s.
Clearly, if we take W to be constantly the one-element set, we obtain the notions from [2].

2. Suppose that the category $\mathscr{S}$ is arbitrary. Then a U-structured source $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$ is exactly the *cylinder* for the diagram

$$\mathscr{S} \xrightarrow{A} \mathscr{A} \xrightarrow{U} \mathscr{X}$$

weighted by $W : \mathscr{S} \longrightarrow \mathsf{Set}$, see Remark 2.7.

Notation 3.4 Suppose $\mathscr{K}$ is any category, $K : \mathscr{S} \longrightarrow \mathscr{K}$ any functor and suppose that the natural transformation $\gamma : W \longrightarrow \mathscr{K}(Z, K-)$ is any "W-indexed family" in $\mathscr{K}$. We denote, for every Z' in $\mathscr{K}$, by

$$\mathsf{hom}_{\mathscr{K}}(Z', \gamma) : \mathscr{K}(Z', Z) \longrightarrow [\mathscr{S}, \mathscr{V}](W, \mathscr{K}(Z', K-))$$

the morphism in $\mathscr{V}_o$ that arises in the following way:

1. Recall the hat-conjugate notation from Notation 2.5. Consider the morphism

$$\delta \equiv \mathscr{K}(Z', Z) \bullet S \xrightarrow{\mathscr{K}(Z',Z)\bullet\gamma} \mathscr{K}(Z', Z) \quad \bullet \; \mathscr{K}(Z, K-)$$
$$\longrightarrow \mathscr{K}(Z', K-)$$

in the ordinary category $[\mathscr{S}, \mathscr{V}]_o$ where the last morphism is the image of

$$\widehat{K}_{Z',Z} : \mathscr{K}(Z', Z) \longrightarrow [\mathscr{S}, \mathscr{V}]\big(\widehat{K}(Z), \widehat{K}(Z')\big)$$

under the isomorphism

$$\mathscr{V}(\mathscr{K}(Z', Z), [\mathscr{S}, \mathscr{V}](\widehat{K}(Z), \widehat{K}(Z'))) \cong [\mathscr{S}, \mathscr{V}]\big(\mathscr{K}(Z', Z) \bullet \widehat{K}(Z), \widehat{K}(Z')\big)$$

from the definition of $\mathscr{K}(Z', Z)$-fold tensor, see Definition 2.2.

2. Now consider the composite

$$I \xrightarrow{\delta} [\mathscr{S}, \mathscr{V}]\big(\mathscr{K}(Z'Z) \bullet S, \mathscr{K}(Z', K-)\big)$$
$$\cong \mathscr{V}\big(\mathscr{K}(Z', Z), [\mathscr{S}, \mathscr{V}](S, \mathscr{K}(Z', K-))\big)$$

and this is how $\mathsf{hom}_{\mathscr{K}}(Z', \gamma)$ is defined. Above, the isomorphism is due to the definition of an $\mathscr{K}(Z'Z)$-fold tensor in $[\mathscr{S}, \mathscr{V}]$.

Example 3.5 Suppose $\mathscr{V} = \mathsf{Set}$ and let $\gamma : W \longrightarrow \mathscr{K}(Z, K-)$ be given. Then the map

$$\mathsf{hom}_{\mathscr{K}}(Z', \gamma) : \mathscr{X}(Z', Z) \longrightarrow [\mathscr{V}, \mathscr{V}](W, \mathscr{K}(Z', K-))$$

is the map that maps $h : Z' \longrightarrow Z$ to the natural transformation having

$$(\gamma_s)(i) \cdot h : Z' \longrightarrow Ks$$

as its s-component evaluated at $i \in Ws$. Hence the above map is "postcomposition with γ".

Lemma 3.6 *Suppose that $U : \mathscr{A} \longrightarrow \mathscr{X}$ is a functor and let $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$ be a U-structured source. Then the following hold:*

1. *A U-lift $\overline{\lambda} : W \longrightarrow \mathscr{A}(\overline{Z}, A-)$ of λ is determined uniquely, provided that $U_{\overline{Z},A-} : \mathscr{A}(\overline{Z}, A-) \longrightarrow \mathscr{X}(Z, U \cdot A-)$ is a monomorphism in $[\mathscr{S}, \mathscr{V}]_o$.*
2. *The square*

$$\begin{array}{ccc} \mathscr{A}(A', \overline{Z}) & \xrightarrow{\mathsf{hom}_{\mathscr{A}}(A', \overline{\lambda})} & [\mathscr{S}, \mathscr{V}](W, \mathscr{A}(A', A-)) \\ {\scriptstyle U_{A',\overline{Z}}}\downarrow & & \downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](W, U_{A',A-})} \\ \mathscr{X}(UA', Z) & \xrightarrow[\mathsf{hom}_{\mathscr{X}}(UA', \lambda)]{} & [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(UA', U \cdot A-)) \end{array} \tag{3.4}$$

commutes in $\mathscr{V}_o$ for any U-lift $\overline{\lambda}$ of λ and any A' in $\mathscr{A}$.

Proof Assertion 1 is trivial, see Eq. 3.2. To prove assertion 2, consider the commutative diagram

$$\begin{array}{ccccc} \mathscr{A}(A', \overline{Z}) \bullet W & \xrightarrow{\mathscr{A}(A',\overline{Z}) \bullet \overline{\lambda}} & \mathscr{A}(A', \overline{Z}) \bullet \mathscr{A}(\overline{Z}, A-) & \longrightarrow & \mathscr{A}(A', A-) \\ {\scriptstyle U_{A',\overline{Z}} \bullet W}\downarrow & & {\scriptstyle U_{A',\overline{Z}} \bullet U_{\overline{Z},A-}}\downarrow & & \downarrow{\scriptstyle U_{A',A-}} \\ \mathscr{X}(UA', Z) \bullet W & \xrightarrow[\mathscr{X}(UA',Z) \bullet \lambda]{} & \mathscr{X}(UA', Z) \bullet \mathscr{X}(Z, U \cdot A-) & \longrightarrow & \mathscr{X}(UA', U \cdot A-) \end{array}$$

where the square on the left commutes since $\overline{\lambda}$ is a U-lift of λ (consider both components of the tensor separately) and the square on the right commutes since U is a functor. Now observe that the square (Eq. 3.4) is the transpose of the above diagram under the adjunction $- \bullet W \dashv [\mathscr{S}, \mathscr{V}](W, -) : [\mathscr{S}, \mathscr{V}] \longrightarrow \mathscr{V}$. □

Remark 3.7 The U-lift of λ is determined uniquely in case $U : \mathscr{A} \longrightarrow \mathscr{X}$ is assumed to be a *faithful* functor. See Remark 2.8.

Definition 3.8 ([3]) The U-lift $\overline{\lambda}$ of λ is called *U-initial*, provided that the diagram (Eq. 3.4) is a pullback in $\mathscr{V}_o$, for every object A' of $\mathscr{A}$.

Remark 3.9 A more general notion of a *semi-initial lift* has been introduced and studied in [3] and [4]. Due to its greater generality, Anghel's definition requires first the introduction of a *factorization* of a U-structured source $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$. Such a factorization is a pair $(e, \overline{\lambda})$, where $e : Z \longrightarrow U\overline{Z}$ is a morphism in $\mathscr{X}_o$, $\overline{\lambda} : W \longrightarrow \mathscr{A}(\overline{Z}, A-)$ is a natural transformation, and the composite

$$W \xrightarrow{\overline{\lambda}} \mathscr{A}(\overline{Z}, A-) \xrightarrow{U_{\overline{Z},A-}} \mathscr{X}(U\overline{Z}, U \cdot A-) \xrightarrow{\mathscr{X}(e,-)} \mathscr{X}(Z, U \cdot A-)$$

equals to the given U-structured source λ.

The notion of a *semi-initial* lift λ then comprises of a factorization $(e, \overline{\lambda})$ satisfying a slightly more general pullback condition than Definition 3.8 above.

Anghel's notion of a semi-initial lift reduces exactly to Definition 3.8 provided that the "epi-part" e of the appropriate semi-initial lift $(e, \overline{\lambda})$ of the given U-structured source λ consists of an isomorphism.

We refer to Section 2.4 of [3] for more details. See also [2], Chapter 25, for the relationship of initial and semi-initial lifts in the case $\mathscr{V} = \mathsf{Set}$.

Definition 3.10 Any U-structured source λ with the property that

$$\mathsf{hom}_{\mathscr{X}}(Z', \lambda) : \mathscr{X}(Z', Z) \longrightarrow [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(Z', U \cdot A-))$$

is a monomorphism in $\mathscr{V}_o$, for all Z' in $\mathscr{X}$, will be called a *monosource* (in $\mathscr{X}$).

Compare the following result with Proposition 10.7 of [2].

Lemma 3.11 *An initial lift of a monosource is a monosource.*

Proof Let A' be an arbitrary object in $\mathscr{A}$. Then, by assumption, the morphism

$$\mathsf{hom}_{\mathscr{X}}(UA', \lambda) : \mathscr{X}(UA', Z) \longrightarrow [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(UA', U \cdot A-))$$

is a monomorphism in $\mathscr{V}_o$. Therefore

$$\mathsf{hom}_{\mathscr{A}}(A', \overline{\lambda}) : \mathscr{A}(A', \overline{Z}) \longrightarrow [\mathscr{S}, \mathscr{V}](W, \mathscr{A}(A', A-))$$

is a monomorphism in $\mathscr{V}_o$ for any U-initial U-lift $\overline{\lambda}$ of λ. This follows from the fact that monomorphisms in $\mathscr{V}_o$ are stable under pulling back. □

Proposition 3.12 below generalizes Lemma 3.1(2) of [18]. Recall that an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is of *descent type*, if the comparison functor $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is fully faithful, where $\mathbb{T}$ is the monad of $F \dashv U$. The category $\mathscr{X}^{\mathbb{T}}$ of Eilenberg-Moore algebras has as objects actions $a : TA \longrightarrow A$ in $\mathscr{X}_o$ satisfying the usual axioms (see, e.g., [2]) and the hom-objects are given by the equalizer

$$\mathscr{X}^{\mathbb{T}}((A,a),(B,b)) \xrightarrow{U^{\mathbb{T}}_{(A,a),(B,b)}} \mathscr{X}(X,Y) \underset{\mathscr{X}(a,Y)}{\overset{}{\rightrightarrows}} \mathscr{X}(TX,Y), \quad \mathscr{X}(X,Y) \xrightarrow{T_{X,Y}} \mathscr{X}(TX,TY) \xrightarrow{\mathscr{X}(TX,b)} \mathscr{X}(TX,Y) \tag{3.5}$$

where the value $U^{\mathbb{T}}_{(A,a),(B,b)}$ of the equalizer then serves as the action on hom-objects of the forgetful functor $U^{\mathbb{T}} : X^{\mathbb{T}} \longrightarrow \mathscr{X}$. In particular, this means that $U^{\mathbb{T}}$ is always a faithful functor.

Proposition 3.12 *Suppose the adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is of descent type. Then any U-lift of a monosource is U-initial.*

Proof Let $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$ be a U-structured monosource, having $\overline{\lambda} : W \longrightarrow \mathscr{A}(\overline{Z}, A-)$ as a U-lift. Let A' be an arbitrary object in $\mathscr{A}$. We need to prove that Eq. 3.4 is a pullback.

Since U is assumed to be of descent type, we have isomorphisms $\mathscr{A}(A', \overline{Z}) \cong \mathscr{X}^{\mathbb{T}}(KA', K\overline{Z})$ and $[\mathscr{S}, \mathscr{V}](W, \mathscr{A}(A', A-)) \cong [\mathscr{S}, \mathscr{V}](W, \mathscr{X}^{\mathbb{T}}(KA', K \cdot A-))$. Therefore we may as well assume that we work within $\mathscr{X}^{\mathbb{T}}$ and we want to prove that the commutative diagram

$$\begin{array}{ccc}
\mathscr{X}^{\mathbb{T}}(KA', K\overline{Z}) & \xrightarrow{\hom_{\mathscr{A}}(A',\overline{\lambda})} & [\mathscr{S}, \mathscr{V}](W, \mathscr{X}^{\mathbb{T}}(KA', K \cdot A-)) \\
{\scriptstyle U^{\mathbb{T}}_{KA',K\overline{Z}}}\downarrow & & \downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](W, U^{\mathbb{T}}_{KA',K\cdot A-})} \\
\mathscr{X}(UA', Z) & \xrightarrow[\hom_{\mathscr{X}}(UA',\lambda)]{} & [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(A', U \cdot A-))
\end{array} \tag{3.6}$$

is a pullback. Bearing in mind that the action of $U^{\mathbb{T}}$ on hom-objects is defined as an equalizer, see Eq. 3.5, we have equalizers

$$\mathscr{X}^{\mathbb{T}}(KA', K\overline{Z}) \xrightarrow{U^{\mathbb{T}}_{KA',K\overline{Z}}} \mathscr{X}(A', Z) \underset{v_{A',Z}}{\overset{u_{A',Z}}{\rightrightarrows}} \mathscr{X}(TA', Z)$$

and

$$\mathscr{X}^{\mathbb{T}}(KA', KA-) \xrightarrow{U^{\mathbb{T}}_{KA',K\cdot A-}} \mathscr{X}(A', Z) \underset{v_{A',K\cdot A-}}{\overset{u_{A',K\cdot A-}}{\rightrightarrows}} \mathscr{X}(TA', KA-)$$

Therefore, the following diagram

$$\begin{array}{ccc}
\mathscr{X}^{\mathbb{T}}(KA', K\overline{Z}) & \xrightarrow{\hom_{\mathscr{A}}(A',\overline{\lambda})} & [\mathscr{S}, \mathscr{V}](W, \mathscr{X}^{\mathbb{T}}(KA', K \cdot A-)) \\
{\scriptstyle U^{\mathbb{T}}_{KA',K\overline{Z}}}\downarrow & & \downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](W, U^{\mathbb{T}}_{KA',K\cdot A-})} \\
\mathscr{X}(UA', Z) & \xrightarrow{\hom_{\mathscr{X}}(UA',\lambda)} & [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(A', U \cdot A-)) \\
{\scriptstyle v_{A',Z}}\downarrow\downarrow{\scriptstyle u_{A',Z}} & & {\scriptstyle [\mathscr{S},\mathscr{V}](W, v_{A',K\cdot A-})}\downarrow\downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](W, u_{A',K\cdot A-})} \\
\mathscr{X}(TA', Z) & \xrightarrow[\hom_{\mathscr{X}}(TA',\lambda)]{} & [\mathscr{S}, \mathscr{V}](W, \mathscr{X}(TA', K \cdot A-))
\end{array} \tag{3.7}$$

commutes (the lower rectangle serially). Now, given a commutative square

$$\begin{array}{ccc} S & \xrightarrow{\quad g \quad} & [\mathscr{S},\mathscr{V}](W,\mathscr{X}^{\mathbb{T}}(KA',K\cdot A-)) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](W,U^{\mathbb{T}}_{KA',K\cdot A-})} \\ \mathscr{X}(UA',Z) & \xrightarrow[\mathsf{hom}_{\mathscr{X}}(UA',\lambda)]{} & [\mathscr{S},\mathscr{V}](W,\mathscr{X}(A',U\cdot A-)) \end{array} \tag{3.8}$$

it is easy to deduce from Eq. 3.7 and the fact that λ is a monosource that h equalizes $u_{A',Z}$, $v_{A',Z}$. The unique $f : S \longrightarrow \mathscr{X}^{\mathbb{T}}(KA', K\overline{Z})$ from the definition of an equalizer then serves as a witness that Eq. 3.6 is indeed a pullback. □

Remark 3.13 The U-initiality of a U-lift $\overline{\lambda} : W \longrightarrow \mathscr{A}(\overline{Z}, A-)$ of $\lambda : W \longrightarrow \mathscr{X}(Z, U \cdot A-)$ allows us to define certain morphisms in $\mathscr{A}_o$ by the universal property of pullbacks.

For example, to define $h : I \longrightarrow \mathscr{A}(A', \overline{Z})$, it suffices to give a commutative square

$$\begin{array}{ccc} I & \xrightarrow{\qquad} & [\mathscr{S},\mathscr{V}](W,\mathscr{A}(A',A-)) \\ \big\downarrow & & \big\downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](W,U_{A',A-})} \\ \mathscr{X}(UA',Z) & \xrightarrow[\mathsf{hom}_{\mathscr{X}}(UA',\lambda)]{} & [\mathscr{S},\mathscr{V}](W,\mathscr{X}(UA',U\cdot A-)) \end{array}$$

4 Enriched Logical Connections

The concept of dual adjunctions will be essential for us. Namely, we will be interested in dual adjunctions that are generated by a special object "living" in both categories that is, therefore, called a *schizophrenic object*. Let us remark that not all dual adjunctions are induced by schizophrenic objects, see [18] or [10] for more details. The adjunctions that do arise in such a way seem to bear an essence of logic (in a rather broad sense). We will call such adjunctions *logical connections* and study them in full generality of enriched category theory in this section.

The Basic Setting

Assumption 4.1 We fix two small categories $\mathscr{A}$ and $\mathscr{S}$ and call them *categories of sorts* (for "algebras" and "spaces", respectively.)

We furthermore fix categories Alg and Spa that are both equipped with *faithful* (generalized) representable functors

$$U = \widetilde{A_0} : \mathsf{Alg} \longrightarrow [\mathscr{A}, \mathscr{V}], \quad V = \widetilde{S_0} : \mathsf{Spa} \longrightarrow [\mathscr{S}, \mathscr{V}]$$

for $A_0 : \mathscr{A}^{op} \longrightarrow \mathsf{Alg}$ and $S_0 : \mathscr{S}^{op} \longrightarrow \mathsf{Spa}$.

Objects of Spa will be called *spaces*, objects of Alg will be called *algebras*.

Remark 4.2 We chose the above setting to allow the greatest possible flexibility for logical connections. Sometimes, one is interested in the case $\mathscr{S} = \mathscr{A}$ (i.e., both "spaces" and "algebras" are sorted in the same way) or, even, $\mathscr{S} = \mathscr{A} = \mathscr{I}$ where $\mathscr{I}$ denotes the unit category.

In the last case, we can identify the functors S_0 and A_0 with mere objects of Spa and Alg, respectively, postulating just the situation

$$V = \mathsf{Spa}(S_0, -) : \mathsf{Spa} \longrightarrow \mathscr{V}, \quad U = \mathsf{Alg}(A_0, -) : \mathsf{Alg} \longrightarrow \mathscr{V}$$

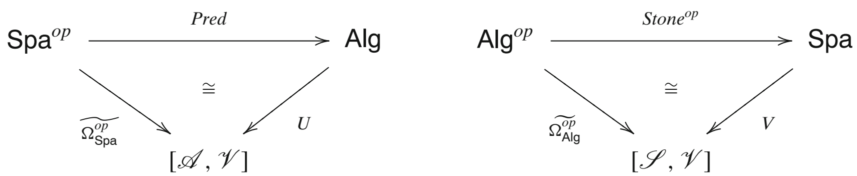
We first prove that, given the setting as in Assumption 4.1, there is an object "living in both Alg and Spa".

Proposition 4.3 *Suppose there is an adjunction*

$$Stone \dashv Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$$

Define $\Omega_{\mathsf{Spa}} := Stone^{op} \cdot A_0^{op} : \mathscr{A} \longrightarrow \mathsf{Spa}$ *and* $\Omega_{\mathsf{Alg}} := Pred \cdot S_0^{op} : \mathscr{S} \longrightarrow \mathsf{Alg}$. *Then the following hold:*

1. *There are natural isomorphisms in*



(4.1)

2. *There exists an isomorphism*

$$\iota_{a,s} : \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s)) \longrightarrow \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a)) \tag{4.2}$$

natural in s and a.

Proof Due to the adjunction $Stone \dashv Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$ we have isomorphisms

$$\begin{aligned}\widetilde{\Omega^{op}_{\mathsf{Spa}}}(X) &= \mathsf{Spa}^{op}(\Omega^{op}_{\mathsf{Spa}}-, X) = \mathsf{Spa}^{op}(Stone \cdot A_0-, X)\\ &\cong \mathsf{Alg}(A_0-, Pred X) = (U \cdot Pred)(X)\end{aligned}$$

natural in X, proving the existence of the isomorphism on the left of Eq. 4.1.

Analogously, but now using the fact that $Pred^{op} \dashv Stone^{op} : \mathsf{Alg}^{op} \longrightarrow \mathsf{Spa}$ holds, we have isomorphisms

$$\begin{aligned}\widetilde{\Omega^{op}_{\mathsf{Alg}}}(A) &= \mathsf{Alg}^{op}(\Omega^{op}_{\mathsf{Alg}}-, A) = \mathsf{Alg}^{op}(Pred^{op} \cdot S_0-, A)\\ &\cong \mathsf{Spa}(S_0-, Stone^{op} A) = (V \cdot Stone^{op})(A)\end{aligned}$$

natural in A, proving the existence of the isomorphism on the right of Eq. 4.1.

In particular, we have

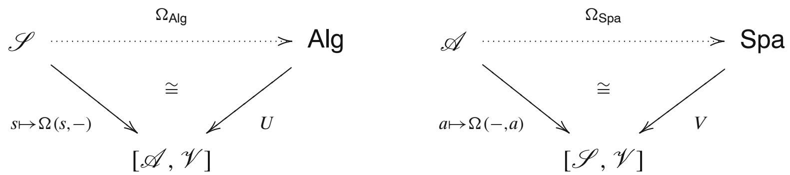
$$\mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s)) = \mathsf{Alg}(A_0(a), Pred \cdot S_0^{op}(s)) \cong \mathsf{Spa}^{op}(Stone \cdot A_0(a), S_0^{op}(s))$$
$$= \mathsf{Spa}(S_0(s), Stone^{op} \cdot A_0^{op}(a)) = \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a))$$

naturally in s and a. □

Remark 4.4 Let us analyze the natural isomorphism ι of Proposition 4.3. It asserts that, given the adjunction $Stone \dashv Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$, there exists a functor

$$\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$$

such that, in the following diagrams, there exist functors denoted by dotted arrows



This is the many-sorted variant of the case when the category of sorts $\mathscr{S}$ is the unit category $\mathscr{I}$, i.e., in the case when there is just one sort. Then Ω is identified with a mere object in $\mathscr{V}$ that has both the structure of a "space" and an "algebra". See also Example 5.1 below.

In fact, the isomorphism ι of Proposition 4.3 also allows us to "interchange" U-structured and V-structured sources having their bases formed on Ω. More precisely, the following result holds.

Lemma 4.5 (Interchange Lemma) *There is an isomorphism*

$$[\mathscr{A}, \mathscr{V}]\big(W, [\mathscr{S}, \mathscr{V}](W', V \cdot \Omega_{\mathsf{Spa}}-)\big) \cong [\mathscr{S}, \mathscr{V}]\big(W', [\mathscr{A}, \mathscr{V}](W, U \cdot \Omega_{\mathsf{Alg}}-)\big)$$

natural W and W′.

Proof Let us compute

$$\begin{aligned}[\mathscr{A}, \mathscr{V}](W, [\mathscr{S}, \mathscr{V}](W', V \cdot \Omega_{\mathsf{Spa}}-)) &\cong \int_a \int_s [W(a), [W'(s), \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a))]] \\ &\cong \int_s \int_a [W'(s), [W(a), \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s))]] \\ &\cong [\mathscr{S}, \mathscr{V}](W', [\mathscr{A}, \mathscr{V}](W, U \cdot \Omega_{\mathsf{Alg}}-))\end{aligned}$$

where the second isomorphism is due to the isomorphism

$$\iota_{a,s} : \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s)) \longrightarrow \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a))$$

and the Fubini Theorem for ends. □

Remark 4.6 In fact, there is a natural common "underlying" object

$$[\mathscr{S} \otimes \mathscr{A}, \mathscr{V}](W' \Box W, \Omega)$$

to both

$$[\mathscr{A}, \mathscr{V}](W, [\mathscr{S}, \mathscr{V}](W', V \cdot \Omega_{\mathsf{Spa}} -)) \quad \text{and} \quad [\mathscr{S}, \mathscr{V}](W', [\mathscr{A}, \mathscr{V}](W, U \cdot \Omega_{\mathsf{Alg}} -))$$

where $W' \Box W : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ is the functor mapping (s, a) to $W'(s) \otimes W(a)$.

More precisely, we have the following isomorphism

$$\begin{aligned}
[\mathscr{S}, \mathscr{V}](W', [\mathscr{A}, \mathscr{V}](W, U \cdot \Omega_{\mathsf{Alg}} -)) &\cong \int_s \int_a [W'(s), [W(a), \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s))]] \\
&\cong \int_{(s,a)} [W'(s) \otimes W(a), \Omega(s, a)] \\
&\cong [\mathscr{S} \otimes \mathscr{A}, \mathscr{V}](W' \Box W, \Omega)
\end{aligned}$$

Canonical Sources The Interchange Lemma 4.5 will allow us to define the structured sources whose initial lifts along U and V we will study.

Assumption 4.7 Suppose that $\Omega_{\mathsf{Spa}} : \mathscr{A}^{op} \longrightarrow \mathsf{Spa}$ and $\Omega_{\mathsf{Alg}} : \mathscr{S}^{op} \longrightarrow \mathsf{Alg}$ are given such that there exists an isomorphism

$$\iota_{a,s} : \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s)) \longrightarrow \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a)) \tag{4.3}$$

natural in s and a.

Notation 4.8 Let us consider, for each A in Alg, the action

$$U_{A, \Omega_{\mathsf{Alg}} -} : \mathsf{Alg}(A, \Omega_{\mathsf{Alg}} -) \longrightarrow [\mathscr{A}, \mathscr{V}](UA, U \cdot \Omega_{\mathsf{Alg}} -)$$

of the forgetful functor $U : \mathsf{Alg} \longrightarrow [\mathscr{A}, \mathscr{V}]$. Due to Lemma 4.5 this defines

$$\alpha_A : UA \longrightarrow [\mathscr{S}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Alg}}}(A), V \cdot \Omega_{\mathsf{Spa}} -\Big) \tag{4.4}$$

Analogously, we define, for each X in Spa

$$\sigma_X : VX \longrightarrow [\mathscr{A}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Spa}}}(X), U \cdot \Omega_{\mathsf{Alg}} -\Big) \tag{4.5}$$

using

$$V_{X, \Omega_{\mathsf{Spa}} -} : \mathsf{Spa}(X, \Omega_{\mathsf{Spa}} -) \longrightarrow [\mathscr{S}, \mathscr{V}](VX, V \cdot \Omega_{\mathsf{Spa}} -)$$

at the start of its formation.

Remark 4.9 Let us unravel the construction of α_A to see that our definition coincides with the definition of canonical sources in [18].

For each a in $\mathscr{A}$, s in $\mathscr{S}$, we consider the action

$$U_{A, \Omega_{\mathsf{Alg}}(s)} : \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)) \longrightarrow [\mathscr{A}, \mathscr{V}](\mathsf{Alg}(A_0(a), A), \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s))$$

that is simply the transpose of the composition morphism

$$\mathsf{Alg}(A_0(a), A) \otimes \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)) \xrightarrow{comp} \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s))$$

The Interchange Lemma 4.5 now forces us to consider the composite

$$\mathsf{Alg}(A_0(a), A) \otimes \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)) \xrightarrow{comp} \mathsf{Alg}(A_0(a), \Omega_{\mathsf{Alg}}(s)) \xrightarrow{\iota_{a,s}} \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a))$$

and its transpose

$$\alpha_{A,a,s} : \mathsf{Alg}(A_0(a), A) \longrightarrow [\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)), \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a))]$$

under $- \otimes \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)) \dashv [\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)), -]$. Observe that the right-hand side of the above is dinatural in s, hence to give the above morphism is the same thing as to give

$$\alpha_{A,a} : \mathsf{Alg}(A_0(a), A) \longrightarrow \int_s [\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}(s)), \mathsf{Spa}(S_0(s), \Omega_{\mathsf{Spa}}(a))]$$

which can be written down as

$$\alpha_{A,a} : \mathsf{Alg}(A_0(a), A) \longrightarrow [\mathscr{S}, \mathscr{V}](\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-), \mathsf{Spa}(S_0-, \Omega_{\mathsf{Spa}}(a)))$$

Furthermore, since $\mathsf{Alg}(A_0(a), A) = U(A)(a)$ and $\mathsf{Spa}(S_0-, \Omega_{\mathsf{Spa}}(a)) = (V \cdot \Omega_{\mathsf{Spa}})(a)$, we see that we have defined a V-structured source

$$\alpha_{A,a} : U(A)(a) \longrightarrow [\mathscr{S}, \mathscr{V}](\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-), (V \cdot \Omega_{\mathsf{Spa}})(a))$$

naturally in a. This is

$$\alpha_A : U(A) \longrightarrow [\mathscr{S}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Alg}}}(A), V \cdot \Omega_{\mathsf{Spa}}-\Big)$$

from Eq. 4.4 and the previous considerations showed that this definition is indeed precisely the definition of a canonical source in [18].

Compare the following lemma with Remark 1.5 of [18].

Lemma 4.10 *All canonical sources α_A and σ_X are monosources.*

Proof We prove that $\alpha_A : UA \longrightarrow [\mathscr{S}, \mathscr{V}](\widetilde{\Omega^{op}_{\mathsf{Alg}}}(A), V \cdot \Omega_{\mathsf{Spa}}-)$ is a monosource, the reasoning for proving that every σ_X is a monosource is similar. By Definition 3.10 we have to prove that, for every Z in $[\mathscr{S}, \mathscr{V}]$, the morphism

$$\mathsf{hom}_{[\mathscr{S},\mathscr{V}]}(Z, \alpha_A) : [\mathscr{S}, \mathscr{V}]\Big(Z, \widetilde{\Omega^{op}_{\mathsf{Alg}}}(A)\Big) \longrightarrow [\mathscr{A}, \mathscr{V}](UA, [\mathscr{S}, \mathscr{V}](Z, V \cdot \Omega_{\mathsf{Spa}}-))$$

is a monomorphism in $\mathscr{V}_o$.

By Lemma 4.5 we know that

$$[\mathscr{A}, \mathscr{V}](UA, [\mathscr{S}, \mathscr{V}](Z, V \cdot \Omega_{\mathsf{Spa}}-)) \cong [\mathscr{S}, \mathscr{V}](Z, [\mathscr{A}, \mathscr{V}](UA, U \cdot \Omega_{\mathsf{Alg}}-))$$

holds. Furthermore we will use that

$$[\mathscr{S}, \mathscr{V}]\Big(Z, \widetilde{\Omega^{op}_{\mathsf{Alg}}}(A)\Big) = [\mathscr{S}, \mathscr{V}](Z, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-))$$

holds.

Then the above morphism $\mathsf{hom}_{[\mathscr{S},\mathscr{V}]}(Z, \alpha_A)$ is just the action

$$[\mathscr{S}, \mathscr{V}](Z, U_{A,\Omega_{\mathsf{Alg}}-}) : [\mathscr{S}, \mathscr{V}](Z, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-)) \longrightarrow [\mathscr{S}, \mathscr{V}](Z, [\mathscr{A}, \mathscr{V}](UA, U \cdot \Omega_{\mathsf{Alg}}-))$$

and that is a monomorphism, since U is assumed to be faithful. □

Remark 4.11 In the one-sorted case when $\mathscr{S} = \mathscr{A} = \mathscr{I}$, the above definitions take the following form: For each A in Alg we define

$$\alpha_A : UA \longrightarrow [\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}), V\Omega_{\mathsf{Spa}}] \tag{4.6}$$

as the transpose under $- \otimes \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}) \dashv [\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}), -]$ of the composite

$$\mathsf{Alg}(A_0, A) \otimes \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}) \xrightarrow{comp} \mathsf{Alg}(A_0, \Omega_{\mathsf{Alg}}) \xrightarrow{\iota} \mathsf{Spa}(S_0, \Omega_{\mathsf{Spa}})$$

Analogously, we define, for each X in Spa

$$\sigma_X : VX \longrightarrow [\mathsf{Spa}(X, \Omega_{\mathsf{Spa}}), U\Omega_{\mathsf{Alg}}] \tag{4.7}$$

as the transpose under $- \otimes \mathsf{Spa}(X, \Omega_{\mathsf{Spa}}) \dashv [\mathsf{Spa}(X, \Omega_{\mathsf{Spa}}), -]$ of the composite

$$\mathsf{Spa}(S_0, X) \otimes \mathsf{Spa}(X, \Omega_{\mathsf{Spa}}) \xrightarrow{comp} \mathsf{Spa}(S_0, \Omega_{\mathsf{Spa}}) \xrightarrow{\iota^{-1}} \mathsf{Alg}(A_0, \Omega_{\mathsf{Alg}})$$

Observe that isomorphisms (Eq. 4.1) assert that, in the presence of the adjunction *Stone* $\dashv$ *Pred*, both α_A and σ_X have lifts, functorial in A and X, respectively.

More precisely, we can define

$$\overline{\alpha}_A : UA \longrightarrow \mathsf{Spa}(\mathit{Stone}\, A, \Omega_{\mathsf{Spa}})$$

to be the action of *Stone* on hom-objects, since, by our assumptions and the definition of Ω_{Spa}, the equalities

$$UA = \mathsf{Alg}(A_0, A) \quad \text{and} \quad \mathsf{Spa}(\mathit{Stone}\, A, \Omega_{\mathsf{Spa}}) = \mathsf{Spa}(\mathit{Stone}\, A, \mathit{Stone}\, A_0)$$

hold. Then the isomorphism from the right-hand side of Eq. 4.1 asserts that $\overline{\alpha}_A$ is a V-lift. The U-lift $\overline{\sigma}_X$ of σ_X can be defined similarly.

This is exactly what Porst and Tholen call a *weak schizophrenic object*, see Conditions (WSO1) and (WSO2) in [18].

We define now weak schizophrenic and schizophrenic objects in our many-sorted setting. Since mere objects need to be replaced by functors of the form $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ and since functors of this form are called *modules* (or *profunctors* or *distributors*) and denoted as $\Omega : \mathscr{S}^{op} \longrightarrow\!\!\!\!\!\!+\,\, \mathscr{A}$ we will call them (weak) schizophrenic *modules*.

Definition 4.12 A triple $(\Omega_{\mathsf{Spa}}, \Omega_{\mathsf{Alg}}, \iota)$ as in Eq. 4.3 is called

1. A *weak schizophrenic module* provided that the following two conditions hold:

   (WSO1) Every V-structured source

   $$\alpha_A : UA \longrightarrow [\mathscr{S}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Alg}}}(A), V \cdot \Omega_{\mathsf{Spa}}-\Big)$$

   has a functorial V-lift

   $$\overline{\alpha}_A : UA \longrightarrow \mathsf{Spa}(Stone\, A, \Omega_{\mathsf{Spa}}-)$$

   (WSO2) Every U-structured source

   $$\sigma_X : VX \longrightarrow [\mathscr{A}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Spa}}}(X), U \cdot \Omega_{\mathsf{Alg}}-\Big)$$

   has a functorial U-lift

   $$\overline{\sigma}_X : VX \longrightarrow \mathsf{Alg}(Pred\, X, \Omega_{\mathsf{Alg}}-)$$

2. A *schizophrenic module* provided that the following two conditions hold:

   (SO1) Every V-structured source

   $$\alpha_A : UA \longrightarrow [\mathscr{S}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Alg}}}(A), V \cdot \Omega_{\mathsf{Spa}}-\Big)$$

   has a V-initial lift

   $$\overline{\alpha}_A : UA \longrightarrow \mathsf{Spa}(Stone\, A, \Omega_{\mathsf{Spa}}-)$$

   (SO2) Every U-structured source

   $$\sigma_X : VX \longrightarrow [\mathscr{A}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Spa}}}(X), U \cdot \Omega_{\mathsf{Alg}}-\Big)$$

   has a U-initial lift

   $$\overline{\sigma}_X : VX \longrightarrow \mathsf{Alg}(Pred\, X, \Omega_{\mathsf{Alg}}-)$$

Remark 4.13 Proposition 4.3 asserts that every adjunction $Stone \dashv Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$ gives rise to a weak schizophrenic module. The reasoning is similar to the discussion of the Set-case in Remark 4.11.

Remark 4.14 In case both $V : \mathsf{Spa} \longrightarrow [\mathscr{S}, \mathscr{V}]$ and $U : \mathsf{Alg} \longrightarrow [\mathscr{A}, \mathscr{V}]$ are of descent type, every weak schizophrenic module $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ is schizophrenic. This follows from Proposition 3.12 and Lemma 4.10 above.

Remark 4.15 In anticipation of Theorem 4.16 we denoted the object-assignments

$$A \mapsto Stone\, A, \quad X \mapsto Pred\, X$$

of initial lifts as if they were functors. That the assignments are indeed functorial is proved in Theorem 4.16 below.

Theorem 4.16 *Every schizophrenic module* $(\Omega_{\mathsf{Spa}}, \Omega_{\mathsf{Alg}}, \iota)$ *induces adjoint functors* $Stone \dashv Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$.

Proof We divide the proof into several parts.

(1) Definition of *Stone* : $\mathsf{Alg} \longrightarrow \mathsf{Spa}^{op}$. Put $Stone\,A$ to be the vertex of a V-initial lift $\overline{\alpha}_A : UA \longrightarrow \mathsf{Spa}(Stone\,A, \Omega_{\mathsf{Spa}}-)$ of α_A, see Eq. 4.4. In particular, the following diagram

$$\begin{array}{ccc} \mathsf{Spa}(X, Stone\,A) & \xrightarrow{\hom_{\mathsf{Spa}}(X,\overline{\alpha}_A)} & [\mathscr{A},\mathscr{V}](UA, \mathsf{Spa}(X,\Omega_{\mathsf{Spa}}-)) \\ {\scriptstyle V_{X,Stone\,A}}\Big\downarrow & & \Big\downarrow{\scriptstyle [\mathscr{A},\mathscr{V}](UA, V_{X,\Omega_{\mathsf{Spa}}-})} \\ [\mathscr{S},\mathscr{V}](VX, \mathsf{Alg}(A,\Omega_{\mathsf{Alg}}-)) & \xrightarrow[\hom_{[\mathscr{S},\mathscr{V}]}(VX,\alpha_A)]{} & [\mathscr{A},\mathscr{V}](UA,[\mathscr{S},\mathscr{V}](VX, V\cdot\Omega_{\mathsf{Spa}}-)) \end{array}$$

is a pullback in $[\mathscr{A}, \mathscr{V}]_o$, for every X in Spa.
To define $Stone_{A,A'} : \mathsf{Alg}(A, A') \longrightarrow \mathsf{Spa}(Stone\,A', Stone\,A)$, put $S := Stone\,A'$ in the above pullback and use the fact that the square

$$\begin{array}{ccc} \mathsf{Alg}(A, A') & \longrightarrow & [\mathscr{A},\mathscr{V}](UA, \mathsf{Spa}(Stone\,A',\Omega_{\mathsf{Spa}}-)) \\ {\scriptstyle \widetilde{\Omega^{op}_{\mathsf{Alg}}}_{A,A'}}\Big\downarrow & & \Big\downarrow{\scriptstyle [\mathscr{A},\mathscr{V}](UA, V_{Stone\,A',\Omega_{\mathsf{Spa}}-})} \\ [\mathscr{S},\mathscr{V}](\mathsf{Alg}(A',\Omega_{\mathsf{Alg}}-), \mathsf{Alg}(A,\Omega_{\mathsf{Alg}}-)) & \xrightarrow[\hom_{[\mathscr{S},\mathscr{V}]}(\mathsf{Alg}(A',\Omega_{\mathsf{Alg}}-),\alpha_A)]{} & [\mathscr{A},\mathscr{V}](UA,[\mathscr{S},\mathscr{V}](\mathsf{Alg}(A',\Omega_{\mathsf{Alg}}-), V\cdot\Omega_{\mathsf{Spa}}-)) \end{array}$$

commutes, where the upper horizontal arrow is the transpose of the composite

$$\mathsf{Alg}(A, A') \bullet \mathsf{Alg}(A_0-, A) \longrightarrow \mathsf{Alg}(A_0-, A')$$

$$\xrightarrow{\overline{\alpha}_{A'}} \mathsf{Spa}(Stone\,A', \Omega_{\mathsf{Spa}}-)$$

under $- \bullet UA \dashv [\mathscr{A}, \mathscr{V}](UA, -)$ (we use that $UA = \mathsf{Alg}(A_0-, A)$).
We define $Stone_{A,A'} : \mathsf{Alg}(A, A') \longrightarrow \mathsf{Spa}(Stone\,A', Stone\,A)$ to be the unique mediating morphism existing by the universal property of pullbacks. It is then easy to see that *Stone* is a functor. Namely, the commutativity of the following two diagrams

$$\begin{array}{ccc} \mathsf{Alg}(A, A) & \xrightarrow{Stone_{A,A}} & \mathsf{Spa}(Stone\,A, Stone\,A) \\ & {\scriptstyle id}\nwarrow \quad \nearrow{\scriptstyle id} & \\ & I & \end{array}$$

and

$$\begin{array}{ccc}
\mathsf{Alg}(A, A') \otimes \mathsf{Alg}(A', A'') & \xrightarrow{Stone_{A,A'} \otimes Stone_{A',A''}} & \mathsf{Spa}(Stone A', Stone A) \otimes \mathsf{Spa}(Stone A'', Stone A') \\
{\scriptstyle comp}\downarrow & & \downarrow{\scriptstyle comp} \\
\mathsf{Alg}(A, A'') & \xrightarrow[Stone_{A,A''}]{} & \mathsf{Spa}(Stone A'', Stone A)
\end{array}$$

is verified using the universal property of pullbacks.

(2) Definition of $Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$. We put $Pred X$ to be the vertex of a U-initial lift $\overline{\sigma}_X : VX \longrightarrow \mathsf{Alg}(Pred X, \Omega_{\mathsf{Alg}}-)$ of σ_X, see Eq. 4.5. In particular, the following diagram

$$\begin{array}{ccc}
\mathsf{Alg}(A, Pred X) & \xrightarrow{\mathsf{hom}_{\mathsf{Alg}}(A, \overline{\sigma}_X)} & [\mathscr{S}, \mathscr{V}](VX, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-)) \\
{\scriptstyle U_{A,Pred X}}\downarrow & & \downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](VX, U_{A,\Omega_{\mathsf{Alg}}-})} \\
[\mathscr{A}, \mathscr{V}](UA, \mathsf{Spa}(X, \Omega_{\mathsf{Spa}}-)) & \xrightarrow[\mathsf{hom}_{[\mathscr{S},\mathscr{V}]}(UA, \sigma_X)]{} & [\mathscr{S}, \mathscr{V}](VX, [\mathscr{A}, \mathscr{V}](UA, U \cdot \Omega_{\mathsf{Alg}}-))
\end{array}$$

is a pullback in $[\mathscr{S}, \mathscr{V}]_o$, for every A in Alg.
The action of *Pred* on hom-objects is defined by the universal property of pullbacks analogously to the definition of *Stone*.

(3) We establish that $Stone \dashv Pred$ holds. We need to prove the existence of morphisms

$$\eta_A : I \longrightarrow \mathsf{Alg}(A, Pred \cdot Stone A), \quad \varepsilon_X : I \longrightarrow \mathsf{Spa}^{op}(Stone \cdot Pred X, X)$$

prove their naturality and verify the triangle equalities, see Remark 2.9.

(a) Definition of η_A.
Define $\ulcorner \overline{\alpha}_A \urcorner : I \longrightarrow [\mathscr{A}, \mathscr{V}](UA, \mathsf{Spa}(Stone A, \Omega_{\mathsf{Spa}}-))$ to be the transpose of

$$\overline{\alpha}_A : UA \longrightarrow \mathsf{Spa}(Stone A, \Omega_{\mathsf{Spa}}-)$$

under $- \bullet UA \dashv [\mathscr{A}, \mathscr{V}](UA, -)$.
Analogously, define $\ulcorner id \urcorner : I \longrightarrow [\mathscr{S}, \mathscr{V}](V Stone A, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-))$ to be the transpose of the identity $id : \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-) \longrightarrow \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-)$ under the adjunction $- \bullet \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}) \dashv [\mathscr{S}, \mathscr{V}](\mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-)$ (we have used that $V Stone A = \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-)$ holds).
Observe that the diagram

$$\begin{array}{ccc}
I & \xrightarrow{\ulcorner id \urcorner} & [\mathscr{S}, \mathscr{V}](V Stone A, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-)) \\
{\scriptstyle \ulcorner \overline{\alpha}_A \urcorner}\downarrow & & \downarrow{\scriptstyle [\mathscr{S},\mathscr{V}](V Stone A, U_{A,\Omega_{\mathsf{Alg}}-})} \\
[\mathscr{A}, \mathscr{V}](UA, \mathsf{Spa}(Stone A, \Omega_{\mathsf{Spa}}-)) & \xrightarrow[\mathsf{hom}_{[\mathscr{A},\mathscr{V}]}(UA, \sigma_{Stone A})]{} & [\mathscr{S}, \mathscr{V}](V Stone A, [\mathscr{A}, \mathscr{V}](UA, U\Omega_{\mathsf{Alg}}-))
\end{array}$$

commutes (use the transpose under $- \bullet VStone A \dashv [\mathscr{S}, \mathscr{V}](VStone A, -)$ and the equality $VStone A = \mathsf{Alg}(A, \Omega_{\mathsf{Alg}} -))$.

Using the universal property of pullbacks we have defined $\eta_A : I \longrightarrow \mathsf{Alg}(A, Pred \cdot Stone A)$. Naturality of η_A is clear.

Observe that one of the defining properties of factorization through a pullback is the commutativity of the triangle

$$\begin{array}{ccc} I & \xrightarrow{\ulcorner id \urcorner} & [\mathscr{S}, \mathscr{V}](VStone A, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}} -)) \\ {\scriptstyle \eta_A} \searrow & & \uparrow {\scriptstyle =} \\ & \mathsf{Alg}(A, Pred \cdot Stone A) \xrightarrow{\mathsf{hom}_{\mathsf{Alg}}(A, \overline{\sigma}_{Stone A})} & [\mathscr{S}, \mathscr{V}](VStone A, \mathsf{Alg}(A, \Omega_{\mathsf{Alg}} -)) \end{array} \tag{4.8}$$

that is exactly (a many-sorted version of) Eq. 11 of [18].

(b) Definition of $\varepsilon_X : I \longrightarrow \mathsf{Spa}(X, Stone \cdot Pred X)$ follows an analogous pattern as that of η_A.

We will need an analogy of Eq. 13 of [18] and that is, similarly to the above, expressed by commutativity of the triangle

$$\begin{array}{ccc} I & \xrightarrow{\ulcorner id \urcorner} & [\mathscr{A}, \mathscr{V}](UPred X, \mathsf{Spa}(X, \Omega_{\mathsf{Spa}} -)) \\ {\scriptstyle \varepsilon_X} \searrow & & \uparrow {\scriptstyle =} \\ & Stone(X, Stone \cdot Pred X) \xrightarrow{\mathsf{hom}_{\mathsf{Spa}}(X, \overline{\alpha}_{Pred X})} & [\mathscr{A}, \mathscr{V}](UPred X, \mathsf{Spa}(X, \Omega_{\mathsf{Spa}} -)) \end{array} \tag{4.9}$$

(c) We prove that the triangle identities hold for η and ε. We only prove that $\varepsilon_{Stone} \cdot Stone\eta = id$ holds in Spa^{op}. The second triangle equality is verified similarly.

Equivalently, we want to prove the equality

$$\begin{array}{ccc} Stone A & \xrightarrow{\varepsilon_{Stone A}} & Stone \cdot Pred \cdot Stone A \\ & \searrow^{=} & \downarrow {\scriptstyle Stone\eta_A} \\ & & Stone A \end{array}$$

holds in Spa, for every A.

Since $V : \mathsf{Spa} \longrightarrow [\mathscr{S}, \mathscr{V}]$ is assumed to be faithful, it suffices to prove that

$$\begin{array}{ccc} V \cdot Stone A & \xrightarrow{V\varepsilon_{Stone A}} & V \cdot Stone \cdot Pred \cdot Stone A \\ & \searrow^{=} & \downarrow {\scriptstyle V \cdot Stone\eta_A} \\ & & V \cdot Stone A \end{array}$$

holds, or, equivalently that

$$\begin{array}{ccc} \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-) & \xrightarrow{V\varepsilon_{Stone\,A}} & \mathsf{Alg}(Pred \cdot Stone\,A, \Omega_{\mathsf{Alg}}-) \\ & \diagdown\!\!\diagdown & \Big\downarrow {\scriptstyle \mathsf{Alg}(\eta_A, \Omega_{\mathsf{Alg}}-)} \\ & & \mathsf{Alg}(A, \Omega_{\mathsf{Alg}}-) \end{array}$$

holds. But the last triangle is just the transpose of Eq. 4.8 above.

□

Remark 4.17 Theorem 4.16 collapses to Theorem 1.7 of [18] in case when $\mathscr{V} = \mathsf{Set}$ and $\mathscr{S} = \mathscr{A} = \mathscr{I}$.

Definition 4.18 Any adjunction $Stone \dashv Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$ induced by a schizophrenic module is called a *logical connection*.

The following result is a generalization of Lemma 2.3 of [18]. We state it for the category Spa, the obvious reformulation to the case of the category Alg is left to the reader.

Lemma 4.19 *Suppose X is an object in* Spa. *Then the following are equivalent:*

1. *The source* $id : \mathsf{Spa}(X, \Omega_{\mathsf{Spa}}-) \longrightarrow \mathsf{Spa}(X, \Omega_{\mathsf{Spa}}-)$ *is a monosource.*
2. *The action*

$$\widetilde{\Omega^{op}_{\mathsf{Spa}}}_{X,X'} : \mathsf{Spa}^{op}(X, X') \longrightarrow [\mathscr{A}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Spa}}}(X), \widetilde{\Omega^{op}_{\mathsf{Spa}}}(X')\Big)$$

of $\widetilde{\Omega^{op}_{\mathsf{Spa}}} : \mathsf{Spa}^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$ *is a monomorphism in* $\mathscr{V}_o$, *for every* X' *in* Spa.

3. *The action*

$$Pred_{X,X'} : \mathsf{Spa}^{op}(X, X') \longrightarrow \mathsf{Alg}(Pred\,X, Pred\,X')$$

of $Pred : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$ *is a monomorphism in* $\mathscr{V}_o$, *for every* X' *in* Spa.

4. *The morphism*

$$\mathsf{Spa}(X', \varepsilon_X) : \mathsf{Spa}(X', X) \longrightarrow \mathsf{Spa}(X', Stone^{op}\,Pred^{op}\,X)$$

is a monomorphism in $\mathscr{V}_o$, *for every* X' *in* Spa.

Proof Suppose that X' in Spa is arbitrary. Then the morphism

$$\hom_{\mathsf{Spa}}(X', id) : \mathsf{Spa}(X', X) \longrightarrow [\mathscr{A}, \mathscr{V}](\mathsf{Spa}(X, \Omega_{\mathsf{Spa}}-), \mathsf{Spa}(X', \Omega_{\mathsf{Spa}}-))$$

is, by Yoneda Lemma, simply the action

$$\widetilde{\Omega^{op}_{\mathsf{Spa}}}_{X,X'} : \mathsf{Spa}^{op}(X, X') \longrightarrow [\mathscr{A}, \mathscr{V}]\Big(\widetilde{\Omega^{op}_{\mathsf{Spa}}}(X), \widetilde{\Omega^{op}_{\mathsf{Spa}}}(X')\Big)$$

of $\widetilde{\Omega^{op}_{\mathsf{Spa}}} : \mathsf{Spa}^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$. This proves the equivalence of 1 and 2.

Since $U \cdot Pred \cong \widetilde{\Omega^{op}_{\mathsf{Spa}}}$ holds by the isomorphism on the left of Eq. 4.1, conditions 2 and 3 are equivalent by the properties of monomorphisms in $\mathscr{V}_o$ and the fact that U is assumed to be faithful.

Finally, to prove the equivalence of 3 and 4, observe that the action

$$Pred_{X,X'} : \mathsf{Spa}^{op}(X, X') \longrightarrow \mathsf{Alg}(Pred X, Pred X')$$

is, in the presence of the adjunction $Stone \dashv Pred$, to within an isomorphism exactly the

$$\mathsf{Spa}^{op}(\varepsilon_X, X') : \mathsf{Spa}^{op}(X, X') \longrightarrow \mathsf{Spa}^{op}(Stone \cdot Pred X, X')$$

or, changing the variance, exactly

$$\mathsf{Spa}(X', \varepsilon_X) : \mathsf{Spa}(X', X) \longrightarrow \mathsf{Spa}(X', Stone^{op} \cdot Pred^{op} X)$$

□

5 Examples of Logical Connections

In this section we gather various interesting examples of logical connections.

Probably the best-known example is the logical connection of the category Set with itself, mediated by the two-element set 2 as a schizophrenic object. More precisely, we put $\mathscr{V} = \mathsf{Set}$ (i.e., we deal with ordinary categories), we further put $\mathsf{Spa} = \mathsf{Set}$ and $\mathsf{Alg} = \mathsf{Set}$, considered as concrete over Set via the identity functor, and we consider the functors

$$Pred : \mathsf{Set}^{op} \longrightarrow \mathsf{Set}, \quad X \mapsto [X, 2], \quad Stone : \mathsf{Set} \longrightarrow \mathsf{Set}^{op}, \quad X \mapsto [X, 2]$$

Then it is straightforward to verify that $Stone \dashv Pred$ holds and that the adjunction is a logical connection given by 2 as the schizophrenic object.

As we show now, the above "prototypical" logical connection is an instance of a more general situation.

Example 5.1 Fix an arbitrary base category $\mathscr{V}$. Recall that, for objects X, Y of $\mathscr{V}$, we denote by $[X, Y]$ the internal hom.

Put $\mathsf{Spa} = \mathscr{V}$, $\mathsf{Alg} = \mathscr{V}$, with $V = Id$ and $U = Id$, and choose Ω to be *any* object of $\mathscr{V}$.

Then the isomorphism

$$\mathscr{V}(X, [Y, \Omega]) \cong \mathscr{V}^{op}([X, \Omega], Y)$$

holds, naturally in X and Y, since $\mathscr{V}$ symmetric monoidal closed. Hence we may put $Stone = [-, \Omega]$ and $Pred = [-, \Omega]$ and we obtain an adjunction

$$[-, \Omega] \dashv [-, \Omega] : \mathscr{V}^{op} \longrightarrow \mathscr{V}$$

Observe that α_A from Notation 4.8 has the form

$$\alpha_A : A \longrightarrow [[A, \Omega], \Omega]$$

having itself as its Id-initial Id-lift. Similarly for σ_X.

Hence the above adjunction is a logical connection. Its special instance is the above logical connection, if we put $\mathscr{V} = \mathsf{Set}$ and $\Omega = 2$.

Another instance of the above is studied in [11]. There: $\mathscr{V} = \mathsf{Pos}$, $\mathsf{Spa} = \mathsf{Pos}$ and $\mathsf{Alg} = \mathsf{DL}$ (the category of bounded distributive lattices and homomorphisms), $V = Id$ and $U : \mathsf{DL} \longrightarrow \mathsf{Pos}$ is the obvious forgetful functor. The schizophrenic object is carried by the poset $\mathbb{2}$ — the two-element chain.

The resulting logical connection works as follows: $Pred\,X$ is the distributive lattice of uppersets of the poset X, and $Stone\,A$ is the poset of prime filters of the distributive lattice A. The underlying *ordinary* adjunction of $Stone \dashv Pred$ serves as a basis for *Priestley duality*, see [19].

Example 5.1 is an instance of a yet more general situation, as the next example shows.

Example 5.2 Let $\mathscr{V}$ be an arbitrary base category and choose any functor $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$, where $\mathscr{S}$ and $\mathscr{A}$ are arbitrary small categories.

Put $\mathsf{Spa} = [\mathscr{S}, \mathscr{V}]$ and $\mathsf{Alg} = [\mathscr{A}, \mathscr{V}]$, considered as concrete categories over $[\mathscr{S}, \mathscr{V}]$ and $[\mathscr{A}, \mathscr{V}]$, respectively, via the identity functor. Recall the notation for weighted limits from Definition 2.6 and define

$$Pred : [\mathscr{S}, \mathscr{V}]^{op} \longrightarrow [\mathscr{A}, \mathscr{V}], \quad X \mapsto (a \mapsto \{X, \Omega(-, a)\})$$

and

$$Stone : [\mathscr{A}, \mathscr{V}] \longrightarrow [\mathscr{S}, \mathscr{V}]^{op}, \quad A \mapsto (s \mapsto \{A, \Omega(s, -)\})$$

Then the isomorphisms

$$\begin{aligned}
[\mathscr{A}, \mathscr{V}](A, Pred\,X) &\cong \int_a \Big[Aa, \{X, \Omega(-, a)\} \Big] \\
&\cong \int_a \Big[Aa, \int_s [Xs, \Omega(s, a)] \Big] \\
&\cong \int_s \Big[Xs, \int_a [Aa, \Omega(s, a)] \Big] \\
&\cong \int_s \Big[Xs, \{A, \Omega(s, -)\} \Big] \\
&\cong [\mathscr{S}, \mathscr{V}]^{op}(Stone\,A, X)
\end{aligned}$$

hold due to the fact how limits are computed and due to monoidal closedness of $\mathscr{V}$.

Hence we have established the adjunction $Stone \dashv Pred$. This adjunction is necessarily a logical connection induced by Ω as a schizophrenic module.

Moreover, by Proposition 4.3, *every* adjunction $Stone \dashv Pred : [\mathscr{S}, \mathscr{V}]^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$ induces a module $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$. The reason for that terminology is, among other facts, given by various instances of the result on logical connections:

1. For $\mathscr{S} = \mathscr{A} = \mathscr{I}$, where $\mathscr{I}$ is the unit category, the functor $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ amounts to a choice of an object $\Omega(*, *)$ in $\mathscr{V}$. The logical connection obtained in this way is exactly the logical connection of Example 5.1 above.
2. Let $\mathscr{V} = \mathsf{Ab}$, the category of Abelian groups and their homomorphisms. It is well-known that, in this case, a unitary ring S can be identified with a one-object

category $\mathscr{S}$. Then $[\mathscr{S}^{op}, \mathscr{V}]$ is exactly the category S-Mod of left S-modules. Similarly, for a unitary ring A the category Mod-A of right A-modules is exactly the category $[\mathscr{A}, \mathscr{V}]$, where $\mathscr{A}$ is the category on one object induced by A.
A functor $\Omega : \mathscr{S}^{op} \otimes \mathscr{A} \longrightarrow \mathsf{Ab}$ is exactly a left-S, right-A module and the induced logical connection $Stone \dashv Pred$ is the well-known example of an adjunction, where Ω is the dualization bimodule, see Chapter 5 of [6] or Exercise 12 in Chapter 10 of [12].

3. Suppose $\mathscr{V} = \mathbb{2}$ (the two-element chain). Categories in this case are exactly the preorders, functors are exactly the monotone maps. Suppose that S and A are small sets, considered as discrete $\mathscr{V}$-categories. Then $[\mathscr{S}, \mathscr{V}]$ and $[\mathscr{A}, \mathscr{V}]$ are just the powersets of S and A, respectively, ordered by inclusion.
Then adjunctions
$$Stone \dashv Pred : [\mathscr{S}, \mathscr{V}]^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$$
are in one-to-one correspondence with modules
$$\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$$
which, in turn, are nothing else than subsets of $S \times A$, i.e., binary relations.
Of course, one may pass to non-discrete preorders $\mathscr{S}$ and $\mathscr{A}$. Then functors $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ are exactly *monotone* relations $\Omega : \mathscr{S}^{op} \mathrel{\longrightarrow\!\!\!\!\!\!\mapsto} \mathscr{A}$ from $\mathscr{S}^{op}$ to $\mathscr{A}$ an they are in one-to-one correspondence with adjunctions
$$Stone \dashv Pred : [\mathscr{S}, \mathscr{V}]^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$$
where $[\mathscr{S}, \mathscr{V}]$ is the poset of uppersets on $\mathscr{S}$ ordered by inclusion and $[\mathscr{A}, \mathscr{V}]$ is the poset of uppersets on $\mathscr{A}$ ordered by inclusion. See [7] for more details.

4. Suppose $\mathscr{V} = \mathsf{Set}$, i.e., suppose we work within the ordinary category theory. Then the functors $\Omega : \mathscr{S} \times \mathscr{A} \longrightarrow \mathsf{Set}$ are in one-to-one correspondence with adjunctions $Stone \dashv Pred : [\mathscr{S}, \mathsf{Set}]^{op} \longrightarrow [\mathscr{A}, \mathsf{Set}]$. See the paper [9].

In fact, the above examples give a full description of logical connections between presheaf categories. To summarize:

Proposition 5.3 *Consider $[\mathscr{S}, \mathscr{V}]$ and $[\mathscr{A}, \mathscr{V}]$ to be concrete over themselves via the identity functor. Then every adjunction Stone $\dashv$ Pred : $[\mathscr{S}, \mathscr{V}]^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$ is a logical connection.*

Proof Suppose that an adjunction $Stone \dashv Pred : [\mathscr{S}, \mathscr{V}]^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$ is given. In the notation of Assumption 4.1, both the functors $S_0 : \mathscr{S}^{op} \longrightarrow [\mathscr{S}, \mathscr{V}]$ and $A_0 : \mathscr{A}^{op} \longrightarrow [\mathscr{A}, \mathscr{V}]$ are Yoneda embeddings.

Form the module $\Omega : \mathscr{S} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ as in the proof of Proposition 4.3. Thus, we have
$$\Omega(s, -) = Pred \cdot S_0^{op}(s) = Pred(\mathscr{S}(s, -))$$
$$\Omega(-, a) = Stone^{op} A_0^{op}(a) = Stone^{op}(\mathscr{A}(-, a))$$

We want to prove that $Stone^{op}(A)(s) \cong \{A, \Omega(s,-)\}$ and $Pred(X)(a) \cong \{X,\} \Omega(-,a)$.

1. For the first isomorphism, consider

$$\begin{aligned} Stone^{op}(A)(s) &\cong [\mathscr{S},\mathscr{V}](S_0 s, Stone^{op}(A)) \\ &= [\mathscr{S},\mathscr{V}]^{op}(Stone(A), S_0^{op}(s)) \\ &\cong [\mathscr{A},\mathscr{V}](A, Pred S_0^{op}(s)) \\ &= [\mathscr{A},\mathscr{V}](A, \Omega(s,-)) \\ &\cong \{A, \Omega(s,-)\} \end{aligned}$$

   where we have used Yoneda Lemma, the adjunction $Stone \dashv Pred$ and the definition of Ω.
2. For the second isomorphism, proceed analogously:

$$\begin{aligned} Pred(X)(a) &\cong [\mathscr{A},\mathscr{V}](A_0(a), Pred(X)) \\ &\cong [\mathscr{S},\mathscr{V}]^{op}(Stone A_0(a), X) \\ &= [\mathscr{S},\mathscr{V}](X, Stone^{op} A_0^{op}(a)) \\ &= [\mathscr{S},\mathscr{V}](X, \Omega(-,a)) \\ &\cong \{X, \Omega(-,a)\} \end{aligned}$$

Hence, using Example 5.2 above, we proved that every adjunction $Stone \dashv Pred : [\mathscr{S},\mathscr{V}]^{op} \longrightarrow [\mathscr{A},\mathscr{V}]$ is a logical connection. □

The next series of examples shows how to generate logical connections provided that Ω is a "bimodel" in a certain sense (see also Remark 5.5). We start with an example of a logical connection between a *pro-completion* and *ind-cocompletion* of a small category mentioned in [15].

Example 5.4 Suppose that $\mathscr{V} = \mathsf{Set}$ in this example. Fix a small category $\mathscr{C}$ that is finitely complete and finitely cocomplete. Put

$$\mathsf{Alg} = \mathsf{Ind}(\mathscr{C})$$

to be the *free cocompletion* of $\mathscr{C}$ under *filtered colimits*. More precisely, let the fully faithful dense

$$A_0 : \mathscr{C} \longrightarrow \mathsf{Ind}(\mathscr{C})$$

exhibit $\mathsf{Ind}(\mathscr{C})$ as such a cocompletion. Since $\mathscr{C}$ is assumed to be finitely cocomplete, by the general theory of locally finitely presentable categories [8] we know that $\mathsf{Ind}(\mathscr{C})$ is complete and cocomplete and that it is a full reflective subcategory of $[\mathscr{C}^{op}, \mathsf{Set}]$ via the full embedding

$$U = \widetilde{A_0} : \mathsf{Ind}(\mathscr{C}) \longrightarrow [\mathscr{C}^{op}, \mathsf{Set}]$$

that preserves filtered colimits.

Suppose further that

$$\mathsf{Spa} = \mathsf{Ind}(\mathscr{C}^{op})$$

or, more precisely, that

$$S_0 : \mathscr{C}^{op} \longrightarrow \mathsf{Ind}(\mathscr{C}^{op})$$

is a free completion of $\mathscr{C}$ under filtered colimits. Since $\mathscr{C}$ has finite limits, using the general theory again, we have a fully faithful reflection

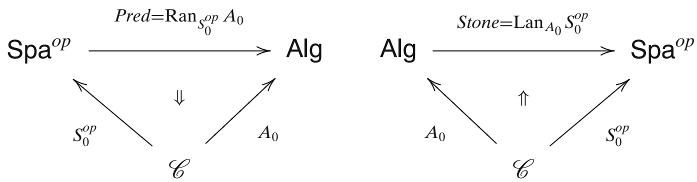
$$V = \widetilde{S}_0 : \mathsf{Ind}(\mathscr{C}^{op}) \longrightarrow [\mathscr{C}, \mathsf{Set}]$$

that preserves filtered colimits.

Hence Assumptions 4.1 are satisfied. Moreover, as it is shown in Section 5 of [15], there is an adjunction

$$\mathit{Stone} \dashv \mathit{Pred} : \mathsf{Spa}^{op} \longrightarrow \mathsf{Alg}$$

given by left and right Kan extensions:



We show now that the above adjunction $\mathit{Stone} \dashv \mathit{Pred}$ is a logical connection, induced by the module

$$\Omega : \mathscr{C}^{op} \times \mathscr{C} \longrightarrow \mathsf{Set}, \quad (s, a) \mapsto \mathscr{C}(s, a)$$

i.e., the logical connection is induced by the hom-functor of $\mathscr{C}$. Clearly, the hom-functor Ω is the weakly schizophrenic object determined by the adjunction $\mathit{Stone} \dashv \mathit{Pred}$, see Proposition 4.3 above. Moreover, since both U and V are monadic functors (they both are full reflections), the module Ω is a schizophrenic object by Remark 4.14.

The above can be applied to Example 5.1 of [15] as follows:

1. Put $\mathscr{C}$ = finite Boolean algebras. Then $\mathsf{Alg} = \mathsf{Ind}(\mathscr{C})$ is the category of all Boolean algebras and $\mathsf{Spa} = \mathsf{Ind}(\mathscr{C}^{op})$ is the category of sets, since $\mathscr{C}^{op}$ is equivalent to the category of finite sets.
2. Put $\mathscr{C}$ = finite distributive lattices. Then $\mathsf{Alg} = \mathsf{Ind}(\mathscr{C})$ is the category of all distributive lattices and $\mathsf{Spa} = \mathsf{Ind}(\mathscr{C}^{op})$ is the category of posets, since $\mathscr{C}^{op}$ is equivalent to the category of finite posets.

Remark 5.5 One can generalize the above example by making the following observation: our assumptions made the hom-functor $\Omega : \mathscr{C}^{op} \times \mathscr{C} \longrightarrow \mathsf{Set}$ preserve finite limits in both arguments (in the first argument this, of course, means that finite colimits in $\mathscr{C}$ are turned into limits). Hence we could replace the above hom-functor by a general module

$$\Omega : \mathscr{S} \times \mathscr{A} \longrightarrow \mathsf{Set}$$

with the property that both $\mathscr{S}$ and $\mathscr{A}$ have finite limits and Ω preserves them in each argument. One would obtain a logical connection of the form

$$Stone \dashv Pred : (\mathsf{Ind}(\mathscr{S}))^{op} \longrightarrow \mathsf{Ind}(\mathscr{A})$$

by the same reasoning as above.

In fact, one can go even farther: one can pass from ind-cocompletions to free cocompletions under *filtered colimits w.r.t. a doctrine* in the sense of [1]. The same reasoning allows one to pass from $\mathscr{V} = \mathsf{Set}$ to a general l.f.p. $\mathscr{V}$ and considering free cocompletions under filtered colimits w.r.t. a doctrine in this setting, see [14] for an account of filteredness notions in this very general setting.

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