1. A category of games.

A (finite) game consists of (finite) sets \( U, X \) of plays for I, II and a pay-off matrix 
\[
A(u, x) \in \mathbb{R} \quad u \in U \land x \in X
\]

(Alternatives to \( \mathbb{R} \): values \( V \) an ordered abelian group (of compact closed category) but later we'll want some convexity structure.)

The \( U \times X \)-matrix \( A \) gives \( \mathbb{R}^{\times X} \to \mathbb{R}^{\times U} \) so write
\[
U \xleftarrow{A} X
\]

A map \( U \xleftarrow{A} X \to V \xleftarrow{B} Y \) is a (circular) reason to prefer B to A, or a proof \( A \to B \) given by
\[
U \xleftarrow{A} X \quad \text{such that for all } \quad v \in U \land y \in Y
\]
\[
f \downarrow \quad \uparrow F
\]
\[
V \xleftarrow{B} Y\quad A(u, f(u)) \leq B(f(u), y).
\]

Whether we = I might play in A we have something to play in B \( \gamma \to f(\gamma) \) with the property that whatever \( \gamma = II \) might play in B we have something for II to play in A where II does better i.e. we do worse.

Theorem: Category of games.

[N.B. We could have maps with \( \mathbb{R} \)-information but map for now.]

A map \( I = \{(1 \leftarrow 0) \to A \) is an element \( u \in U \) such that \( A(u, n) \geq 0 \) all \( n \in X \).
2. Duality

Given \( A = (U \leftarrow X) \) we have its dual

\[
A^\perp = (X \leftarrow A^* U)
\]

Fact

\[
\begin{array}{c}
A \\ \downarrow \\
\frac{A}{B} \\
\downarrow \\
B^\perp \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\rightarrow \\
A^\perp
\end{array}
\]

A map \( I \rightarrow A^\perp \circ A \rightarrow \perp \) is \( x \in X \) such that \( A(u, x) \leq 0 \) if \( u \in U \).

3. Multiplicative Structure

The (linear) function space \( A \rightarrow B \) should satisfy

\[
\begin{array}{c}
I \\
\downarrow
\end{array} \quad \begin{array}{c}
A \rightarrow B
\end{array} \\
\begin{array}{c}
\frac{A}{B}
\end{array} \quad \begin{array}{c}
A \rightarrow B
\end{array}
\]

So I plays are \( U \rightarrow V \times Y \rightarrow X \)

II plays are \( U \times Y \)

and

\[
A \rightarrow B \left( (\phi, \varphi), (u, y) \right) = B(\phi(u, y)) - A(u, \varphi(y)).
\]

Interpretation of this game.

This gives a closed structure with corresponding multiplicative conjunction \( A \otimes B \) given by

\[
\begin{array}{c}
I \\
\downarrow
\end{array} \quad \begin{array}{c}
U \times V
\end{array} \\
\begin{array}{c}
\frac{U \rightarrow X}{V \rightarrow Y}
\end{array} \quad \begin{array}{c}
U \rightarrow \otimes Y
\end{array}
\]

\[
A \otimes B \left( (u, v), (\phi, \psi) \right) = A(u, \phi(v)) + B(v, \psi(u)).
\]

Interpretation of this game.
3. Additive structure

\begin{align*}
\text{Product} & \quad \begin{array}{c}
\mathbb{I} \times \mathbb{V} \\
\mathbb{I} + \mathbb{Y}
\end{array} \\
\mathbb{I} & \quad x \times y \quad \begin{array}{c}
\mapsto \alpha(u,v) \\
\mapsto \beta(v,y)
\end{array}
\end{align*}

\text{II gets to choose which game to play; I = we have to be prepared for either.}

\text{Sum \quad evident dual: we = I get to choose.}
4. Exponential structure

\[ A \Rightarrow I \Rightarrow U \Rightarrow M(x) \]

We have well-behaved maps

\[ !A \rightarrow I \]

\[ !A \rightarrow !A \otimes !A \]

and internal structure

\[ !A \rightarrow A \]

\[ !A \rightarrow !!A \]

Then we have

**Fact**

Good \( ! (A \times B) \cong !A \otimes !B \) (more or less automatic)

and \( !B \)

This is the Kleisli category for \( !A \) cartesian closed.

Map in the Kleisli \( A \rightarrow B \) is:

\[ A \Rightarrow X \Rightarrow M(x) \Rightarrow U \xrightarrow{f} V \Rightarrow U \times Y \xrightarrow{m} M(x) \]

such that for all \( u, y \)

\[ \sum_{x \in M(u, y)} A(u, x) \leq B(f(u), y) \]
5. **Dialectica Modality**

Gödel's Dialectica Interpretation

**Procechu 1942**

**Dialectica 1958**

\[ A \rightarrow QA \]

\[ \frac{I}{II} \quad U \rightarrow \exists X \quad (u, \varphi) \rightarrow A(u, \varphi(u)) \]

**Comment**

Interpretation of game. I plays but II plays knowing what I plays.

So traditional single value is evident viz:

\[ \max_u \min_X A(u, x) \]

Dually a monad

\[ FA \quad X = \exists U \quad X \quad \psi, \phi \rightarrow A(\psi(u), x) \]

Interpretation of game. II plays but I plays knowing II's play. Value

\[ \min_u \max_X A(u, x) \]

The trivial fact

\[ \max_u \min_X A(u, x) \leq \min_u \max_X A(u, x) \]

follows from

\[ QA \rightarrow A \rightarrow FA \quad or \quad (\rightarrow QA \rightarrow A \rightarrow QA) \]

The proof

\[ U \rightarrow U \rightarrow X \quad \psi \mapsto \text{确真} \]

\[ X = \exists U \rightarrow X \]

\[ \forall u \quad \forall x \quad \exists u \quad \phi \quad \forall x \quad \forall u \quad \exists u \quad \phi \]
Interpretation of Kleisli map

\[
\begin{array}{c}
A \\
\downarrow \\
\downarrow \\
B
\end{array}
\quad \begin{array}{c}
\xrightarrow{u} \\
\xrightarrow{\text{X}} \\
\text{V} \\
\text{Y}
\end{array}
\]

For each \( u \) we might play \( u \in A \), we get \( f(u) \) to play in \( B \) and (dependent on \( u \)) a map from possible plays \( y \) of \( \Pi \) to plays \( F(u,y) \) in \( A \).

Showing we do better in \( B \).

Analogous to problem reduction Bluss [computer reduction (Budiu, Galen, Platen)]
Part 2 Mixed strategies

6. Change of base

Let $D(-)$ be the collection of probability distributions on $-$. Apply $D$ to each term-set.

Get a category enriched in convex spaces.

Multiplicative structure: fine

Additive structure: small problem eg

\[
\begin{array}{c}
W \uparrow \\
\downarrow \\
U \times V \quad x+y
\end{array}
\quad \text{Prob distr on}
\]

\[
W = 7 U \times W \Rightarrow V \times x = 7 Z \times y = 7 Z
\]

\[
\text{Prob distr s on }
\]

\[
(W = 7 U \times x = 7 Z) \quad \& \quad m(W = 7 V \times y = 7 Z)
\]

not the same through there is a retraction.

Exponentials: more complicated (but perhaps we don't care).

Instruct: it's boring;

something else: we are not exploiting the idea of the value of a mixed strategy !!!!
7. Birkeland Category

We have (restricted) \( (\text{Comm}_n, \text{Mnd}_n) \):

\[
\begin{align*}
(U \rightarrow X) & \rightarrow U \mathbin{\text{Ac}} D(X) & (U \mathbin{\text{Ac}} X) & \rightarrow D(U) \mathbin{\text{Ac}} X \\
(U, \mu) & \mapsto \sum_{n} A(n, \mu_1) \mu_{n+1} & (A, \mu) & \mapsto \sum_{n} \lambda(n) A(n, \mu)
\end{align*}
\]

These commute so \( \mathbb{H} \) is a bimodal distribution law + we get the entry

\[
\begin{array}{ccc}
\text{d}(U \mathbin{\text{Ac}} X) & \leq & \text{d}(D(U) \mathbin{\text{Ac}} X) \\
\phi^\downarrow & \uparrow & \text{d}^\uparrow \\
D(\text{d}) & \leq & Y
\end{array}
\]

Good as now we see the mixed strategies we couldn't see before.

\[
\begin{align*}
\sum_{x} A(n, x) \overline{\phi}_{y}(x) & \leq \sum_{y} \phi_{y}(x) A(y, y) \\
\sum_{y} B(v, y) \overline{\phi}_{z}(y) & \leq \sum_{z} \psi_{v}(y) C(w, z)
\end{align*}
\]

+ deduce

\[
\sum_{x} A(n, x) \left( \sum_{y} \overline{\psi}_{z}(y) \overline{\psi}_{y}(y) \right) \leq \sum_{y} \phi_{y}(x) B(v, y) \overline{\phi}_{z}(y)
\]

\[
\leq \sum_{w} \left( \sum_{v} \phi_{v}(w) \overline{\psi}_{w}(w) \right) C(w, z)
\]
Multiplicative problems

Compare $U \times V$  $D(V=x, U=y)$:

\[
\begin{array}{c}
U \\
D(U) \\
V \times U
\end{array}
\]

Problems:

$D(M) \times D(N)$

$\Delta$

$D(M \times N)$

and

Additive problems:

$W$

$D(Z)$

$\uparrow$

$D(W \times V)$

$X + Y$

What to do?

Feels like we have a result up to retracts: so try splitting them?

\[ D \] such that \( D(x) \) consists of formal sums \( \sum \lambda_i x_i \) with \( \lambda_i \geq 0 \) \( \sum \lambda_i = 1 \), i.e., formal convex sums: \( D \) is convex.

\( \text{Distributivity} \) \( \lambda (x + (1 - \lambda)y) \).

The \( D \)-algebra is convex space.

\( D \)-convex \( \iff \text{there is a mounded closed structure}: \text{convex hull} \text{complicated} \)

There are \( R \)-products:

\[ R \]

\[ S \]

The free \( D \)-algebra on \( n \) generators is \( D(n) \).

The \( (n-1) \)-dimensional simplicial set: \( D(0) = \emptyset \)

has dimension \( = \infty \).

Observe \( D(n) \oplus D(m) = D(n+m) \)

\[ a = n-1 \quad b = m-1 \quad a+b+1 = n+m-1 \]

\[ \dim (x \times Y) = \dim x + \dim Y \]

\[ D(n) \otimes D(m) = D(n \times m) \]

(And \( D(1) = 1 \) is the unit for \( \otimes \).

\[ a = n-1 \quad b = m-1 \quad ab + a + b = nm - 1 \]

\[ R \otimes S = \text{convex maps } R \to S \text{ into convex structure} \]

\[ \text{In } D(n) \otimes S = S^n \]

\[ \text{Dim } (D(n) \otimes D(S)) = R(s-1). \quad \text{(Special case } D(0) \text{)} \]
New category of games

**Notation**

$U, X$ the new convex spaces (finite)

**Idea**

Generalised spaces of probabilities / strategies, replacing $D(m), D(m)$ etc.

Remark: $\sigma$-additive: there is no absolute size.

**Games**

$U \times X \xrightarrow{A} \mathbb{R}$  

$(A$ was bilinear in mixed strategies$)$

**Maps**

$f \downarrow \quad f$  

$s.t. 

U \otimes \Gamma \xrightarrow{A} \mathbb{R}$  

for $V \otimes \Gamma \xrightarrow{B}$

**Multiplicative structure**

Eg $A \rightarrow B \quad \left[(U-OV) \times (Y-OX)\right] \otimes (U \otimes V) \rightarrow \mathbb{R}$

**Additive structure**

$A \times B \quad (U \times V) \otimes (X+Y) \rightarrow \mathbb{R}$

(Using $I = 1$ in convex spaces)

**Exponential structure**

In progress: probably restrict convex spaces to those with a matrix command structure.
Following the mathematics gives an extended version of game in which players can be restricted to using strategies constructed in simple ways (without any issue of complexity).

Misleading example

\[
\begin{array}{c}
\Delta \\
\rightarrow \\
\Box
\end{array}
\]

Instead of an arbitrary strategy in \( 4 = 2 \times 2 \) we need to play a product strategy i.e. a pair of strategies on 2 and an 2.

(Generally the geometry is more complicated.)