Coalgebra and Modal Logic  
Notes from a Research Program

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my sincere thanks to Alexander Kurz
My Goals

I looked at the other titles/abstracts while preparing this talk, and also the TANCL program. My goals are to present

- a big picture on the whole subject and beyond.
- a somewhat-detailed look at a problem area involving interactions with measure theory and probability.
- another somewhat-detailed look at an application area: revisiting modal weak completeness theorems.
I started talking about coalgebra as a successor to work I had been
doing with Jon Barwise on non-wellfounded sets.
There were, and still are, some recurring complaints:
Problems, problems

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There’s no computation.

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It’s not going to last, anyways.
## The big picture

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On the set theory connection

<table>
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**Theorem (Turi; Turi and Rutten; implicit in Aczel)**

The *Foundation Axiom* is equivalent to the assertion that the universe $V$ together with $id : \mathcal{P}V \rightarrow V$ is an *initial algebra* of $\mathcal{P}$ on the category of classes.

The *Anti-Foundation Axiom* is equivalent to the assertion that the universe $V$ together with $id : V \rightarrow \mathcal{P}V$ is a *final coalgebra* of $\mathcal{P}$ on the category of classes.
On coalgebraic treatments of recursion

- recursion: map out of an initial algebra
- corecursion: map into a final coalgebra

recursion on well-founded relations

- rec’n on N → interpreted recursive program schemes
- on “cpos”

interpretations in Elgot algebras (includes, e.g., fractal sets)
Where did coalgebraic logic come from?

Let’s consider the functor on sets $F(w) = \{a, b\} \times w \times w$. The final coalgebra $F^*$ consists of infinite binary trees such as

```
  a
 / \  
 a  b
 / \  
 b  a  b  b
 / \  
 b  a  b  b
 /   
 .   
```

A (finitary) logic to probe coalgebras of $F$

$\varphi \in L : \ a | b | \text{left} : \varphi | \text{right} : \varphi |$
Here are some formulas satisfied by our tree:

\[
\begin{array}{ccc}
  & a & \\
  a & b & \\
  & b & a & b & b \\
  & : & : & : \\
\end{array}
\]

It’s easy in this case to see that the trees correspond to certain theories (sets of formulas) in this logic. It is not so easy to connect the logic back to the functor $F(w) = \{a, b\} \times w \times w$. 
Another try

We are dealing with $F(w) = \{a, b\} \times w \times w$. Let’s try the least fixed point of $F$

\[
L = \{a, b\} \times L \times L.
\]
Another try

We are dealing with $F(w) = \{a, b\} \times w \times w$.
Ok, it’s empty.
Let’s try the least fixed point plus a trivial sentence to start:

$$L = (\{a, b\} \times L \times L) + \{true\}.$$

Or, we could add a conjunction operation, with $\bigwedge \emptyset = true$.
Either way, we get formulas like

$$\langle b, \langle a, true, true \rangle, \langle a, true, true \rangle \rangle$$
$$\langle a, true, \langle b, \langle a, true, true \rangle, \langle a, true, true \rangle \rangle \rangle$$
We want to define $t \models \varphi$ for $t$ a tree and $\varphi \in L$. Note that $\models \subseteq F^* \times L$. We treat this as an object, applying $F$ to it. In fact, we also have

$$
\begin{align*}
\pi_1 : \models &\rightarrow F^* & \pi_2 : \models &\rightarrow L \\
F \pi_1 : F(\models) &\rightarrow F^* & F \pi_2 : F(\models) &\rightarrow F(L) \hookrightarrow L
\end{align*}
$$

$$t \models \langle a, \varphi, \psi \rangle \iff (\exists u, v)t = \langle a, u, v \rangle \& (\langle u, \varphi \rangle \in \models) \& (\langle v, \psi \rangle \in \models)$$
iff 
$$\begin{align*}
(\exists x \in F(\models)) &
\quad x \text{ is } \langle a, \varphi, \psi \rangle \\
F \pi_1(x) &\Rightarrow t, \\
\text{and } F \pi_2(x) &\Rightarrow \langle a, \varphi, \psi \rangle
\end{align*}$$
What are we trying to do?

\[
\text{Modal logic} \quad = \quad \text{the functor } K(a) = \mathcal{P}(a) \times \mathcal{P}(\text{AtProp}) \\
\text{an arbitrary (??) functor } F
\]

The logic ??? should be interpreted on all coalgebras of \( F \). It should characterize points in (roughly) the sense that

points in a coalgebra have the same \( L \) theory
iff they are bisimilar
iff they are mapped to the same point in the final coalgebra
What has been done?

The first paper constructed logics $L_F$ from functors $F$ and gives semantics so that

$$\frac{\text{the } \nabla \text{ fragment}}{L_F} = \frac{\text{the functor } K}{\text{a functor } F \text{ meeting some conditions}}$$

But $L_F$ often has an unfamiliar syntax, and in general one needs an infinitary boolean operations. There’s no logical system around.

(In fact, it was only this year that Palmigiano and Venema axiomatized the $\nabla$ fragment of standard modal logic.)
What has been done?

A more influential line of work constructs logics $L_F$ so that

$$\frac{\text{standard modal logic}}{L_F} = \frac{\text{the functor } K}{\text{a functor } F \text{ which is polynomial in } P_{\text{fin}}}$$

Here we have nicer syntaxes, and complete logical systems. The class of functors is smaller, but it contains everything of interest. The logics are not constructed just from the functors. This is the result of many people’s work, including Rößiger, Kurz, Pattinson, Jacobs, and others.
Suppose we liked the Kripke semantics and then asked *where did modal logic come from?* This line of work would suggest an answer; compare with van Benthem’s Theorem. In addition, it would give many other logical languages and systems with similar features.

Points in the final coalgebra of $F$ “are” the $L_F$ theories of all points in all coalgebras. So if we have some independent reason to consider $L_F$, we can use it to study the final coalgebra, *or to get our hands on it in the first place.*

One such case concerned *universal Harsanyi type spaces*, a semantic modeling space originating in game theory.
The category \textit{Meas}

A \textbf{measurable space} is a pair $M = (M, \Sigma)$, where $M$ is a set and $\Sigma$ is a $\sigma$-algebra of subsets of $M$. Usually $\Sigma$ contains all singletons $\{x\}$, but this is not needed here. A \textbf{morphism} of measurable spaces $f : (M, \Sigma) \rightarrow (N, \Sigma')$ is a function $f : M \rightarrow N$ such that for each $A \in \Sigma'$, $f^{-1}(A) \in \Sigma$. This gives a category which is often called \textit{Meas}. \textit{Meas} has products and coproducts.
The functor $\Delta$ on **Meas**

A probability measure on $M$ is a $\sigma$-additive function $\mu : \Sigma \rightarrow [0, 1]$ such that $\mu(\emptyset) = 0$, and $\mu(M) = 1$.

There is an endofunctor $\Delta : \text{Meas} \rightarrow \text{Meas}$ defined by:

$$\Delta(M)$$

is the set of probability measures on $M$

endowed with the $\sigma$-algebra generated by

$$\{ B^p(E) \mid p \in [0, 1], E \in \Sigma \},$$

where

$$B^p(E) = \{ \mu \in \Delta(M) \mid \mu(E) \geq p \}.$$

Here is how $\Delta$ acts on morphisms.

If $f : M \rightarrow N$ is measurable, then for $\mu \in \Delta(M)$ and $A \in \Sigma'$,

$$(\Delta f)(\mu)(A) = \mu(f^{-1}(A)).$$

That is, $(\Delta f)(\mu) = \mu \circ f^{-1}$. 
For each $p \in [0, 1]$, $B^p$ may be regarded as a predicate lifting. $B^p$ takes measurable subsets of each space $M$ to measurable subsets of $\Delta M$.

It is natural in the sense that if $f : M \to N$, then the diagram below commutes:

$$
\begin{array}{ccc}
\mathcal{P}_{meas}(M) & \xrightarrow{B_M^p} & \mathcal{P}_{meas}(\Delta M) \\
\downarrow f^{-1} & & \downarrow (\Delta f)^{-1} \\
\mathcal{P}_{meas}(N) & \xrightarrow{B_N^p} & \mathcal{P}_{meas}(\Delta N)
\end{array}
$$
I am not going to say what Harsanyi type spaces are. They are “multi-player” versions of coalgebras of

\[ F(M) = \Delta(M \times S), \]

where \( S \) is a fixed space.
The universal space “is” a final coalgebra.
**Prior work**

Much of the prior work on this topic used the final sequence

\[ 1 \leftarrow ! F_1 \leftarrow F! F F_1 \leftarrow \cdots \]

But in this category, the functors involved usually don’t preserve the colimits.

So the literature primarily considered subcategories of \( \text{Meas} \) where one had additional results (Kolmogorov’s Theorem).

An alternative approach was initiated by Heifetz and Samet: see “Topology-free typology of beliefs” *Journal of Economic Theory, 1998.*

Their work essentially used coalgebraic modal logic(!)

So it was not so hard to believe that it would generalize.
The class of *measure polynomial functors* is the smallest class of functors on *Meas* containing the identity, the constant functor $M$ for each measurable space $M$ and closed under products, coproducts, and $\Delta$.

**Theorem (with Ignacio Viglizzo 2004)**

*Every MPF has a final coalgebra.*

The point for this talk is that the proof used developments in coalgebraic modal logic and also was related to the point of this talk. Especially important was the work of Rößiger (1999,2001) and Jacobs (2001).
For a measure polynomial functor $T$, we define a finite set $\text{Ing}(T)$ of functors by the following recursion:

For the identity functor, $\text{Ing}(\text{Id}) = \{\text{Id}\}$;

for a constant space $M$, $\text{Ing}(M) = \{M, \text{Id}\}$,

$\text{Ing}(U \times V) = \{U \times V\} \cup \text{Ing}(U) \cup \text{Ing}(V)$,

and similarly for $U + V$;

$\text{Ing}(\Delta S) = \{\Delta S\} \cup \text{Ing}(S)$.

We call $\text{Ing}(T)$ the set of ingredients of $T$.

Each measure polynomial functor $T$ has only finitely many ingredients.

**Example**

Let $[0, 1]$ be the unit interval of the reals, endowed with the usual Borel $\sigma$-algebra, and $T = [0, 1] \times (\Delta X + \Delta X)$. Then

$$\text{Ing}(T) = \{\text{Id}, [0, 1], \Delta \text{Id}, \Delta \text{Id} + \Delta \text{Id}, [0, 1] \times (\Delta \text{Id} + \Delta \text{Id})\}.$$
The notation \( \varphi :: S \) means that for every constant functor \( M \in \text{Ing}(T) \), every subformula of \( \varphi \) of sort \( M \) is a measurable set.
Let $c : X \to TX$ be a coalgebra of $T$.
The semantics assigns to each $S \in \text{Ing}(T)$ and each $\varphi : S$ a subset $[[\varphi]]_S^c \subseteq SX$.

\[
\begin{align*}
[[\text{true}]]_S^c &= SX \\
[[A]]_M^c &= A \\
[[\varphi \land \psi]]_S^c &= [[\varphi]]_S^c \cap [[\psi]]_S^c \\
[[\langle \varphi, \psi \rangle]]_{U \times V}^c &= [[\varphi]]_U^c \times [[\psi]]_V^c \\
[[\text{inl } \varphi]]_{U+V}^c &= \text{inl}([[[\varphi]]_U^c) \\
[[\text{inr } \varphi]]_{U+V}^c &= \text{inr}([[[\varphi]]_V^c) \\
[[B^p \varphi]]_{\Delta S}^c &= B^p([[[\varphi]]_S^c) \\
[[\text{next } \varphi]]_{Id}^c &= c^{-1}([[[\varphi]]_T^c)
\end{align*}
\]
Coalgebra morphisms preserve the semantics

That is, if $f : b \to c$ is a morphism of coalgebras $b : X \to TX$ and $c : Y \to TY$, and if $\varphi : S$, then

$$(Sf)^{-1}(\llbracket \varphi \rrbracket^c_S) = \llbracket \varphi \rrbracket^b_S.$$
For each coalgebra $c : X \to TX$ and each $x \in SX$, we define

$$d^c_S(x) = \{ \varphi : S \mid x \in \llbracket \varphi \rrbracket^c_S \}.$$ 

We call each such set $d^c_S(x)$ a satisfied theory.

**The canonical sets $S^*$ for $S \in \text{Ing}(T)$**

by $S^* = \{ d^c_S(x) \mid x \in SX \text{ for some coalgebra } c : X \to TX \}$.

**The sets $|\varphi|_S$**

$|\varphi|_S = \{ s \in S^* \mid \varphi \in s \}$.

$\varphi \in d^c_S(x)$ iff $d^c_S(x) \in |\varphi|_S$.

**The canonical spaces $S^*$ for $S \in \text{Ing}(T)$**

Each $S^*$ is a measurable space, via the $\sigma$-algebra generated by the family of sets $|\varphi|_S$ for $\varphi :: S$. 
The main work

There are maps as shown in blue below

\[
\begin{array}{cccccc}
X & \xrightarrow{c} & TX & \xrightarrow{Td_{Id}^c} & T(Id^*) \\
\downarrow{d_{Id}^c} & & \downarrow{d_T^c} & & \\
Id^* & \xrightarrow{\text{[next]}^{-1}} & T^* & \xrightarrow{r_T} & T(Id^*) \\
\end{array}
\]

and then \( Id^*, r_T \circ \text{[next]}^{-1} \) is a final coalgebra of \( T \).

I'm skipping all the hard stuff.
The Dynkin \( \lambda - \pi \) Lemma is used, for example.
My newly-finished Ph.D. student Chunlai Zhou has axiomatized the logic of Harsanyi types spaces. His work is finitary and improves on earlier systems (Heifetz & Mongin, Meier). His work makes essential use of linear programming.
Summary so far

We built final coalgebras from the satisfied theories in independently-motivated logics. This strengthens the motivation for both the logics and the final coalgebras.
One of the goals of this TANCL workshop is to investigate treatments of logics that go beyond rank 1 axiomatizations. My contribution here is a coalgebraic re-working of the basic weak completeness results for various standard modal logics.
One of the goals of this TANCL workshop is to investigate treatments of logics that go beyond rank 1 axiomatizations. My contribution here is a coalgebraic re-working of the basic weak completeness results for various standard modal logics. I have to confess that my work here originally had different motivations: I wanted to teach the weak completeness results to students who lacked the background to really understand maximal consistent sets and filtration. Also, I wanted a more “semantic” method than tableaux.
We start with a set AtProp of atomic sentences.

\[ \varphi \in L_\nabla : \quad p \mid \neg p \mid \varphi \land \psi \mid \nabla S \text{ for } S \subseteq L_\nabla \]

The semantics is

\[ w \models \nabla S \quad \text{iff} \quad \text{every } y \leftarrow x \text{ satisfies some } \varphi \in S \]

and every \( \varphi \in S \) is satisfied by some \( y \leftarrow x \)

So \( \nabla S \) “abbreviates” \( \bigwedge_{\varphi \in S} \lozenge \varphi \land \square \bigvee_{\varphi \in S} \varphi \).
Which modal sentences are the smartest?

Let $\mathcal{A}$ be any Kripke model. Fix a number $n$. For every $a \in A$ and every $h$, we define the sentence $\varphi^h_a$. The definition is by recursion on $h$ (simultaneously for all $a \in A$) as follows:

$$\varphi^0_a = \bigwedge \{ p : a \models p \} \land \bigwedge \{ \neg p : a \models \neg p \}.$$ 

Given $\varphi^h_b$ for all $b \in A$, we define

$$\varphi^{h+1}_a = \nabla \{ \varphi^h_b : a \rightarrow b \} \land \varphi^0_a.$$ 

Each $\varphi^h_a$ belongs to $C_{h,n}$.

The idea is that $\varphi^n_a$ gives us as much information as possible about the points reachable from $a$ in $\leq h$ steps.
We define the \textit{height} and \textit{order} of an arbitrary sentence $\varphi$ of modal logic. The height measures the maximum nesting depth of boxes, and the order gives the largest subscript on any atomic proposition occurring. For example,

\begin{align*}
ht(\lozenge p_3 \land \square \lozenge p_2) &= 2 \\
ord(\lozenge p_3 \land \square \lozenge p_2) &= 3
\end{align*}

\[ \mathcal{L}_{h,n} = \{ \varphi : \ht(\varphi) \leq h, \ord(\varphi) \leq n \} . \]
The sets $C_{h,n}$

We define the sets $C_{h,n}$ of *canonical sentences of height $h$ and order $n* as follows:

$C_{0,n} =$ the complete conjunctions of order $n$.

$C_{h+1,n}$ is the collection of sentences of the form $\nabla S \land \hat{T}$, where

$$S \subseteq C_{h,n}$$

$$T \subseteq \{p_1, \ldots, p_n\}$$

$$\hat{T} = (\land_{p_i \in T} p_i) \land (\land_{p_i \notin T} \neg p_i)$$

In other words, $\alpha \in C_{h+1,n}$ is of the form

$$(\land_{\psi \in S} \diamond \psi) \land (\Box \lor S) \land (\land T) \land (\land_{p_i \notin T} \neg p_i)$$

for some $S \subseteq C_{h,n}$ and some $T \subseteq \{p_1, \ldots, p_n\}$. 
Examples: $C_{0,1}$ and $C_{1,1}$

$C_{0,1} = \{p_1, \neg p_1\}$. Henceforth we drop the subscript.

$C_{1,1} = \{\alpha_1, \ldots, \alpha_8\}$, where

\[
\begin{align*}
\alpha_1 &= \nabla\emptyset \land p \\
\alpha_2 &= \nabla\emptyset \land \neg p \\
\alpha_3 &= \nabla\{p\} \land p \\
\alpha_4 &= \nabla\{p\} \land \neg p \\
\alpha_5 &= \nabla\{\neg p\} \land p \\
\alpha_6 &= \nabla\{\neg p\} \land \neg p \\
\alpha_7 &= \nabla C_{0,1} \land p \\
\alpha_8 &= \nabla C_{0,1} \land \neg p
\end{align*}
\]

Note that $\nabla\emptyset \equiv \Box \text{false}$.

$C_{0,2}$ has $2^2 = 4$ elements. $C_{1,2}$ has $2 \times 2^4 = 32$ elements.

And $C_{2,2}$ has $2 \times 2^{32} = 8,589,934,592$ elements.
THE MODELS \( C_{h,n}(L) \)

Let \( L \) be a normal modal logic.
We define \((C_{h,n}(L), \sim\mapsto)\), the canonical model of consistent sentences of \( L \) of height \( h \) and order \( n \):

- The points of \( C_{h,n}(L) \) are the elements of \( C_{h,n} \) which happen to be consistent in the logic \( L \).
- \( \alpha \sim\mapsto \beta \iff \alpha \land \Box \beta \) is consistent in \( L \). This comes from the Kozen-Parikh completeness theorem for PDL.
- \( \alpha \models p \iff \vdash \alpha \rightarrow p \) in \( L \).

This part of the definition is what we are exploring here.
The points satisfying $p$ are exactly those on the left side of the figure: $\alpha_1$, $\alpha_3$, $\alpha_5$, and $\alpha_7$.

A sentence $\varphi$ in one proposition $p$ and of height 1 is valid iff $\varphi$ holds at all points of the model above.
Coalgebra and Modal Logic
Weak completeness in modal logic

\[ \alpha_1 \quad \alpha_2 \]

\[ \alpha_3 \quad \alpha_4 \]

\[ \alpha_5 \quad \alpha_6 \]

\[ \alpha_7 \quad \alpha_8 \]

\[ K \]

\[ K_4 \]

\[ KB \]

\[ S_4 \]

\[ KB_4 \]

\[ S_5 \]
\[ C_{2,1}(S4) \]

\[
\begin{align*}
\beta_2 &= \nabla\{\alpha_3, \alpha_6, \alpha_7\} \land p \\
\beta_3 &= \nabla\{\alpha_3, \alpha_7, \alpha_8\} \land p \\
\beta_4 &= \nabla\{\alpha_6, \alpha_7, \alpha_8\} \land p \\
\beta_5 &= \nabla\{\alpha_6, \alpha_7\} \land p \\
\beta_6 &= \nabla\{\alpha_7, \alpha_8\} \land p \\
\beta_7 &= \nabla\{\alpha_3\} \land p \\
\beta_8 &= \nabla\{\alpha_3, \alpha_6, \alpha_7, \alpha_8\} \land \neg p \\
\beta_9 &= \nabla\{\alpha_3, \alpha_6, \alpha_8\} \land \neg p \\
\beta_{10} &= \nabla\{\alpha_3, \alpha_7, \alpha_8\} \land \neg p \\
\beta_{11} &= \nabla\{\alpha_6, \alpha_7, \alpha_8\} \land \neg p \\
\beta_{12} &= \nabla\{\alpha_3, \alpha_8\} \land \neg p \\
\beta_{13} &= \nabla\{\alpha_7, \alpha_8\} \land \neg p \\
\beta_{14} &= \nabla\{\alpha_6\} \land \neg p
\end{align*}
\]

These are the elements of \( C_{2,1} \) consistent in \( S4 \). The structure as always is given by \( \beta_i \rightsquigarrow \beta_j \) iff \( \beta_i \land \lozenge \beta_j \) is consistent in \( S4 \).
For each $h$ and $n$, $C_{h,n}$ is a finite subset of $\mathcal{L}_{h,n}$.

**Lemma**

Let $\chi \in \mathcal{L}_{h,n}$ and $\alpha \in C_{h,n}$. Then in $K$,

either $\vdash \alpha \rightarrow \chi$ or else $\vdash \alpha \rightarrow \neg \chi$.

**Lemma**

$\vdash \bigvee C_{h,n}$.

And for $\alpha \neq \beta$,

$\vdash \alpha \rightarrow \neg \beta$. 
More properties

**Lemma**

The following hold for all $h$ and $n$:

1. If $KT \leq L$, $C_{h,n}(L)$ is reflexive.
2. If $KD \leq L$, $C_{h,n}(L)$ is serial.
3. If $KB \leq L$, $C_{h,n}(L)$ is symmetric.

(One interesting failure is that if $L = K$ with $\Diamond \varphi \to \Box \varphi$, then $C_{h,n}$ is not a partial function.)

**Lemma (Existence Lemma)**

Let $\psi \in \mathcal{L}_{h,n}$, let $\varphi$ be arbitrary, and suppose that $\varphi \land \Diamond \psi$ is consistent in $L$. Then there is some $\alpha \in C_{h,n}(L)$ such that $\varphi \land \Diamond \alpha$ is consistent in $L$, and $\vdash \alpha \to \psi$ in $K$. 
Easy weak completeness results

**Lemma (Truth Lemma for $C_{h,n}(L)$)**

For all $\alpha \in C_{h,n}(L)$ and all $\psi \in L_{h,n}$,

$$(C_{h,n}(L), \alpha) \models \psi \iff \vdash \alpha \rightarrow \psi \text{ in } K.$$

**Theorem**

We have the following completeness/decidability results:

1. $KT$ for reflexive models. ($T$ is $\Box \varphi \rightarrow \varphi$.)
2. $KD$ for serial models. ($D$ is $\Diamond \text{true}$.)
3. $KB$ for symmetric models. ($B$ is $\varphi \rightarrow \Box \Diamond \varphi$.)
4. etc.

With a trick, one can also get the result for partial functions.
Completeness for classes of transitive models

**Lemma**

\( C_{h,n}(K4) \) is transitive.

**Theorem**

\( K4 \) is complete for transitive models. Other results for all combinations of \( B, D, T, 4, \) and \( 5. \)
Completeness for classes of transitive models, continued

**Theorem** \((K4McK = K \text{ with the McKinsey axioms})\)

\[\Box \diamond \varphi \rightarrow \diamond \Box \varphi.\]

\(C_{h,n}(K4McK)\) is transitive, and each point has a successor with at most one successor.

Thus \(K4McK\) is weakly complete for this class.

**Theorem** \((KL = K \text{ with the Löb axioms})\)

\[\Box (\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi.\]

\(C_{h,n}(KL)\) is transitive and converse well-founded.

Thus \(KL\) is weakly complete for this class.
Completeness for classes of transitive models, continued

**Theorem** \((K4McK = K \text{ with the McKinsey axioms})\)

\[ \Box \Diamond \varphi \to \Diamond \Box \varphi. \]

\(C_{h,n}(K4McK)\) is transitive, and each point has a successor with at most one successor. Thus \(K4McK\) is weakly complete for this class.

**Theorem** \((KL = K \text{ with the Löb axioms})\)

\[ \Box (\Box \varphi \to \varphi) \to \Box \varphi. \]

\(C_{h,n}(KL)\) is transitive and converse well-founded. Thus \(KL\) is weakly complete for this class.

**Open Question**

If \(K4 \leq L\), then is \((C_{h,n}(L), \rightsquigarrow)\) transitive?
Recall that $\mathcal{L}_{h,n}$ is the (infinite) set of modal formulas of height $\leq h$ and of order $\leq n$.

Let $\text{Can}(L)$ be the canonical model of a logic $L$.
Consider the equivalence on $\text{Can}(L)$ induced by $\mathcal{L}_{h,n}(L)$.

**Theorem**

$\mathcal{C}_{h,n}(L)$ is isomorphic to the minimal filtration of $\text{Can}(L)$.

Fine defined a model $\mathcal{C}_{h,n}$ in connection with K4.

**Theorem**

$\mathcal{C}_{h,n}(K4) \cong \mathcal{C}_{h,n}$. 
Coalgebra and Modal Logic
Weak completeness in modal logic

Mix \[ \square^* \varphi \rightarrow (\varphi \land \Box \square^* \varphi) \]

Induction \[ (\varphi \land \Box^* (\varphi \rightarrow \Box \varphi)) \rightarrow \square^* \varphi \]

We build $C_{h,n}(K\square^*)$ the same way we built $C_{h,n}$ except that we use

\[
(\bigwedge_{\psi \in R} \Diamond \psi) \land (\Box \bigvee R) \land \\
(\bigwedge_{\psi \in S} \Diamond^* \psi) \land (\Box^* \bigvee S) \land \\
(\bigwedge T) \land (\bigwedge_{p, i \notin T} \neg p_i)
\]

and we also are only interested in sentences of this form which are consistent in $K\square^*$.

The $ht$ function works as before, except we also say that \[ ht(\square^* \varphi) = 1 + ht(\varphi). \]

The analogs of general Lemmas on $C_{h,n}$ hold.
Completeness for $K\square^*$

**Lemma**

Let $\alpha, \beta \in C_{h,n}(K\square^*)$ and $\Diamond^*\varphi \in \mathcal{L}_{h,n}$. Suppose that $\alpha \rightsquigarrow \beta$ and $\vdash \beta \rightarrow \Diamond^*\varphi$. Then $\vdash \alpha \rightarrow \Diamond^*\varphi$ as well.

**Lemma**

Let $X \subseteq C_{h,n}(K\square^*)$ be closed under $\rightsquigarrow$. Then $\vdash \bigvee X \rightarrow \Box^* \bigvee X$.

**Lemma (Truth Lemma for $C_{h,n}(K\square^*)$)**

For all $\alpha \in C_{h,n}(K\square^*)$ and all $\psi \in \mathcal{L}_{h,n}$, $(C_{h,n}(K\square^*), \alpha) \models \psi$ iff $\vdash \alpha \rightarrow \psi$ in $K\square^*$.

**Theorem**

$K\square^*$ is complete and decidable.
1. Coalgebraic versions of modal logic are connected to exploration of other issues.
2. One can construct a final coalgebra by taking as the carrier the satisfied theories in an associated logic.
3. One can prove modal weak completeness/decidability results using models built from sentences in the same logics.

**Issues**

The definition of the models $C_{h,n}$ is not as *principled* as one would like, especially if we are to generalize these models to coalgebraic settings.

Even more, why does all this work?