# A Logical Tour of Judgment Aggregation Theory

Philippe Mongin (CNRS & HEC School of Management)

Lorentz Center
Leiden

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# General Introduction

- Contemporary aggregation theories derive from two remote historical sources, (i) mathematical politics, as in 18th century France (Condorcet), and (ii) utilitarianism, as in 18th and 19th century Britain (Bentham).
- Classical and neo-classical economists have paid much attention to the second current, reformulating utilitarianism and providing alternatives, s.t. 20th century welfare economics, but ignored the first current.
- Arrow (1951) reconciled the two, thus reviving the neglected area of mathematical politics. Henceforth, the formal work of SOCIAL CHOICE THEORY has developed with a dual interpretation.
- The key was to bring to the fore the *preference* concept, which is "represented" by the economic utility function and is "revealed" by choices, hence votes as a particular case. *A brillant generalization, but not yet sufficient*.

- Conceptually, preferences are but a species of *judgments*, and *the aggregation problem arises at this higher level of generality*. Even for the purposes of politics and economics, witness:
- "I prefer a positive inflation rate to a zero or negative one in the euro zone" means "I judge ... to be preferable to ...". A *comparative* and *evaluative* judgment made from a *speficic* (= the preferability) viewpoint. Typically grounded in other judgments of a different logical form: "Zero or negative inflation rate is bad for the economy", "It is conducive to postponement of consumption", etc.
- If the individuals express both their preferences and the underlying judgments, the aggregative theory should take both into account, social choice theory notwithstanding.

- JUDGMENT AGGREGATION THEORY (JAT), as initiated by List & Pettit (2002), makes the next generalization. Its concepts and formalism permit relating individual and collective judgments, regardless of their semantic content.
- Mongin (2012a) argues that "logical aggregation theory" would be an appropriate label because: (i) JAT does not include PROBABILITY AGGREGATION THEORY, which also aggregates judgments; (ii) it connects with a *specifically* logical analysis of judgments.

- This analysis represents someone's *judgment* as the acceptance or rejection of a *proposition*, and then investigates the proposition for itself.
- Common to old (Aristotelian) and new (post-Fregean) logic with more emphasis on the judgment in the first, and more emphasis on the proposition in the second (where "assertion" is Frege's word for approval).
- This tradition is present in Condorcet (1785), who aggregates "opinions" rather than preferences directly.
- After rediscovering Condorcet, Guilbaud (1952) proved a judgment aggregation theorem *before anybody else*; see Eckert & Monjardet (2009) and Mongin (2012b) on this history.

# Aim and plan of the lecture

- We will review **some** of the JAT work having a **logical bias**, and moreover emphasizing **syntax** within logic.
- Logic is taken here in a broad (and technically modest) sense: the so-called *general logic* of Dietrich (2007), which covers (a) the elementary propositional calculus, (b) various propositional modal logics, and (c) the first-order predicate calculus.
- *No application of JAT thus far has needed more than (a), (b), (c)*.
- Some contributors actually claim that logic, in whatever sense, is unnecessary to JAT, and they approach it by purely combinatorial techniques. Thus the present "logical bias" is in fact contentious in the field!

- (1) The DISCURSIVE DILEMMA of Pettit (2002) and List & Pettit (2002) (briefly).
- (2) A FORMAL SET-UP for JAT, with more on the general logic.
- (3) Some JAT AXIOMS with a first theorem accounting for the discursive dilemma (briefly again).

- (4) The CANONICAL THEOREM of JAT (Dokow & Holzman, 2010a, with significant contributions of Nehring & Puppe, 2002, 2010a, and Dietrich & List, 2007a, and relevant earlier results by Guilbaud, 1952, Wilson, 1975, and Fishburn & Rubinstein, 1986).
- (5) Some EARLY RESULTS (irreducible to the canonical theorem) by Pauly & van Hees (2006), Dietrich (2006), and Mongin (2008).
- (6) Recent work by Dietrich & Mongin (2010, with a modal logic application) and by Dokow and Holzman (2010c, following the combinatorial approach).

# (1) The discursive dilemma

In Kornhauser & Sager (1993), three judges must decide a breach-of-contract case against a defendant. A unanimously agreed legal doctrine says that *a compensation is due* ($c$) *iff the contract was broken ($a$) and was valid in the first place ($b$)*.

| | $a$ 'contract broken' | $b$ 'contract valid' | $c$ 'compensation due' | $c \leftrightarrow a \wedge b$ (the legal doctrine) |
|---|---|---|---|---|
| Judge 1 | Y | Y | Y | Y |
| Judge 2 | N | Y | N | Y |
| Judge 3 | Y | N | N | Y |
| Court, premiss-based | Y | Y | Y | Y |
| Court, conclusion-based | | | N | |

# The discursive dilemma (cont.)

- Kornhauser & Sager (1993) emphasize the contradiction betwee two sensible applications of majority voting: the *conclusion-based* way aggregates only the judges' votes on the case, and the *premiss-based* way aggregates only their votes on the issues and then draws the consequences using the legal doctrine.
- Pettit (2002) and List & Pettit (2002) reformulate the problem more simply as the logical contradiction that arise from voting *on all propositions at once*:

$\{a, b, \neg c, a \wedge b \longleftrightarrow c\}$ is a propositional contradiction.

# The discursive dilemma (end)

- Kornhauser & Sager's (1993) analysis of the problem (called the "doctrinal paradox" by them) is valuable and it has led to a subbranch of JAT not pursued here (Bovens & Rabinowicz, 2006, Pigozzi, 2006, Mongin, 2008, Nehring & Puppe, 2008, Dietrich & Mongin, 2010, Hartman, Pigozzi & Sprenger, 2010,....)
- By reformulating the problem, List & Pettit (2002) have given rise to the main branch of JAT, the only one considered here.
- For them, the problem is a "discursive dilemma" because two normative considerations clash with each other: "individual responsiveness" (as captured by majority voting on all propositions) and "collective rationality" (as captured by propositional consistency).

# (2) The formal set-up of JAT

- Here a *logical language* means a non-empty set of logical formulas, $\mathbf{L}$, which is closed under negation, i.e., if $p \in \mathbf{L}$ then $\neg p \in \mathbf{L}$. There may be further Boolean connectives $\wedge$, $\vee$, $\rightarrow$,$\leftrightarrow$, as well as modal, non-Boolean ones, such as $\Box$ for necessity, $\rightsquigarrow$ for conditional implication, etc.
- The logical language may be (1) an object-language of the propositional type (including a *modal propositional language*); (2) a set of designators for relevant formulas of some other object-language. The major application of (2) takes a *first-order predicative language* and considers for $\mathbf{L}$ all of its closed formulas (i.e., those with constants or quantifiers bounding all variables).

- There is a *logic* defined by either an entailment relation $S \vdash p$, defined for all $S \subseteq \mathbf{L}$ and $p \in \mathbf{L}$, or (equivalently) a set $\mathcal{I}$ of subsets of $\mathbf{L}$ representing the logical inconsistencies. Both $\vdash$ and $\mathcal{I}$ can make sense in both interpretations (1) and (2). The GENERAL LOGIC (Dietrich, 2007, slightly improved in Dietrich & Mongin, 2010) states the (weak) axiomatic constraints on either $\vdash$ or $\mathcal{I}$.

# The formal set-up: General Logic

What the GL leaves out are:

- the *non-monotonic logics*, which capture inductive rather than deductive reasoning, and are arguably out of scope here,
- the *paraconsistent logics*, which are deductive, hence within scope, but impossible to handle here.

The GL envisages *compactness* as an optional condition.

Dietrich (2007) has introduced the GL in order to overcome the restriction of the earlier papers to the proposition calculus. The main results here are stated within the general logic. Compactness is needed for one direction of the canonical theorem (necessity).

# The formal set-up: General Logic (cont.)

The conditions on $S \vdash p$ are:

(E1) There is no $p \in \mathbf{L}$ such that $\emptyset \vdash p$ and $\emptyset \vdash \neg p$ (non-triviality).

(E2) For all $p \in \mathbf{L}$, $p \vdash p$ (reflexivity).

(E3) For all $S \subseteq \mathbf{L}$ and all $p, q \in \mathbf{L}$, if $S \nvdash q$, then $S \cup \{p\} \nvdash q$ or $S \cup \{\neg p\} \nvdash q$ (one-step completability).

(E4) For all $S \subseteq S' \subseteq \mathbf{L}$ and all $p \in \mathbf{L}$, if $S \vdash p$, then $S' \vdash p$ (monotonicity).

(E5) For all $S \subseteq \mathbf{L}$ and all $p \in \mathbf{L}$, if $S \vdash p$, there is a finite subset $S_0 \subseteq S$ such that $S_0 \vdash p$ (compactness).

(E6) For all $S \subseteq \mathbf{L}$, if there is a $p \in \mathbf{L}$ such that $S \vdash p$ and $S \vdash \neg p$, then for all $q \in \mathbf{L}$, $S \vdash q$ (non-paraconsistency).

# The formal set-up: General Logic (end)

The following is implied:
(E7) For all $S, T \subseteq \mathbf{L}$ and all $q \in \mathbf{L}$, if $T \vdash q$ and $S \vdash p$ for all $p \in T$, then $S \vdash q$ (transitivity).

A characterization is also available in terms of the set $\mathcal{I}$ of inconsistent sets. The equivalence with (E1)-(E6) is checked by adding obvious connecting rules:

- if $\vdash$ is the primitive, $S \in \mathcal{I}$ iff for all $\psi \in \mathbf{L}$, $S \vdash \psi$;
- if $\mathcal{I}$ is the primitive, $S \vdash \varphi$ iff $S \cup \{\neg\varphi\} \in \mathcal{I}$.

## The formal set-up: the agenda and the judgment sets

The *agenda* is a non-empty set of logical formulas representing the propositions of interest. It is closed by negation, i.e.,

$$X = \{p, \neg p, q, \neg q, ...\} = \{p, q, ...\}^{\pm}$$

with $p, q...$ being unnegated. Contradictions and logical truths are excluded for convenience.

A *judgment set* (JS) is any $B \subseteq X$ representing the propositions accepted by an individual or the collective.

$D$ is the set of all JS that are *consistent* and *complete* (i.e. contain a member of each pair $p, \neg p \in X$), and $D^*$ is the set of all JS that are consistent and *deductively closed* . A weaker rationality notion, but still strong (cf. the "logical omniscience problem" in Kripkean epistemic logic).

*All logical notions here are taken in the General Logic sense.*

# The formal framework: the social judgment function

- A *profile* is a vector $(A_1, ..., A_n)$ of JS, one for each individual $i = 1, ..., n$. Only some work allows for an infinite population $N$ (Dietrich & Mongin, 2007; Herzberg, 2010; Eckert & Herzberg, 2012).
- A *social judgment function* is a mapping $F$ from profiles $(A_1, \ldots, A_n)$ to collective judgment sets

$$F(A_1, \ldots, A_n) = A \subseteq X.$$

- All theorems below make a Universal Domain assumption (but see Dietrich & List, 2010, on restricted domains).
- There will be three cases: (i, for reference) $F : D^n \longrightarrow 2^X$; (ii, main case) $F : D^n \longrightarrow D$; (iii, occasionally) $F : D^n \longrightarrow D^*$.

Example: *Propositionwise majority voting* is a $F$ of type (i) defined by

$$F_{maj}(A_1, ..., A_n) = \{p \in X : |\{i : p \in A_i\}| > n/2\}.$$

# (3) Axioms and a first theorem

The choice of $D$ or $D^*$ covers the "collective rationality" horn of the discursive dilemma. The following covers the "individual responsiveness" horn more generally than propositionwise majority voting.

**Anonymity**. For all $A_1, ..., A_n$ and all permutations $\sigma$ of $N$, $F(A_1, ..., A_n) = F(A_{\sigma 1}, ..., A_{\sigma n})$.

**Non Oligarchy**. There are no oligarchs, i.e., no $M \subseteq N$ s.t. $F(A_1, ..., A_n) = \cap_{i \in M} A_i$ for all $(A_1, ..., A_n)$.

**Non Dictatorship**. There is no dictator, i.e., no $i$ such that $F(A_1, ..., A_n) = A_i$ for all $(A_1, ..., A_n)$.

**Unanimity Preservation.** For all $p \in X$ and all $(A_1, ..., A_n)$, if $p \in A_i$ for all $i$, then $p \in F(A_1, ..., A_n)$.

# Axioms (cont.)

A *Dictatorship* (a $F$ with a dictator) is a $F : D^n \longrightarrow D$.

An *Oligarchy* (a $F$ with oligarchs) is a $F : D^n \longrightarrow D^*$.

Nehring & Puppe (2008) redefine oligarchy to secure a $F : D^n \longrightarrow D$ (a default rule applies when the $i \in M$ disagree).

**Anonymity $\Longrightarrow$Non-Oligarchy (except for $M = N$), Non-Oligarchy $\Longrightarrow$Non-Dictatorship**

**Unanimity Preservation**, like a Pareto condition, is compatible with each of the three and its negate.

All the above are satisfied by $F_{maj}$.

# Axioms (cont.)

Generalizing $F_{maj}$ in still another direction gives:

**Systematicity.** For all $p, p' \in X$ and all $(A_1, ..., A_n), (A'_1, ..., A'_n)$, if $p \in A_i \Leftrightarrow p' \in A'_i$ for all $i$, then

$$p \in F(A_1, ..., A_n) \Leftrightarrow p' \in F(A'_1, ..., A'_n).$$

**Independence.** For all $p \in X$ and all $(A_1, ..., A_n)$, $(A'_1, ..., A'_n)$, if $p \in A_i \Leftrightarrow p \in A'_i$ for all $i$, then

$$p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A'_1, ..., A'_n).$$

# Axioms (end)

Clearly, **Systematicity** $\Rightarrow$ **Independence.**
Independence says that aggregation proceeds *propositionwise*.
Systematicity adds that the content of the formula does not matter, *only the pattern of acceptance* does.
As a normative principle, systematicity is even more dubious than Independence, but it is worth considering because most voting rules imply it.

The "discursive dilemma" is in fact a *trilemma*: Systematicity or Independence is the third horn. Thm 1 brings this out.

# A first theorem

**Theorem 1.** *If the agenda X is weakly connected (see below), no $F : D^n \to D$ is systematic, unanimity-preserving and non-dictatorial. Otherwise, there exist $F$ with these properties.*

A weak form of the canonical theorem to come. An even weaker form, with Anonymity instead of Non-Dictatorship, appeared as the first result in JAT (List & Pettit, 2002) .

An agenda $X$ is *WEAKLY CONNECTED* if:

**(a)** There is a minimally inconsistent $X' \subseteq X$ of size at least 3 ('non-simplicity').

**(b)** There are a minimally inconsistent $Y \subseteq X$ and a $Z \subseteq Y$ of even size s.t. $Y_{\neg Z} = (Y \backslash Z) \cup \{\neg z : z \in Z\}$ is consistent ('even-number-negatability' or 'non-affineness').

"Of size 2" can actually replace "of even size" ($Y$ becomes consistent after two formulas are negated).

# A first theorem (end)

**Example 1:** The language $\mathbf{L}$ and the entailment relation $\vdash$ are propositional.

The agenda of the discursive dilemma $X = \{a, b, c, c \leftrightarrow a \wedge b\}^{\pm}$ is WEAKLY CONNECTED.

*Proof*:

$X$ **satisfies (a):** $X \supset X' = \{a, b, \neg c, c \leftrightarrow a \wedge b\}$ is minimally inconsistent.

$X$ **satisfies (b):** $X \supset X' \supset Z = \{a, b\}$, and $X'_{\neg Z} = \{\neg a, \neg b, \neg c, c \leftrightarrow a \wedge b\}$ is consistent.

Thm 1 accounts for (and generalizes) the discursive dilemma by showing what properties of $F_{maj}$ and what properties of $\overline{X}$ are responsible for the contradiction.

# (4) The canonical theorem of JAT

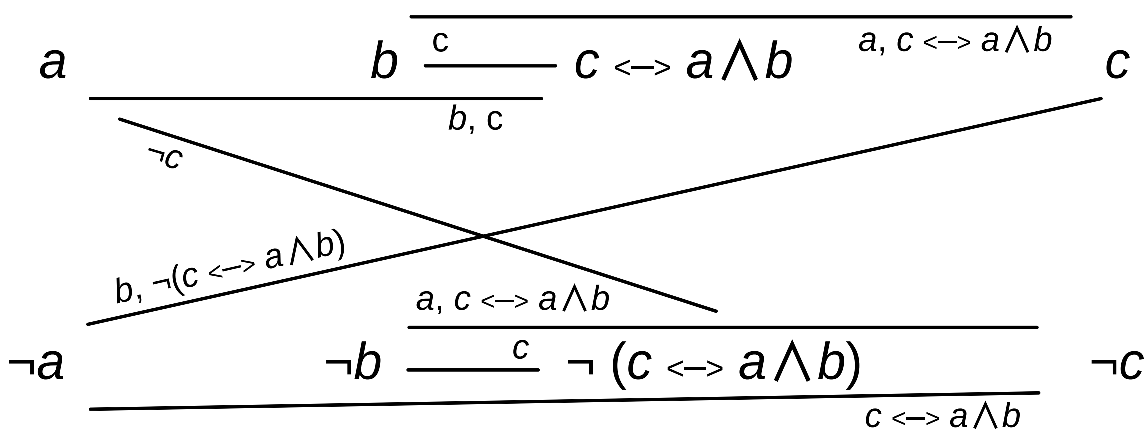
**Theorem 2.** *If the agenda X is strongly connected (see below), no* $F : D^n \longrightarrow D$ *is independent, unanimity-preserving and non-dictatorial. Otherwise, there exist F with these properties.*

THE CANONICAL THEOREM OF CURRENT JAT. Proved in Dokow & Holzman (2010a), but others have contributed towards it. Thus, for a weaker form of necessity and sufficiency, Nehring & Puppe (2002, 2010a), after Fishburn & Rubinstein (1986) and Wilson (1975); for sufficiency, Dietrich & List (2007a); for a weaker form of sufficiency, Guilbaud (1952).

An agenda $X$ is *STRONGLY CONNECTED* if it satisfies **(b)** and a new condition **(c)**.
First, say that $p, q \in X$, $p$ *conditionally entails* $q$ (‘$p \vdash^* q$’) if $\{p\} \cup Y \vdash q$ for some $Y \subseteq X$ that is consistent with $p$ and with $\neg q$.
Second, define **(c)**: for all $p, q \in X$, there are $p_1, ..., p_k \in X$ such that $p = p_1 \vdash^* p_2 \vdash^* ... \vdash^* p_k = q$.
($p$ and $q$ are 'path-connected', and $X$ is 'totally blocked').

# Example 1: the discursive dilemma agenda

**Example 1**: the agenda $X = \{a, b, c, c \leftrightarrow a \wedge b\}^{\pm}$ is STRONGLY CONNECTED. *Proof:*




paths between any two propositions !

(not all conditional entailments shown)

agenda totally blocked

# A monotonic variant of the canonical theorem

Compare Thm 1 and Thm 2: **(b)** is common and **(a)** vanishes because **(c)**$\Longrightarrow$**(a)**.

The difference between **(c)** and **(a)** matches the difference between Systematicity and Independence, an example of a *metatheoretical equivalence between agenda conditions and axiomatic conditions*. Social choice theory has nothing comparable to offer.

The present metaequivalence: **The monotonicity axiom matches agenda condition (b).**

**Monotonicity.** For all $p \in X$ and all $(A_1, ..., A_n)$, $(A'_1, ..., A'_n)$, if $p \in A_i \Rightarrow p \in A'_i$ for all $i$, and $p \notin A_i, p \in A'_i$ for at least one $i$, then

$$p \in F(A_1, ..., A_n) \Rightarrow p \in F(A'_1, ..., A'_n).$$

**Theorem 3.** *If the agenda X satisfies (c), no $F : D^n \longrightarrow D$ is independent, monotonic, unanimity-preserving and non-dictatorial. Otherwise, there exist F with these properties.*

# A monotonic variant (cont.)

Thm 3 is Nehring and Puppe's version of the canonical theorem (generally, they assume Monotonicity as a basic axiom along with Independence and Unanimity Preservation).

Since $F_{maj}$ satisfies Monotonicity and Systematicity, the following holds (also directly provable):

**Proposition 1.** *Propositionwise majority voting is $D^n \to D$ if and only if (a) does not hold*.

Much more can be said on voting rules In JAT; see Nehring & Puppe (2002, 2008, 2010b), Dietrich & List (2007b), Pivato & Nehring (2010), Nehring, Puppe & Pivato (2012).

# Example 2: the strict preference agenda

**Example 2**: the strict preference agenda is STRONGLY CONNECTED (Dietrich & List, 2007a).

*Logical preliminaries (simplified form)*:

- Given a finite set $X = \{x, y, ...\}$ with $|X \geq \mathbf{3}|$, define $\mathbf{L}$ to be the smallest set containing formulas $xPy$ for all $x, y \in X$ , $x \neq y$, and closed for the Boolean operations. $X$ represents the *set of options* and $P$ a *preference relation*.
- Define $Z$ to be the (finite) set of formulas stating that the preference relation is asymmetric, transitive and complete.
- Define the entailment relation $S \vdash p$, for all $S \subseteq \mathbf{L}$ and $p \in \mathbf{L}$, by $S \cup Z$ *entails* $p$ *in the propositional sense*. The General Logic applies to $\vdash$.
- Define the *strict preference agenda* to be the (finite) set $X \subset \mathbf{L}$ of formulas $xPy$ and $\neg xPy$, for all $x, y \in X$ , $x \neq y$.

**Example 2 (cont.)**

*Proof of strong connectedness:*

- $X$ satisfies (b): for pairwise distinct $x, y, z \in X$ , $Y = \{xPy, yPz, zPx\}$ is minimally inconsistent, and negating any two formulas leads to a consistent set.
- $X$ satisfies (c): by asymmetry and completeness, negated formulas $\neg xPy$ are equivalent to nonnegated formulas $yPx$, so it is enough to check by transitivity that any two $xPy$ and $x'Py'$ satisfy the chain condition (c).

**Corollary 1 of Thm 2:** a version of Arrow's theorem, as restricted to complete strict preference.

Arrow's theorem is *more powerful*, since it allows for indifferences, and only a more complicated JAT argument can recover it (see Dokow & Holzman, 2010b, Dietrich, forthcoming).

*Proof of Corollary 1* (trivial, but instructive):

$\Pi_{strict}$ is the set of complete strict preferences over $X$.

To be shown that $G : (\pi_1, ..., \pi_n) \longmapsto \pi$, with $\pi_1, ..., \pi_n, \pi \in \Pi_{strict}$, cannot satisfy Independence of irrelevant alternatives, Weak Pareto and Non Dictatorship together.

Define $D$, the set of complete and consistent JS, from the above.

To each $\pi^* \in \Pi_{strict}$ associate the JS $A^* \in D$ s.t. for all $x, y \in X$,

$$x\pi^* y \Leftrightarrow xPy \in A^*$$

and define $F_G : D^n \to D$ accordingly.

Check that if $G$ satisfies IIA, WP and ND, $F_G$ satisfies Independence, Unanimity Preservation and Nondictatorship, and conclude.

(*Warning: The step from IIA to Independence will fail in some more complicated examples.*)

# (Counter)example 3: the equivalence relation agenda

**Example 3:** the equivalence relation agenda is NOT STRONGLY CONNECTED

*Logical preliminaries*:

- Given a finite $X = \{x, y, ...\}$ with $|X \geq \mathbf{3}|$, define $\mathbf{L}$ to be the smallest set containing formulas $x \sim y$ for all $x, y \in X$ , $x \nsim y$ (denoting $\neg x \sim y$) and closed for the Boolean operations.
- Define $Z$ to be the (finite) set of formulas stating that the relation represented by $\sim$ is an equivalence relation.
- Define $S \vdash p$, for all $S \subseteq \mathbf{L}$ and $p \in \mathbf{L}$, by $S \cup Z$ *entails* $p$ *in the propositional sense*.
- Define $X \subset \mathbf{L}$ to be the set of formulas $x \sim y$ , $x \nsim y$ for all $x, y \in X$ , $x \neq y$.

**Example 3 (cont.)**: Strong connectedness does not hold (*see picture*).

Consistently, a theorem by Fishburn & Rubinstein (1986) on aggregating equivalence relations delivers *an oligarchy, not a dictatorship*.

Define $\mathcal{E}$ to be the set of all equivalence relations on $X$, and $G : (E_1, ..., E_n) \longmapsto E$, where $E_1, ..., E_n, E \in \mathcal{E}$. Suppose that $G$ satisfies: for all $x, y \in X$, for all $(E_1, ..., E_n), (E'_1, ..., E'_n) \in \mathcal{E}^n$,

(C1) If $xE_iy$ for all $i$, then $xEy$; if $not\ xE_iy$ for all $i$, then $not$ $xEy$.

(C2) If $xE_iy \Longleftrightarrow xE'_iy$ for all $i$, then $xEy \Longleftrightarrow xE'y$.

Then there is a non-empty $M \subseteq \{1, ..., n\}$ s.t. for all $x, y \in X$,

$$xEy \Longleftrightarrow xE_iy \text{ for all } i \in M.$$

(F. & R. think of $i$ as an *attribute* and $G$ as aggregating attribut-ewise classifications. In a social context, the oligarchic conclusion is more problematic.)

**Example 3 (end)**: To restore strong connectedness,
(i) impose exactly 2 nonempty equivalence classes *(see picture)*
(ii) constrain the number of options in the equivalence classes

Solution (i) leads to:
**Corollary 2 to Thm 2:** suppose $\mathcal{E}$ is the set equivalence relations with exactly 2 nonempty equivalence classes; then the set $M$ in the theorem above reduces to a singleton, i.e., $G$ is a dictatorship. This is illustrated by Kasher & Rubinstein (1997) in a social context. The members of society must divide themselves into two groups, the $J$ and non-$J$ members, so $X = \{1, ..., n\}$. Then, if $n \geq 3$, $G$ satisfying (C1) and (C2) is a dictatorship.

# The canonical theorem (end)

*AN EXCURSUS IN OLIGARCHIES*

Variant results exist with $D^*$ instead of $D$, either everywhere or just at the collective level, and they deliver oligarchies (however, Fishburn & Rubinstein's theorem is not derivable from them).

**Theorem 4.** *If the agenda X is strongly connected, no $F : D^n \longrightarrow D^*$ is independent, unanimity-preserving and non-oligarchic. Otherwise, there exist F with these properties.*

Thus, incomplete, but still coherent social judgment sets only weaken Dictatorship into Oligarchy. If one dislikes Oligarchy, one is faced with the trilemma again: "rationality", "individual responsiveness", and Independence (the initially neglected third horn).

# (5) Early theorems in JAT

Thm 2 is canonical because it sets *a standard* for impossibility theorems in JAT (i.e., to fully characterize the impossibility agendas $X$ corresponding to a given list of axioms on $F$), not because it unifies the existing theory (its given list is specific).

The three theorems below came early and do not yet comply with the standard, stating only *sufficient* conditions for an impossibility agenda. They have distinctive lists of axioms and are *not* special cases of Thm 2.

They are stated in the *propositional calculus*. $\mathbf{L}$ is constructed from a set $\mathbf{PV}$ of propositional variables by closure under all Boolean connectives or some sufficient set of them, and $\vdash$ is the entailment of the propositional calculus.

A *literal* is some $a \in \mathbf{PV}$ or its negation.The agenda $X$ is sometimes required to be *closed by propositional variables*: if $p \in X$, then for all $a \in \mathbf{PV}$ occurring in $p$, $a \in X$. We assume $|\mathbf{PV}| \geq 2$.

# Early theorems (cont.)

**Theorem 5.** *Take* $\mathbf{L}$ *constructed from* $\mathbf{PV}$ *and the sufficient set of connectives* $\{\neg, \wedge\}$*. Suppose* $X$ *is closed by propositional variables, contains at least two of them, and is such that if two literals* $a', b' \in X$*, then* $a' \wedge b' \in X$*. Then, no* $F : D^n \longrightarrow D$ *is independent, non-constant and non-dictatorial.*

Here, Pauly & van Hees (2006) for the first time use Independence instead of Systematicity (unlike List & Pettit, 2002) and are able to replace Unanimity Preservation by the non-constancy of $F$.

# Early theorems (cont.)

**Theorem 6.** *Take* $\mathbf{L}$ *constructed from a finite* $\mathbf{PV}$ *and the sufficient set of connectives* $\{\neg, \wedge\}$. *Suppose* $X$ *includes the set* $AT$ *of the atoms of* $\mathbf{L}$. *Then, no* $F : D^n \to D$ *is independent on* $AT$, *non-constant and non-dictatorial.*

Here, Dietrich (2006) also replaces Unanimity Preservation by non-constancy. Independence is restricted to the atomic components of $X$ (or applies globally to JS since an atom is like a complete and consistent JS).

# Early theorems (end)

**Theorem 7.** *Take* $\mathbf{L}$ *constructed from* $\mathbf{PV}$ *and any set of connectives. Suppose* $X$ *is closed by propositional variables, contains at least two of them, and satisfies strong connectedness for propositional variables (a variant of STRONG CONNECTEDNESS not stated here). Then, no* $F : D^n \rightarrow D$ *is independent on* $\mathbf{PV}$*, unanimity-preserving, and non-dictatorial.*

Here, Mongin (2008) restricts Independence so drastically that Unanimity Preservation is required again for the impossibility. Arguably, $\mathbf{PV}$ *is the largest set on which Independence should apply*. If Unanimity Preservation is also restricted to $\mathbf{PV}$, a well-behaved solution ensues (first collectively decide on $\mathbf{PV}$, and then draw the implications for $X \setminus \mathbf{PV}$).

Thm 7 is extended by Dietrich & Mongin (2010) to the General Logic and the canonical form.

# Early theorems (technical)

*Strong connectednees for propositional variables* holds if

(b$_{\mathbf{PV}}$) There are a minimally inconsistent $Y \subseteq X$ and $a, b \in Y \cap \mathbf{PV}$ s.t. $Y_{\neg\{a,b\}}$ is consistent.

(c$_{\mathbf{PV}}$) For all $a, b \in \mathbf{PV}$, there are $a_1, ..., a_m \in \mathbf{PV}$ s.t. $a = a_1 \vdash^* a_2 \vdash^* ... \vdash^* a_m = b$

(As usual, $a \vdash^* b$ means $Y \cup \{a\} \vdash^* q$ for some $Y \subset X$**,** so formulas in $X \smallsetminus \mathbf{PV}$ may be involved.)

The generalized version applies the same conditions to any $P \subseteq \mathbf{L}$ instead of $PV$, just adding:

(d$_P$) for all JS $B \in D$,

$$B = \{p \in X \mid B \cap P \vdash p\}\,.$$

# (6) Some recent work and problems

## 6.1 A modal logic application

As in Dietrich and Mongin (2010), we propose to **dissolve the discursive dilemma by replacing the Boolean implication $\rightarrow$ by a non-Boolean implication $\hookrightarrow$ of conditional logic**. Strong connectedness does not hold anymore, so by the necessity part of the canonical thm, there exist nondictatorial social judgment functions satisfying Independence and Unanimity Preservation.

This strategy is potentially general (the move to modalities often destroys either **(b)** or **(c)**).

Easily justified in legal examples, where doctrinal implications cannot be plausibly captured by material implications.

# A modal logic application (cont.)

Take the agenda $\overline{X}$ and list all minimally inconsistent subsets $Y \subseteq \overline{X}$ containing $q = c \longleftrightarrow a \wedge b$ or $\neg q$.

$$Y_1 = \{\neg a, c, c \longleftrightarrow a \wedge b\}, Y_2 = \{\neg b, c, c \longleftrightarrow a \wedge b\},$$
$$Y_3 = \{a, b, \neg c, c \longleftrightarrow a \wedge b\}, Y_4 = \{c \longleftrightarrow a \wedge b, \neg(c \longleftrightarrow a \wedge b)\},$$
$$Y_5 = \{a, b, c, \neg(c \longleftrightarrow a \wedge b)\},$$
$$Y_6 = \{\neg a, \neg c, \neg(c \longleftrightarrow a \wedge b)\}, Y_7 = \{\neg b, \neg c, \neg(c \longleftrightarrow a \wedge b)\}.$$

$Y_3 = \{v, b, \neg d, c \longleftrightarrow a \wedge b\}$ corresponds to the initial example, and the other sets to possible variants.

**With $c \hookleftarrow\!\hookrightarrow a \wedge b$ instead of $c \longleftrightarrow a \wedge b$, the list reduces to** $Y_1, Y_2, Y_3, Y_4$, $Y_5$. Some conditional theorists even exclude $Y_5$.

*Another application of the General Logic which departs from the elementary propositional calculus.*

# A modal logic application (cont.)

The court agenda $\overline{X}' = \{a, b, c, q'\}^{\pm}$ with $q' = c \hookleftarrow\!\hookrightarrow a \wedge b$

violates the conditions

*(see picture)*

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No path from negative to positive formulas because $Y_6, Y_7$ are now consistent. (If $Y_5$ were also consistent, the picture would not change.) **By Canonical Thm possibilities exist**.

# 6.2 Nonbinary judgments

JAT has derived relatively few social choice theorems (either new or existing, see list in Mongin, 2012a).

Why this disappointing record?

A reason is that the core theory - presented here - is *binary*, in the sense of

- (1) considering only binary judgements: approval and disapproval are the only possible attitudes ($p \in B$ or $p \notin B$ in syntactical format);
- (2) applying an Independence condition which is only suited for binary judgments.

To illustrate the less obvious (2), consider Arrow's theorem in actual form: given two options, there are *three* attitudes - two strict preferences and indifference - and this makes *IIA discrepant from Independence*, blocking any easy derivation of the thm.

# Nonbinary judgments

Part of JAT has overcome binariness, but mostly in a *combinatorial* framework (Dokow & Holzman, 2010c).

The primitives are now:

$J = \{1, ..., m\}$, the set of *issues*, $m \geq 2$.

$P = \{1, ..., p\}$, the set of *positions*, with $p \geq 3$ (the nonbinary case)

$N = \{1, ..., n\}$, the set of *individuals*, $n \geq 1$.

An *evaluation* is a vector $\mathbf{x} = (x_1, ..., x_m) \in P^m$. The *set of feasible evaluations* is $X \subseteq P^m$ (typically, $X \subsetneq P^m$). We may assume that $X_j$, the $j$-projection of $X$, is $P$ for all $j$. An *aggregator* is some $F : X^n \rightarrow X$,

$$((x_1^1, ..., x_m^1), ..., (x_1^n, ..., x_m^n)) \longmapsto (x_1, ..., x_m)$$

# Nonbinary (cont.)

$F$ satisfies *Independence* (*I*) if for all $(\mathbf{x}^1, ..., \mathbf{x}^n)$, $(\mathbf{y}^1, ..., \mathbf{y}^n) \in X^n$, and for all $j \in J$,

$$x_j^i = y_j^i \text{ for all } i \in N \ \Rightarrow x_j = y_j \ .$$

$F$ satisfies *Unanimity* (*U*) if for all $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$, for all $j \in J$, for all $u \in P$,

$$x_j^i = u \text{ for all } i \in N \ \Rightarrow x_j = u.$$

$F$ satisfies *Supportiveness* (*S*) if for all $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$, and for all $j \in J$, for all $u \in P$,

$$x_j = u \ \Rightarrow \ \exists i : x_j^i = u.$$

$F$ satisfies *Dictatorship* (*D*) if there is $i \in N$ such that for all $(\mathbf{x}^1, ..., \mathbf{x}^n) \in X^n$ and for all $j \in J$,

$$x_j^i = x_j.$$

By Dokow and Holzman (2010c):

**Theorem 8:** If $X$ is multiply constrained and totally blocked, every $F$ satisfying *I* and *S* satisfies *D*.

# Nonbinary (end)

The (difficult) conditions "multiply constrained" and "totally blocked" generalize conditions **(b)** and **(c)** of the canonical impossibility theorem to the nonbinary case .

Maniquet & Mongin (2014) apply Thm 8 to generalize Kasher and Rubinstein's (1997) dictatorship result on aggregating binary classifications:
**Proposition 2:** $p \geq 3$ classes, $m \geq p$ objects, and each individual and the collective must leave no class empty (an "ontoness" domain). Then, every $F$ satisfying *I* and *U* satisfies *D*.
*An application where logic is prominently absent.* Should be connected with related JAT work with logical flavour (Duddy & Piggins, 2013, Dietrich, forthcoming) but this is not yet done.

# Two words of conclusion

- There are conceptual reasons for preferring the *logical* (and even *syntactical*) form of judgment aggregation to the combinatorial form just illustrated. This makes it possible to distinguish the contraints due to the connections between the propositions (as captured by the inference relation) and those due to the nature of the agenda. Generally, this permits a relatively precise rendering of the activity of judgment making (along classic philosophical lines).
- It is not clear whether very advanced logical tools are needed *at this point*, since much remains to be done either by directly applying familiar nonclassical logics (e.g., conditional logics) or by connecting the combinatorial work with such logics (e.g., multivalued propositional logic to restate the nonbinary work).