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Equational presentations of functors and monads

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We study equational presentations of functors and monads defined on a category $\mathscr{K}$ that is equipped by an adjunction $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ of descent type. We present a class of functors/monads that admit such an equational presentation that involves finitary signatures in $\mathscr{X}$.
We apply these results to an equational description of functors arising in various areas of theoretical computer science.

1. Introduction

In categorical universal algebra, it is well known that a *finitary equational presentation* of algebras in a finitary variety $\mathscr{K}$ amounts to the existence of a coequaliser of a pair of morphisms between finitely generated free algebras.

The essence of a *universal-algebraic flavour* of a finitary functor $L : \mathscr{K} \longrightarrow \mathscr{K}$ is that L is determined by its behaviour on finitely generated free algebras. In fact, such functors also admit an equational presentation, but this time the coequaliser is more complex, though it again involves functors freely generated from Set-functors.

Recently, such functors L have appeared naturally in the study of *modal algebras* for coalgebraic modal logic, see, for example, Bonsangue and Kurz (2006) or Kurz and Rosický (2006). In fact, for the case of one-sorted varieties $\mathscr{K}$, functors L that are determined by their values on finitely generated free algebras are exactly the functors preserving a class of colimits and referred to as *sifted* or, equivalently, they are exactly the class of functors admitting an equational presentation (Kurz and Rosický 2006).

In the current paper we study presentations of functors/monads on $\mathscr{K}$ that are determined by finitely generated free algebras, but we want the requirements on $\mathscr{K}$ to be as relaxed as possible in view of Kurz *et al.* (2010). Hence we again study functors/monads $L : \mathscr{K} \longrightarrow \mathscr{K}$, but $\mathscr{K}$ is now only required to be a full subcategory of a variety, though $\mathscr{K}$ must still contain free objects on finitely many generators.

Thus, the initial setting is now given by a finitary adjunction $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ that is of descent type, that is, such that $\mathscr{K}$ embeds fully into the Eilenberg–Moore category

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$\mathscr{X}^{\mathbb{T}}$ where $\mathbb{T}$ is the finitary monad on $\mathscr{X}$ generated by $F \dashv U$. In order that we can speak of finitely generated free algebras, we further require both $\mathscr{K}$ and $\mathscr{X}$ to be locally finitely presentable. We say that (necessarily finitary) functors/monads on $\mathscr{K}$ that are determined by their values on finitely generated free objects are *finitely based*.

Our main results are the following:

(1) We prove that finitely based functors/monads on $\mathscr{K}$ can be equationally presented using finitary signatures on $\mathscr{X}$. This is the content of Theorems 3.18 and 4.4 below.
(2) In Theorem 4.1 we prove that algebras for finitely based monads on a monadic category also form a monadic category.

Since we expect applications in coalgebraic modal logic in enriched category theory, we state and prove all results in the enriched setting.

Organisation of the paper

We gather together the necessary definitions and notational conventions of enriched category theory in Section 2. We introduce the notion of *finitely based* functors in Section 3 and prove the presentation result in Theorem 3.18. In Section 4, we prove the presentation result for finitely based *monads*. Finally, we give various applications of our results in Section 5 by showing presentations of functors/monads arising in various areas of theoretical computer science.

Related work

The fact that every finitary endofunctor of Set can be equationally presented is stated in Adámek and Trnková (1990) – see also Example 3.19. This presentation result was generalised in Kurz and Rosický (2006) to a certain class of finitary endofunctors of a (one-sorted) finitary variety.

2. Preliminaries

In this section, we recall the necessary notions from enriched category theory we will use in the rest of the paper. Our standard reference for enriched category theory is the book Kelly (1982a); for the details of locally finitely presentable categories enriched in $\mathscr{V}$, see Kelly (1982b).

We assume that $\mathscr{V} = (\mathscr{V}_o, \otimes, I)$ is a symmetric monoidal closed category that is locally finitely presentable as a closed category. The latter means that the (ordinary) category $\mathscr{V}_o$ is locally finitely presentable, that I is a finitely presentable object in $\mathscr{V}_o$ and that the tensor product $X \otimes Y$ is a finitely presentable object in $\mathscr{V}_o$ whenever X and Y are. See Gabriel and Ulmer (1971) and Adámek and Rosický (1994) for the details on locally finitely presentable categories.

Our assumptions on $\mathscr{V}$ allow us to develop category theory enriched in $\mathscr{V}$, and in the following we work with $\mathscr{V}$-categories, $\mathscr{V}$-functors, and so on. We will frequently omit the prefix $\mathscr{V}$- and just write categories, functors, and so on. Whenever we work with $\mathscr{V} = \mathsf{Set}$, we will say the categories, functors, and so on, are *ordinary*.

In enriched category theory we need to consider weighted colimits as the basic colimit concept. Recall that a *colimit* of a diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$ *weighted* by $W : \mathscr{D}^{op} \longrightarrow \mathscr{V}$ is an object $W * D$ in $\mathscr{X}$ together with an isomorphism

$$\mathscr{X}(W * D, X) \cong [\mathscr{D}^{op}, \mathscr{V}](W, \mathscr{X}(D-, X))$$

natural in X.

The dual notion is that of a *limit* of $D : \mathscr{D} \longrightarrow \mathscr{X}$ *weighted* by $W : \mathscr{D} \longrightarrow \mathscr{V}$, which is an object $\{W, D\}$ in $\mathscr{X}$ together with an isomorphism

$$\mathscr{X}(X, \{W, D\}) \cong [\mathscr{D}, \mathscr{V}](W, \mathscr{X}(X, D-))$$

natural in X.

In fact, the assumptions on $\mathscr{V}$ allow us to consider *locally finitely presentable* (l.f.p.) categories in the enriched sense. For these we only need to define the concept of being a *filtered* colimit in the enriched setting, and then we can proceed as in the classical ordinary case. We define *filtered colimits* as those weighted by *flat* weights, where a weight $W : \mathscr{D}^{op} \longrightarrow \mathscr{V}$ is flat whenever the functor

$$W * (-) : [\mathscr{D}, \mathscr{V}] \longrightarrow \mathscr{V}$$

preserves *finite* limits. In enriched category theory a limit $\{K, C\}$ is finite if $K : \mathscr{C} \longrightarrow \mathscr{V}$ is a finite weight. The last assertion means that $\mathscr{C}$ has finitely many objects, and that every hom-object $\mathscr{C}(c, c')$ and every value Kc are finitely presentable in $\mathscr{V}_o$.

Then one defines the notions of being a finitely presentable object, finitary functor, and so on, in the usual manner. See Kelly (1982b) for more details.

3. Finitely based functors and their presentations

In this section we introduce the class of finitary functors that are fully determined by their values on 'finitely generated' free algebras and refer to them as *finitely based* (see Definition 3.8). We will show using examples in Section 5 that such functors arise naturally and that they enjoy nice properties: for example, one can give their equational presentation just using operations and equations coming from the category from which we pick the generators of the algebras. This is the main result of this section – see Theorem 3.18.

The idea of being determined by values on free algebras suggests that we need to work relative to a certain fixed adjunction.

Assumption 3.1. We fix a *finitary adjunction*

$$F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$$

between locally finitely presentable categories and assume that the adjunction is of *descent type*.

Remark 3.2. Being of descent type means that the *comparison functor* $K : \mathscr{K} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is fully faithful, or, equivalently, that every commutative diagram

$$FUFUA \underset{FU\varepsilon A}{\overset{\varepsilon FUA}{\rightrightarrows}} FUA \xrightarrow{\varepsilon A} A \tag{3.1}$$

is a coequaliser, where ε denotes the counit of $F \dashv U$. We will say the above coequaliser is the *canonical resolution* of A.

Any adjunction of descent type can be considered as the existence of an *equational presentation*, as will become clearer in Theorems 3.18 and 4.4. In other words, the parallel pair in a coequaliser (3.1) can be considered as a system of equations that the object A satisfies. See Kelly and Power (1993) for more details of equational presentations and more properties characterising adjunctions of descent type.

Notation 3.3. We use

$$J : \mathscr{A} \longrightarrow \mathscr{K}$$

to denote the embedding of the full subcategory spanned by objects of the form Fn, with n finitely presentable in $\mathscr{X}$.

Remark 3.4. The category $\mathscr{A}$ consists of finitely presentable objects in $\mathscr{K}$. Indeed, due to the adjunction $F \dashv U$, we have that for every finitely presentable n, the isomorphism

$$\mathscr{K}(Fn, -) \cong \mathscr{X}(n, U-) \cong \mathscr{X}(n, -) \cdot U$$

holds. The latter functor preserves filtered colimits since both U and $\mathscr{X}(n, -)$ do.

We prove now that objects in $\mathscr{A}$ suffice to reconstruct all objects of $\mathscr{K}$. More precisely, we prove that the inclusion J of $\mathscr{A}$ in $\mathscr{K}$ is a *dense* functor. Recall that a functor $J : \mathscr{A} \longrightarrow \mathscr{K}$ is dense if the left Kan extension of J along itself is (naturally isomorphic to) the identity functor on $\mathscr{K}$. This statement means that every object X of $\mathscr{K}$ can be expressed as a *canonical* colimit

$$\mathscr{K}(J-, X) * J$$

see Kelly (1982a, Chapter 5).

Lemma 3.5. The functor $J : \mathscr{A} \longrightarrow \mathscr{K}$ is dense.

Proof. The idea of the proof is essentially contained in Bird (1984, Theorem 6.9). Since J is fully faithful, we can use Kelly (1982a, Theorem 5.19(v)): J is dense, if there is a class of colimits such that the category $\mathscr{K}$ is the closure of $\mathscr{A}$ under these colimits, and every such colimit is preserved by the functor

$$\widetilde{J} : \mathscr{K} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}], \quad A \mapsto \mathscr{K}(J-, A).$$

We will derive the required class of colimits in two steps:

(1) For every object of the form FX, we do the following.
Consider the canonical colimit

$$\mathscr{X}(E-, X) * E$$

expressing X as a filtered colimit of finitely presentable objects in $\mathscr{X}$, where $E : \mathscr{X}_{fp} \longrightarrow \mathscr{X}$ denotes the full embedding of the subcategory representing all finitely presentable objects of $\mathscr{X}$.
Then FX can be expressed as a colimit

$$\mathscr{X}(E-, X) * FE$$

since F is a left adjoint, and thus preserves (all) colimits.
We still need to prove that the colimit $\mathscr{X}(E-,X) * FE$ is preserved by $\widetilde{J} : \mathscr{K} \longrightarrow [\mathscr{A}^{op}, \mathscr{V}]$. But this is clear since $\widetilde{J}$ preserves filtered colimits (= colimits weighted by flat weights) because objects of $\mathscr{A}$ are finitely presentable. And the weight $\mathscr{X}(E-,X)$ is flat, since $\mathscr{X}_{fp}$ consists of finitely presentable objects.

(2) For every object A of $\mathscr{K}$, we do the following.
Express A as a coequaliser (3.1). This is possible because the adjunction $F \dashv U$ is assumed to be of descent type.
We claim that the coequaliser (3.1) is preserved by $\widetilde{J}$. First observe that for every finitely presentable object n in $\mathscr{X}$, we have

$$\mathscr{K}(Fn, -) \cong \mathscr{X}(n, U-) = \mathscr{X}(n, -) \cdot U.$$

Since the coequaliser (3.1) is U-absolute (it is known to be a U-split coequaliser – see Mac Lane (1998)), the image of it under any $\mathscr{X}(n,-) \cdot U$ is also a coequaliser. Hence, the image of (3.1) under every $\mathscr{K}(Fn,-)$ is a coequaliser, so $\widetilde{J}$ preserves the coequaliser (3.1). □

Remark 3.6. In fact, in the above result we have obtained the *density presentation* of $J : \mathscr{A} \longrightarrow \mathscr{K}$. The density presentation is (see the comments just before Proposition 5.20 in Kelly (1982a)) a family

$$\Phi = \langle W_\gamma : \mathscr{D}_\gamma^{op} \longrightarrow \mathscr{V}, D_\gamma : \mathscr{D}_\gamma \longrightarrow \mathscr{K} \mid \gamma \in \Gamma \rangle$$

such that each colimit $W_\gamma * D_\gamma$ exists and is preserved by $\widetilde{J}$, and $\mathscr{K}$ is the closure of $\mathscr{A}$ under these colimits.

From the above remark and the fact that $\mathscr{A}$ consists of finitely presentable objects, we immediately obtain the following corollary from Kelly (1982a, Theorem 5.29).

Corollary 3.7. For $L : \mathscr{K} \longrightarrow \mathscr{K}$, the following are equivalent:

(1) L is of the form $\mathrm{Lan}_J LJ$.
(2) L is finitary and preserves all canonical resolutions (3.1).

Definition 3.8. We will say a finitary endofunctor of $\mathscr{K}$ that preserves canonical resolutions (3.1) is *finitely based*. The category of all finitely based functors is denoted by $\mathsf{FinB}(\mathscr{K}, \mathscr{K})$.

Remark 3.9. By passing from l.f.p. categories to *locally λ-presentable* categories (in the enriched sense – see Bird (1984)), where λ is a regular cardinal, we obtain the obvious generalisation of finitely based functors: a functor is *λ-based* (relative to a λ-accessible adjunction $F \dashv U$ of descent type) if it preserves λ-filtered colimits and the canonical resolutions (3.1).

Remark 3.10. By Remark 3.6, we know that there is an equivalence

$$\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \simeq [\mathscr{A}, \mathscr{K}]$$

of categories that we will often use below. Hence the category $\mathsf{FinB}(\mathscr{K}, \mathscr{K})$ is locally finitely presentable; in particular, it is complete and cocomplete.

Example 3.11. The following are examples of finitely based functors:

(1) Suppose $\mathscr{K}$ is a finitary, one-sorted variety. Thus we work within ordinary category theory, that is, $\mathscr{V}$ is the category Set of sets and mappings.
The fact that $\mathscr{K}$ is a finitary one-sorted variety is equivalent to the existence of a finitary monadic adjunction $F \dashv U : \mathscr{K} \longrightarrow \mathsf{Set}$. Suppose $L : \mathscr{K} \longrightarrow \mathscr{K}$ has the form
$$LA = \coprod_n \mathsf{Set}(n, UA) \bullet \Delta n$$
where the coproduct is taken over all finite ordinals, $n \mapsto \Delta n$ is the assignment of an object of $\mathscr{K}$ to each finite ordinal, and $\mathsf{Set}(n, UA) \bullet \Delta n$ is the coproduct in $\mathscr{K}$ of $\mathsf{Set}(n, UA)$-many copies of Δn. We will show that L is finitely based.
It is clear that L preserves filtered colimits since every $\mathsf{Set}(n, U-)$ does and colimits commute with colimits. To prove that L preserves canonical resolutions, observe that each $\mathsf{Set}(n, U-)$ preserves canonical resolutions since they are U-absolute coequalisers.
As a special case, observe that $Id \dashv Id : \mathsf{Set} \longrightarrow \mathsf{Set}$ is clearly a finitary monadic adjunction and functors L of the form described above are exactly the *polynomial functors*. Hence every polynomial functor is finitely based.

(2) The above can be extended to all l.f.p. base categories $\mathscr{V}$ as follows.
Suppose $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ is a finitary monadic adjunction. Then every functor $L : \mathscr{K} \longrightarrow \mathscr{K}$ having the form
$$LA = \coprod_n \mathscr{X}(n, UA) \bullet \Delta n$$
is finitely based. The reasoning is the same as in (1) above.
Above, the coproduct is taken over all finitely presentable objects of $\mathscr{X}$, $n \mapsto \Delta n$ is the assignment of an object of $\mathscr{K}$ to each finitely presentable n, and $\mathscr{X}(n, UA) \bullet \Delta n$ is the $\mathscr{X}(n, UA)$-th tensor of Δn in $\mathscr{K}$ (see (3.4) for the definition of a tensor).

Example 3.12. As an example of a finitary functor that is not finitely based, consider the finitary variety Ab of Abelian groups and their homomorphisms, and its full reflective subcategory $I^* \dashv I : \mathsf{TorFree} \longrightarrow \mathsf{Ab}$ of torsion-free groups. Then the composite $L = I \cdot I^* : \mathsf{Ab} \longrightarrow \mathsf{Ab}$ is a finitary functor that does not preserve the canonical resolutions (3.1). The reason is that L coincides with the identity functor on every finitely generated free Abelian group, but L is clearly not isomorphic to the identity functor on Ab.

Every finitary endofunctor has a finitely based coreflection.

Lemma 3.13. The full embedding $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K})$ has a right adjoint. In particular, the category $\mathsf{FinB}(\mathscr{K}, \mathscr{K})$ is closed in $\mathsf{Fin}(\mathscr{K}, \mathscr{K})$ under colimits.

Proof. We use $J' : \mathscr{A} \longrightarrow \mathscr{K}_{fp}$ to denote the full embedding of $\mathscr{A}$ into the category representing all finitely presentable objects. Then we have an adjunction
$$\mathrm{Lan}_{J'}(-) \dashv [J', \mathscr{K}] : [\mathscr{K}_{fp}, \mathscr{K}] \longrightarrow [\mathscr{A}, \mathscr{K}].$$
Since, using the fact that $\mathsf{Fin}(\mathscr{K}, \mathscr{K}) \cong [\mathscr{K}_{fp}, \mathscr{K}]$ and $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \simeq [\mathscr{A}, \mathscr{K}]$, the full embedding $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{K}, \mathscr{K})$ is given by the left Kan extension along J', and the result then follows. $\square$

We will employ the following technical lemma in proving the presentation results on finitely based functors in Proposition 3.15 and Theorem 3.18.

Lemma 3.14. Suppose $S : \mathscr{B} \longrightarrow \mathscr{C}$ is a functor surjective on objects, where the categories $\mathscr{B}$ and $\mathscr{C}$ are small. Then the composite

$$[\mathscr{C}, \mathscr{K}] \xrightarrow{[S,\mathscr{K}]} [\mathscr{B}, \mathscr{K}] \xrightarrow{[\mathscr{B},U]} [\mathscr{B}, \mathscr{X}]$$

has a left adjoint and the resulting adjunction is of descent type.

Proof. The functor $[S, \mathscr{K}]$ is monadic. This follows from the fact that S is surjective on objects since then $[S, \mathscr{K}]$ is faithful and reflects isomorphisms, and since $[S, \mathscr{K}]$ has both left and right adjoints given by Kan extensions, we can conclude that $[S, \mathscr{K}]$ is monadic by Beck's Theorem.

The functor $[\mathscr{B}, U]$ has a left adjoint $[\mathscr{B}, F]$, this adjunction being the image of the adjunction $F \dashv U$ under the 2-functor $[\mathscr{B}, -]$. Moreover, $[\mathscr{B}, F] \dashv [\mathscr{B}, U]$ is an adjunction of descent type since $F \dashv U$ is.

Since $[S, \mathscr{K}]$ sends coequalisers to epimorphisms (in fact, it preserves coequalisers, since it is a left adjoint), the composite $[S, U] = [\mathscr{B}, U] \cdot [S, \mathscr{K}]$ is of descent type by Kelly and Power (1993, Proposition 3.5). $\square$

We will now establish the first presentation result for finitely based functors and endofunctors of the 'base' category $\mathscr{X}$.

Proposition 3.15. The functor

$$[F, U] : \mathsf{FinB}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{X}, \mathscr{X}), \quad L \mapsto ULF$$

has a left adjoint $H \mapsto \widehat{H}$, and the adjunction is of descent type.

Proof. Using the identifications $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \simeq [\mathscr{A}, \mathscr{K}]$ and $\mathsf{Fin}(\mathscr{X}, \mathscr{X}) \simeq [\mathscr{X}_{fp}, \mathscr{X}]$, observe that the forgetful functor $[F, U] : \mathsf{FinB}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Fin}(\mathscr{X}, \mathscr{X})$ can be written as the composite

$$[F', U] \equiv [\mathscr{A}, \mathscr{K}] \xrightarrow{[F',\mathscr{K}]} [\mathscr{X}_{fp}, \mathscr{K}] \xrightarrow{[\mathscr{X}_{fp},U]} [\mathscr{X}_{fp}, \mathscr{X}] \tag{3.2}$$

where we use $F' : \mathscr{X}_{fp} \longrightarrow \mathscr{A}$ to denote the restriction of $F : \mathscr{X} \longrightarrow \mathscr{K}$. Now use Lemma 3.14 with $S = F'$. $\square$

Remark 3.16. The finitely based functor $\widehat{H} : \mathscr{A} \longrightarrow \mathscr{K}$ free on H is given explicitly at $F'm$ in $\mathscr{A}$ by the formula

$$\begin{aligned}
\widehat{H}(F'm) &= (\mathrm{Lan}_{F'}(FH))(F'm) \\
&\cong \int^n \mathscr{A}(F'n, F'm) \bullet FHn \\
&\cong \int^n \mathscr{K}(Fn, Fm) \bullet FHn \\
&\cong \int^n \mathscr{X}(n, UFm) \bullet FHn \\
&\cong FHU(Fm),
\end{aligned}$$

where the last isomorphism is by the Yoneda Lemma.

Remark 3.17. As always, in the presence of an adjunction of descent type, we have a presentation result: Proposition 3.15 states that every finitely based $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be expressed as a coequaliser

$$\widehat{H_1} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \widehat{H_2} \xrightarrow{\gamma} L$$

for some suitable finitary functors $H_1, H_2 : \mathscr{X} \longrightarrow \mathscr{X}$. In the following we want to improve this coequaliser presentation to involve finitary signatures rather than endofunctors.

Before we state our main presentation result, recall the monadic adjunction

$$\mathrm{Lan}_E(-) \dashv [E, \mathscr{X}] : \mathsf{Fin}(\mathscr{X}, \mathscr{X}) \longrightarrow [|\mathscr{X}_{fp}|, \mathscr{X}] \tag{3.3}$$

where $E : |\mathscr{X}_{fp}| \longrightarrow \mathscr{X}$ is the inclusion of the discrete underlying category of $\mathscr{X}_{fp}$ into $\mathscr{X}$.

The functor category $[|\mathscr{X}_{fp}|, \mathscr{X}]$ is best perceived as the *category of finitary signatures* on $\mathscr{X}$ (Kelly and Power 1993), and we denote it by

$$\mathsf{Sig}_{fin}(\mathscr{X}).$$

Such a signature Σ is a collection (Σn) indexed by finitely presentable objects of $\mathscr{X}$. The object Σn is then an object of n-ary *operations* for each finitely presentable object n in $\mathscr{X}$.

The left adjoint in (3.3) sends each signature Σ to its corresponding *polynomial endofunctor*

$$H_\Sigma X = \coprod_n \mathscr{X}(n, X) \bullet \Sigma n,$$

where the coproduct is taken over objects in $|\mathscr{X}_{fp}|$ and $\mathscr{X}(n, X) \bullet \Sigma n$ is the $\mathscr{X}(n, X)$-th *tensor* of Σn in $\mathscr{X}$ defined by the isomorphism

$$\mathscr{X}(\mathscr{X}(n, X) \bullet \Sigma n, X') \cong \mathscr{V}(\mathscr{X}(n, X), \mathscr{X}(\Sigma n, X')) \tag{3.4}$$

natural in X'.

Theorem 3.18. The composite

$$\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \xrightarrow{[F,U]} \mathsf{Fin}(\mathscr{X}, \mathscr{X}) \xrightarrow{[E,\mathscr{X}]} \mathsf{Sig}_{fin}(\mathscr{X}) \tag{3.5}$$

has a left adjoint and the resulting adjunction is of descent type.

Proof. We first recall that $F' : \mathscr{X}_{fp} \longrightarrow \mathscr{A}$ denotes the restriction of $F : \mathscr{X} \longrightarrow \mathscr{K}$, and we will use $E' : |\mathscr{X}_{fp}| \longrightarrow \mathscr{X}_{fp}$ to denote the restriction of the inclusion $E : |\mathscr{X}_{fp}| \longrightarrow \mathscr{X}$. Then, using the identifications $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \simeq [\mathscr{A}, \mathscr{K}]$ and $\mathsf{Fin}(\mathscr{X}, \mathscr{X}) \simeq [\mathscr{X}_{fp}, \mathscr{X}]$, the composite (3.5) can be written as the composite

$$[\mathscr{A}, \mathscr{K}] \xrightarrow{[F'E', \mathscr{K}]} [|\mathscr{X}_{fp}|, \mathscr{K}] \xrightarrow{[|\mathscr{X}_{fp}|, U]} [|\mathscr{X}_{fp}|, \mathscr{X}]$$

Putting $S = F' \cdot E'$ in Lemma 3.14 completes the proof. □

The above theorem states that every finitely based functor $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be expressed as a coequaliser in the spirit of Bonsangue and Kurz (2006), and it generalises results of Kurz and Rosický (2006) from finitary varieties (over Set) to finitary adjunctions of descent type (over an arbitrary l.f.p. category $\mathscr{X}$, enriched in $\mathscr{V}$).

Indeed, Theorem 3.18 states that every finitely based endofunctor $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be written as a coequaliser

$$\widehat{H_\Gamma} \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \widehat{H_\Sigma} \xrightarrow{\gamma} L \tag{3.6}$$

for some suitable finitary signatures Σ and Γ. In fact, the pair λ, ρ in (3.6) can equivalently be given by a parallel pair

$$\Gamma \overset{\lambda^\flat}{\underset{\rho^\flat}{\rightrightarrows}} U\widehat{H_\Sigma}F$$

of signature morphisms. In fact, by Remark 3.16, we have

$$U\widehat{H_\Sigma}Fn \cong UFH_\Sigma UFn,$$

so the purpose of Γ and the pair $\lambda^\flat$, $\rho^\flat$ is to pick up, for every n, Γn-many pairs of 'terms' in $UFH_\Sigma UFn$ to be equal. This is exactly the type of equation treated in Bonsangue and Kurz (2006) and Kurz and Rosický (2006) – see Section 5 below for more details.

Example 3.19. Suppose $\mathscr{V} = \mathsf{Set}$ in this example and also let $\mathscr{K} = \mathscr{X} = \mathsf{Set}$. In this setting, Theorem 3.18 reduces to equational presentations of finitary endofunctors $L : \mathsf{Set} \longrightarrow \mathsf{Set}$ by the *basic* (or, *flat*) equations of Adámek and Trnková (1990):

> For every finitary endofunctor $L : \mathsf{Set} \longrightarrow \mathsf{Set}$, there exists a finitary signature Σ and a set E of equations between Σ-terms of depth $\leqslant 1$, such that every LX can be obtained from $H_\Sigma X$ by quotienting by equations in E.

In fact, one can find a *canonical* equational presentation of the above form as follows:

(1) Since L is finitary, it has a canonical coend representation

$$LX \cong \int^n \mathsf{Set}(n, X) \bullet Ln.$$

(2) Since a coend is a colimit, it can be represented using coequalisers and coproducts as follows:

$$\textstyle\coprod_{n,m} \mathsf{Set}(m, n) \bullet (\mathsf{Set}(n, X) \bullet Lm) \overset{\lambda_X}{\underset{\rho_X}{\rightrightarrows}} \coprod_n \mathsf{Set}(n, X) \bullet Ln \xrightarrow{\gamma_X} LX$$

where:

— λ_X sends $(h : m \longrightarrow n, x : n \longrightarrow X, \sigma \in Lm)$ to $(x \cdot h : m \longrightarrow X, \sigma \in Lm)$;

— ρ_X sends (h, x, σ) to $(x, Lh(\sigma))$; and

— γ_X sends $(x : n \longrightarrow X, \tau \in Ln)$ to $Lx(\tau)$.

See, for example, (the dual of) Formula (3.68) of Kelly (1982a).

After shuffling the coproducts, we can write down the above coequaliser as

$$\textstyle\coprod_n \mathsf{Set}(n, X) \bullet (\coprod_m \mathsf{Set}(m, n) \bullet Lm) \overset{\lambda_X}{\underset{\rho_X}{\rightrightarrows}} \coprod_n \mathsf{Set}(n, X) \bullet Ln \xrightarrow{\gamma_X} LX.$$

(3) Define two finitary signatures

$$\Sigma n = Ln$$
$$\Gamma n = \coprod_m \mathsf{Set}(m,n) \bullet Lm$$

where n and m range through finite sets. Then the above coequaliser takes the form

$$\textstyle\coprod_n \mathsf{Set}(n,X) \bullet \Gamma n \underset{\rho_X}{\overset{\lambda_X}{\rightrightarrows}} \coprod_n \mathsf{Set}(n,X) \bullet \Sigma n \xrightarrow{\gamma_X} LX$$

or, more suggestively, the form

$$H_\Gamma X \underset{\rho_X}{\overset{\lambda_X}{\rightrightarrows}} H_\Sigma X \xrightarrow{\gamma_X} LX,$$

which is exactly an equational presentation of L in the spirit of Theorem 3.18.

We can illustrate this procedure using the example of the finitary powerset functor P_{fin}. Its canonical equational presentation is given by signatures

$$\Sigma n = P_{fin} n$$
$$\Gamma n = \coprod_m \mathsf{Set}(m,n) \bullet P_{fin} m,$$

and the above pair λ, ρ of natural transformations is induced uniquely by the pair

$$\lambda_k^\flat : \Gamma k \longrightarrow H_\Sigma k$$
$$\rho_k^\flat : \Gamma k \longrightarrow H_\Sigma k$$

indexed by finite sets k, or, when writing Γ and H_Σ explicitly, by the pair

$$\textstyle\coprod_m \mathsf{Set}(m,k) \bullet P_{fin} m \underset{\rho_k^\flat}{\overset{\lambda_k^\flat}{\rightrightarrows}} \coprod_n \mathsf{Set}(n,k) \bullet P_{fin} n,$$

that represents the system of equations of the form

$$(h,\sigma) \approx (id_k, P_{fin} h(\sigma))$$

for every m, k, every $h : m \longrightarrow k$ and every $\sigma \in P_{fin} m$.

Such an equation 'holds' in $P_{fin} X$, that is, after employing an interpretation $x : k \longrightarrow X$ of variables, the mapping γ_X sends both sides of the above equation to the same element of $P_{fin} X$, that is, we have an actual identity

$$P_{fin}(x \cdot h)(\sigma) = P_{fin} x \cdot P_{fin} h(\sigma)$$

in $P_{fin} X$.

Note that one usually wants to find a more 'effective' equational presentation since the above canonical one expresses exactly the information that L is a finitary functor, and it does not care about any possible special properties of L.

For example, a more effective equational presentation of P_{fin} consists of signatures

$$\Sigma k = \{\sigma_k\}$$
$$\Gamma k = \coprod_m \mathsf{Set}(m,k) \bullet \{\sigma_m\}$$

with

$$\lambda_k^\flat(h : m \longrightarrow k, \sigma_m) = (h, \sigma_m)$$
$$\rho_k^\flat((h : m \longrightarrow k, \sigma_m)) = (id_\ell, \sigma_\ell)$$

where ℓ is the direct image $h[m] \subseteq k$.

This more economical presentation presents elements of $P_{fin}X$ as flat terms, subject to equations $\sigma_n(x) \approx \sigma_m(y)$ for all pairs n, m of finite sets and all $x : n \longrightarrow X$, $y : m \longrightarrow X$ with $\{x(i) \mid i \in n\} = \{y(j) \mid j \in m\}$. See also Adámek *et al.* (2009, Example 3.8).

The above canonical presentation of a finitary functor $L : \mathscr{K} \longrightarrow \mathscr{K}$ can be found in the same manner as in Example 3.19 for *any* l.f.p. base category $\mathscr{V}$ and *any* l.f.p. category $\mathscr{K}$.

We will give some more examples of presentations in Section 5.

4. Presentations of finitely based monads

A monad $\mathbb{M} = (M, \eta, \mu)$ on $\mathscr{K}$ is said to be *finitely based* if its underlying functor $M : \mathscr{K} \longrightarrow \mathscr{K}$ is finitely based. The category of all finitely based monads on $\mathscr{K}$ is denoted by

$$\mathsf{Mnd}_{finb}(\mathscr{K}).$$

Before we turn to equational presentations of finitely based monads, note that such monads fulfil the 'monadic composition' property. More precisely, the following result holds.

Theorem 4.1. Suppose $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ is monadic. Consider a finitely based monad $\mathbb{M} = (M, \eta, \mu)$ on $\mathscr{K}$. Then the composite

$$\mathscr{K}^{\mathbb{M}} \xrightarrow{U^{\mathbb{M}}} \mathscr{K} \xrightarrow{U} \mathscr{X}$$

is monadic, where $U^{\mathbb{M}}$ is the forgetful functor from the category of Eilenberg–Moore algebras for $\mathbb{M}$.

Proof. By Beck's Theorem (see, for example, Mac Lane (1998, Chapter VI.7, Exercise 2)), we need to prove that $\mathscr{K}^{\mathbb{M}}$ has and $UU^{\mathbb{M}}$ preserves and reflects $UU^{\mathbb{M}}$-absolute coequalisers.

Since the monad $\mathbb{M}$, being finitely based, is finitary, the category $\mathscr{K}^{\mathbb{M}}$ is locally finitely presentable (Bird 1984, Theorem 6.9). In particular, $\mathscr{K}^{\mathbb{M}}$ has coequalisers.

Take a pair

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \tag{4.7}$$

in $\mathscr{K}^{\mathbb{M}}$ that has a $UU^{\mathbb{M}}$-absolute coequaliser. Consider its image under $U^{\mathbb{M}}$:

$$U^{\mathbb{M}}A \underset{U^{\mathbb{M}}g}{\overset{U^{\mathbb{M}}f}{\rightrightarrows}} U^{\mathbb{M}}B\,. \tag{4.8}$$

The above pair (4.8) in $\mathscr{K}$ has a U-absolute coequaliser

$$UU^{\mathbb{M}}A \underset{UU^{\mathbb{M}}g}{\overset{UU^{\mathbb{M}}f}{\rightrightarrows}} UU^{\mathbb{M}}B \xrightarrow{\ q\ } Z \tag{4.9}$$

in $\mathscr{X}$. Since U is assumed to be monadic, there is a coequaliser

$$U^{\mathbb{M}}A \underset{U^{\mathbb{M}}g}{\overset{U^{\mathbb{M}}f}{\rightrightarrows}} U^{\mathbb{M}}B \xrightarrow{\ c\ } X \tag{4.10}$$

with $Uc \cong q$.

To complete the proof we only need to endow X in (4.10) with the structure of an $\mathbb{M}$-algebra such that $c : U^{\mathbb{M}}B \longrightarrow X$ is an $\mathbb{M}$-algebra homomorphism. To do this, it suffices to prove that both M and MM preserve the coequaliser (4.10).

Consider the following 3×3 scheme:

$$\begin{tikzcd}
FUFUU^{\mathbb{M}}A \arrow[r, shift left, "FUFUU^{\mathbb{M}}f"] \arrow[r, shift right, "FUFUU^{\mathbb{M}}g"'] \arrow[d, shift right, "\varepsilon FUU^{\mathbb{M}}A"'] \arrow[d, shift left, "FU\varepsilon U^{\mathbb{M}}A"] & FUFUU^{\mathbb{M}}B \arrow[r, "FUFUc"] \arrow[d, shift right, "\varepsilon FUU^{\mathbb{M}}B"'] \arrow[d, shift left, "FU\varepsilon U^{\mathbb{M}}B"] & FUFUX \arrow[d, shift right, "\varepsilon FUX"'] \arrow[d, shift left, "FU\varepsilon X"] \\
FUU^{\mathbb{M}}A \arrow[r, shift left, "FUU^{\mathbb{M}}f"] \arrow[r, shift right, "FUU^{\mathbb{M}}g"'] \arrow[d, "\varepsilon U^{\mathbb{M}}A"'] & FUU^{\mathbb{M}}B \arrow[r, "FUc"] \arrow[d, "\varepsilon U^{\mathbb{M}}B"'] & FUX \arrow[d, "\varepsilon X"'] \\
U^{\mathbb{M}}A \arrow[r, shift left, "U^{\mathbb{M}}f"] \arrow[r, shift right, "U^{\mathbb{M}}g"'] & U^{\mathbb{M}}B \arrow[r, "c"] & X
\end{tikzcd} \tag{4.11}$$

Observe that, by assumption, the first two 'rows' are absolute coequalisers and all three 'columns' are coequalisers preserved by M.

Hence, by applying M to (4.11), we obtain a 3×3 scheme where all 'columns' and the first two 'rows' are coequalisers. Therefore the bottom 'row' must be a coequaliser, which is exactly what we wanted.

The argument for MM is analogous.

Having proved that M and MM preserve the coequaliser (4.10), we define the $\mathbb{M}$-algebra structure $x : MX \longrightarrow X$ on X as the unique mediating morphism in

$$\begin{tikzcd}
MU^{\mathbb{M}}A \arrow[r, shift left, "MU^{\mathbb{M}}f"] \arrow[r, shift right, "MU^{\mathbb{M}}g"'] \arrow[d, "a"] & MU^{\mathbb{M}}B \arrow[r, "Mc"] \arrow[d, "b"] & MX \arrow[d, dotted, "x"] \\
U^{\mathbb{M}}A \arrow[r, shift left, "U^{\mathbb{M}}f"] \arrow[r, shift right, "U^{\mathbb{M}}g"'] & U^{\mathbb{M}}B \arrow[r, "c"] & X
\end{tikzcd} \tag{4.12}$$

It is now just a standard proof (Linton 1969, Page 67, Proposition 3) to show that $x : MX \longrightarrow X$ is indeed an $\mathbb{M}$-algebra structure (for this one needs that MM preserves (4.10)) and that $c : U^{\mathbb{M}}B \longrightarrow X$ is a coequaliser in the category of $\mathbb{M}$-algebras. □

Remark 4.2. Observe that the assumption that U and $U^{\mathbb{M}}$ are finitary is irrelevant in the above theorem. Indeed, the theorem holds more generally:

> Suppose $F \dashv U : \mathscr{K} \longrightarrow \mathscr{X}$ is an *arbitrary* monadic adjunction. Suppose further that $\mathbb{M} = (M, \eta, \mu)$ is an *arbitrary* monad on $\mathscr{K}$ such that the functor M preserves canonical resolutions (3.1). Then the composite $UU^{\mathbb{M}} : \mathscr{K}^{\mathbb{M}} \longrightarrow \mathscr{X}$ is monadic.

Proposition 4.3. The forgetful functor $\mathsf{Mnd}_{finb}(\mathscr{K}) \longrightarrow \mathsf{FinB}(\mathscr{K}, \mathscr{K})$ is monadic.

Proof. It suffices to prove that the free finitary monad $\mathbb{F}_L$ on a finitely based functor L is finitely based. Recall from Adámek (1974) that the underlying finitary functor F_L of $\mathbb{F}_L$ is given by a colimit of the countable chain

$$W_0^L \xrightarrow{w_{0,1}^L} W_1^L \xrightarrow{w_{1,2}^L} \cdots$$

where $W_0^L = Id$ and $W_{k+1}^L = L \cdot W_k^L + Id$, $w_{0,1}^L = \mathsf{inr}$ and $w_{k+1,k+2}^L = Lw_{k,k+1}^L + id$.

To prove that F_L is finitely based, observe that each W_k is, and then use the fact that finitely based functors are closed in the category of finitary functors under colimits (see Lemma 3.13). □

We will now prove that finitely based monads on $\mathscr{K}$ can be presented using finitary signatures on $\mathscr{X}$.

Theorem 4.4. The functor $\mathsf{Mnd}_{finb}(\mathscr{K}) \longrightarrow \mathsf{Sig}_{fin}(\mathscr{X})$ sending $\mathbb{M} = (M, \eta, \mu)$ to the signature $n \mapsto UMFn$ has a left adjoint, and the resulting adjunction is of descent type.

Proof. For the purposes of this proof, we introduce the following notation:

(1) The forgetful monadic functor $\mathsf{Mnd}_{finb}(\mathscr{K}) \longrightarrow \mathsf{FinB}(\mathscr{K}, \mathscr{K})$ is denoted by W and its left adjoint is denoted by $\mathbb{F}$.
(2) The forgetful functor $\mathsf{FinB}(\mathscr{K}, \mathscr{K}) \longrightarrow \mathsf{Sig}_{fin}(\mathscr{X})$ of Theorem 3.18 is denoted by V. By the same theorem, V has a left adjoint, denoted by G, and the adjunction $G \dashv V$ is of descent type.

We need to prove that the composite adjunction $\mathbb{F}G \dashv VW$ is of descent type. We will closely follow the proof of Kelly and Power (1993, Theorem 5.1).

We use α to denote the counit of the adjunction $\mathbb{F}G \dashv VW$. We will prove that $\alpha_{\mathbb{T}}$ is W-final, for every finitely based monad $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$. This amounts to proving the following:

> For every finitely based monad $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ and every natural transformation $\tau : T \longrightarrow S$ such that the composite $\tau \cdot \alpha_{\mathbb{T}} : \mathbb{F}GVW(\mathbb{T}) \longrightarrow \mathbb{S}$ is a monad morphism, τ is a monad morphism.

Therefore, we assume that the perimeters of the following two diagrams commute:

$$
\begin{tikzcd}
F_{GV(T)} \cdot F_{GV(T)} \arrow[r, "\alpha_{\mathbb{T}} * \alpha_{\mathbb{T}}"] \arrow[d, "\mu^{\mathbb{F}_{GV(T)}}"'] & T \cdot T \arrow[r, "\tau * \tau"] \arrow[d, "\mu^{\mathbb{T}}"] & S \cdot S \arrow[d, "\mu^{\mathbb{S}}"] \\
F_{GV(T)} \arrow[r, "\alpha_{\mathbb{T}}"'] & T \arrow[r, "\tau"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{GV(T)} \arrow[r, "\alpha_{\mathbb{T}}"] & T \arrow[r, "\tau"] & S \\
& Id \arrow[ul, "\eta^{\mathbb{F}_{GV(T)}}"] \arrow[u, "\eta^{\mathbb{T}}"'] \arrow[ur, "\eta^{\mathbb{S}}"'] &
\end{tikzcd}
$$

There is nothing to prove for the right-hand triangle since the equality $\tau \cdot \eta^{\mathbb{T}} = \eta^{\mathbb{S}}$ clearly holds. To prove that the right-hand rectangle commutes, we will show that $\alpha_{\mathbb{T}} * \alpha_{\mathbb{T}}$, or, more precisely, $W(\alpha_{\mathbb{T}}) * W(\alpha_{\mathbb{T}})$ is an epimorphism in $\mathsf{FinB}(\mathscr{K}, \mathscr{K})$.

We will prove that both

$$W(\alpha_{\mathbb{T}}) * id_{W(\mathbb{F}_{GV(T)})} : W(\mathbb{F}_{GV(T)}) \cdot W(\mathbb{F}_{GV(T)}) \longrightarrow T \cdot W(\mathbb{F}_{GV(T)})$$

and

$$id_{W(\mathbb{T})} * W(\alpha_{\mathbb{T}}) : T \cdot W(\mathbb{F}_{GV(T)}) \longrightarrow T \cdot T$$

are epimorphic, and the transformation $W(\alpha_{\mathbb{T}}) * W(\alpha_{\mathbb{T}})$ will then be a composition of two epimorphisms.

(1) The natural transformation $W(\alpha_{\mathbb{T}}) * id_{W(\mathbb{F}_{GV(T)})}$ is an epimorphism.
To prove this, observe that $VW(\alpha_{\mathbb{T}})$ is a split epimorphism in $\mathsf{Sig}_{fin}(\mathscr{X})$ by the triangle equality for $\mathbb{F}G \dashv VW$. Since V is faithful (being of descent type by Theorem 3.18), $W(\alpha_{\mathbb{T}})$ is an epimorphism. Now the functor $- \cdot W(\mathbb{F}_{GV(T)})$ is a left adjoint, so it preserves epimorphisms. Therefore $W(\alpha_{\mathbb{T}}) * id_{W(\mathbb{F}_{GV(T)})}$ is an epimorphism.

(2) The natural transformation $id_{W(\mathbb{T})} * W(\alpha_{\mathbb{T}})$ is an epimorphism.
We again use the fact that $VW(\alpha_{\mathbb{T}})$ is a split epimorphism in $\mathsf{Sig}_{fin}(\mathscr{X})$. Then from Kelly and Power (1993, Lemma 5.2), it follows that $id_{W(\mathbb{T})} * W(\alpha_{\mathbb{T}})$ is an epimorphism. □

Theorem 4.4 states that, for every finitely based monad $\mathbb{M}$, there is a coequaliser of the form

$$
\begin{tikzcd}
\mathbb{F}_{\Gamma} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\rho"'] & \mathbb{F}_{\Sigma} \arrow[r, "\gamma"] & \mathbb{M}
\end{tikzcd}
$$

in the category $\mathsf{Mnd}_{finb}(\mathscr{K})$ for some suitable finitary signatures Γ and Σ on the category $\mathscr{X}$. The above coequaliser then represents an equational presentation of $\mathbb{M}$ in the same way as discussed for functors at the end of the previous section.

5. Examples

Theorems 3.18 and 4.4 are new in two respects: they generalise previous results from 'ordinary' categories to $\mathscr{V}$-categories and from monadic categories to categories of descent type. In this section we give some examples to show where and how our results can be applied.

5.1. *Presenting functors on varieties: modal algebras*

We will first illustrate Theorem 3.18 in the case of $\mathscr{K} = \mathsf{BA}$ and show how we can use it to generalise the notion of a modal algebra (Blackburn *et al.* 2001) from Kripke frames $X \longrightarrow \mathscr{P}X$ to arbitrary set-coalgebras $\xi : X \longrightarrow TX$.

Recall that there are contravariant functors $P : \mathsf{Set} \longrightarrow \mathsf{BA}$ and $S : \mathsf{BA} \longrightarrow \mathsf{Set}$, which are adjoint on the right. P maps a set to the Boolean algebra of subsets and S maps a Boolean algebra to the set of ultrafilters. On arrows, both act as the inverse image, and we use F to denote the left adjoint of the forgetful functor $U : \mathsf{BA} \longrightarrow \mathsf{Set}$.

Given $\mathscr{P}$, we define $L : \mathsf{BA} \longrightarrow \mathsf{BA}$ by $LFn = P\mathscr{P}SFn$ on finitely generated free algebras, and then extend L continuously to all of BA. By Theorem 3.18, L has a presentation where n-ary operations are given by $ULFn = UPTSFn \cong 2^{T(2^n)}$.

We call a $\boxdot \in 2^{T(2^n)}$ an n-ary modal operator since it gives rise to an operation on n-tuples of subsets of X taking $\varphi : X \longrightarrow 2^n$ to the predicate

$$X \xrightarrow{\xi} TX \xrightarrow{T\varphi} T(2^n) \xrightarrow{\boxdot} 2 \tag{5.13}$$

on X. In the particular case of $T = \mathscr{P}$, it can be shown that, together with the Boolean operations, all n-ary modal operators can be generated from the particular unary one given by

$$\mathscr{P}(2) \xrightarrow{\Box} 2 \qquad \{0,1\} \mapsto 0, \{0\} \mapsto 0, \{1\} \mapsto 1, \{\} \mapsto 1,$$

which, using (5.13), does indeed reveal itself to be the usual $\Box$ of modal logic (Blackburn *et al.* 2001). Furthermore, the equations (3.6) defining L amount to the usual axioms of modal logic, namely that $\Box$ preserves finite meets.

To summarise, we recalled from Kurz and Rosický (2006) how to recover the classic modal algebras from the powerset-functor $\mathscr{P}$ and, at the same time, how to generalise the notion of a modal algebra from Kripke frames to coalgebras for an arbitrary set-functor T. The generalisation of Theorem 3.18 opens the way to transfer the same analysis to coalgebras over enriched categories such as posets, ω-cpo's and certain kinds of metric space. We leave this for future work (first steps in this direction were taken in Kapulkin *et al.* (2010)) and will restrict ourselves here to some examples of such functors and their presentations.

5.2. *Presenting monads on nominal sets*

The category Nom of nominal sets (Gabbay and Pitts 1999) plays an important role in the modelling of calculi involving the notion of name-binding, be it first-order logic, λ-calculus or process algebras such as the π-calculus. It is well known, see, for example, Fiore and Staton (2006) or Gadducci *et al.* (2006), that Nom embeds into the presheaf category $[\mathbb{I}, \mathsf{Set}]$, where $\mathbb{I}$ is the category of finite sets with injections. Since $[\mathbb{I}, \mathsf{Set}]$ is monadic over the category of many-sorted sets $\mathsf{Set}^{|\mathbb{I}|}$ (where $|\mathbb{I}|$ denotes the set of objects of $\mathbb{I}$), this suggests we do universal algebra over nominal sets as standard many-sorted universal algebra over $\mathsf{Set}^{|\mathbb{I}|}$. Indeed, as shown in Kurz and Petrişan (2010), it is possible to translate the logics of Gabbay (2009) and Clouston and Pitts (2007) into many-sorted equational logic and then make use of classical results of universal algebra such as Birkhoff's HSP-theorem on equationally definable classes of algebras.

Theorems 4.1 and 4.4 give a conceptual explanation of this approach as follows.

The full and faithful embedding $I : \mathsf{Nom} \longrightarrow [\mathbb{I}, \mathsf{Set}]$ gives rise to a forgetful functor $\mathsf{Nom} \longrightarrow \mathsf{Set}^{|\mathbb{I}|}$, which is of descent type. According to Theorem 4.4, every finitely based monad $\mathbb{M}$ on Nom has a presentation. Since it takes arities from $\mathsf{Set}^{|\mathbb{I}|}$, it is also a presentation of a monad $\mathbb{M}'$ on $\mathsf{Set}^{\mathbb{I}}$, which is also finitely based. By Theorem 4.1, the category of $\mathbb{M}'$-algebras is monadic not only over $\mathsf{Set}^{\mathbb{I}}$ but also over $\mathsf{Set}^{|\mathbb{I}|}$, that is, it falls into the realm of standard many-sorted set-based universal algebra. In particular, we can study $\mathbb{M}$-algebras over Nom by studying the standard many-sorted universal $\mathbb{M}'$-algebras and their representation by operations and equations. For further details, see Kurz and Petrişan (2010) and Kurz *et al.* (2010).

5.3. *Presenting functors on posets*

In this section $\mathscr{V} = \mathsf{Pos}$, the category of all posets and monotone maps. It is an l.f.p. cartesian closed category (see, for example, Adámek and Rosický (1994)), hence it is a suitable base category.

Convex powerset. Consider a signature Σ where Σn has one operation symbol for each finite discrete poset n and is empty otherwise. The induced functor

$$\begin{aligned} H_\Sigma &: \mathsf{Pos} \longrightarrow \mathsf{Pos} \\ H_\Sigma X &= \coprod_n \mathsf{Pos}(n, X) = \{[x_1, \ldots, x_n] \mid x_i \in X, n < \omega\} \end{aligned}$$

maps a poset X to the poset of lists, ordered pointwise, over X. Also consider equations Γ that quotient lists to sets (expressing the fact that the order and repetition of elements can be ignored). We write $\{x_1, \ldots, x_n\}$ for the equivalence class of $[x_1, \ldots, x_n]$.

Note that the presentation Σ, Γ above is exactly the one of Example 3.19, except for the fact that the coequaliser (3.6) defining L is now computed over Pos instead of Set.

It follows from the equations Γ that the pointwise order on $[x_1, \ldots, x_n]$ turns into the Egli–Milner order on LX: the inequality

$$\{x_0, \ldots, x_{n-1}\} \leqslant \{y_0, \ldots, y_{m-1}\}$$

holds in LX if and only if for all $i \in n$ there exists $j \in m$ such that $x_i \leqslant y_j$ and for all $j \in m$ there exists $i \in n$ such that $x_i \leqslant y_j$.

Furthermore, it follows from the way coequalisers are computed in Pos that the functor L presented by Σ, Γ identifies the sets that are equal according to the Egli–Milner order.

Moreover, calculating in LX and assuming $x_1 \leqslant x \leqslant x_2$,

$$\begin{aligned} \{x_1, x_2 \ldots, x_{n-1}\} &= \{x_1, x_1, x_2 \ldots, x_{n-1}\} \\ &\leqslant \{x_1, x, x_2 \ldots, x_{n-1}\} \\ &\leqslant \{x_1, x_2, x_2 \ldots, x_{n-1}\} \\ &= \{x_1, x_2 \ldots, x_{n-1}\} \end{aligned}$$

shows that we can represent the elements of LX as the finitely generated convex subsets of X. To summarise, we have shown the following proposition.

Proposition 5.1. The functor L presented by Σ, Γ maps a poset X to the poset of finitely generated convex subsets ordered by the Egli–Milner order.

Remark 5.2. The above functor L is a 'Pos-enriched analogy' of the finitary powerset functor P_{fin} of Example 3.19. We will show one more example of such an analogy in Proposition 5.4.

Note that by definition, L is a Pos-functor, that is, L preserves the order on the homsets. It is this condition that is responsible for interesting poset phenomena (convexity) arising even though the arities and the sets of operation symbols (co-arities) are discrete. We will now look at two examples where arities and co-arities are non-discrete posets.

Arities that are posets. Consider the functor $\mathsf{Pos} \longrightarrow \mathsf{Pos}$ that maps a poset to the discrete poset of its connected components. It is presented by:

— arities: $1, \mathbb{2}$
— operations: $\Sigma 1$ is the (po)set $\{\Box\}$. $\Sigma \mathbb{2}$ is the (po)set $\{\Diamond\}$
— equations: $\Box(x_1) = \Diamond(x_1, x_2), \Box(x_2) = \Diamond(x_1, x_2)$

(where $\mathbb{2}$ is $\{0 < 1\}$). Intuitively, $\Box$ just makes a copy of the poset and $\Diamond$ is used to identify all elements that are related in the order.

Co-arities that are posets. Consider the functor $\mathsf{Pos} \longrightarrow \mathsf{Pos}$, $X \mapsto \mathbb{2} \times X$. It is presented by:

— arities: 1
— operations: $\Sigma 1$ is the poset $\{l, r\}$ with $l < r$.
— no equations.

Here, each of l, r makes a copy and $l < r$ guarantees that x on the left is smaller than x on the right.

5.4. *Presenting functors on metric spaces*

The category CUMet of all complete 1-bounded ultrametric spaces and non-expanding maps is cartesian closed and locally $\aleph_1$-presentable. A *1-bounded ultrametric space* is a set X with a function $d : X \times X \longrightarrow [0, 1]$ such that:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$.

A *non-expanding map* $f : (X_1, d_1) \longrightarrow (X_2, d_2)$ is a map $f : X_1 \longrightarrow X_2$ satisfying $d_2(f(x), f(y)) \leqslant d_1(x, y)$, for all x, y. See de Bakker and de Vink (1996).

Hence, $\mathscr{V} = \mathsf{CUMet}$ can serve as a base category and our theory applies (see Remark 3.9 for the shift from finitely based to $\aleph_1$-based).

Consider a signature Σ where ΣX has one operation symbol for each finite discrete X, one operation symbol for the countable discrete space ω, and Σn is empty otherwise. The induced functor

$$H_\Sigma : \mathsf{CUMet} \longrightarrow \mathsf{CUMet}$$
$$H_\Sigma X = \coprod_{n \leqslant \omega} \mathsf{CUMet}(n, X)$$

maps a metric space X to the metric space of (finite or infinite) lists over X. The distance between two lists l, l' of the same length λ is

$$\sup\{d(l_i, l'_i) \mid i < \lambda\}. \tag{5.14}$$

Now consider adding equations Γ that quotient lists to sets (expressing the fact that the order and repetition of elements can be ignored) and denote by L the functor presented by Σ, Γ as in (3.6).

It follows from the equations Γ that the distance (5.14) between lists turns into the Hausdorff distance between sets. In detail, we consider two lists l, l' of elements from X and use $\bar{l}, \bar{l}'$ to denote the corresponding sets. We write $C(l, l')$ for the set of pairs (k, k') of lists of length ω such that we have $\bar{k} = \bar{l}$ and $\bar{k}' = \bar{l}'$. It then follows from the way colimits are calculated in CUMet that

$$d(\bar{l}, \bar{l}') = \inf\{d(k, k') \mid (k, k') \in C(l, l')\}.$$

Lemma 5.3. The equality

$$\begin{aligned}&\inf\{d(k, k') \mid (k, k') \in C(l, l')\}\\ &\qquad = \inf\{\alpha > 0 \mid \ \forall x \in \bar{l}.\exists y \in \bar{l}'.d(x, y) < \alpha \ \& \forall y \in \bar{l}'.\exists x \in \bar{l}.d(x, y) < \alpha\}\end{aligned}$$

holds.

The lemma shows that $d(\bar{l}, \bar{l}')$ is equal to the Hausdorff distance (de Bakker and de Vink 1996, Definition 2.2), which can also be expressed as

$$\max\{\sup_{x \in \bar{l}} \inf_{y \in \bar{l}'} d(x, y), \sup_{y \in \bar{l}'} \inf_{x \in \bar{l}} d(x, y)\}.$$

Subsets with distance 0 are identified in LX. It is not difficult to see that two subsets have distance 0 if and only if they have the same completion if and only if they have the same topological closure. In particular, on closed subsets, the Hausdorff distance becomes a metric. Also recall that a (subset of a) topological space is said to be separable if it has a dense, countable subset; for example, finite dimensional Euclidean space is separable. We can now summarise with the following proposition.

Proposition 5.4. The functor $L : \mathsf{CUMet} \longrightarrow \mathsf{CUMet}$ presented by Σ, Γ maps a space X to the the space of its closed and separable subsets with the Hausdorff metric.

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