Part I: Coalgebraic Logics: Motivation and Some Results
Coalgebraic Logics: Describe *computational phenomena* with *modal logics*

- State Transition Systems → Hennessy-Milner Logic
- Probabilistic Effects → Probabilistic Modal Logic
- Games → Coalition Logic
- Ontologies . . . → Description Logic . . .

Logical Aspects
- completeness
- complexity
- cut elimination
- interpolation . . .

Computer Science Aspects
- *Genericity*: development of uniform proofs/algorithms/tools?
- *Modularity*: synthesis of complex systems from simple building blocks
A Cook’s Tour Through Modal Semantics

**Kripke Models.**

\[ C \rightarrow \mathcal{P}(C) \times \mathcal{P}(A) \]

**Multigraphs.**

\[ C \rightarrow \mathcal{B}(C) \times \mathcal{P}(A) \]

\[ \mathcal{B}(X) = \{ f : X \rightarrow \mathbb{N} \mid \text{supp}(f) \text{ finite} \} \]

**Probabilistic Systems.**

\[ C \rightarrow \mathcal{D}(C) \times \mathcal{P}(A) \]

\[ \mathcal{D}(X) = \{ \mu : X \rightarrow [0, 1] \mid \sum_{x \in X} \mu(x) = 1 \} \]
Unifying Feature: Coalgebraic Semantics

All examples are instances of **Coalgebras**

\[(C, \gamma : C \to TC)\]

where \(T : \text{Set} \to \text{Set}\) is an endofunctor, the **signature functor**.

(Dually, **\(T\)-algebras** are pairs \((A, \alpha : TA \to A)\))

**Intuition.**

- **coalgebras** are generalised transition systems
- **morphisms** of coalgebras are generalised \(p\)-morphisms

**Computer Science Concerns**

- **Genericity**: Prove things once and for all, **parametric in** \(T\)
- **Modularity**: Construct complex functors from simple ingredients
Coalgebraic Semantics of Modal Logics

Given: \( T : \text{Set} \rightarrow \text{Set} \)

Question: What’s the “right” logic for \( T \)-coalgebras?

- should generalise well-known cases, e.g. K, probabilistic/graded modal logic, coalition logic
- theory should be *parametric* in \( T \)
  \( \sim \) *uniform* theorems that apply to a large class of logics

Semantically: What’s a modal operator, or: what is \( [\Box \phi] \)?
Moss’ Coalgebraic Logic I

Kripke Frames: \( C \rightarrow \mathcal{P}(C) \)

\[ \begin{align*}
\text{Concrete Syntax} & \quad \phi, \psi \in L \quad \Phi \in \mathcal{P}(L) \\
\bot & \in L \quad \phi \rightarrow \psi \in L \\
\n\end{align*} \]

Modal Semantics
\[ c \models \nabla \Phi \iff (\gamma(c), \Phi) \in \mathcal{P}(\models) \]

Abstract Syntax:
\[ L \cong F(L) = 1 + L^2 + \mathcal{P}(L) \]

Algebraic Semantics
\[ F(L) \xrightarrow{\gamma} F(\mathcal{P}(C)) \]

\[ \nabla \Phi = \Box \bigvee \Phi \land \Diamond \Phi \quad \text{Need: } F\text{-algebra structure } F(\mathcal{P}(C)) \rightarrow \mathcal{P}(C) \]

\[ T\text{-coalgebras: } C \rightarrow T(C) \]

\[ \begin{align*}
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Algebraic Semantics
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Algebraic Semantics of Coalgebraic Logic:

\[ 1 + L^2 + TL \rightarrow 1 + (\mathcal{P}C)^2 + T(\mathcal{P}C) \]

\[ L \rightarrow [\cdot] \rightarrow \mathcal{P}(C) \]

where \( \hat{\gamma} : T(\mathcal{P}C) \xrightarrow{\delta} \mathcal{P}(TC) \xrightarrow{\gamma^{-1}} \mathcal{P}(C) \)

distributive law

Representation Theorem: \( \sum_n A_n \times X^n \rightarrow TX \), e.g. \( X \xrightarrow{M} TX \)
gives algebraic semantics of Unary Modalities:

\[ \mathcal{P}(C) \xrightarrow{M} T(\mathcal{P}C) \xrightarrow{\delta} \mathcal{P}(TC) \xrightarrow{\gamma^{-1}} \mathcal{P}(C) \]

unary modality
Coalgebraic Semantics of Modal Logics

**Structures** for $T$ coalgebras determine the semantics of modal operators: they

assign a *nbhd frame translation* or, equivalently, a *predicate lifting*

$$[[M]] : TC \to \mathcal{P}(C')$$  \quad $$[[M]] : \mathcal{P}(C) \to \mathcal{P}(TC)$$

to every modal operator $M$ of the language, parametric in $C$.

Together with a $T$-coalgebra $(C, \gamma)$ this gives a

**neighbourhood frame**  \quad **boolean algebra with operator**

$$C \xrightarrow{\gamma} TC \xrightarrow{[[M]]} \mathcal{P}(C')$$  \quad $$\mathcal{P}(C') \xrightarrow{[[M]]} \mathcal{P}(TC') \xrightarrow{\gamma^{-1}} \mathcal{P}(C')$$

**Induced Coalgebraic Semantics** $[[\phi]] \subseteq C$ of a modal formula

<table>
<thead>
<tr>
<th>from a <em>modal perspective</em></th>
<th>equivalent <em>algebraic viewpoint</em></th>
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<tbody>
<tr>
<td>$c \in [[M \phi]]$ iff $[[\phi]] \in [[M]] \circ \gamma([[\phi]])$</td>
<td>$c \in [[M \phi]] \iff \gamma(c) \in <a href="%5B%5B%5Cphi%5D%5D">[M]</a>$</td>
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Examples

**Neighbourhood Frames**, i.e. coalgebras $C \to \mathcal{P}\mathcal{P}(C)$

$$\square \mathcal{C} = \text{id} : \mathcal{P}\mathcal{P}(C) \to \mathcal{P}\mathcal{P}(C)$$

(identical nbhd frame translation)

**Kripke Frames**, i.e. coalgebras $C \to \mathcal{P}(C)$

viewed as neighbourhood frames via boolean algebras with operators

$$\square : \mathcal{P}(C) \to \mathcal{P}\mathcal{P}(C)$$

$c \mapsto \{c' : c' \supseteq c\}$

**Probabilistic Transition Systems**, i.e. coalgebras $C \to \mathcal{D}C$

$$\square \mathcal{L}_p : \mathcal{P}(C) \to \mathcal{P}\mathcal{D}(C)$$

(algebraic perspective)

$c \mapsto \{\mu : C \to [0, 1] : \mu(c) \geq p\}$
Genericity I: Expressivity

**Easy, but important:** Coalgebraic Logics are bisimulation invariant.

**Hennessy-Milner Property:**
Bisimulation coincides with logical equivalence over *image finite* transition systems.

- what is *image finite* for $T$-coalgebras?
- additional condition(s) on the logic (e.g. exclude empty set of operators)

**Theorem** (P, 2001)
If $T$ is $\omega$-accessible and the modal structure is *separating*, i.e. for predicate liftings

$$TC \ni t \mapsto \{[[M]](c) : c \subseteq C, M \text{ modal op}\}$$

is injective, then the induced logic has the Hennessy-Milner property.

**Theorem** (Schroeder, 2005)
Admitting polyadic modalities, the structure that comprises *all* predicate liftings is separating.
Genericity II: Completeness

**Deduction** for Coalgebraic Logics: propositional logic plus a set \( \mathcal{R} \) of

*one-step rules* \( \phi / \psi \): \( \phi \) propositional, \( \psi \) clause over \( Ma, a \in V \)

**Intuition.** Rules axiomatise those nbhd frames that come from coalgebras

**One Step Derivability** of \( \chi \) (propositional over \( \{ Mx : x \subseteq X \} \) over a set \( X \))

- \( TX \models \chi \) defined inductively by \( \llbracket Mx \rrbracket = \llbracket M \rrbracket(x) \)
- \( \mathcal{R}X \vdash \chi \) iff \( \{ \psi \sigma : X \models \phi \sigma, \phi / \psi \in \mathcal{R} \} \vdash_{\text{PL}} \chi \)

\( \mathcal{R} \) is one-step sound (complete) if \( TX \models \chi \) whenever (only if) \( \mathcal{R}X \vdash \chi \)

**Theorem** (P, 2003, Schroeder 2006)

Soundness and weak completeness are implied by their one-step counterparts.

**Theorem** (Schroeder 2006)

The set of axioms that is one-step sound is one-step complete.
Genericity III: Complexity

**Shallow Model Construction** for $T$-coalgebras: inductively strip off modalities

$$\forall \phi/\psi \in R. \psi \sigma \rightarrow \chi \implies \neg \phi \sigma \text{ satisfiable}$$

Countermodel of $\phi \sigma$’s

$$\uparrow$$

$\neg \chi$ satisfiable

Countermodel of $\chi$

Crucial Requirement is **Resolution Closure** of $R$:

derivable consequences are derivable using a *single* rule.

**Theorem.** (Schroeder/P, 2006)

If $R$ is resolution closed and rule matching is in NP, then satisfiability is in PSPACE.

**Example.** K, KD, Coalition Logic, GML, PML, Majority Logic are in PSPACE.
Construction of Resolution Closed Sets

**Example:** \( K \) axiomatised by rules

\[
\begin{align*}
\frac{a}{\Box a} & \quad \frac{a \land b \rightarrow c}{\Box a \land \Box b \rightarrow \Box c} \\
\end{align*}
\]

**Rule Resolution:**

\[
\begin{align*}
\frac{a \land b \rightarrow c}{\Box a \land \Box b \rightarrow \Box c} & \quad \frac{c \land d \rightarrow e}{\Box c \land \Box d \rightarrow \Box e} \\
\end{align*}
\]

Resolving the conclusions at \( c \)

\[
\frac{(a \land b \rightarrow c) \land (c \land d \rightarrow e)}{\Box a \land \Box b \land \Box d \rightarrow \Box e}
\]

Eliminating \( c \) from the premise:

\[
\frac{a \land b \land d \rightarrow e}{\Box a \land \Box b \land \Box d \rightarrow \Box e}
\]

(This converges to a cut-free sequent-calculus . . .)
Modularity

Example. Combining Probabilities and Non-Determinism

Simple Segala Systems

Alternating Systems

Coalgebraic Interpretation

Semantics of Combination. Functor Composition – ingredients represent features.

Logic Combinations. Mimic Functor Composition
Logics for Combined Systems

Simple Segala Systems: \( C \rightarrow \mathcal{P}(A \times \mathcal{D}(C)) \)

\[ \mathcal{L}_n \ni \phi ::= \top \mid \phi_1 \land \phi_2 \mid \neg \phi \mid \Box_a \psi \quad \text{(nondeterministic formulas; } \psi \in \mathcal{L}_u, a \in A) \]
\[ \mathcal{L}_u \ni \psi ::= \top \mid \psi_1 \land \psi_2 \mid \neg \psi \mid L_p \phi \quad \text{(probabilistic formulas; } \phi \in \mathcal{L}_n, p \in [0, 1] \cap \mathbb{Q}). \]

Alternating Systems: \( C \rightarrow \mathcal{P}(A \times C) + \mathcal{D}(C) \)

\[ \mathcal{L}_o \ni \rho ::= \top \mid \rho_1 \land \rho_2 \mid \neg \rho \mid \phi + \psi \quad \text{(alternating formulas; } \phi \in \mathcal{L}_u, \psi \in \mathcal{L}_n) \]
\[ \mathcal{L}_u \ni \phi ::= \top \mid \phi_1 \land \phi_2 \mid \neg \phi \mid L_p \rho \quad \text{(probabilistic formulas; } \rho \in \mathcal{L}_o, p \in [0, 1] \cap \mathbb{Q}) \]
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Semantics by Example: given \( \gamma : C \rightarrow \mathcal{P}(A \times C) + \mathcal{D}(C) \)

- \((\mathcal{L}_o)\) \( [\phi + \psi] = \gamma^{-1}(\llbracket \phi \rrbracket + \llbracket \psi \rrbracket) \subseteq C \)
- \((\mathcal{L}_u)\) \( [L_p \rho] = \llbracket L_p \rrbracket(\llbracket \rho \rrbracket) \subseteq \mathcal{D}C \)
- \((\mathcal{L}_n)\) \( [\Box_a \rho] = \llbracket \Box_a \rrbracket(\llbracket \rho \rrbracket) \subseteq \mathcal{P}(A \times C) \)
Modularity I: Expressivity

**Features:** Basic Building Blocks comprising

- an endofunctor $F : \text{Set}^n \rightarrow \text{Set}$
- typed modal operators $M : i_1, \ldots, i_k$
- predicate liftings $[M] : \mathcal{P}(X_1) \times \cdots \times \mathcal{P}(X_k) \rightarrow \mathcal{P}F(X_1, \ldots, X_k)$

**Example 1: Uncertainty**
- $\mathcal{D} : \text{Set} \rightarrow \text{Set}$
- $L_p : 1 (p \in [0, 1] \cap \mathbb{Q})$
- $[L_p]$ as before

**Example 2: Binary Choice**
- $\coprod : \text{Set}^2 \rightarrow \text{Set}$
- $+ : 1, 2$
- $[+] : (x, y) \mapsto x + y$

**Theorem** (Cirstea, 2000)
The logic associated with any combination of features that are $\omega$-accessible and separating has the Hennessy-Milner property.
Deduction for combined logics: Extend features with typed one-step rules

**Example 1:** Uncertainty

\[
\frac{\sum_{1 \leq i \leq n} r_ia_i \geq k}{\bigvee_{1 \leq i \leq n} \text{sgn}(r_i)L_p, a_i}\]  

(plus side conditions)

**Example 2:** Binary Choice

\[
\frac{\bigwedge_{i=1}^{m} \alpha_i \rightarrow \bigvee_{j=1}^{n} \beta_j : 1}{\bigwedge_{i=1}^{m} (\alpha_i + \gamma_i) \rightarrow \bigvee_{j=1}^{n} (\beta_j + \delta_j) : 2} \quad (m, n \geq 0)
\]

**Deduction for Combined Logics:** type correct application of deduction rules

**Theorem.** (Cirstea/P, 2003)

One-step completeness of all features implies weak completeness of combinations.

**Theorem.** (Schroeder/P, 2007)

Satisfiability for combined logics is in PSPACE provided rule matching for all features is in NP.
Part II: Extensions and Open Problems
Frequently Asked Questions

Coalgebraic Completeness Theorem

Referee: This is nice, but can you also do S4?
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Complexity of Coalgebraic Logics

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So maybe it’s time to go beyond rank 1 . . .
Frame Conditions

Recall: Coalgebraic Logics can always be axiomatised by rank-1 axioms.

Our Setting: Rank-1 axioms $\mathcal{A} +$ Frame Conditions $\Phi$, i.e.

- $\mathcal{A}$ is rank-1, sound and complete w.r.t. all $T$-coalgebras
- $\Phi$ is a set of additional axioms (not necessarily rank 1), e.g. $T$ or 4.

Kripke Frame Analogy. $\mathcal{A} = K$ and e.g. $\Phi = 4$

Semantic Consequence and Deduction

- $T\Phi \models \phi$ iff $C \models \phi$ whenever $C \models \Phi$, for all $T$-coalgebras $C$
- $\mathcal{A}\Phi \vdash \phi$ iff $\phi$ is derivable from $\mathcal{A} \cup \Phi$

Question. For which $\phi$ do we have completeness, i.e. $T\Phi \models \phi \iff \mathcal{A}\Phi \vdash \phi$?
Partial Answers and Open Questions

**Frame Completeness.** \( T\Phi \models \phi \iff A\Phi \vdash \phi \) holds, for example, if

- if \( \Phi \) is a collection of *positive* formulas
- if \( \Phi \) is any collection of rank 0/1 formulas (e.g. \( T \))
- if \( \Phi = \emptyset \) or \( \Phi = T, 4 \)

**Open Questions.**

- semantical characterisation of admissible frame conditions?
- syntactical characterisation? Sahlquist completeness theorem?
Observation I. Rule Resolution \textit{seems} to lead to sequent calculus presentations, but:

Observation II. General Rule Premises are of the form

\[ \bigwedge_{J \subseteq I} \left( \bigwedge_{j \in J} a_j \rightarrow \bigvee_{j \notin J} b_j \right) \]

Open Questions

\begin{itemize}
  \item can we \textit{systematically} derive sequent calculi?
  \item are they cut-free?
  \item and have interpolation and/or subformula properties?
\end{itemize}
Decidability and Complexity

Decidability via finite models: by-product of completeness via fmp

Challenge Question: Complexity

In a setting *without* frame conditions . . .

**Semantically**

- coalgebraic *shallow models*
- based on *extended rulesets*

**Syntactically**

- cut-free sequent calculus
- induced by extended rulesets

Ruleset extension is algorithmic: *resolution closure*

**Open Questions**

- is resolution closure meaningful outside rank-1, and when? (yes, e.g. for S4)
- does either the syntactical or the semantic method extend?
Fixpoint Formulas

**Application Pull.** Reasoning about *ongoing behaviour*: safety and liveness

Language Extension: **flat fixpoint formulas**

\[ M^* \phi \equiv \nu x. \phi \land M x \quad \text{and} \quad M_* \phi \equiv \mu x. \phi \lor M x \]

(many possible variations)

New (Fixpoint) Axioms, e.g.

\[ F \equiv M^* p \rightarrow p \land M M^* p \quad (p \land M^* (p \rightarrow M p)) \rightarrow M^* p \]

**Trivial Theorem:**

\[ \mathcal{A} S \vdash \phi \iff T \models \phi \text{ if } \mathcal{A} \text{ is sound w.r.t. all } T\text{-coalgebras} \]

**Hard Problem:** Completeness.

**Even Harder Problem:** Complexity
Any Answers?