Institutions for Notions of Behaviour and Reachable-Part (Abstract)

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Notions of behaviour and reachable-part are defined and conditions under which they give rise to institutions are investigated. We proceed in three steps.

Step 1. Let $\mathcal{C}$ be a category. Given an operation B on the objects of $\mathcal{C}$ and a family of arrows $(\eta_M : M \to BM)_{M \in \mathcal{C}}$, we call (B, η) a *notion of behaviour* for $\mathcal{C}$ iff

$$\text{all } \eta_M \text{ are epi} \tag{1}$$

$$\forall N, M \in \mathcal{C} \text{ there is } (-)^\sharp : \mathcal{C}(N, BM) \to (BN, BM) \text{ such that } f^\sharp \circ \eta_N = f \tag{2}$$

(In case that $\mathcal{C}$ is a concrete category) we call two elements of M *behaviourally equivalent* iff they are identified by η_M. Intuitively, (1) expresses that every model M has a quotient BM which we call the behaviour of M; (2) expresses that $f : N \to BM$ identifies all elements which are identified in BN, hence morphisms in $\mathcal{C}$ preserve behavioural equivalence.

Define $\mathcal{C}^B$ as the full subcategory of $\mathcal{C}$ consisting of all objects isomorphic to a behaviour. We think of $\mathcal{C}$ as the category of implementations and of $\mathcal{C}^B$ as the category of behaviours. Define $\mathcal{C}_B$ as the category having the same objects as $\mathcal{C}$ and morphisms $\mathcal{C}_B(N, M) = \mathcal{C}(N, BM)$.

(B, η) induces an idempotent monad on $\mathcal{C}$ (idempotence means that $BBM \cong BM$). This means that $\mathcal{C}, \mathcal{C}^B, \mathcal{C}_B$ are related as shown in

$$\begin{tikzcd}
 & \mathcal{C}^B \arrow[dl, hook', bend right, "I'"'] \\
\mathcal{C} \arrow[loop left, "B"] \arrow[ur, bend right, "B'"'] \arrow[ur, phantom, "\top"] \arrow[dr, bend right, "H"'] \arrow[dr, phantom, "\top"] & \\
 & \mathcal{C}_B \arrow[ul, bend right, "G"'] \arrow[uu, "B''"']
\end{tikzcd}$$

where, in particular, B'' is an equivalence of categories. Hence $\mathcal{C}_B$ has objects from $\mathcal{C}$ and morphisms from $\mathcal{C}^B$. For an example consider observational logic [2]: If $\mathcal{C}$ is the category of observational algebras and standard morphisms, then $\mathcal{C}^B$ is the category of fully abstract algebras and $\mathcal{C}_B$ is the category of observational algebras and observational morphisms; behavioural equivalence coincides with observational equality.

Dually, (R, ε) is a notion of reachable-part for $\mathcal{C}$ iff $(R^{\mathrm{op}}, \varepsilon^{\mathrm{op}})$ is a notion of behaviour for $\mathcal{C}^{\mathrm{op}}$. (1) says that all ε_M are mono and we think of $\varepsilon_M : RM \to M$ as the inclusion of the reachable-part into M. (2) states that $\varepsilon_N \circ f^\sharp = f$ for $f : RM \to N$, hence morphisms in $\mathcal{C}$ preserve the reachable-part. Note that $\mathcal{C}_R(M, N) = \mathcal{C}(RM, N)$. (R, ε) is an idempotent comonad and the diagram above remains valid (replacing B with R), just the adjunctions have to be turned around, ie $I' \dashv R', G \dashv H$.

Step 2. Given a functor $\mathit{Mod} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ we call a family $\mathsf{B} = (B_\Sigma, \eta_\Sigma)_{\Sigma \in \mathsf{Sig}}$ a notion of behaviour for Mod iff each (B_Σ, η_Σ) is a notion of behaviour for $\mathit{Mod}(\Sigma)$ and for all arrows $\sigma : \Sigma \to \Sigma'$ and all $M' \in \mathit{Mod}(\mathsf{Sig}')$

$$\mathit{Mod}(\sigma)(\mathsf{B}(M')) \cong \mathsf{B}(\mathit{Mod}(\sigma)(M')). \tag{3}$$

where we write B for $B_{\Sigma'}, B_\Sigma$, respectively, the subscripts being clear from the context. (3) states that $\mathit{Mod}(\sigma)$ preserves behaviours.

A notion of behaviour B for a functor $\mathit{Mod} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ gives rise to functors $\mathit{Mod}^{\mathsf{B}} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ and $\mathit{Mod}_{\mathsf{B}} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ defined as $\mathit{Mod}^{\mathsf{B}}(\Sigma) = (\mathit{Mod}(\Sigma))^{\mathsf{B}_\Sigma}$ and $\mathit{Mod}_{\mathsf{B}}(\Sigma) = (\mathit{Mod}(\Sigma))_{\mathsf{B}_\Sigma}$ (cf. the diagram with $\mathit{Mod}(\Sigma)$ for $\mathcal{C}$ and B_Σ for B). This follows from a stronger result: In an appropriate category, we have the diagram above with $\mathcal{C}$ replaced by Mod and B replaced by B.

Equivalently to the definition above, one can say that B *is a notion of behaviour for* $\mathit{Mod} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$ *iff* B *is a fibred monad on the fibration* p_{Mod} *obtained from Mod via the Grothendieck construction.*[1] *Dually, we say that* R *is a notion of reachable-part for Mod iff* R *is a fibred comonad for the cofibration obtained from Mod via the co-Grothendieck construction. This allows dualising the results proved for notions of behaviours for* $\mathit{Mod} : \mathsf{Sig}^{\mathrm{op}} \to \mathsf{CAT}$.

Step 3. Suppose we have an institution $(\mathit{Mod}^{\mathsf{B}}, \mathit{Sig}, \models)$. Then with

$$M \Vvdash \varphi \;\Leftrightarrow\; \mathsf{B}M \models \varphi, \tag{4}$$

$(\mathit{Mod}, \mathit{Sig}, \Vvdash)$ and $(\mathit{Mod}_{\mathsf{B}}, \mathit{Sig}, \Vvdash)$ are institutions. Moreover, these are *behavioural institutions* in the sense that formulas are invariant under behavioural equivalence, ie $M \Vvdash \varphi \Leftrightarrow \mathsf{B}M \Vvdash \varphi$ for all models M.

For the dual case just replace B *by* R *in (4).*

Applications. First, the proofs that observational logic [2], its variant [4], and constructor-based logic [1] are institutions are shown to be instances of a uniform framework. Second, we apply the framework to show how the categorical formulation of observational logic and constructor-based logic in [1] give rise to institutions.

Remark. In all applications the verification of (1), (2) is straightforward. Similarly $\Vvdash$ is usually designed to satisfy (4). So the point is always to carefully choose Sig in order to guarantee (3).

Conclusion. I set the approach of [4] to behavioural institutions in a wider context, show that it can be dualised to obtain constructor-based institutions, and provide new applications.

[1] M. Bidoit, R. Hennicker, and A. Kurz. On the duality between observability and reachability. In F. Honsell and M. Miculan, editors, *Foundations of Software Science and Computation Structures (FOSSACS'01)*, volume 2030 of *LNCS*, pages 72–87, 2001.

[2] R. Hennicker and M. Bidoit. Observational logic. In Armando Haeberer, editor, *Algebraic Methodology and Software Technology (AMAST'98)*, volume 1548 of *LNCS*. Springer, 1999.

[3] Bart Jacobs. *Categorical Logic and Type Theory.* Elsevier, 1998.

[4] A. Kurz and R. Hennicker. On institutions for modular coalgebraic specifications. *Theoretical Computer Science.* To appear.

[1] For fibred categories see [3].