BITOPOLOGICAL DUALITY FOR DISTRIBUTIVE LATTICES AND HEYTING ALGEBRAS

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Abstract. We introduce pairwise Stone spaces as a natural bitopological generalization of Stone spaces—the duals of Boolean algebras—and show that they are exactly the bitopological duals of bounded distributive lattices. The category $\mathbf{PStone}$ of pairwise Stone spaces is isomorphic to the category $\mathbf{Spec}$ of spectral spaces and to the category $\mathbf{Pries}$ of Priestley spaces. In fact, the isomorphism of $\mathbf{Spec}$ and $\mathbf{Pries}$ is most naturally seen through $\mathbf{PStone}$ by first establishing that $\mathbf{Pries}$ is isomorphic to $\mathbf{PStone}$, and then showing that $\mathbf{PStone}$ is isomorphic to $\mathbf{Spec}$. We provide the bitopological and spectral descriptions of many algebraic concepts important for the study of distributive lattices. We also give new bitopological and spectral dualities for Heyting algebras, thus providing two new alternatives to Esakia’s duality.

1. Introduction

It is widely considered that the beginning of duality theory was Stone’s groundbreaking work in the mid 30s on the dual equivalence of the category $\mathbf{Bool}$ of Boolean algebras and Boolean algebra homomorphism and the category $\mathbf{Stone}$ of compact Hausdorff zero-dimensional spaces, which became known as Stone spaces, and continuous functions. In 1937 Stone [33] extended this to the dual equivalence of the category $\mathbf{DLat}$ of bounded distributive lattices and bounded lattice homomorphisms and the category $\mathbf{Spec}$ of what later became known as spectral spaces and spectral maps. Spectral spaces provide a generalization of Stone spaces. Unlike Stone spaces, spectral spaces are not Hausdorff (not even $T_1$)\(^1\), and as a result, are more difficult to work with. In 1970 Priestley [25] described another dual category of $\mathbf{DLat}$ by means of special ordered Stone spaces, which became known as Priestley spaces, thus establishing that $\mathbf{DLat}$ is also dually equivalent to the category $\mathbf{Pries}$ of Priestley spaces and continuous order-preserving maps. Since $\mathbf{DLat}$ is dually equivalent to both $\mathbf{Spec}$ and $\mathbf{Pries}$, it follows that the categories $\mathbf{Spec}$ and $\mathbf{Pries}$ are equivalent. In fact, more is true: as shown by Cornish [6] (see also Fleisher [11]), $\mathbf{Spec}$ is actually isomorphic to $\mathbf{Pries}$.

Spectral spaces are more natural to work with from the point of view of pointfree topology, as demonstrated by Johnstone [17]. In addition, spectral spaces only have a topological structure, while Priestley spaces also have an order structure on top of topology, thus their signature is more complicated than that of spectral spaces. However, Priestley spaces arise more naturally in relation with logics, as Priestley spaces incorporate the now widely used Kripke semantics in them. As a result, Priestley’s duality became rather popular among logicians, and most dualities for distributive lattices with operators have been performed in

\(^{1}\text{In fact, a spectral space } X \text{ is a Stone space iff } X \text{ is } T_1.\)
terms of Priestley spaces. Here we only mention Esakia’s duality for Heyting algebras [8], which is a restricted version of Priestley’s duality.\(^2\)

Another way to represent distributive lattices is by means of bitopological spaces, as demonstrated by Jung and Moshier [20]. In fact, bitopological spaces provide a natural medium in establishing the isomorphism between Priestley spaces and Spectra: with each Priestley space \((X, \tau, \leq)\), there are two natural topologies associated with it; the upper topology \(\tau_1\) consisting of open upsets of \((X, \tau, \leq)\), and the lower topology \(\tau_2\) consisting of open downsets of \((X, \tau, \leq)\). Consequently, \((X, \tau_1, \tau_2)\) is a bitopological space. Moreover, both topologies \(\tau_1\) and \(\tau_2\) are spectral topologies, the Priestley topology \(\tau\) is in fact the join of \(\tau_1\) and \(\tau_2\), and the spectral space associated with \((X, \tau, \leq)\) is obtained from \((X, \tau_1, \tau_2)\) by simply forgetting \(\tau_2\).

In this paper we provide an explicit axiomatization of the class of bitopological spaces obtained this way. We call these spaces *pairwise Stone spaces*. On the one hand, pairwise Stone spaces provide a natural generalization of Stone spaces as each of the three conditions defining a Stone space naturally generalizes to the bitopological setting: compact becomes pairwise compact, Hausdorff – pairwise Hausdorff, and zero-dimensional – pairwise zero-dimensional. On the other hand, pairwise Stone spaces provide a natural medium in moving from Priestley spaces to spectral spaces and backwards, thus Cornish’s isomorphism of *Priestley* and Spectra can be established more naturally by first showing that *Priestley* is isomorphic to the category *PStone* of pairwise Stone spaces and bicontinuous maps, and then showing that *PStone* is isomorphic to Spectra. Thirdly, the signature of pairwise Stone spaces naturally carries the symmetry present in Priestley spaces (and distributive lattices), but hidden in spectral spaces. Moreover, the proof that *DLat* is dually equivalent to *PStone* is simpler than the existing proofs of the dual equivalence of *DLat* with Spectra and *Priestley*. Lastly, the isomorphism of *Priestley*, *PStone*, and Spectra fits nicely in a more general isomorphism of the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces described in [14, Ch. VI-6] (see also [30] and [24]).

The dualities described above have many applications in logic and computer science. In fact, the basic idea underlying completeness results of (propositional) logics is based on duality theory as the canonical model of a propositional logic is the dual of the Lindenbaum-Tarski algebra of the logic. Duality theory also provides a framework for understanding the relationship between denotational semantics of programs and program logics. In particular, as was shown by Abramsky [1], the denotational semantics and the corresponding program logic are duals of each other. For a recent application of these ideas to the \(\pi\)-calculus see [4]. For an application of duality theory to regular languages we refer to Gehrke et al. [12]. For a variety of applications of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces in probabilistic systems, we refer to the work of Jung, Moshier, and their collaborators [18, 19, 3, 20]. Here we only mention that there is a dual equivalence between these categories and the category of proximity lattices [32, 21], which are a generalization of distributive lattices, thus providing an interesting generalization of the duality for distributive lattices. We view our pairwise Stone spaces as a particular case of pairwise compact pairwise regular bitopological spaces, and our isomorphism of the categories of Priestley spaces, pairwise Stone spaces, and spectral spaces as a particular case

\(^2\)We note that Esakia’s work was independent of Priestley’s; a proof that Esakia spaces are Priestley spaces can be found in [10, p. 62].
of the isomorphism of the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces.

One of the advantages of Priestley’s duality is that many algebraic concepts important for the study of distributive lattices can be easily described by means of Priestley spaces. In addition, we show that they have a natural dual description by means of pairwise Stone spaces. We also give their dual description by means of spectral spaces, which at times is less transparent than the order topological and bitopological descriptions.

Finally, we introduce the subcategories of \( \text{PStone} \) and \( \text{Spec} \), which are isomorphic to the category \( \text{Esa} \) of Esakia spaces and dually equivalent to the category \( \text{Heyt} \) of Heyting algebras. This provides an alternative to Esakia’s duality in the setting of bitopological spaces and spectral spaces.

The paper is organized as follows. In Section 2 we recall some basic facts about bitopological spaces, introduce pairwise Stone spaces, and study their basic properties. In Section 3 we prove that the category \( \text{PStone} \) of pairwise Stone spaces is isomorphic to the category \( \text{Pries} \) of Priestley spaces. In Section 4 we prove that \( \text{PStone} \) is isomorphic to the category \( \text{Spec} \) of spectral spaces, thus establishing that all three categories are isomorphic to each other. In Section 5 we give a direct proof that the category \( \text{DLat} \) of distributive lattices is dually equivalent to \( \text{PStone} \), thus providing an alternative of Stone’s and Priestley’s dualities. In Section 6 we give the dual description of many algebraic concepts important for the study of distributive lattices by means of Priestley spaces, pairwise Stone spaces, and spectral spaces. In particular, we give the dual description of filters, prime filters, maximal filters, ideals, prime ideals, maximal ideals, homomorphic images, sublattices, complete lattices, McNeille completions, and canonical completions. At the end of the section we list all the obtained results in one table, which can be viewed as a dictionary of duality theory for distributive lattices, complementing the dictionary given in [27]. Finally, in Section 7 we develop new bitopological and spectral dualities for Heyting algebras, thus providing an alternative to Esakia’s duality, and give a table similar to the one given at the end of Section 6, which can be viewed as a dictionary of duality theory for Heyting algebras.

2. Pairwise Stone spaces

We recall that a bitopological space is a triple \((X, \tau_1, \tau_2)\), where \(X\) is a nonempty set and \(\tau_1\) and \(\tau_2\) are two topologies on \(X\). Ever since Kelly [22] introduced them, bitopological spaces have been subject of intensive investigation of many topologists. In particular, there has been a lot of research on the “correct” generalization of the basic topological properties to the bitopological setting. A large number of results obtained in this direction is collected in a recent monograph [7]. For our purposes it is important to find the right generalization of the concept of a Stone space. Therefore, we are interested in the bitopological versions of compactness, Hausdorffness, and zero-dimensionality.

There are several ways to generalize a topological property to the bitopological setting. Let \((X, \tau_1, \tau_2)\) be a bitopological space and let \(\tau = \tau_1 \lor \tau_2\). For a topological property \(P\), we say that \((X, \tau_1, \tau_2)\) is \(\text{bi}-P\) if both \((X, \tau_1)\) and \((X, \tau_2)\) are \(P\), and we say that \((X, \tau_1, \tau_2)\) is \(\text{join} P\) if \((X, \tau)\) is \(P\). For example, \((X, \tau_1, \tau_2)\) is \(\text{bi}-T_0\), \(\text{bi}-T_1\), or \(\text{bi}-T_2\) if both \((X, \tau_1)\) and \((X, \tau_2)\) are \(T_0\), \(T_1\), or \(T_2\), respectively; and \((X, \tau_1, \tau_2)\) is \(\text{join} T_0\), \(\text{join} T_1\), or \(\text{join} T_2\) if \((X, \tau)\) is \(T_0\), \(T_1\), or \(T_2\), respectively. However, for our purposes, neither bi-Stone nor join Stone turns out to be the right generalization of the concept of a Stone space to the bitopological setting.

**Definition 2.1.** Let \((X, \tau_1, \tau_2)\) be a bitopological space.
Remark 2.2. We have chosen [29] as our primary source of reference, although the concepts of a pairwise $T_0$ space and a pairwise $T_1$ space have appeared earlier in the literature.

Remark 2.3. It would be more in the vein of Definition 2.1(1) and 2.1(2) if we defined a pairwise $T_2$ space as a bitopological space satisfying the following condition: For any two distinct points $x, y \in X$ there exist disjoint $U, V \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$. Obviously if $(X, \tau_1, \tau_2)$ is pairwise $T_2$, then it satisfies the condition above, but the converse is not true in general. Nevertheless, we will show below that in the realm of pairwise zero-dimensional spaces the two conditions are equivalent.

For a bitopological space $(X, \tau_1, \tau_2)$, let $\delta_1$ denote the collection of closed subsets of $(X, \tau_1)$ and $\delta_2$ denote the collection of closed subsets of $(X, \tau_2)$. The next definition generalizes the notion of zero-dimensionality to bitopological spaces.

Definition 2.4. [28, p. 127] We call a bitopological space $(X, \tau_1, \tau_2)$ pairwise zero-dimensional if opens in $(X, \tau_1)$ closed in $(X, \tau_2)$ form a basis for $(X, \tau_1)$ and opens in $(X, \tau_2)$ closed in $(X, \tau_1)$ form a basis for $(X, \tau_2)$; that is, $\beta_1 = \tau_1 \cap \delta_2$ is a basis for $\tau_1$ and $\beta_2 = \tau_2 \cap \delta_1$ is a basis for $\tau_2$.

We point out that if $(X, \tau_1, \tau_2)$ is pairwise zero-dimensional, then $\beta_2 = \{U^c \mid U \in \beta_1\}$ and $\beta_1 = \{V^c \mid V \in \beta_2\}$. Moreover, both $\beta_1$ and $\beta_2$ contain $\emptyset, X$ and are closed with respect to finite unions and intersections.

Lemma 2.5. Suppose that $(X, \tau_1, \tau_2)$ is pairwise zero-dimensional. Then the following conditions are equivalent:

1. $(X, \tau_1)$ is $T_0$.
2. $(X, \tau_2)$ is $T_0$.
3. $(X, \tau_1, \tau_2)$ is pairwise $T_2$.
4. For any two distinct points $x, y \in X$ there exist disjoint $U, V \in \tau_1 \cup \tau_2$ such that $x \in U$ and $y \in V$.
5. $(X, \tau_1, \tau_2)$ is join $T_2$.
6. $(X, \tau_1, \tau_2)$ is bi-$T_0$.

Proof. (1)$\Rightarrow$(2): Suppose that $(X, \tau_1)$ is $T_0$ and $x, y$ are two distinct points of $X$. Then there exists $U \in \tau_1$ containing exactly one of $x, y$. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since $(X, \tau_1, \tau_2)$ is pairwise zero-dimensional, there exists $V \in \beta_1$ such that $x \in V \subseteq U$. Therefore, $V^c \in \beta_2$, $y \in V^c$, and $x \notin V^c$. Thus, $(X, \tau_2)$ is $T_0$.

(2)$\Rightarrow$(3): Suppose that $(X, \tau_2)$ is $T_0$ and $x, y$ are two distinct points of $X$. Then there exists $U \in \tau_2$ containing exactly one of $x, y$. Without loss of generality we may assume that $x \in U$ and $y \notin U$. Since $(X, \tau_1, \tau_2)$ is pairwise zero-dimensional, there exists $V \in \beta_2$ such that $x \in V \subseteq U$. Then $x \in V \in \beta_2$, $y \in V^c \in \beta_1$, and $V, V^c$ are disjoint. Thus, $(X, \tau_1, \tau_2)$ is pairwise $T_2$. 


are disjoint. If \( y \not\in X \), then there exists \( U' \in \beta_1 \) such that \( x \in U' \subseteq U \). Let \( V' = X - U' \). Then \( V \subseteq V' \), so \( y \in V' \in \beta_2 \), and so there exist two disjoint \( \tau \)-open sets \( U', V' \) such that \( x \in U' \) and \( y \not\in V' \). Thus, \( (X, \tau_1, \tau_2) \) is join \( T_2 \).

(5)\( \Rightarrow \)(6): Suppose that \( (X, \tau_1, \tau_2) \) is join \( T_2 \). We show that \( (X, \tau_1) \) is \( T_0 \). Let \( x, y \) be two distinct points of \( X \). Since \( (X, \tau_1, \tau_2) \) is pairwise zero-dimensional and pairwise \( U \), \( \tau \), compact subsets of \( (X, \tau_1, \tau_2) \) are disjoint. If \( y \not\in U_1 \), then there is \( U_1 \in \tau_1 \) containing exactly one of \( x, y \). If \( y \in U_1 \), then \( y \not\in V_1 \). Therefore, \( y \in U_2 \cap V_1 \). Clearly \( U_2 \cap V_1 \in \beta_1 \). Moreover, \( y \not\in U_2 \cap V_1 \) as \( x \not\in V_1 \). Thus, there exists \( U_2 \cap V_1 \in \tau_1 \) containing exactly one of \( x, y \). In either case, we separate \( x, y \) by a \( \tau_1 \)-open set, and so \( (X, \tau_1) \) is \( T_0 \). That \( (X, \tau_2) \) is \( T_0 \) is proved similarly. Consequently, \( (X, \tau_1, \tau_2) \) is bi-\( T_0 \).

(6)\( \Rightarrow \)(1) is obvious.

On the other hand, \( (X, \tau_1, \tau_2) \) may be pairwise zero-dimensional and pairwise \( T_2 \) without either of \( \tau_1, \tau_2 \) being even \( T_1 \) as the following simple example shows.

**Example 2.6.** Let \( X = \{0, 1\} \), \( \tau_1 = \{\emptyset, \{1\}, X\} \) and \( \tau_2 = \{\emptyset, \{0\}, X\} \). Then both \( \tau_1 \) and \( \tau_2 \) are the Sierpinski topologies on \( X \), thus both are \( T_0 \), but not \( T_1 \). Nevertheless, \( (X, \tau_1, \tau_2) \) is pairwise zero-dimensional and pairwise \( T_2 \).

The next definition generalizes the notion of compactness to bitopological spaces.

**Definition 2.7.** [29, Def. 2.2.17] We call a bitopological space \( (X, \tau_1, \tau_2) \) pairwise compact if for each cover \( \{U_i \mid i \in I\} \) of \( X \) with \( U_i \in \tau_1 \cup \tau_2 \), there exists a finite subcover.

**Remark 2.8.** In [29, Def. 2.2.17] Salbany defines a bitopological space \( (X, \tau_1, \tau_2) \) to be pairwise compact if \( (X, \tau) \) is compact, where \( \tau = \tau_1 \vee \tau_2 \). In our terminology this means that \( (X, \tau_1, \tau_2) \) is join compact. But it is a consequence of Alexander’s Lemma—a classical result in general topology—that the two notions of pairwise compact and join compact coincide.

It is obvious that if \( (X, \tau_1, \tau_2) \) is pairwise compact, then both \( (X, \tau_1) \) and \( (X, \tau_2) \) are compact; that is, \( (X, \tau_1, \tau_2) \) is bi-compact. On the other hand, it was observed by Salbany [29, p. 17] that the converse is not true in general. Let \( \sigma_1 \) and \( \sigma_2 \) denote the collections of compact subsets of \( (X, \tau_1) \) and \( (X, \tau_2) \), respectively.

**Proposition 2.9.** A bitopological space \( (X, \tau_1, \tau_2) \) is pairwise compact iff \( \delta_1 \subseteq \sigma_2 \) and \( \delta_2 \subseteq \sigma_1 \).

**Proof.** \( \Rightarrow \) Suppose that \( (X, \tau_1, \tau_2) \) is pairwise compact. We show that \( \delta_1 \subseteq \sigma_2 \). Let \( A \in \delta_1 \) and let \( A \subseteq \bigcup\{U_i \mid i \in I\} \) with \( \{U_i \mid i \in I\} \subseteq \tau_2 \). Then the collection \( \{U_i \mid i \in I\} \cup \{A^c\} \) is a cover of \( X \). Since \( A^c \in \tau_1 \) and \( (X, \tau_1, \tau_2) \) is pairwise compact, there exist \( i_1, \ldots, i_n \in I \) such that \( U_{i_1} \cup \cdots \cup U_{i_n} \cup A^c = X \). It follows that \( A \subseteq U_{i_1} \cup \cdots \cup U_{i_n} \), and so \( A \in \sigma_2 \). Thus, \( \delta_1 \subseteq \sigma_2 \). That \( \delta_2 \subseteq \sigma_1 \) is proved similarly.

\( \Leftarrow \) Suppose that \( \delta_1 \subseteq \sigma_2 \) and \( \delta_2 \subseteq \sigma_1 \). To show that \( (X, \tau_1, \tau_2) \) is pairwise compact let \( \{U_i \mid i \in I\} \subseteq \tau_1 \) and \( \{V_j \mid j \in J\} \subseteq \tau_2 \) with \( \bigcup\{U_i \mid i \in I\} \cup \bigcup\{V_j \mid j \in J\} = X \). We set \( U = \bigcup\{U_i \mid i \in I\} \). Clearly \( U \in \tau_1 \) and \( U \cup \bigcup\{V_j \mid j \in J\} = X \), so \( U^c \subseteq \bigcup\{V_j \mid j \in J\} \). Since \( U^c \in \delta_1 \) and \( \delta_1 \subseteq \sigma_2 \), we have that \( U^c \in \sigma_2 \). Therefore, there exist \( j_1, \ldots, j_n \in J \) such that \( U^c \subseteq V_{j_1} \cup \cdots \cup V_{j_n} \). We set \( V = V_{j_1} \cup \cdots \cup V_{j_n} \). Then \( U \cup V = X \), so \( V^c \subseteq U = \bigcup\{U_i \mid i \in I\} \).
Since $V^c \in \delta_2$ and $\delta_2 \subseteq \sigma_1$, we have that $V^c \in \sigma_1$. Therefore, there exist $i_1, \ldots, i_m \in I$ such that $V^c \subseteq U_{i_1} \cup \cdots \cup U_{i_m}$. Clearly the finite collection $\{V_{j_1}, \ldots, V_{j_n}, U_{i_1}, \ldots, U_{i_m}\}$ is a cover of $X$. Thus, $X$ is pairwise compact.

Now we generalize the notion of a Stone space to that of a pairwise Stone space.

**Definition 2.10.** We call $(X, \tau_1, \tau_2)$ a pairwise Stone space if it is pairwise compact, pairwise Hausdorff, and pairwise zero-dimensional.

**Remark 2.11.** In the definition of a pairwise Stone space, pairwise Hausdorff can be replaced by any of the equivalent conditions of Lemma 2.5, and that pairwise compact can be replaced by $\delta_1 \subseteq \sigma_2$ and $\delta_2 \subseteq \sigma_1$, as follows from Proposition 2.9.

Let $\textbf{PStone}$ denote the category of pairwise Stone spaces and bi-continuous functions; that is functions which are continuous with respect to both topologies.

### 3. Priestley spaces and pairwise Stone spaces

Let $(X, \leq)$ be a poset. We recall that $A \subseteq X$ is an upset if $x \in A$ and $x \leq y$ imply $y \in A$, and that $A$ is a downset if $x \in A$ and $y \leq x$ imply $y \in A$. For $Y \subseteq X$ let $\uparrow Y = \{x \mid \exists y \in Y \text{ with } y \leq x\}$ and $\downarrow Y = \{x \mid \exists y \in Y \text{ with } x \leq y\}$. Let $\text{Up}(X)$ denote the set of upsets and $\text{Do}(X)$ denote the set of downsets of $(X, \leq)$.

Let $(X, \tau, \leq)$ be an ordered topological space. We denote by $\text{OpUp}(X)$ the set of open upsets, by $\text{ClUp}(X)$ the set of closed upsets, and by $\text{CpUp}(X)$ the set of clopen upsets of $(X, \tau, \leq)$. Similarly, let $\text{OpDo}(X)$ denote the set of open downsets, $\text{ClDo}(X)$ denote the set of closed downsets, and $\text{CpDo}(X)$ denote the set of clopen downsets of $(X, \tau, \leq)$. The next definition is well-known.

**Definition 3.1.** An ordered topological space $(X, \tau, \leq)$ is a Priestley space if $(X, \tau)$ is compact and whenever $x \not\leq y$, there exists a clopen upset $A$ such that $x \in A$ and $y \not\in A$.

The second condition in the above definition is known as the Priestley separation axiom (PSA for short). The next lemma is well-known.

**Lemma 3.2.** Let $(X, \tau, \leq)$ be an ordered topological space.

1. If $(X, \tau, \leq)$ is a Priestley space, then $(X, \tau)$ is a Stone space.
2. If $(X, \tau, \leq)$ is a Priestley space, then $\uparrow F$ and $\downarrow F$ are closed for each closed subset $F$ of $X$.
3. In a Priestley space, every open upset is the union of clopen upsets, every closed upset is the intersection of clopen upsets, every open downset is the union of clopen downsets, and every closed downset is the intersection of clopen downsets.
4. In a Priestley space, clopen upsets and clopen downsets form a subbasis for the topology.
5. $(X, \tau, \leq)$ is a Priestley space iff $(X, \tau)$ is compact and for closed subsets $F$ and $G$ of $X$, whenever $\uparrow F \cap \downarrow G = \emptyset$, there exists a clopen upset $A$ of $X$ such that $F \subseteq A$ and $G \subseteq A^c$.

We will refer to condition (5) in the lemma as the strong Priestley separation axiom (SPSA for short). Let $\textbf{Pries}$ denote the category of Priestley spaces and continuous order-preserving maps. We show that the categories $\textbf{Pries}$ and $\textbf{PStone}$ are isomorphic. To this end, we will define two functors $\Phi : \textbf{PStone} \rightarrow \textbf{Pries}$ and $\Psi : \textbf{Pries} \rightarrow \textbf{PStone}$ which will set the required isomorphism.
For a topological space $(X, \tau)$, let $\leq$ denote the specialization order of $(X, \tau)$; that is,

$$x \leq y \iff x \in \text{Cl}(y) \text{ iff } (\forall U \in \tau)(x \in U \implies y \in U).$$

It is well-known that $\leq$ is reflexive and transitive, and that $\leq$ is antisymmetric iff $(X, \tau)$ is $T_0$.

**Lemma 3.3.** Let $(X, \tau_1, \tau_2)$ be a bitopological space, $\leq_1$ be the specialization order of $(X, \tau_1)$, and $\leq_2$ be the specialization order of $(X, \tau_2)$. If $(X, \tau_1, \tau_2)$ is pairwise zero-dimensional, then $\leq_1 = \geq_2$.

**Proof.** Let $(X, \tau_1, \tau_2)$ be pairwise zero-dimensional; that is, $\beta_1 = \tau_1 \cap \delta_2$ is a basis for $\tau_1$ and $\beta_2 = \tau_2 \cap \delta_1$ is a basis for $\tau_2$. Then, for each $x, y \in X$, we have:

$$x \leq_1 y \iff (\forall U \in \tau_1)(x \in U \implies y \in U)$$

$$\quad \iff (\forall U \in \beta_1)(x \in U \implies y \in U)$$

$$\quad \iff (\forall U \in \beta_1)(y \in U^c \implies x \in U^c)$$

$$\quad \iff (\forall V \in \beta_2)(y \in V \implies x \in V)$$

$$\quad \iff (\forall V \in \tau_2)(y \in V \implies x \in V)$$

$$\quad \iff y \leq_2 x.$$  

For a pairwise Stone space $(X, \tau_1, \tau_2)$, let $\tau = \tau_1 \lor \tau_2$, and let $\leq = \leq_1$ be the specialization order of $(X, \tau_1)$.

**Proposition 3.4.** If $(X, \tau_1, \tau_2)$ is a pairwise Stone space, then $(X, \tau, \leq)$ is a Priestley space. Moreover:

(i) $\text{CpUp}(X, \tau, \leq) = \beta_1$.

(ii) $\text{OpUp}(X, \tau, \leq) = \tau_1$.

(iii) $\text{ClUp}(X, \tau, \leq) = \delta_2$.

(iv) $\text{CpDo}(X, \tau, \leq) = \beta_2$.

(v) $\text{OpDo}(X, \tau, \leq) = \tau_2$.

(vi) $\text{CIDo}(X, \tau, \leq) = \delta_1$.

**Proof.** Since $(X, \tau_1, \tau_2)$ is pairwise compact, $(X, \tau_1, \tau_2)$ is join compact, and so $(X, \tau)$ is compact. Also, as $(X, \tau_1, \tau_2)$ is pairwise Hausdorff, it follows from Lemma 2.5 that $(X, \tau_1)$ is $T_0$. Therefore, $\leq = \leq_1$ is a partial order. We show that $(X, \tau, \leq)$ satisfies PSA. If $x \not\leq y$, then $x \not\leq_1 y$, so there exists $U \in \beta_1$ such that $x \in U$ and $y \not\in U$. Since $\leq_1$ is the specialization order of $(X, \tau_1)$, $U$ is an $\leq_1$-upset. From $U \in \beta_1$ it follows that $U^c \in \beta_2 \subseteq \tau$. So both $U$ and $U^c$ are open in $(X, \tau)$, and so $U$ is clopen in $(X, \tau)$. Therefore, $U$ is a clopen upset of $(X, \tau, \leq)$, implying that $(X, \tau, \leq)$ satisfies PSA. Thus, $(X, \tau, \leq)$ is a Priestley space.

(i) We already showed that $\beta_1 \subseteq \text{CpUp}(X, \tau, \leq)$. Let $A \in \text{CpUp}(X, \tau, \leq)$. We show that $A = \bigcup\{U \in \beta_1 \mid U \subseteq A\}$. That $\bigcup\{U \in \beta_1 \mid U \subseteq A\} \subseteq A$ is obvious. Let $x \in A$. Since $A$ is an upset, for each $y \in A^c$ we have $x \not\leq y$. Therefore, $x \not\leq_1 y$, and as $\beta_1$ is a basis for $(X, \tau_1)$, there exists $U_y \in \beta_1$ such that $x \in U_y$ and $y \not\in U_y$. It follows that $A^c \cap \bigcap\{U_y \mid y \in A^c\} = \emptyset$. Thus, $\{A^c\} \cup \{U_y \mid y \in A^c\}$ is a family of closed subsets of $(X, \tau)$ with the empty intersection, and as $(X, \tau)$ is compact, there are $U_1, \ldots, U_n \in \beta_1$ with $A^c \cap U_1 \cap \cdots \cap U_n = \emptyset$. Therefore, $x \in U_1 \cap \cdots \cap U_n \subseteq A$. Since $\beta_1$ is closed under finite intersections, we obtain that there is $U \in \beta_1$ such that $x \in U \subseteq A$. Thus, $A = \bigcup\{U \in \beta_1 \mid U \subseteq A\}$. Now since $A$ is a closed subset of a compact space, $A$ is compact, so it is a finite union of elements of $\beta_1$, thus $A \in \beta_1$. 


Proposition 3.5. Let \( \text{complements of unions of elements of } \beta \tau \) Consequently, \( \text{ClUp}(X, \tau, \leq) = \delta_2 \).

Proof. Since \( \text{f} \) is bi-continuous, the \( \text{f} \) inverse image of every element of \( \tau_1 \cup \tau_2 \) is an element of \( \tau_1 \cup \tau_2 \). As \( \tau_1 \cup \tau_2 \) is a subbasis for \((X, \tau')\), it follows that \( f : (X, \tau) \to (X', \tau') \) is continuous. Also, since the \( \text{f} \) inverse image of an element of \( \tau_1 \) is an element of \( \tau_1 \) and \( \leq' = \leq_1' \), it follows that \( f : (X, \leq) \to (X', \tau', \leq') \) is order-preserving. Thus, \( f : (X, \tau, \leq) \to (X', \tau', \leq') \) is continuous and order-preserving. We define the functor \( \Phi : \text{PStone} \to \text{Pries} \) as follows. For \((X, \tau_1, \tau_2)\) a pairwise Stone space, we put \( \Phi(X, \tau_1, \tau_2) = (X, \tau, \leq) \), and for \( f : (X, \tau, \leq) \to (X', \tau', \leq') \) a bi-continuous map, we put \( \Phi(f) = f \). It follows from Propositions 3.4 and 3.5 that \( \Phi \) is well-defined.

For \((X, \tau, \leq)\) a Priestley space, let \( \tau_1 = \text{OpUp}(X, \tau, \leq) \) and \( \tau_2 = \text{OpDo}(X, \tau, \leq) \). Clearly \( \tau_1 \) and \( \tau_2 \) are topologies on \( X \).

Proposition 3.6. If \((X, \tau, \leq)\) is a Priestley space, then \((X, \tau_1, \tau_2)\) is a pairwise Stone space. Moreover:

(i) \( \beta_1 = \text{CpUp}(X, \tau, \leq) \).

(ii) \( \beta_2 = \text{CpDo}(X, \tau, \leq) \).

(iii) \( \leq = \leq_1 = \geq_2 \).

Proof. Since \((X, \tau)\) is compact and \( \tau_1 \cup \tau_2 \subseteq \tau \), it follows that \((X, \tau_1, \tau_2)\) is pairwise compact. To show that \((X, \tau_1, \tau_2)\) is pairwise Hausdorff, let \( x, y \) be two distinct points of \( X \). Since \( \leq \) is a partial order, we have \( x \not\leq y \) or \( y \not\leq x \). In either case, by PSA, one of the points has a clopen upset neighborhood \( U \) not containing the other. Clearly \( U^c \) is a clopen downset. Therefore, \( U \in \tau_1 \) and \( U^c \in \tau_2 \) separate \( x \) and \( y \). Thus, \((X, \tau_1, \tau_2)\) is pairwise Hausdorff. That \((X, \tau_1, \tau_2)\) is pairwise zero-dimensional follows from (i), (ii), and the fact that open upsets are unions of clopen upsets and open downsets are unions of clopen downsets (see Lemma 3.2(3)). Consequently, \((X, \tau_1, \tau_2)\) is a pairwise Stone space.

(i) For \( U \subseteq X \) we have:

\[
A \in \beta_1 \iff (A \in \tau_1 \text{ and } A^c \in \tau_2) \text{ and } A \in \text{OpUp}(X, \tau, \leq) \text{ and } A^c \in \text{OpDo}(X, \tau, \leq)
\]

Thus, \( \beta_1 = \text{CpUp}(X, \leq) \).

(ii) is proved similarly to (i).

(iii) For \( x, y \in X \), by PSA, we have:
$x \leq y$ if $(\forall U \in \mathsf{OpUp}(X, \tau))(x \in U \Rightarrow y \in U)$

iff $(\forall U \in \tau_1)(x \in U \Rightarrow y \in U)$

iff $x \leq_1 y$.

Thus, $\leq = \leq_1$. That $\leq = \geq_2$ is proved similarly.

Proposition 3.7. If $f : (X, \tau, \leq) \to (X', \tau', \leq')$ is continuous and order-preserving, then $f : (X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2)$ is bi-continuous.

Proof. Since $f$ is continuous and order-preserving, $U \in \mathsf{OpUp}(X', \tau', \leq')$ implies $f^{-1}(U) \in \mathsf{OpUp}(X, \tau, \leq)$ and $U \in \mathsf{OpDo}(X', \tau', \leq')$ implies $f^{-1}(U) \in \mathsf{OpDo}(X, \tau, \leq)$. By the definition of the topologies, $\mathsf{OpUp}(X, \tau, \leq) = \tau_1$, $\mathsf{OpUp}(X', \tau', \leq') = \tau'_1$, $\mathsf{OpDo}(X, \tau, \leq) = \tau_2$, and $\mathsf{OpDo}(X', \tau', \leq') = \tau'_2$. Thus, $f : (X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2)$ is bi-continuous.

Now we define $\Psi : \mathsf{Pries} \to \mathsf{PStone}$ as follows. For $(X, \tau, \leq)$ a Priestley space, we put $\Psi(X, \tau, \leq) = (X, \tau_1, \tau_2)$, and for $f : (X, \tau, \leq) \to (X', \tau', \leq')$ continuous and order-preserving, we put $\Psi(f) = f$. It follows from Propositions 3.6 and 3.7 that $\Psi$ is well-defined.

Theorem 3.8. The functors $\Phi$ and $\Psi$ establish an isomorphism between the categories $\mathsf{PStone}$ and $\mathsf{Pries}$.

Proof. We already verified that $\Phi$ and $\Psi$ are well-defined. That they are natural is easy to see. Moreover, for each pairwise Stone space $(X, \tau_1, \tau_2)$, by Proposition 3.4, we have $\Phi(X, \tau_1, \tau_2) = (X, \mathsf{OpUp}(X, \tau, \leq), \mathsf{OpDo}(X, \tau, \leq)) = (X, \tau_1, \tau_2)$. Also, for each Priestley space $(X, \tau, \leq)$, by Lemma 3.2(4) and Proposition 3.6, we have $\Phi\Psi(X, \tau, \leq) = (X, \tau_1 \vee \tau_2, \leq_1) = (X, \tau, \leq)$. Thus, $\Phi$ and $\Psi$ establish an isomorphism between $\mathsf{PStone}$ and $\mathsf{Priest}$.

4. Pairwise Stone spaces and spectral spaces

For a topological space $(X, \tau)$, let $\mathcal{E}(X, \tau)$ denote the set of compact open subsets of $(X, \tau)$. We recall that $(X, \tau)$ is coherent if $\mathcal{E}(X, \tau)$ is closed under finite intersections and forms a basis for the topology. We also recall that a subset $A$ of $X$ is irreducible if $A = F \cup G$, with $F, G$ closed, implies that $A = F$ or $A = G$, and that $(X, \tau)$ is sober if every irreducible closed subset of $(X, \tau)$ is the closure of a point. Clearly a closed subset of $X$ is irreducible if it is a join-prime element in the lattice of closed subsets of $(X, \tau)$. We will use this fact in the proof of Proposition 4.2.

Definition 4.1. [16, p. 43] A topological space $(X, \tau)$ is called a spectral space if $(X, \tau)$ is compact, $T_0$, coherent, and sober.

Let $(X, \tau)$ and $(X', \tau')$ be two spectral spaces. We recall [16, p. 43] that a map $f : (X, \tau) \to (X', \tau')$ is a spectral map if $U \in \mathcal{E}(X', \tau')$ implies $f^{-1}(U) \in \mathcal{E}(X, \tau)$. Clearly every spectral map is continuous.

Let $\mathsf{Spec}$ denote the category of spectral spaces and spectral maps. It follows from [6] that $\mathsf{Spec}$ is isomorphic to $\mathsf{Pries}$. Thus, by Theorem 3.8, $\mathsf{Spec}$ is isomorphic to $\mathsf{PStone}$. Nevertheless, we give a direct proof of this result. On the one hand, it will underline the utility of sobriety in the definition of a spectral space; on the other hand, it will provide a more natural proof of Cornish’s result that $\mathsf{Pries}$ and $\mathsf{Spec}$ are isomorphic, by first establishing the intermediate isomorphisms of $\mathsf{Pries}$ and $\mathsf{PStone}$ and $\mathsf{PStone}$ and $\mathsf{Spec}$.

Proposition 4.2. If $(X, \tau_1, \tau_2)$ is a pairwise Stone space, then $(X, \tau_1)$ is a spectral space. Moreover, $\mathcal{E}(X, \tau_1) = \beta_1$. 

Proof. Since \((X, \tau_1, \tau_2)\) is pairwise compact, it is immediate that \((X, \tau_1)\) is compact. It follows from Lemma 2.5 that \((X, \tau_1)\) is \(T_0\). We show that \(E(X, \tau_1) = \beta_1\). By Proposition 2.9, \(\beta_1 = \tau_1 \cap \delta_2 \subseteq \tau_1 \cap \sigma_1 = E(X, \tau_1)\). Conversely, suppose that \(U \in E(X, \tau_1)\). Since \(\beta_1\) is a basis for \((X, \tau_1)\), we have \(U\) is the union of elements of \(\beta_1\). As \(U\) is compact, it is a finite union of elements of \(\beta_1\), thus belongs to \(\beta_1\) because \(\beta_1\) is closed under finite unions. Therefore, \(E(X, \tau_1) = \beta_1\). It follows that \(E(X, \tau_1)\) is closed under finite intersections and forms a basis for the topology. Therefore, \((X, \tau_1)\) is coherent. To show that \((X, \tau_1)\) is sober, let \(F\) be a join-prime element in the lattice of closed subsets of \((X, \tau_1)\). We show that \(F\) is equal to the closure in \((X, \tau_1)\) of a point of \(F\). If not, then for each \(x \in F\) there exists \(y \in F\) such that \(y \notin \text{Cl}_1(x)\). Therefore, there exists \(U_y \in \beta_1\) such that \(y \in U_y\) and \(x \notin U_y\). Let \(U_x = U_y^c\). Then \(x \in U_x \in \beta_2\), \(y \notin U_x\), and \(F\) is covered by the family \(\{U_x \mid x \in F\}\). Since \(F \subseteq U_{x_1} \cup \cdots \cup U_{x_n}\). As \(F\) is join-prime in \(\delta_1\) and for each \(i\) we have \(U_{x_i} \in \beta_2 \subseteq \delta_1\), there exists \(k\) such that \(F \subseteq U_{x_k}\). On the other hand, the \(y_k\) corresponding to \(x_k\) belongs to \(F\) and does not belong to \(U_{x_k}\), a contradiction. Thus, there is \(x \in F\) such that \(F = \text{Cl}_1(x)\). Consequently, \((X, \tau_1)\) is sober, and so \((X, \tau_1)\) is a spectral space.

Proposition 4.3. Let \((X, \tau_1, \tau_2)\) and \((X', \tau_1', \tau_2')\) be two pairwise Stone spaces. If \(f : (X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')\) is bi-continuous, then \(f : (X, \tau_1) \rightarrow (X', \tau_1')\) is spectral.

Proof. Since \(f\) is bi-continuous, by Proposition 4.2, we have:

\[
U \in E(X', \tau_1') \Rightarrow U \in \beta_1' \\
\Rightarrow U \in \tau_1' \cap \delta_2' \\
\Rightarrow f^{-1}(U) \in \tau_1 \cap \delta_2 \\
\Rightarrow f^{-1}(U) \in \beta_1 \\
\Rightarrow f^{-1}(U) \in E(X, \tau_1).
\]

Thus, \(f\) is spectral.

We define the functor \(F : \text{PStone} \rightarrow \text{Spec}\) as follows. For a pairwise Stone space \((X, \tau_1, \tau_2)\), we put \(F(X, \tau_1, \tau_2) = (X, \tau_1)\), and for \(f : (X, \tau_1, \tau_2) \rightarrow (X', \tau_1', \tau_2')\) bi-continuous, we put \(F(f) = f\). It follows from Propositions 4.2 and 4.3 that \(F\) is well-defined. Note that \(F\) is a forgetful functor, forgetting the topology \(\tau_2\).

For \((X, \tau)\) a spectral space, let \(\tau_1 = \tau\) and \(\tau_2\) be the topology generated by the basis \(\Delta(X, \tau) = \{U^c \mid U \in E(X, \tau)\}\).

Remark 4.4. Let \((X, \tau)\) be a topological space. We recall (see, e.g., [23, Def. 4.4]) that the de Groot dual of \(\tau\) is the topology \(\tau^*\) whose closed sets are generated by compact saturated sets of \((X, \tau)\). Since in a spectral space \((X, \tau)\) the compact saturated sets are exactly the intersections of compact open sets, we obtain that the topology generated by \(\Delta(X, \tau)\) is exactly the de Groot dual \(\tau^*\) of \(\tau\).

Proposition 4.5. If \((X, \tau)\) is a spectral space, then \((X, \tau_1, \tau_2)\) is a pairwise Stone space. Moreover:

(i) \(\beta_1 = E(X, \tau)\).

(ii) \(\beta_2 = \Delta(X, \tau)\).

Proof. First we show that \((X, \tau_1, \tau_2)\) is pairwise compact. For this it suffices to show that any collection \(K \subseteq E(X, \tau) \cup \Delta(X, \tau)\) with the FIP (Finite Intersection Property) has a nonempty intersection. Let \(\delta = \{F \mid F^c \in \tau\}\) denote the collection of closed subsets of...
Conversely, since $\Delta(X, \tau) \subseteq \delta$, we have that $K \subseteq E(X, \tau) \cup \delta$. To show that $\bigcap K \neq \emptyset$, by Zorn’s Lemma, we extend $K$ to a maximal subset $M$ of $E(X, \tau) \cup \delta$ with the FIP. Let $C$ denote the intersection of all $\tau$-closed sets in $M$; that is, $C = \bigcap \{F \mid F \in M \cap \delta\}$. Since $(X, \tau)$ is compact, $C \in \delta$ is nonempty. Because $E(X, \tau)$ is closed under finite intersections, it is easy to see that the collection $M \cup \{C\}$ has the FIP, and as $M$ is maximal, we have $C \in M$. We show that $C$ is irreducible. Suppose that $C = A \cup B$ and $A, B \in \delta$. If $M \cup \{A\}$ and $M \cup \{B\}$ do not have the FIP, then there exist $A_1, \ldots, A_n \in M$ with $A_1 \cap \cdots \cap A_n \cap A = \emptyset$ and $B_1, \ldots, B_m \in M$ with $B_1 \cap \cdots \cap B_m \cap B = \emptyset$. This implies that $A_1 \cap \cdots \cap A_n \cap B_1 \cap \cdots \cap B_m \cap C = \emptyset$, which is a contradiction. Therefore, either $M \cup \{A\}$ or $M \cup \{B\}$ has the FIP. Since $M$ is maximal, either $A \in M$ or $B \in M$. Because of the choice of $C$, this implies that either $C \subseteq A$ or $C \subseteq B$, and so either $C = A$ or $C = B$. Thus, $C$ is irreducible. As $(X, \tau)$ is sober, $C = \text{Cl}(x)$ for some $x \in X$. It is clear that $x$ belongs to all $F \in M \cap \delta$ since $C \subseteq F$ for all such $F$. Moreover, for each $U \in M \cap E(X, \tau)$, we have $U \cap \text{Cl}(x) = U \cap C \neq \emptyset$. Since $U$ is open in $(X, \tau)$, this implies that $x \in U$. Therefore, $x \in \bigcap M$, so $x \in \bigcap K$, as $K \subseteq M$, and so $\bigcap K \neq \emptyset$. Consequently, $(X, (\tau_1, \tau_2))$ is pairwise compact.

We show that $\beta_1 = E(X, \tau)$ and $\beta_2 = \Delta(X, \tau)$, which establishes that $(X, (\tau_1, \tau_2))$ is pairwise zero-dimensional. By the definition of $\tau_2$ we have $E(X, \tau) \subseteq \tau_2$, and so $E(X, \tau) \subseteq \beta_1$. Conversely, since $(X, \tau_1, \tau_2)$ is pairwise compact, by Proposition 2.9, we have $\beta_1 = \tau_1 \cap \delta_2 \subseteq \tau_1 \cap \sigma_1 = E(X, \tau)$. Therefore, $\beta_1 = E(X, \tau)$. Moreover, $U \in \Delta(X, \tau) \iff U^c \in E(X, \tau) = \beta_1 = \tau_1 \cap \delta_2 \iff U \in \delta_1 \cap \tau_2 = \beta_2$. Thus, $\beta_2 = \Delta(X, \tau)$.

Lastly, we have for granted that $(X, \tau_1)$ is $T_0$. Therefore, by Lemma 2.5, $(X, (\tau_1, \tau_2))$ is pairwise $T_2$, so a pairwise Stone space, which concludes the proof. 

**Proposition 4.6.** Let $(X, \tau)$ and $(X', \tau')$ be two spectral spaces. If $f : (X, \tau) \to (X', \tau')$ is a spectral map, then $f : (X, \tau_1, \tau_2) \to (X', \tau'_1, \tau'_2)$ is bi-continuous.

**Proof.** Since $f$ is spectral, $f : (X, \tau_1) \to (X', \tau'_1)$ is continuous. Moreover, for $U \in \beta'_2$ we have $U^c \in \beta'_1$. Therefore, $f^{-1}(U) = f^{-1}(U^c)^c = f^{-1}(U^c) \in \beta_1$ as $f$ is spectral. Consequently, $f : (X, \tau_2) \to (X', \tau'_2)$ is continuous, and so $f : (X, (\tau_1, \tau_2)) \to (X', (\tau'_1, \tau'_2))$ is bi-continuous.

Now we define the functor $G : \text{Spec} \to \text{PStone}$ as follows. For a spectral space $(X, \tau)$, we put $G(X, \tau) = (X, \tau_1, \tau_2)$, and for $f : (X, \tau) \to (X', \tau')$ a spectral map, we put $G(f) = f$. It follows from Propositions 4.5 and 4.6 that $G$ is well-defined.

**Theorem 4.7.** The functors $F$ and $G$ establish an isomorphism between the categories $\text{PStone}$ and $\text{Spec}$.

**Proof.** We already verified that $F$ and $G$ are well-defined. That they are natural is easy to see. Moreover, for each pairwise Stone space $(X, \tau_1, \tau_2)$ we have $GF(X, \tau_1, \tau_2) = G(X, \tau_1) = (X, \tau_1, \tau_2)$, by Proposition 4.2. Also, for each spectral space $(X, \tau)$ we have $FG(X, \tau) = F((X, \tau_1, \tau_2)) = (X, \tau_1) = (X, \tau)$. Thus, $F$ and $G$ establish an isomorphism between $\text{PStone}$ and $\text{Spec}$.

Putting Theorems 3.8 and 4.7 together, we obtain that the three categories $\text{Pries}$, $\text{PStone}$, and $\text{Spec}$ are isomorphic. As we pointed out in the introduction, this can be viewed as a particular case of a more general result of [14, Ch. VI-6] that the categories of compact order-Hausdorff spaces, pairwise compact pairwise regular bitopological spaces, and stably compact spaces are isomorphic. It appears to be an interesting question to investigate how far the above isomorphisms can be pushed. In other words, what are the largest categories of ordered topological spaces, bitopological spaces, and sober spaces which are still isomorphic?
5. Distributive lattices and pairwise Stone spaces

Since P\text{Stone} is isomorphic to Spec and Spec is dually equivalent to D\text{Lat}, it follows that P\text{Stone} is also dually equivalent to D\text{Lat}. We give an explicit proof of this result. It will show that of the dual equivalences of D\text{Lat} with Spec, Pries, and P\text{Stone}, the dual equivalence of D\text{Lat} with P\text{Stone} is the easiest to establish. Indeed, as we will see below, the proof of compactness of the bitopological dual of a bounded distributive lattice \( L \) does not require the use of Alexander's Lemma, hence is simpler than in the Priestley case; moreover, the complicated proof of sobriety of the dual spectral space of \( L \) is completely avoided in the bitopological setting.

Let \( L \) be a bounded distributive lattice and let \( X = \text{pf}(L) \) be the set of prime filters of \( L \). We define \( \phi_+, \phi_- : L \to \varphi(X) \) by

\[
\phi_+(a) = \{ x \in X \mid a \in x \} \quad \text{and} \quad \phi_-(a) = \{ x \in X \mid a \not\in x \}.
\]

If we think of \( L \) as a Lindenbaum algebra and of \( a \in L \) as (an equivalence class of) a formula, then we can think of \( \phi_+(a) \) as the set of points \( a \) is true at, and of \( \phi_-(a) \) as the set of points \( a \) is false at. It is easy to check that \( \phi_+(a) = \phi_-(a)^c \), and that the following identities hold:

\[
\begin{align*}
1_+ & : \phi_+(0) = \emptyset, & 1_- : \phi_-(0) = X, \\
2_+ & : \phi_+(1) = X, & 2_- : \phi_-(1) = \emptyset, \\
3_+ & : \phi_+(a \land b) = \phi_+(a) \land \phi_+(b), & 3_- : \phi_-(a \land b) = \phi_-(a) \lor \phi_-(b), \\
4_+ & : \phi_+(a \lor b) = \phi_+(a) \lor \phi_+(b), & 4_- : \phi_-(a \lor b) = \phi_-(a) \land \phi_-(b).
\end{align*}
\]

Let \( \beta_+ = \phi_+[L] = \{ \phi_+(a) \mid a \in L \} \), \( \beta_- = \phi_-[L] = \{ \phi_-(a) \mid a \in L \} \), \( \tau_+ \) be the topology generated by \( \beta_+ \), and \( \tau_- \) be the topology generated by \( \beta_- \).

**Proposition 5.1.** \((X, \tau_+, \tau_-)\) is a pairwise Stone space.

**Proof.** We start by showing that \((X, \tau_+, \tau_-)\) is pairwise Hausdorff. Suppose that \( x \neq y \). Without loss of generality we may assume that \( x \not\subseteq y \). Therefore, there exists \( a \in L \) with \( a \in x \) and \( a \not\in y \). Thus, \( x \in \phi_+(a) \in \tau_+ \) and \( y \in \phi_-(a) \in \tau_- \). Since \( \phi_-(a) = \phi_+(a)^c \), \( \phi_+(a) \) and \( \phi_-(a) \) are disjoint. Consequently, \((X, \tau_+, \tau_-)\) is pairwise Hausdorff.

Next we show that \((X, \tau_+, \tau_-)\) is pairwise compact. For this it is sufficient to show that for each cover of \( X \) by elements of \( \beta_+ \cup \beta_- \), there is a finite subcover. Suppose that \( X = \bigcup \{ \phi_+(a_i) \mid i \in I \} \cup \bigcup \{ \phi_-(b_j) \mid j \in J \} \) for some \( a_i, b_j \in L \). Let \( \Delta \) be the ideal generated by \( \{ a_i \mid i \in I \} \) and \( \nabla \) be the filter generated by \( \{ b_j \mid j \in J \} \). If \( \Delta \cap \nabla = \emptyset \), then by the prime filter lemma, there is a prime filter \( x \) of \( L \) such that \( x \subseteq \nabla \) and \( x \cap \Delta = \emptyset \). Therefore, \( x \in \phi_+(b_j) \) and \( x \in \phi_-(a_i) \) for each \( j \in J \) and \( i \in I \). Thus, \( x \not\in \phi_-(b_j) \) and \( x \not\in \phi_+(a_i) \) for each \( j \in J \) and \( i \in I \). Consequently, \( \{ \phi_+(a_i) \mid i \in I \} \cup \{ \phi_-(b_j) \mid j \in J \} \) is not a cover of \( X \), a contradiction. This shows that \( \nabla \cap \Delta = \emptyset \), and so there exist \( b_{j_1}, \ldots, b_{j_n} \) and \( a_{i_1}, \ldots, a_{i_m} \) such that \( b_{j_1} \land \cdots \land b_{j_n} \leq a_{i_1} \lor \cdots \lor a_{i_m} \). Therefore, \( \phi_+(b_{j_1}) \cap \cdots \cap \phi_+(b_{j_n}) \subseteq \phi_+(a_{i_1}) \cup \cdots \cup \phi_+(a_{i_m}) \subseteq X \), implying that \( \phi_-(b_{j_1}) \cup \cdots \cup \phi_-(b_{j_n}) \cup \phi_+(a_{i_1}) \cup \cdots \cup \phi_+(a_{i_m}) = X \). Thus, \( \{ \phi_+(a_{i_1}), \ldots, \phi_+(a_{i_m}), \phi_-(b_{j_1}), \ldots, \phi_-(b_{j_n}) \} \) is a finite subcover of \( \{ \phi_+(a_i) \mid i \in I \} \cup \{ \phi_-(b_j) \mid j \in J \} \), and so \((X, \tau_+, \tau_-)\) is pairwise compact.

Let \( \delta_+ \) denote the set of closed subsets and \( \sigma_+ \) denote the set of compact subsets of \((X, \tau_+)\); \( \delta_- \) and \( \sigma_- \) are defined similarly. We show that \( \delta_+ = \tau_+ \cap \delta_- \). If \( U \in \beta_+ \), then it is clear that \( U \in \tau_+ \). Moreover, since \( U = \phi_+(a) \) for some \( a \in L \), we have \( U^c = \phi_-(a) \), and so \( U^c \in \beta_- \). Thus, \( U \in \delta_- \), so \( U \in \tau_+ \cap \delta_- \), and so \( \beta_+ \subseteq \tau_+ \cap \delta_- \). Conversely, let \( U \in \tau_+ \cap \delta_- \). Since \((X, \tau_+, \tau_-)\) is pairwise compact, by Proposition 2.9, \( U \in \tau_+ \cap \sigma_+ \). As \( \beta_+ \) is a basis for \( \tau_+ \), we have that \( U \) is a union of elements of \( \beta_+ \). Because \( U \) is compact, it is a finite such union,
thus an element of $\beta_+$ as $\beta_+$ is closed under finite unions. Consequently, $\tau_+ \cap \delta_- \subseteq \beta_+$, and so $\beta_+ = \tau_+ \cap \delta_-$. A similar argument shows that $\beta_- = \tau_- \cap \delta_+$. It follows that $(X, \tau_+, \tau_-)$ is pairwise zero-dimensional, and so $(X, \tau_+, \tau_-)$ is a pairwise Stone space.

For a bounded lattice homomorphism $h : L \to L'$, let $f_h : \text{pf}(L') \to \text{pf}(L)$ be given by $f_h(x) = h^{-1}(x)$. It is easy to check that $f_h$ is well-defined.

**Proposition 5.2.** The map $f_h$ is bi-continuous.

**Proof.** Let $a \in L$. Then it is easy to verify that $f_h^{-1}(\phi_+(a)) = \phi_+'(ha)$ and $f_h^{-1}(\phi_-(a)) = \phi_-'(ha)$. Therefore, the inverse image of each element of $\beta_+$ is in $\beta_+'$ and the inverse image of each element of $\beta_-$ is in $\beta_-'$. Thus, $f_h$ is bi-continuous. □

This allows us to define the contravariant functor $(-)_* : \text{DLat} \to \text{PStone}$ as follows. For a bounded distributive lattice $L$, we let $L_* = (X, \tau_+, \tau_-)$, where $X = \text{pf}(L)$, $\tau_+$ is the topology generated by the basis $\beta_+$ and $\tau_-$ is the topology generated by the basis $\beta_- = \phi_-[L]$. For $h \in \text{hom}(L, L')$, we let $h_* = h^{-1}$. It follows from Propositions 5.1 and 5.2 that the functor $(-)_*$ is well-defined.

For a pairwise Stone space $(X, \tau_1, \tau_2)$ it is easy to see that $(\beta_1, \cap, \cup, \emptyset, X)$ is a bounded distributive lattice. (Note that $(\beta_2, \cap, \cup, \emptyset, X)$ is also a bounded distributive lattice dually isomorphic to $(\beta_1, \cap, \cup, \emptyset, X)$.) If $f : X \to X'$ is a bi-continuous map, then for each $U \in \beta'_1$, we have $U \in \tau'_1 \cap \delta'_2$. Since $f$ is bi-continuous, $f^{-1}(U) \in \tau_1 \cap \delta_2$. Therefore, $f^{-1}(U) \in \beta_1$. Moreover, it is clear that $f^{-1} : \beta'_1 \to \beta_1$ is a bounded lattice homomorphism. We define the contravariant functor $(-)^* : \text{PStone} \to \text{DLat}$ as follows. For a pairwise Stone space $(X, \tau_1, \tau_2)$, we let $(X, \tau_1, \tau_2)^* = (\beta_1, \cap, \cup, \emptyset, X)$, and for $f \in \text{hom}(X, X')$, we let $f^* = f^{-1}$. Then the functor $(-)^*$ is well-defined.

**Theorem 5.3.** The functors $(-)_*$ and $(-)^*$ establish a dual equivalence between $\text{DLat}$ and $\text{PStone}$.

**Proof.** For a bounded distributive lattice $L$, we have $L_*^* = \phi_+[L]$, and so $\phi_+$ is a lattice isomorphism from $L$ to $L_*^*$. For a pairwise Stone space $(X, \tau_1, \tau_2)$, let $\psi : X \to X_*^*$ be given by $\psi(x) = \{U \in X_*^* \mid x \in U\}$. It is easy to see that $\psi$ is well-defined. Since $X$ is pairwise Hausdorff, $\psi$ is 1-1. To see that $\psi$ is onto, let $P$ be a prime filter of $\beta_1$. We let $Q = \{V \in \beta_2 \mid Q^c \notin P\}$. It is easy to see that $Q$ is a prime filter of $\beta_2$, and that $P \cup Q$ has the FIP. Since $X$ is pairwise compact and pairwise Hausdorff, there is $x \in X$ such that $\bigcap(P \cup Q) = \{x\}$. Therefore, $\psi(x) = P$, and so $\psi$ is onto. Moreover, for $U \in \beta_1$ we have $\psi^{-1}(\phi_+(U)) = U \in \beta_1$ and $\psi^{-1}(\phi_-(U)) = U^c \in \beta_2$. Therefore, $f$ is bi-continuous. Furthermore, for $U \in \beta_1$, because $\psi$ is a bijection, $\psi^{-1}(\phi_+(U)) = U$ implies $\psi(U) = \phi_+(U)$, and $\psi^{-1}(\phi_-(U)) = U^c$ implies $\psi(U^c) = \phi_-(U)$. Thus, $f$ is bi-open, and so $f$ is a bi-homeomorphism from $X$ to $X_*^*$. That the functors $(-)_*$ and $(-)^*$ are natural is standard to prove. Consequently, $(-)_*$ and $(-)^*$ establish a dual equivalence between $\text{DLat}$ and $\text{PStone}$. □

**Remark 5.4.** It is worth pointing out that as in the case of the spectral and Priestley dualities, the dual equivalence between $\text{DLat}$ and $\text{PStone}$ is also induced by the *schizophrenic object* $2 = \{0, 1\}$. It has many lives: In $\text{DLat}$ it is the two-element lattice; in $\text{Spec}$ it is the *Sierpinski space* with the spectral topology $\tau_1 = \{\emptyset, \{1\}, \{0, 1\}\}$; in $\text{Pries}$ it is the two-element ordered topological space with the discrete topology and the order $\leq$ given by $x \leq y$ if $x = y$ or $x = 0$ and $y = 1$; finally in $\text{PStone}$ it is the two element bitopological space with two Sierpinski topologies $\tau_1$ and $\tau_2 = \{\emptyset, \{0\}, \{0, 1\}\}$.

\[ \text{BITOPOLITICAL DUALITY FOR DISTRIBUTIVE LATTICES AND HEYTING ALGEBRAS 13] \]
6. Duality

In this section we use the isomorphism of Pries, PStone, and Spec, and their dual equivalence to DLat to obtain the dual description of the algebraic concepts important for the study of distributive lattices. In particular, we give the dual descriptions of filters, ideals, homomorphic images, sublattices, canonical completions, and MacNeille completions of bounded distributive lattices. We also give the dual description of complete distributive lattices. The dual description of these concepts by means of Priestley spaces is known. Some of these concepts have also been described by means of spectral spaces. We complete the picture by giving the spectral description of the remaining concepts as well as describing them all by means of pairwise Stone spaces. At the end of the section we give a table, which serves as a dictionary of duality theory for distributive lattices, complementing the dictionary given in [27].

6.1. Filters and ideals. We start by the dual description of filters, prime filters, and maximal filters, as well as ideals, prime ideals, and maximal ideals of bounded distributive lattices by means of Priestley spaces.

Let \( L \) be a bounded distributive lattice and let \((X, \tau, \subseteq)\) be the Priestley space of \( L \). We recall that the poset \((\text{Fi}(L), \supseteq)\) of filters of \( L \) is isomorphic to the poset \((\text{ClUp}(X), \subseteq)\) of closed upsets of \( X \), that the poset \((\text{Id}(L), \subseteq)\) of ideals of \( L \) is isomorphic to the poset \((\text{OpUp}(X), \subseteq)\) of open upsets of \( X \), and that the isomorphisms are obtained as follows. With each filter \( F \) of \( L \) we associate the closed upset \( C_F = \bigcap \{ \varphi(a) \mid a \in F \} \subseteq X \), and with each closed upset \( C \) of \( X \) we associate the filter \( F_C = \{ a \in L \mid C \subseteq \varphi(a) \} \subseteq L \). Then \( F \subseteq G \) if \( F_C \supseteq G_C \). Let \((\text{Fi}(L), \supseteq)\) be isomorphic to \((\text{ClUp}(X), \subseteq)\). Also, with each ideal \( I \) of \( L \) we associate the open upset \( U_I = \bigcup \{ \varphi(a) \mid a \in I \} \subseteq X \), and with each open upset \( U \) of \( X \) we associate the ideal \( I_U = \{ a \in L \mid \varphi(a) \subseteq U \} \subseteq L \). Then \( I \subseteq J \) if \( U_I \subseteq U_J \), \( U_{I_U} = U \). Thus, \((\text{Id}(L), \subseteq)\) is isomorphic to \((\text{OpUp}(X), \subseteq)\). Let \((X, \tau_1, \tau_2)\) be the Priestley Stone space corresponding to \((X, \tau, \subseteq)\). By Proposition 3.6, \( \beta_1 = \text{CpUp}(X) \) and \( \beta_2 = \text{CpDo}(X) \). Therefore, \( \tau_1 = \text{OpUp}(X) \) and \( \tau_2 = \text{OpDo}(X) \), and so \( \delta_1 = \text{ClDo}(X) \) and \( \delta_2 = \text{ClUp}(X) \). Thus, \((\text{Fi}(L), \supseteq)\) is isomorphic to \((\delta_2, \subseteq)\) and \((\text{Id}(L), \subseteq)\) is isomorphic to \((\tau_1, \subseteq)\). Let \((X, \tau_1)\) be the spectral space corresponding to \((X, \tau_1, \tau_2)\). Then clearly \((\text{Id}(L), \subseteq)\) is isomorphic to the poset of \( \tau_1 \)-open sets. In order to characterize \((\text{Fi}(L), \supseteq)\) in terms of \((X, \tau_1)\), we recall [14, Def. O.5.3] that a subset \( A \) of a topological space is saturated if it is an intersection of open subsets of the space; alternatively, \( A \) is saturated if it is an upset in the specialization order. We define \( A \) to be co-saturated if \( A \) is a union of closed subsets; alternatively, \( A \) is co-saturated if it is a downset in the specialization order.

Let \((X, \tau, \subseteq)\) be a Priestley space, \((X, \tau_1, \tau_2)\) be the corresponding pairwise Stone space, and \((X, \tau_1)\) be the corresponding spectral space. Then it is clear that for \( A \subseteq X \), we have that the following four conditions are equivalent: (i) \( A \) is an upset of \((X, \tau, \subseteq)\), (ii) \( A \) is a \( \tau_1 \)-saturated subset of \((X, \tau_1, \tau_2)\), (iii) \( A \) is a \( \tau_2 \)-co-saturated subset of \((X, \tau_1, \tau_2)\), and (iv) \( A \) is a saturated subset of \((X, \tau_1)\). Similarly, for \( B \subseteq X \), we have that the following four conditions are equivalent: (i) \( B \) is a downset of \((X, \tau, \subseteq)\), (ii) \( B \) is a \( \tau_1 \)-co-saturated subset of \((X, \tau_1, \tau_2)\), (iii) \( B \) is a \( \tau_2 \)-saturated subset of \((X, \tau_1, \tau_2)\), and (iv) \( B \) is a co-saturated subset of \((X, \tau_1)\).

For a pairwise Stone space \((X, \tau_1, \tau_2)\) and for \( i = 1, 2 \), let \( S_i(X) \) denote the set of \( \tau_i \)-saturated sets and \( \text{CS}_i(X) \) denote the set of \( \tau_i \)-co-saturated sets. Then \( \text{Up}(X) = S_1(X) = \text{CpUp}(X) \) and \( \text{Id}(X) = \text{CpDo}(X) \).
\( \text{CS}_2(X) \) and \( \text{Do}(X) = \text{CS}_1(X) = \text{S}_2(X) \). This gives us the following characterization of closed upsets and closed downsets of \((X, \tau, \leq)\).

**Theorem 6.1.** Let \((X, \tau, \leq)\) be a Priestley space, \((X, \tau_1, \tau_2)\) be the corresponding pairwise Stone space, and \((X, \tau_1)\) be the corresponding spectral space. For \(C \subseteq X\), the following conditions are equivalent:

1. \(C\) is a closed upset of \((X, \tau, \leq)\).
2. \(C\) is a \(\tau_2\)-closed set of \((X, \tau_1, \tau_2)\).
3. \(C\) is a compact saturated set of \((X, \tau_1)\).

**Proof.** As we already observed, \((1) \Leftrightarrow (2)\) follows from Proposition 3.6. Next we show that \((1) \Rightarrow (3)\). Since \(C\) is an upset of \(X\), \(C\) is saturated in \((X, \tau_1)\). As \(C\) is closed in \((X, \tau)\) and \((X, \tau)\) is Hausdorff, \(C\) is a compact subset of \((X, \tau)\). Therefore, \(C\) is also compact in \((X, \tau_1)\). Thus, \(C\) is compact and saturated in \((X, \tau_1)\). Finally, we show that \((3) \Rightarrow (1)\). Since \(C\) is saturated in \((X, \tau_1)\), \(C\) is an upset of \(X\). We show that \(C\) is closed in \((X, \tau)\). Let \(x \notin C\). Then for each \(c \in C\) we have \(c \notin x\). Therefore, there is a clopen upset \(U_c\) of \(X\) such that \(c \in U_c\) and \(x \notin U_c\). Thus, \(C \subseteq \bigcup \{U_c \mid c \in C\}\). By Propositions 3.6 and 4.2, each \(U_c\) belongs to \(\mathcal{E}(X, \tau_1)\). Since \(C\) is compact, there are \(c_1, \ldots, c_n \in C\) such that \(C \subseteq U_{c_1} \cup \cdots \cup U_{c_n}\). But then \(V = U_{c_1} \cap \cdots \cap U_{c_n}\) is a clopen downset of \(X\) containing \(x\) and having the empty intersection with \(C\). Thus, \(C\) is closed. 

A similar argument gives us:

**Theorem 6.2.** Let \((X, \tau, \leq)\) be a Priestley space, \((X, \tau_1, \tau_2)\) be the corresponding pairwise Stone space, and \((X, \tau_1)\) be the corresponding spectral space. For \(D \subseteq X\), the following conditions are equivalent:

1. \(D\) is a closed downset of \((X, \tau, \leq)\).
2. \(D\) is a \(\tau_1\)-closed set of \((X, \tau_1, \tau_2)\).
3. \(D\) is a compact saturated set of \((X, \tau_2)\).

For a pairwise Stone space \((X, \tau_1, \tau_2)\) and \(i = 1, 2\), let \(\text{KS}_i(X)\) denote the set of compact saturated subsets of \(X\). Then the following characterization of filters and ideals of a bounded distributive lattice is an immediate consequence of the results obtained above.

**Corollary 6.3.** Let \(L\) be a bounded distributive lattice, \((X, \tau, \leq)\) be its Priestley space, \((X, \tau_1, \tau_2)\) be its pairwise Stone space, and \((X, \tau_1)\) be its spectral space. Then:

1. \((\text{Fi}(L), \supseteq) \cong (\text{ClUp}(X), \subseteq) = (\delta_2, \subseteq) = (\text{KS}_1(X), \subseteq)\).
2. \((\text{Id}(L), \subseteq) \cong (\text{OpUp}(X), \subseteq) = (\tau_1, \subseteq)\).

**Remark 6.4.** Corollary 6.3(1) is a particular case of the celebrated Hofmann-Mislove theorem. To see this, let \(X\) be a sober space. We recall that a filter \(F\) of the lattice \(\tau\) of open subsets of \(X\) is Scott open if for a family \(\{U_i \mid i \in I\}\) of open subsets of \(X\), from \(\bigcup \{U_i \mid i \in I\} \in F\) it follows that there exist \(i_1, \ldots, i_n \in I\) such that \(U_{i_1} \cup \cdots \cup U_{i_n} \in F\). Let \(\text{SFi}(\tau)\) denote the set of Scott open filters of \(\tau\). Then the Hofmann-Mislove theorem states that \((\text{SFi}(\tau), \supseteq)\) is isomorphic to \((\text{KS}(X), \subseteq)\). Observing that if \(X\) is spectral, then \((\text{SFi}(\tau), \supseteq)\) is actually isomorphic to \((\text{Fi}(\mathcal{E}(X)), \supseteq)\), we see that Corollary 6.3(1) expresses the Hofmann-Mislove theorem in the particular case of spectral spaces.

Now we turn to the dual description of prime filters and prime ideals of \(L\). Let \((X, \tau, \leq)\) be the Priestley space of \(L\). It is well-known that a filter \(F\) of \(L\) is prime iff \(C_F = \uparrow x\) for
some $x \in X$, and that an ideal $I$ of $L$ is prime iff $U_I = (\downarrow x)^c$ for some $x \in X$. Now we give the dual description of prime filters and prime ideals of $L$ by means of pairwise Stone and spectral spaces of $L$.

**Lemma 6.5.** Let $(X, \tau, \leq)$ be a Priestley space, $(X, \tau_1, \tau_2)$ be the corresponding pairwise Stone space, and $(X, \tau_1)$ be the corresponding spectral space. Then for each $A \subseteq X$ we have:

1. $\text{Cl}_1(A) = \downarrow \text{Cl}(A)$.
2. $\text{Cl}_2(A) = \uparrow \text{Cl}(A)$.

**Proof.** (1) We have $\text{Cl}_1(A) = \bigcap \{B \in \delta_1 \mid A \subseteq B\} = \bigcap \{B \in \text{ClDo}(X) \mid A \subseteq B\}$. By Lemma 3.2(2), $\downarrow \text{Cl}(A)$ is a closed downset, and clearly $A \subseteq \downarrow \text{Cl}(A)$. Therefore, $\text{Cl}_1(A) \subseteq \downarrow \text{Cl}(A)$. Conversely, suppose that $x \notin \text{Cl}_1(A)$. Then there is $U \in \tau_1$ such that $x \in U$ and $U \cap A = \emptyset$. Since $\tau_1 = \text{OpUp}(X)$, then $U$ is an open upset of $X$. As $U$ is open in $(X, \tau)$, from $U \cap A = \emptyset$ it follows that $U \cap \text{Cl}(A) = \emptyset$. Because $U$ is an upset, $U \cap \text{Cl}(A) = \emptyset$ implies $U \cap \downarrow \text{Cl}(A) = \emptyset$. Thus, $x \notin \downarrow \text{Cl}(A)$, and so $\text{Cl}_1(A) = \downarrow \text{Cl}(A)$.

(2) is proved similarly. \hfill –

Let $(X, \tau_1, \tau_2)$ be a bitopological space. Following [14, Def. O-5.3], for $A \subseteq X$ and $i = 1, 2$, we define the $\tau_i$-saturation of $A$ as $\text{Sat}_i(A) = \bigcap \{U \in \tau_i \mid A \subseteq U\}$. Obviously $\text{Sat}_1(A) = \uparrow_1 A$ and $\text{Sat}_2(A) = \downarrow_2 A$. This immediately gives us the following corollary to Lemma 6.5.

**Corollary 6.6.** Let $(X, \tau, \leq)$ be a Priestley space, $(X, \tau_1, \tau_2)$ be the corresponding pairwise Stone space, and $(X, \tau_1)$ be the corresponding spectral space. Then for each closed set $A$ of $(X, \tau)$ we have:

1. $\downarrow A = \text{Cl}_1(A) = \text{Sat}_2(A)$.
2. $\uparrow A = \text{Cl}_2(A) = \text{Sat}_1(A)$.

In particular, for each $x \in X$ we have:

1. $\downarrow x = \text{Cl}_1(x) = \text{Sat}_2(x)$.
2. $\uparrow x = \text{Cl}_2(x) = \text{Sat}_1(x)$.

Putting these results together, we obtain the following dual description of prime filters and prime ideals of $L$.

**Corollary 6.7.** Let $L$ be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, $(X, \tau_1, \tau_2)$ be its pairwise Stone space, and $(X, \tau_1)$ be its spectral space. For a filter $F$ of $L$, the following conditions are equivalent:

1. $F$ is a prime filter of $L$.
2. $C_F = \uparrow x$ for some $x \in X$.
3. $C_F = \text{Cl}_2(x)$ for some $x \in X$.
4. $C_F = \text{Sat}_1(x)$ for some $x \in X$.

Also, for an ideal $I$ of $L$, the following conditions are equivalent:

1. $I$ is a prime ideal of $L$.
2. $U_I = (\downarrow x)^c$ for some $x \in X$.
3. $U_I = [\text{Cl}_1(x)]^c$ for some $x \in X$.
4. $U_I = [\text{Sat}_2(x)]^c$ for some $x \in X$.

Another consequence of our results is the dual description of maximal filters and maximal ideals of $L$. Let $(X, \tau, \leq)$ be the Priestley space of $L$. We let $\max X$ and $\min X$ denote the sets of maximal and minimal points of $X$, respectively. From the dual description of prime filters
and prime ideals of \( L \) it immediately follows that a filter \( F \) of \( L \) is maximal if and only if \( C_F = \{ x \} = \uparrow x \) for some \( x \in \text{max}X \), and that an ideal \( I \) of \( L \) is maximal if and only if \( U_I = \{ x \}^c = (\downarrow x)^c \) for some \( x \in \text{min}X \). This together with the above corollary immediately give us:

**Corollary 6.8.** Let \( L \) be a bounded distributive lattice, \((X, \tau, \leq)\) be its Priestley space, \((X, \tau_1, \tau_2)\) be its pairwise Stone space, and \((X, \tau_1)\) be its spectral space. For a filter \( F \) of \( L \), the following conditions are equivalent:

1. \( F \) is a maximal filter of \( L \).
2. \( C_F = \{ x \} \) for some \( x \in X \) with \( \uparrow x = \{ x \} \).
3. \( C_F = \{ x \} \) for some \( x \in X \) with \( \text{Cl}_1(x) = \{ x \} \).
4. \( C_F = \{ x \} \) for some \( x \in X \) with \( \text{Sat}_1(x) = \{ x \} \).

Also, for an ideal \( I \) of \( L \), the following conditions are equivalent:

1. \( I \) is a maximal ideal of \( L \).
2. \( U_I = \{ x \}^c \) for some \( x \in X \) with \( \downarrow x = \{ x \} \).
3. \( U_I = \{ x \}^c \) for some \( x \in X \) with \( \text{Cl}_1(x) = \{ x \} \).
4. \( U_I = \{ x \}^c \) for some \( x \in X \) with \( \text{Sat}_2(x) = \{ x \} \).

6.2. Homomorphic images. It is well-known (see, e.g., [27, Cor. 2.5]) that homomorphic images of a bounded distributive lattice \( L \) are in 1-1 correspondence with closed subsets of the Priestley space \((X, \tau, \leq)\) of \( L \). Now we give the dual description of homomorphic images of \( L \) in terms of the pairwise Stone space and spectral space of \( L \).

**Lemma 6.9.** Let \((X, \tau, \leq)\) be a Priestley space and let \((X, \tau_1, \tau_2)\) be its corresponding pairwise Stone space. For \( C \subseteq X \), the following conditions are equivalent.

1. \( C \) is closed in \((X, \tau, \leq)\).
2. \( C \) is compact in \((X, \tau, \leq)\).
3. \( C \) is pairwise compact in \((X, \tau_1, \tau_2)\).

**Proof.** That (1)\(\Leftrightarrow\) (2) is obvious since \((X, \tau)\) is compact and Hausdorff. That (2)\(\Rightarrow\) (3) is straightforward. To see that (3)\(\Rightarrow\) (2), it follows from (3) that each cover \( \{ U_i \mid i \in I \} \) of \( C \), with \( U_i \in \tau_1 \cup \tau_2 \), has a finite subcover. Now use Alexander’s Lemma.

For a topological space \((X, \tau)\) and a subset \( Y \) of \( X \), let \( \tau^Y \) denote the subspace topology on \( Y \); that is, \( \tau^Y = \{ U \cap Y \mid U \in \tau \} \).

**Definition 6.10.** Let \((X, \tau)\) be a spectral space. We call a subset \( Y \) of \( X \) a spectral subset of \( X \) if \((Y, \tau^Y)\) is a spectral space and \( U \in \mathcal{E}(X, \tau) \) implies \( U \cap Y \in \mathcal{E}(Y, \tau^Y) \).

**Theorem 6.11.** Let \((X, \tau_1, \tau_2)\) be a pairwise Stone space and let \((X, \tau_1)\) be its corresponding spectral space. For \( Y \subseteq X \), the following conditions are equivalent.

1. \( Y \) is pairwise compact in \((X, \tau_1, \tau_2)\).
2. \( Y \) is a spectral subset of \((X, \tau_1)\).

**Proof.** (1)\(\Rightarrow\) (2): Since \( Y \) is pairwise compact, by Theorem 6.9, \( Y \) is closed in the corresponding Priestley space \((X, \tau, \leq)\). Let \( \leq^Y \) denote the restriction of \( \leq \) to \( Y \). Then \((Y, \tau^Y, \leq^Y)\) is a Priestley space. By Propositions 3.6 and 4.2, \((Y, \tau_1^Y)\) is a spectral space. Let \( U \in \mathcal{E}(X) \). Again using Propositions 3.6 and 4.2 we obtain \( U \in \text{CpUp}(X, \tau, \leq) \). Therefore, \( U \cap Y \in \text{CpUp}(Y, \tau^Y, \leq^Y) \). Thus, \( U \cap Y \in \mathcal{E}(Y, \tau^Y) \), and so \( Y \) is a spectral subset of \((X, \tau_1)\).

(2)\(\Rightarrow\) (1): Let \( Y \) be a spectral subset of \((X, \tau_1)\) and let \( \Delta(Y, \tau_1^Y) = \{ Y - U \mid U \in \mathcal{E}(Y, \tau^Y) \} \). We show that \( \tau_2^Y \) is the topology generated by \( \Delta(Y, \tau_1^Y) \). For this we show that \( \mathcal{E}(Y, \tau^Y) = \)
{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)}$. Since $Y$ is a spectral subset, we have \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1}\} \subseteq \mathcal{E}(Y, \tau_Y)$. Conversely, suppose that $U \in \mathcal{E}(Y, \tau_Y)$. Then there is $V \in \tau_1$ such that $U = V \cap Y$. From $V \in \tau_1$ it follows that $V = \bigcup \{V_i \mid i \in I\}$ for some family $\{V_i \mid i \in I\} \subseteq \mathcal{E}(X, \tau_1)$. Then $U = \bigcup \{V_i \mid i \in I\} \cap Y = \bigcup \{V_i \cap Y \mid i \in I\}$. Since $U$ is compact and $V_i \cap Y$ are open in $(Y, \tau_Y)$, there exist $i_1, \ldots, i_n \in I$ such that $U = (V_{i_1} \cap Y) \cup \cdots \cup (V_{i_n} \cap Y) = (V_{i_1} \cup \cdots \cup V_{i_n}) \cap Y$. Let $W = V_{i_1} \cup \cdots \cup V_{i_n}$. Since $\mathcal{E}(X, \tau_1)$ is closed under finite unions, $W \in \mathcal{E}(X, \tau_1)$. Therefore, $U = W \cap Y$ for some $W \in \mathcal{E}(X, \tau_1)$. Thus, $\mathcal{E}(Y, \tau_Y) \subseteq \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$, and so $\mathcal{E}(Y, \tau_Y) = \{U \cap Y \mid U \in \mathcal{E}(X, \tau_1)\}$. Consequently, $\Delta(Y, \tau_Y) = \{Y - U \mid U \in \mathcal{E}(Y, \tau_Y)\} = \{Y - (V \cap Y) \mid V \in \mathcal{E}(X, \tau_1)\} = \{Y - V \mid V \in \mathcal{E}(X, \tau_1)\}$, and so $\tau_Y$ is the topology generated by $\Delta(Y, \tau_Y)$. Now, since $(Y, \tau_Y)$ is a spectral space, by Proposition 4.5, $(Y, \tau_Y)$ is pairwise compact. It follows that $Y$ is pairwise compact in $(X, \tau_1, \tau_2)$.

Now putting the above results together, we obtain the following dual description of homomorphic images of $L$ by means of all three dual spaces of $L$.

**Corollary 6.12.** Let $L$ be a bounded distributive lattice, $(X, \tau, \leq)$ be its Priestley space, $(X, \tau_1, \tau_2)$ be its pairwise Stone space, and $(X, \tau_1)$ be its spectral space. Then there is a 1-1 correspondence between (i) homomorphic images of $L$, (ii) closed subsets of $(X, \tau, \leq)$, (iii) pairwise compact subsets of $(X, \tau_1, \tau_2)$, and (iv) spectral subsets of $(X, \tau_1)$.

**Proof.** As follows from [27, Cor. 2.5], homomorphic images of $L$ are in 1-1 correspondence with closed subsets of $(X, \tau, \leq)$. Lemma 6.9 and Theorem 6.11 imply that closed subsets of $(X, \tau, \leq)$ are in 1-1 correspondence with pairwise compact subsets of $(X, \tau_1, \tau_2)$, which are in 1-1 correspondence with spectral subsets of $(X, \tau_1)$. The result follows.

We conclude this subsection by giving an example of a subset $Y$ of a spectral space $(X, \tau)$ such that $(Y, \tau_Y)$ is a spectral space, but there exists $U \in \mathcal{E}(X, \tau)$ such that $U \cap Y \notin \mathcal{E}(Y, \tau_Y)$. Therefore, the condition “$U \in \mathcal{E}(X, \tau)$ implies $U \cap Y \in \mathcal{E}(Y, \tau_Y)$” can not be omitted from Definition 6.10.

**Example 6.13.** Let $(X, \tau)$ be the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the interval topology. Then each $n \in \omega$ is an isolated point of $X$ and $\omega$ is the only limit point of $X$. For $x, y \in X$ we set $x \leq y$ iff $x = y$ or $x = 0$ and $y = \omega$ (see Figure 1). It is easy to verify that $(X, \tau, \leq)$ is a Priestley space. Let $(X, \tau_1, \tau_2)$ be the corresponding pairwise Stone space and $(X, \tau_1)$ be the corresponding spectral space. We let $Y = X - \{\omega\}$. Then $(Y, \tau_Y)$ is a spectral space. On the other hand, $U = X - \{0\}$ is compact open in $(X, \tau_1)$, however $U \cap Y = \omega - \{0\}$ is not compact in $(Y, \tau_Y)$. Therefore, $Y$ is not a spectral subset of $(X, \tau_1)$.

### 6.3. Sublattices.

The dual description of bounded sublattices of a bounded distributive lattice by means of its Priestley space can be found in [2, 5, 31]. We will rephrase it in our terminology. We recall that a *quasi-order* $Q$ on a set $X$ is a reflexive and transitive relation on $X$. We call the pair $(X, Q)$ a *quasi-ordered set*. For a quasi-ordered set $(X, Q)$, we call $A \subseteq X$ a *$Q$-upset* of $X$ if $x \in A$ and $xQy$ imply $y \in A$. 

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**Figure 1**

![Diagram showing the relationship between different types of spaces in a lattice](image-url)
**Definition 6.14.** Let $X$ be a topological space and $Q$ be a quasi-order on $X$. We call $Q$ a Priestley quasi-order on $X$ if for each $x, y \in X$ with $x \not\leq y$ there exists a clopen $Q$-upset $A$ of $X$ such that $x \in A$ and $y \notin A$.

**Theorem 6.15.** [31, Thm. 3.7] Let $L$ be a bounded distributive lattice and $(X, \tau, \leq)$ be the Priestley space of $L$. Then there is a dual isomorphism between the poset $(S_L, \subseteq)$ of bounded sublattices of $L$ and the poset $(Q_X, \subseteq)$ of Priestley quasi-orders on $X$ extending $\leq$.

**Proof.** (Sketch) For $S \in S_L$, we define $Q_S$ on $X$ by $xQ_S y$ if $x \cap S \subseteq y \cap S$. Then $Q_S \in Q_X$, and $S \subseteq K$ implies $Q_K \subseteq Q_S$ for each $S, K \in S_L$. Therefore, $S \mapsto Q_S$ is an order-reversing map from $S_L$ to $Q_X$. For $Q \in Q_X$, we let $S_Q = \{ a \in L \mid \phi(a) \in Q \}$, where $\phi: L \to Q$ is a $Q$-upset of $X$. Then $S_Q$ is a bounded sublattice of $L$, and $Q \subseteq R$ implies $S_R \subseteq S_Q$ for each $Q, R \in Q_X$. Thus, $Q \mapsto S_Q$ is an order-reversing map from $Q_X$ to $S_L$. Moreover, $Q_{S_Q} = S$ and $Q_{S_Q} = Q$ for each $S \in S_L$ and $Q \in Q_X$. It follows that the order-reversing maps $S \mapsto Q_S$ and $Q \mapsto S_Q$ are inverses of each other. Consequently, $(S_L, \subseteq)$ is dually isomorphic to $(Q_X, \subseteq)$.

Now we characterize Priestley quasi-orders extending $\leq$ by means of pairwise Stone spaces and spectral spaces.

**Definition 6.16.** Let $(\tau_1, \tau_2)$ and $(\tau'_1, \tau'_2)$ be two bitopologies on $X$. We say that $(\tau_1, \tau_2)$ is finer than $(\tau'_1, \tau'_2)$ and that $(\tau'_1, \tau'_2)$ is coarser than $(\tau_1, \tau_2)$ if $\tau'_1 \subseteq \tau_1$ and $\tau'_2 \subseteq \tau_2$.

**Lemma 6.17.** Let $(X, \tau, \leq)$ be a Priestley space and $(X, \tau_1, \tau_2)$ be the corresponding pairwise Stone spaces. Then the poset $(Q_X, \subseteq)$ of Priestley quasi-orders on $X$ is dually isomorphic to the poset $(Z_X, \subseteq)$ of pairwise zero-dimensional bi-topologies on $X$ coarser than $(\tau_1, \tau_2)$.

**Proof.** For a Priestley quasi-order $Q$ on $X$, let $\tau^Q_1$ be the set of open $Q$-upsets and $\tau^Q_2$ be the set of open $Q$-downsets of $X$. Clearly $(\tau^Q_1, \tau^Q_2)$ is a bitopology on $X$ coarser than $(\tau_1, \tau_2)$. Moreover, $\beta^Q_1 = \tau^Q_1 \cap \delta_2^Q$ is exactly the set of clopen $Q$-upsets of $X$ and $\beta^Q_2 = \tau^Q_2 \cap \delta_2^Q$ is exactly the set of clopen $Q$-downsets of $X$. Since $Q$ is a Priestley quasi-order, clopen $Q$-upsets are a basis for open $Q$-upsets and clopen $Q$-downsets are a basis for open $Q$-downsets. Therefore, $(\tau^Q_1, \tau^Q_2)$ is pairwise zero-dimensional. For two Priestley quasi-orders $Q$ and $R$ on $X$, we show $Q \subseteq R$ implies $\tau^R_1 \subseteq \tau^Q_1$ and $\tau^R_2 \subseteq \tau^Q_2$. Let $U \in \tau^R_1$. Then $U$ is an open $R$-upset of $X$. Since $Q \subseteq R$, $U$ is also a $Q$-upset of $X$. Thus, $U \in \tau^Q_1$. That $\tau^R_2 \subseteq \tau^Q_2$ is proved similarly. It follows that $Q \mapsto (\tau^Q_1, \tau^Q_2)$ is an order-reversing map from $Q_X$ to $Z_X$.

Let $(\tau'_1, \tau'_2)$ be a pairwise zero-dimensional bitopology on $X$ coarser than $(\tau_1, \tau_2)$. We define $Q_{(\tau'_1, \tau'_2)}$ to be the specialization order of $\tau'_1$. Since $(\tau'_1, \tau'_2)$ is pairwise zero-dimensional, $Q_{(\tau'_1, \tau'_2)}$ is the dual of the specialization order of $\tau'_2$. Because $Q_{(\tau'_1, \tau'_2)}$ is a specialization order, it is clear that $Q_{(\tau'_1, \tau'_2)}$ is a quasi-order. From $\tau'_1 \subseteq \tau_1$ it follows that $Q_{(\tau'_1, \tau'_2)}$ extends the specialization order of $\tau_1$. Consequently, $Q_{(\tau'_1, \tau'_2)}$ extends $\leq$. We show that $Q_{(\tau'_1, \tau'_2)}$ is a Priestley quasi-order. If $xQ_{(\tau'_1, \tau'_2)} y$, then there exists $U \in \tau'_1$ such that $x \in U$ and $y \notin U$. Since $(\tau'_1, \tau'_2)$ is pairwise zero-dimensional, we may assume that $U \in \beta'_1$. Therefore, $U$ is clopen in $\tau$. Clearly each $U \in \tau'_1$ is a $Q_{(\tau'_1, \tau'_2)}$-upset. Thus, there exists a clopen $Q_{(\tau'_1, \tau'_2)}$-upset $U$ of $X$ such that $x \in U$ and $y \notin U$. For $(\tau'_1, \tau'_2)$, $(\tau''_1, \tau''_2) \in Z_X$, we show $(\tau'_1, \tau'_2) \subseteq (\tau''_1, \tau''_2)$ implies $Q_{(\tau''_1, \tau''_2)} \subseteq Q_{(\tau'_1, \tau'_2)}$. Let $xQ_{(\tau''_1, \tau''_2)} y$. Then $x \in U$ implies $y \in U$ for each $U \in \tau''_1$. Therefore, $x \in U$ implies $y \in U$ for each $U \in \tau'_1$. Thus, $xQ_{(\tau'_1, \tau'_2)} y$. It follows that $(\tau'_1, \tau'_2) \mapsto Q_{(\tau'_1, \tau'_2)}$ is an order-reversing map from $Z_X$ to $Q_X$.

We show that $Q_{(\tau'_1, \tau'_2)} = Q$ and $(\tau^Q_{(\tau'_1, \tau'_2)}, \tau^Q_{(\tau'_1, \tau'_2)}) = (\tau'_1, \tau'_2)$ for each $Q \in Q_X$ and $(\tau'_1, \tau'_2) \in Z_X$. Indeed, $xQ_{(\tau'_1, \tau'_2)} y$ if $(\forall U \in \tau'_1)(x \in U \Rightarrow y \in U)$, which is equivalent to $xQy$ since
$Q$ is a Priestley quasi-order. Thus, $Q_{(\tau_1, \tau_2^o)} = Q$. Moreover, $U \in \tau_1^{Q(\tau_1, \tau_2^o)}$ iff $U$ is an open $Q_{(\tau_1, \tau_2^o)}$-upset of $X$. Clearly $U \in \tau_1'$ implies $U$ is an open $Q_{(\tau_1', \tau_2^o)}$-upset of $X$. Conversely, let $U$ be an open $Q_{(\tau_1, \tau_2^o)}$-upset of $X$. We show that $U = \bigcup \{ V \in \tau_1' \mid V \subseteq U \}$. Clearly $\bigcup \{ V \in \tau_1' \mid V \subseteq U \} \subseteq U$. Let $x \in U$. Since $U$ is a $Q_{(\tau_1, \tau_2^o)}$-upset, for each $y \in U^c$ we have $x \notin Q_{(\tau_1, \tau_2^o)} y$. Therefore, there exists $V_y \in \tau_1'$ such that $x \in V_y$ and $y \notin V_y$. Since $\beta_1'$ is a basis for $\tau_1'$, we may assume that $V_y \in \beta_1'$. Thus, $\bigcap \{ V \in U \mid y \in U \} \cap U^c = \emptyset$. Since $U^c$ and each $V_y$ is closed in $\tau$ and $\tau$ is compact, there exist $V_1, \ldots, V_n \in \beta_1'$ such that $V_1 \cap \ldots \cap V_n \cap U^c = \emptyset$. So $x \in V_1 \cap \ldots \cap V_n \subseteq U$, and so $U \subseteq \bigcup \{ V \in \tau_1' \mid V \subseteq U \}$.

Consequently, $U \in \tau_1'$, which implies that $\tau_1^{Q(\tau_1, \tau_2^o)} = \tau_1'$. A similar argument shows that $\tau_1^{Q(\tau_1', \tau_2^o)} = \tau_1''$. Thus, $(\tau_1^{Q(\tau_1, \tau_2^o)}, \tau_1^{Q(\tau_1', \tau_2^o)}) \cong (\tau_1', \tau_2^o)$. It follows that the order-reversing maps $Q \hookrightarrow (\tau_1', \tau_2^o)$ and $(\tau_1', \tau_2^o) \hookrightarrow Q_{(\tau_1, \tau_2^o)}$ are inverses of each other. Thus, $(Q_X, \subseteq)$ is dually isomorphic to $(Z_X, \subseteq)$.

**Definition 6.18.** Let $\tau$ be a spectral topology on $X$ and let $\tau'$ be a coherent topology on $X$ coarser than $\tau$. We call $\tau'$ strongly coherent if the set $E(X, \tau')$ of compact open subsets of $(X, \tau')$ is equal to the set $\tau' \cap \sigma$ of open subsets of $(X, \tau')$ that are compact in $(X, \tau)$.

**Lemma 6.19.** Let $(X, \tau_1, \tau_2)$ be a pairwise Stone space and $(X, \tau_1)$ be the corresponding spectral space. Then the poset $(Z_X, \subseteq)$ of pairwise zero-dimensional bitopologies $\tau_1, \tau_2)$ on $X$ coarser than $(\tau_1, \tau_2)$ is isomorphic to the poset $(SC_X, \subseteq)$ of strongly coherent topologies $\tau_1'$ on $X$ coarser than $\tau_1$.

**Proof.** Let $(\tau_1', \tau_2')$ be a pairwise zero-dimensional bitopology on $X$ coarser than $(\tau_1, \tau_2)$. Then $\tau_1'$ is a topology on $X$ coarser than $\tau_1$. Let $\beta_1' = \tau_1' \cap \delta_1'$. We show that $E(X, \tau_1') = \beta_1' = \tau_1' \cap \sigma_1$. Let $U \in E(X, \tau_1')$. Since $\beta_1'$ is a basis for $\tau_1'$, $U$ is the union of elements of $\beta_1'$ contained in $U$. As $U$ is compact in $(X, \tau_1')$, $U$ is a finite union of elements of $\beta_1'$, so $U$ is an element of $\beta_1'$, and so $E(X, \tau_1') \subseteq \beta_1'$. Now let $U \in \beta_1'$. Because $(X, \tau_1, \tau_2)$ is pairwise compact, $\delta_2 \subseteq \sigma_1$. Therefore, $\delta_2' \subseteq \delta_2 \subseteq \sigma_1$, and so $\beta_1' \subseteq \tau_1' \cap \delta_2 \subseteq \tau_1' \cap \sigma_1$. Finally, let $U \in \tau_1' \cap \sigma_1$. Since $U \in \tau_1'$ and $E(X, \tau_1')$ is a basis for $\tau_1'$, $U$ is the union of elements of $E(X, \tau_1')$ contained in $U$. Because $U \in \sigma_1$ and $\tau_1' \subseteq \tau_1$, $U$ is a finite union of elements of $E(X, \tau_1')$. Therefore, $U \in E(X, \tau_1')$, and so $\tau_1' \cap \sigma_1 \subseteq E(X, \tau_1')$. Thus, $E(X, \tau_1') = \beta_1' = \tau_1' \cap \sigma_1$, implying that $\tau_1'$ is a strongly coherent topology. For $(\tau_1', \tau_2') \in Z_X$, if $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$, then it is obvious that $\tau_1' \subseteq \tau_1''$. It follows that $(\tau_1', \tau_2') \mapsto \tau_1'$ is an order-preserving map from $Z_X$ to $SC_X$.

For a strongly coherent topology $\tau_1'$ on $X$ coarser than $\tau_1$, let $\tau_2'$ be the topology generated by the basis $\Delta(X, \tau_1') = \{ U^c \mid U \in E(X, \tau_1') \}$. Let $\delta_1'$ denote the set of closed subsets of $(X, \tau_1')$ and $\delta_2'$ denote the set of closed subsets of $(X, \tau_2')$. We set $\beta_1' = \tau_1' \cap \delta_2'$ and $\beta_2' = \tau_2' \cap \delta_1'$. We show that $\beta_1' = E(X, \tau_1')$ and $\beta_2' = \Delta(X, \tau_1')$. It follows from the definition that $E(X, \tau_1') \subseteq \beta_1'$. Conversely, $\beta_1' = \tau_1' \cap \delta_2' \subseteq \tau_1' \cap \delta_2 \subseteq \tau_1' \cap \sigma_1 = E(X, \tau_1')$. Therefore, $\beta_1' = E(X, \tau_1')$. Also, $U \in E(X, \tau_1')$ if $U^c \in E(X, \tau_1')$ if $U \in \beta_1'$ if $U \in \beta_2'$ if $U \in \delta_1' \cap \tau_2'$. Thus, $\beta_2' = \Delta(X, \tau_1')$. Consequently, $\beta_1'$ is a basis for $\tau_1'$ and $\beta_2'$ is a basis for $\tau_2'$, so $(\tau_1', \tau_2')$ is pairwise zero-dimensional. For $\tau_1', \tau_2' \in SC_X$, we show $\tau_1' \subseteq \tau_2'$ implies $(\tau_1', \tau_2') \subseteq (\tau_1'', \tau_2'')$. Let $U \in \Delta(X, \tau_1')$. Then $U^c \in E(X, \tau_1')$. Therefore, $U \in \tau_1' \cap \sigma_1 \subseteq \tau_1'' \cap \sigma_1$, and so $U \in E(X, \tau_1'')$. Thus, $U \in \Delta(X, \tau_1'')$, so $\Delta(X, \tau_1') \subseteq \Delta(X, \tau_1'')$, and so $\tau_1' \subseteq \tau_1'''$. It follows that $\tau_1' \mapsto (\tau_1'', \tau_2')$ is an order-preserving map from $SC_X$ to $Z_X$.

Finally, if $(\tau_1', \tau_2') \in Z_X$, then $E(X, \tau_1') = \beta_1'$, so $\Delta(X, \tau_1') = \beta_2'$, and so the composition $Z_X \to SC_X \to Z_X$ is an identity. Moreover, it is clear that the composition $SC_X \to Z_X \to SC_X$ is also an identity. Thus, $(Z_X, \subseteq)$ is isomorphic to $(SC_X, \subseteq)$. 

\[\blacksquare\]
Putting Theorem 6.15 and Lemmas 6.17 and 6.19 together, we obtain the following dual description of bounded sublattices of $L$ by means of all three dual spaces of $L$.

**Corollary 6.20.** Let $L$ be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of $L$, $(X, \tau_1, \tau_2)$ be the pairwise Stone space of $L$, and $(X, \tau_1)$ be the spectral space of $L$. Then the poset $(S_L, \subseteq)$ of bounded sublattices of $L$ is dually isomorphic to the poset $(Q_X, \subseteq)$ of Priestley quasi-orders on $X$ extending $\leq$, and is isomorphic to the poset $(Z_X, \subseteq)$ of pairwise zero-dimensional bitopologies on $X$ coarser than $(\tau_1, \tau_2)$, and to the poset $(\mathcal{SC}_X, \subseteq)$ of strongly coherent topologies on $X$ coarser than $\tau_1$.

### 6.4. Canonical completions, MacNeille completions, and complete lattices.

In the theory of completions of lattices, or more generally of posets, the MacNeille and canonical completions play a prominent role. Let $L$ be a lattice. We recall that a subset $S$ of $L$ is join-dense in $L$ if for each $a \in L$ we have $a = \bigvee \{ a \cap S \}$, and that $S$ is meet-dense in $L$ if for each $a \in L$ we have $a = \bigwedge \{ a \cap S \}$. We further recall that the MacNeille completion of $L$ is a unique up to isomorphism complete lattice $\overline{L}$, together with a lattice embedding $i : L \rightarrow \overline{L}$ such that $i[L]$ is both join-dense and meet-dense in $L$. Furthermore, we recall that the canonical completion of $L$ is a unique up to isomorphism complete lattice $L'$ together with a lattice embedding $j : L \rightarrow L'$ such that (i) for each filter $F$ and ideal $I$ of $L$, from $F \cap I = \emptyset$, it follows that $\bigwedge j[F] \nleq \bigvee j[I]$, (ii) the set $K_L = \{ \bigwedge j[S] \mid S \subseteq L \}$ of closed elements of $L'$ is join-dense in $L'$, and (iii) the set $O_L = \{ \bigvee j[S] \mid S \subseteq L \}$ of open elements of $L'$ is meet-dense in $L'$.

For a Priestley space $(X, \tau, \leq)$, following [15, Sec. 3], we define two maps $D : \text{OpUp}(X) \rightarrow \text{ClUp}(X)$ and $J : \text{ClUp}(X) \rightarrow \text{OpUp}(X)$ by $D(U) = \uparrow \text{Cl}(U)$ and $J(K) = (\downarrow \text{Int}(K))^c$, for $U \in \text{OpUp}(X)$ and $K \in \text{ClUp}(X)$. Then it follows from [15, Lemma 3.4] that $D$ and $J$ form a Galois connection between $(\text{OpUp}(X), \subseteq)$ and $(\text{ClUp}(X), \supseteq)$. Let $\text{RgOpUp}(X)$ denote the set of fixpoints of $J \circ D$; that is, $\text{RgOpUp}(X) = \{ U \in \text{OpUp}(X) \mid JDU = U \}$. The next theorem is well-known. The first half of it can be found in [15, Thm. 3.5], and the second half in [13, Sec. 2].

**Theorem 6.21.** Let $L$ be a bounded distributive lattice and $(X, \tau, \leq)$ be the Priestley space of $L$. Then $L$ is isomorphic to $\text{RgOpUp}(X)$ and $L'$ is isomorphic to $\text{Up}(X)$.

Let $L$ be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of $L$, $(X, \tau_1, \tau_2)$ be the pairwise Stone space of $L$, and $(X, \tau_1)$ be the spectral space of $L$. Since $\text{Up}(X) = S_1(X) = \text{CS}_2(X)$, we immediately obtain the following dual description of the canonical completion of $L$.

**Theorem 6.22.** Let $L$ be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of $L$, $(X, \tau_1, \tau_2)$ be the pairwise Stone space of $L$, and $(X, \tau_1)$ be the spectral space of $L$. Then $L'$ is isomorphic to $\text{Up}(X) = S_1(X) = \text{CS}_2(X)$.

Let $L$ be a bounded distributive lattice, $(X, \tau, \leq)$ be the Priestley space of $L$, and $(X, \tau_1, \tau_2)$ be the pairwise Stone space of $L$. Since $\text{OpUp}(X) = \tau_1$, $\text{ClUp}(X) = \delta_2$, $D(U) = \text{Cl}_2(U)$, and $J(U) = \text{Int}_1(U)$ for $U \subseteq X$, we obtain that $\text{Cl}_2 : \tau_1 \rightarrow \delta_2$ and $\text{Int}_1 : \delta_2 \rightarrow \tau_1$ form a Galois connection between $(\tau_1, \subseteq)$ and $(\delta_2, \supseteq)$, and so the MacNeille completion $\overline{L}$ of $L$ is isomorphic to the fixpoints of $\text{Int}_1 \circ \text{Cl}_2$, we denote by $\text{RgOp}_{12}(X)$.

Let $(X, \tau_1)$ be the spectral space corresponding to the pairwise Stone space $(X, \tau_1, \tau_2)$. Then $\delta_2 = K_{S_1}(X)$ and $\text{Cl}_2(U) = \text{Sat}_1 \text{Cl}(U)$ for $U \subseteq X$. Let $S_1 = \text{Sat}_1 \circ \text{Cl}$. Then $S_1 : \tau_1 \rightarrow K_{S_1}(X)$ and $\text{Int}_2 : K_{S_1}(X) \rightarrow \tau_1$ form a Galois connection between $(\tau_1, \subseteq)$ and
(KS1(X), ⊇), and so the MacNeille completion \( \overline{L} \) of \( L \) is isomorphic to the fixpoints of \( \text{Int}_1 \circ \text{S}_1 \), we denote by \( \text{SatOp}_1(X) \). Consequently, we obtain the following dual description of the MacNeille completion of \( L \).

**Theorem 6.23.** Let \( L \) be a bounded distributive lattice, \((X, \tau, \leq)\) be the Priestley space of \( L \), \((X, \tau_1, \tau_2)\) be the pairwise Stone space of \( L \), and \((X, \tau_1)\) be the spectral space of \( L \). Then \( \overline{L} \) is isomorphic to \( \text{RgOpUp}(X) = \text{RgOp}_{12}(X) = \text{SatOp}_1(X) \).

The bitopological description of \( \overline{L} \) provides a nice generalization of the characterization of the MacNeille completion of a Boolean algebra \( B \) by means of the regular open subsets of the Stone space \((X, \tau)\) of \( B \). We recall that the regular open subsets of \((X, \tau)\) are exactly the fixpoints of the composition of the maps \( \text{Cl} : \tau \to \delta \) and \( \text{Int} : \delta \to \tau \). When working with a pairwise Stone space \((X, \tau_1, \tau_2)\), we consider the fixpoints of the composition of the maps \( \text{Cl}_2 \) and \( \text{Int}_1 \) between \( \tau_1 \) and \( \delta_2 \), respectively. Therefore, whenever \( \tau_1 = \tau_2 \), the pairwise Stone space \((X, \tau_1, \tau_2)\) becomes the Stone space \((X, \tau)\), where \( \tau = \tau_1 = \tau_2 \). So \( \tau_1 = \tau \), \( \delta_2 = \delta \), \( \text{Cl}_2 = \text{Cl} \), \( \text{Int}_1 = \text{Int} \), and the fixpoints of \( \text{Int}_1 \circ \text{Cl}_2 \) are exactly the regular open subsets of \((X, \tau)\). As a corollary, we obtain the well-known dual description of the MacNeille completion of a Boolean algebra:

**Corollary 6.24.** Let \( B \) be a Boolean algebra and \( X \) be the Stone space of \( B \). Then the MacNeille completion \( \overline{B} \) of \( B \) is isomorphic to the regular open subsets \( \text{RgOp}(X) \) of \( X \).

Since \( L \) is a complete lattice iff \( L \) is isomorphic to \( \overline{L} \), it follows from the construction of \( \overline{L} \) that \( L \) is complete iff in the dual Priestley space \((X, \tau, \leq)\) of \( L \) we have \( \text{RgOpUp}(X) = \text{CpUp}(X) \) (see [26, Prop. 16] and [15, p. 948]). Such Priestley spaces were called extremally order disconnected in [26, p. 521]. This together with Theorem 6.23 immediately give us the following dual description of complete distributive lattices.

**Theorem 6.25.** Let \( L \) be a bounded distributive lattice, \((X, \tau, \leq)\) be the Priestley space of \( L \), \((X, \tau_1, \tau_2)\) be the pairwise Stone space of \( L \), and \((X, \tau_1)\) be the spectral space of \( L \). Then the following conditions are equivalent:

1. \( L \) is complete.
2. \( \text{RgOpUp}(X) = \text{CpUp}(X) \).
3. \( \text{RgOp}_{12}(X) = \beta_1 \).
4. \( \text{SatOp}_1(X) = \mathcal{E}(X, \tau_1) \).

In Table 1 we gather together the dual descriptions of different algebraic concepts for bounded distributive lattices by means of their Priestley spaces, pairwise Stone spaces, and spectral spaces obtained in this section. This can be thought of as a dictionary of duality theory for bounded distributive lattices, complementing the dictionary given in [27].

### 7. Duality for Heyting algebras

A rather natural subclass of the class of distributive lattices is the class of Heyting algebras, which plays an important role in the study of superintuitionistic logics. The first duality for Heyting algebras was developed by Esakia [8]. It is a restricted version of Priestley’s duality. In this section we develop duality for Heyting algebras by means of pairwise Stone spaces and spectral spaces, thus providing the bitopological and spectral alternatives to the Esakia duality.
Let Heyt denote the category of Heyting algebras and Heyting algebra homomorphisms. For a topological space $(X, \tau)$, let $Cp(X)$ denote the set of clopen subsets of $X$.

**Definition 7.1.** Let $(X, \tau, \leq)$ be a Priestley space. We call $(X, \tau, \leq)$ an Esakia space if $A \in Cp(X)$ implies $\downarrow A \in Cp(X)$.

Let $(X, \leq)$ and $(X', \leq')$ be two posets. We recall that a map $f : X \rightarrow X'$ is a $p$-morphism if it is order-preserving and for each $x \in X$ and $x' \in X'$, from $f(x) \leq x'$ it follows that there is $y \in X$ such that $x \leq y$ and $f(y) = x'$. For two Esakia spaces $(X, \tau, \leq)$ and $(X', \tau', \leq')$, we call a map $f : X \rightarrow X'$ an Esakia morphism if it is a continuous $p$-morphism. Let $Esa$ denote the category of Esakia spaces and Esakia morphisms. Then we have the following theorem established in [8]:

**Theorem 7.2.** Heyt is dually equivalent to $Esa$.

In fact, the same functors establishing the dual equivalence of $DLat$ and $Pries$ restricted to Heyt and Esa, respectively, establish the required dual equivalence. In order to describe the pairwise Stone spaces and spectral spaces dual to Heyting algebras, it is sufficient to characterize those pairwise Stone spaces and spectral spaces that correspond to Esakia spaces. As an immediate consequence of Lemma 6.9 and Theorem 6.11, we obtain:

**Lemma 7.3.** Let $(X, \tau, \leq)$ be a Priestley space, $(X, \tau_1, \tau_2)$ be the corresponding pairwise Stone space, and $(X, \tau_1)$ be the corresponding spectral space. For $Y \subseteq X$, the following conditions are equivalent:

1. $Y$ is clopen in $(X, \tau, \leq)$.
2. $Y$ and $Y^c$ are pairwise compact in $(X, \tau_1, \tau_2)$.
3. $Y$ and $Y^c$ are spectral subsets of $(X, \tau_1)$.

Let $(X, \tau_1, \tau_2)$ be a pairwise Stone space. We call $Y \subseteq X$ pairwise clopen if both $Y$ and $Y^c$ are pairwise compact in $(X, \tau_1, \tau_2)$. Let $PC(X)$ denote the set of pairwise clopen subsets of $(X, \tau_1, \tau_2)$.

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**Table 1.** Dictionary for $DLat$, $Pries$, $PStone$, and $Spec$.

<table>
<thead>
<tr>
<th>$DLat$</th>
<th>$Pries$</th>
<th>$PStone$</th>
<th>$Spec$</th>
</tr>
</thead>
<tbody>
<tr>
<td>filter</td>
<td>closed upset</td>
<td>$\tau_2$-closed set</td>
<td>compact saturated set</td>
</tr>
<tr>
<td>ideal</td>
<td>open upset</td>
<td>$\tau$-open set</td>
<td>open set</td>
</tr>
<tr>
<td>prime filter</td>
<td>${x}$</td>
<td>$Cl(x)$</td>
<td>$Sat(x)$</td>
</tr>
<tr>
<td>prime ideal</td>
<td>${x}$</td>
<td>$Cl(x)^c$</td>
<td>$Sat(x)^c$</td>
</tr>
<tr>
<td>maximal filter</td>
<td>${x}$</td>
<td>$Cl(x) = {x}$</td>
<td>$Sat(x) = {x}$</td>
</tr>
<tr>
<td>homomorphic image</td>
<td>closed subset</td>
<td>pairwise compact subset</td>
<td>spectral subset</td>
</tr>
<tr>
<td>subalgebra</td>
<td>$Q \in Q_X$</td>
<td>$(\tau_1, \tau_2) \in Z_X$</td>
<td>$\tau' \in SC_X$</td>
</tr>
<tr>
<td>canonical completion</td>
<td>$Up(X)$</td>
<td>$S_L(X) = CS_2(X)$</td>
<td>$S(L)$</td>
</tr>
<tr>
<td>MacNeille completion</td>
<td>$RgOpUp_1(X)$</td>
<td>$RgOp_{12}(X)$</td>
<td>$SatOp(X)$</td>
</tr>
<tr>
<td>complete lattice</td>
<td>$RgOpUp(X) = CpUp(X)$</td>
<td>$b_1 = RgOp_{12}(X)$</td>
<td>$\mathcal{E}(X) = SatOp(X)$</td>
</tr>
</tbody>
</table>
**Definition 7.4.** Let \((X, \tau_1, \tau_2)\) be a pairwise Stone space. We call \((X, \tau_1, \tau_2)\) a bitopological Esakia space if \(A \in \text{PC}(X)\) implies \(\text{Cl}_1(A) \in \text{PC}(X)\).

For a pairwise Stone space \((X, \tau_1, \tau_2)\), we recall that \(\delta_1\) denotes the collection of closed subsets of \((X, \tau_1)\), that \(\delta_2\) denotes the collection of closed subsets of \((X, \tau_2)\), that \(\beta_1 = \tau_1 \cap \delta_2\), and that \(\beta_2 = \tau_2 \cap \delta_1\).

**Theorem 7.5.** Let \((X, \tau_1, \tau_2)\) be a pairwise Stone space. Then \((X, \tau_1, \tau_2)\) is a bitopological Esakia space iff for each \(A \in \beta_1\) and \(B \in \beta_2\) we have \(\text{Cl}_1(A \cap B) \in \beta_2\).

**Proof.** Let \((X, \tau, \leq)\) be the Priestley space corresponding to \((X, \tau_1, \tau_2)\). Suppose that \((X, \tau_1, \tau_2)\) is a bitopological Esakia space, \(A \in \beta_1\), and \(B \in \beta_2\). Then \(A \in \delta_2\) and \(A^c \in \delta_1\). Therefore, both \(A\) and \(A^c\) are closed in \((X, \tau, \leq)\). A similar argument shows that both \(B\) and \(B^c\) are closed in \((X, \tau, \leq)\). Thus, both \(A \cap B\) and \((A \cap B)^c = A^c \cup B^c\) are closed in \((X, \tau, \leq)\). By Lemma 6.9, both \(A \cap B\) and \((A \cap B)^c\) are pairwise compact in \((X, \tau, \leq)\), implying that \(A \cap B \in \text{PC}(X)\). Since \((X, \tau_1, \tau_2)\) is a bitopological Esakia space, we have \(\text{Cl}_1(A \cap B) \in \text{PC}(X)\). By Lemma 7.3, \(\text{Cl}_1(A \cap B)\) is clopen in \((X, \tau, \leq)\). Moreover, since \(\leq\) is the specialization order of \((X, \tau)\), we have that \(\text{Cl}_1(A \cap B)\) is a downset of \((X, \tau, \leq)\). Therefore, \(\text{Cl}_1(A \cap B) \in \text{CpDo}(X)\). By Proposition 3.4, \(\text{CpDo}(X) = \beta_2\). Thus, \(\text{Cl}_1(A \cap B) \in \beta_2\).

Conversely, suppose that \((X, \tau_1, \tau_2)\) is a pairwise Stone space and for each \(A \in \beta_1\) and \(B \in \beta_2\) we have \(\text{Cl}_1(A \cap B) \in \beta_2\). Let \(A \in \text{PC}(X)\). By Lemma 7.3, \(A\) is clopen in \((X, \tau, \leq)\). Since \(\text{CpUp}(X) \cup \text{CpDo}(X)\) is a subbasis for \(\tau\) and \(A\) is compact in \((X, \tau)\), we have \(A = (U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)\) for some \(U_1, \ldots, U_n \in \text{CpUp}(X)\) and \(V_1, \ldots, V_n \in \text{CpDo}(X)\). By Proposition 3.4, \(\text{CpUp}(X) = \beta_1\) and \(\text{CpDo}(X) = \beta_2\). Therefore, for each \(i = 1, \ldots, n\) we have \(\text{Cl}_1(U_i \cap V_i) \in \beta_2\). Thus, \(\text{Cl}_1(A) = \text{Cl}_1[(U_1 \cap V_1) \cup \cdots \cup (U_n \cap V_n)] = \text{Cl}_1(U_1 \cap V_1) \cup \cdots \cup \text{Cl}_1(U_n \cap V_n) \in \beta_2 = \text{CpDo}(X)\). This implies that \(\text{Cl}_1(A)\) is clopen in \((X, \tau, \leq)\), so by Lemma 7.3, \(\text{Cl}_1(A) \in \text{PC}(X)\), and so \((X, \tau_1, \tau_2)\) is a bitopological Esakia space.

From now on we will call a pairwise Stone space a bitopological Esakia space if it satisfies the condition of Theorem 7.5.

**Theorem 7.6.** Let \((X, \tau, \leq)\) be a Priestley space and \((X, \tau_1, \tau_2)\) be the corresponding pairwise Stone space. Then \((X, \tau, \leq)\) is an Esakia space iff \((X, \tau_1, \tau_2)\) is a bitopological Esakia space.

**Proof.** Since \(\text{Cp}(X) = \text{PC}(X)\) and for \(A \in \text{PC}(X)\) we have \(\text{Cl}_1(A) = \downarrow A\), the result follow.

In order to characterize morphisms between bitopological Esakia spaces, we recall the following characterization of \(p\)-morphisms.

**Lemma 7.7.** [10, pp. 17-18] For two posets \((X, \leq)\) and \((X', \leq')\) and a map \(f : X \to X'\), the following conditions are equivalent:

1. \(f\) is a \(p\)-morphism.
2. For each \(x \in X\) we have \(f(\uparrow x) = \uparrow f(x)\).
3. For each \(x' \in X'\) we have \(f^{-1}(\downarrow x') = \downarrow f^{-1}(x')\).

**Definition 7.8.** Let \((X, \tau_1, \tau_2)\) and \((X', \tau'_1, \tau'_2)\) be two bitopological Esakia spaces. We call a map \(f : X \to X'\) a bitopological Esakia morphism if \(f\) is \(p\)-continuous and \(f(\text{Cl}_2(x)) = \text{Cl}'_2(f(x))\) for each \(x \in X\).

Let \((X, \tau, \leq)\) and \((X', \tau', \leq')\) be two Esakia spaces, \((X, \tau_1, \tau_2)\) and \((X', \tau'_1, \tau'_2)\) be the corresponding bitopological Esakia spaces, and \(f : X \to X'\) be \(p\)-continuous. By Corollary 6.6,
for each \( x \in X \) we have \( \uparrow x = \text{Cl}_2(x) \) and \( \downarrow x = \text{Cl}_1(x) \). Therefore, by Lemma 7.7, \( f \) is an Esakia morphism iff \( f \) is a bitopological Esakia morphism iff \( f^{-1}(\text{Cl}_1(x')) = \text{Cl}_1(f^{-1}(x')) \).

Let \( \mathbf{BEsa} \) denote the category of bitopological Esakia spaces and bitopological Esakia morphisms. Clearly \( \mathbf{BEsa} \) is a proper subcategory of \( \mathbf{PStone} \). Moreover, putting the results obtained above together, we obtain:

**Theorem 7.9.** The categories \( \mathbf{Esa} \) and \( \mathbf{BEsa} \) are isomorphic. Consequently, \( \mathbf{Heyt} \) is dually equivalent to \( \mathbf{BEsa} \).

Let \((X, \tau)\) be a spectral space. We call \( Y \subseteq X \) a doubly spectral subset of \((X, \tau)\) if both \( Y \) and \( Y^c \) are spectral subsets of \((X, \tau)\). Let \( \mathbf{DS}(X) \) denote the set of doubly spectral subsets of \( X \).

**Definition 7.10.** Let \((X, \tau)\) be a spectral space. We call \((X, \tau)\) a spectral Esakia space if \( A \in \mathbf{DS}(X) \) implies \( \text{Cl}(A) \in \mathbf{DS}(X) \).

**Theorem 7.11.** Let \((X, \tau_1, \tau_2)\) be a pairwise Stone space and \((X, \tau_1)\) be the corresponding spectral space. Then \((X, \tau_1, \tau_2)\) is a bitopological Esakia space iff \((X, \tau_1)\) is a spectral Esakia space.

**Proof.** By Lemma 7.3, \( \mathbf{PC}(X) = \mathbf{DS}(X) \). The result follows.

For two spectral Esakia spaces \((X, \tau)\) and \((X', \tau')\), we call a map \( f : X \to X' \) a spectral Esakia morphism if \( f \) is spectral and \( f(\text{Sat}(x)) = \text{Sat}'(f(x)) \).

Let \((X, \tau_1, \tau_2)\) and \((X', \tau_1', \tau_2')\) be two bitopological Esakia spaces and \((X, \tau_1)\) and \((X', \tau_1')\) be the corresponding spectral Esakia spaces. By Corollary 6.6, for each \( x \in X \) we have \( \text{Cl}_2(x) = \text{Sat}_1(x) \) and \( \text{Cl}_1(x) = \text{Sat}_2(x) \). Therefore, a bi-continuous map \( f : X \to X' \) is a bi-Esaki morphism iff \( f \) is a spectral Esakia morphism iff \( f^{-1}(\text{Cl}_1(x')) = \text{Cl}_1(f^{-1}(x')) \).

Let \( \mathbf{SpecE} \) denote the category of spectral Esakia spaces and spectral Esakia morphisms. Clearly \( \mathbf{SpecE} \) is a proper subcategory of \( \mathbf{Spec} \). Moreover, putting the results obtained above together, we obtain:

**Theorem 7.12.** The categories \( \mathbf{Esa}, \mathbf{BEsa}, \) and \( \mathbf{SpecE} \) are isomorphic. Consequently, \( \mathbf{Heyt} \) is also dually equivalent to \( \mathbf{SpecE} \).

**Remark 7.13.** In Remark 5.4 we pointed out that the duality between \( \mathbf{DLat} \) and the categories \( \mathbf{Pries}, \mathbf{PStone}, \) and \( \mathbf{Spec} \) can be obtained through the schizophrenic object \( \mathbf{2} \). On the other hand, there is no schizophrenic object that induces the duality for Heyting algebras. To see this, let there exist a schizophrenic object \( S \) in \( \mathbf{Heyt} \) such that the duality between \( \mathbf{Heyt} \) and, say, \( \mathbf{Esa} \) is obtained through \( S \). Then \( S \) is also an object of \( \mathbf{Esa} \) and the functors \((-)_* : \mathbf{Heyt} \to \mathbf{Esa} \) and \((-)^* : \mathbf{Esa} \to \mathbf{Heyt} \) can be described through \( S \); that is, for each object \( A \) of \( \mathbf{Heyt} \), the carrier of \( A_* \) is the set \( \text{Hom}_{\mathbf{Heyt}}(A, S) \) and for each object \( X \) of \( \mathbf{Esa} \), the carrier of \( X^* \) is the set \( \text{Hom}_{\mathbf{Esa}}(X, S) \). Therefore, the isomorphism \( \varphi : A \to A_*^* \) is given by \( \varphi(a)(h) = h(a) \) for each \( a \in A \) and \( h \in A_* \). Thus, if \( a \neq b \) in \( A \), then there exists \( h \in \text{Hom}_{\mathbf{Heyt}}(A, S) \) such that \( h(a) \neq h(b) \). We show that this leads to a contradiction. Let \( A \) be a linearly ordered Heyting algebra with the second largest element \( a \). Then \( a \neq 1 \). We observe that each \( h \in \text{Hom}_{\mathbf{Heyt}}(A, S) \) for which \( h(a) \neq 1 \) is injective. Indeed, let \( b < c \leq a \). If \( h(b) = h(c) \), then \( h(b) = h(c \to b) = h(c) \to h(b) = 1 \). This together with \( h(b) \leq h(a) \) imply \( h(a) = 1 \), a contradiction. Consequently, such an \( S \) cannot exist because it would contain a subset of an arbitrarily large cardinality. Clearly this argument does not depend on the category \( \mathbf{Esa} \). In fact, it shows that there is no co-generating object in \( \mathbf{Heyt} \), and
hence the duality for Heyting algebras can not be induced by a schizophrenic object. For a general discussion of co-generators and dualities which are obtained through schizophrenic objects we refer to Johnstone [17, p. 254].

The dual description of algebraic concepts important for the study of Heyting algebras is similar to that of bounded distributive lattices. The dual description of filters, prime filters, and maximal filters as well as ideals, prime ideals, and maximal ideals is exactly the same. So is the dual description of the canonical completions. On the other hand, the dual description of the MacNeille completions gets simplified [15, Sec. 3] because in the case of Heyting algebras, we have $D = \text{Cl}$.

It is well-known that homomorphic images of a Heyting algebra $A$ are characterized by its filters. Consequently, unlike the case of bounded distributive lattices, homomorphic images of a Heyting algebra $A$ dually correspond to closed upsets of the Esakia space of $A$. Therefore, homomorphic images of $A$ dually correspond to $\tau_2$-closed subsets of the bitopological Esakia space of $A$, and to compact saturated subsets of the spectral Esakia space of $A$.

We give the dual description of subalgebras of a Heyting algebra. For a quasi-ordered set $(X, Q)$, we define an equivalence relation $E$ on $X$ by $xQy$ if $xQy$ and $yQx$.

**Definition 7.14.** Let $(X, \tau, \leq)$ be a Priestley space and $Q$ be a Priestley quasi-order on $X$ extending $\leq$. We call $Q$ an Esakia quasi-order if for each $x, y \in X$, from $xQy$ it follows that there exists $z \in X$ such that $x \leq z$ and $zEy$.

**Remark 7.15.** Let $(X, \tau, \leq)$ be a Priestley space and $E$ be an equivalence relation on $X$. We call $E$ an Esakia equivalence relation if $E$ is a quasi-order that is a Priestley quasi-order on $X$ and $\uparrow E(x) \subseteq E(\downarrow x)$. It is easy to see that if $Q$ is an Esakia quasi-order, then $E$ is an Esakia equivalence relation. For an Esakia equivalence relation $E$, we define $Q$ on $X$ by $xQy$ if there exists $z \in X$ such that $x \leq z$ and $zEy$. Then for an Esakia space $X$, it is easy to see that $Q$ is an Esakia quasi-order. Thus, for an Esakia space $X$, there is an isomorphism between Esakia quasi-orders on $X$ ordered by inclusion and Esakia equivalence relations on $X$ ordered by inclusion.

**Theorem 7.16.** Let $A$ be a Heyting algebra and $(X, \tau, \leq)$ be the Esakia space of $A$. Then the poset $(\text{HS}_A, \subseteq)$ of Heyting subalgebras of $A$ is dually isomorphic to the poset $(\text{EQ}_X, \subseteq)$ of Esakia quasi-orders on $X$.

**Proof.** In view of Theorem 6.15, it is sufficient to show that if $S \in \text{HS}_A$, then $Q_S \in \text{EQ}_X$, and that if $Q \in \text{EQ}_X$, then $S_Q \in \text{HS}_A$. Let $S \in \text{HS}_A$. By Theorem 6.15, $Q$ is a Priestley quasi-order on $X$ extending $\leq$. Suppose that $xQ_Sy$. Then $x \land S \subseteq y \land S$. Let $F$ be the filter of $A$ generated by $x \cup (y \land S)$. Then $F$ is a proper filter of $A$ with $x \subseteq F$ and $F \cap S = y \land S$. By Zorn’s lemma we can extend $F$ to a maximal such filter $z$. The standard argument shows that $z$ is prime. Therefore, there exists $z \in X$ such that $x \leq z$ and $zE_Sy$. Thus, $Q_S \in \text{EQ}_X$. Now let $Q \in \text{EQ}_X$. By Theorem 6.15, $S_Q$ is a bounded distributive sublattice of $A$. For $a, b \in S_Q$ we have $\phi(a), \phi(b)$ are $Q$-upsets of $X$. We show that $\phi(a \rightarrow b) = \phi(a) \rightarrow \phi(b) = [[[\phi(a) - \phi(b)]^c] = \{x \in X \mid \uparrow x \cap \phi(a) \subseteq \phi(b)\}$ is also a $Q$-upset of $X$. Let $x \in \phi(a \rightarrow b)$ and $xQy$. We show that $\uparrow y \cap \phi(a) \subseteq \phi(b)$. Let $u \in \uparrow y \cap \phi(a)$. Then $y \leq u$ and $u \in \phi(a)$. Therefore, $xQu$, and so there exists $z \in X$ such that $x \leq z$ and $zEu$. Since $zEu$, $u \in \phi(a)$, and $\phi(a)$ is a $Q$-upset, we have $z \in \phi(a)$. This implies that $z \in \uparrow x \cap \phi(a)$ as $\uparrow x \cap \phi(a) \subseteq \phi(b)$, we obtain $z \in \phi(b)$. Now $zEu$ and $\phi(b)$ being a $Q$-upset imply that $u \in \phi(b)$. Consequently, $\uparrow y \cap \phi(a) \subseteq \phi(b)$, so $y \in \phi(a \rightarrow b)$, and so $\phi(a \rightarrow b)$ is a $Q$-upset. It follows that $a, b \in S_Q$ implies $a \rightarrow b \in S_Q$, and so $S_Q \in \text{HS}_A$. 

As a consequence of Remark 7.15 and Theorem 7.16, we obtain the following well-known dual description of subalgebras of Heyting algebras [8, Thm. 4]: The poset of Heyting subalgebras of a Heyting algebra \( A \) is dually isomorphic to the poset of Esakia equivalence relations on the Esakia space \( X \) of \( A \).

Now we give the dual description of subalgebras of Heyting algebras by means of bitopological Esakia spaces and spectral Esakia spaces. Let \( (X, \tau_1, \tau_2) \) be a bitopological Esakia space. We call a bitopology \((\tau_1', \tau_2')\) an Esakia bitopology on \( X \) if \((\tau_1', \tau_2')\) is pairwise zero-dimensional and \( A \in \beta_1', B \in \beta_2' \) imply \( \text{Cl}(A \cap B) \in \beta_2' \). Let \((\mathcal{EB}_X, \subseteq)\) denote the poset of Esakia bitopologies on \( X \) coarser than \((\tau_1, \tau_2)\).

**Lemma 7.17.** Let \((X, \tau, \leq)\) be an Esakia space and \((X, \tau_1, \tau_2)\) be the corresponding bitopological Esakia space. Then \((\mathcal{EQ}_X, \subseteq)\) is dually isomorphic to \((\mathcal{EB}_X, \subseteq)\).

**Proof.** In view of Lemma 6.17, we only need to show that if \( Q \in \mathcal{EQ}_X \), then \((\tau_1^Q, \tau_2^Q) \in \mathcal{EB}_X \), and that if \((\tau_1', \tau_2') \in \mathcal{EB}_X \), then \( Q_{(\tau_1', \tau_2')} \in \mathcal{EQ}_X \). Let \( Q \in \mathcal{EQ}_X \). By Lemma 6.17, \((\tau_1^Q, \tau_2^Q)\) is a zero-dimensional bitopology coarser than \((\tau_1, \tau_2)\). Moreover, \( \beta_1^\mathcal{Q} \) coincides with the set of clopen \( Q \)-upsets and \( \beta_2^\mathcal{Q} \) coincides with the set of clopen \( Q \)-downsets of \((X, \tau, \leq)\). Therefore, for \( A \in \beta_1^\mathcal{Q} \) and \( B \in \beta_2^\mathcal{Q} \) we have that \( A \) is a clopen \( Q \)-set and \( B \) is a clopen \( Q \)-set of \((X, \tau, \leq)\). Since \( Q \) is an Esakia quasi-order, by Theorem 7.16, the lattice of clopen \( Q \)-upsets of \((X, \tau, \leq)\) is a Heyting subalgebra of the Heyting algebra of all clopen upsets of \((X, \tau, \leq)\). Thus, \( \downarrow(A \cap B) \) is a clopen \( Q \)-downset of \((X, \tau, \leq)\), and so \( \downarrow(A \cap B) \in \beta_2^\mathcal{Q} \). By Corollary 6.6, \( \text{Cl}(A \cap B) = \downarrow(A \cap B) \). Consequently, \( \text{Cl}(A \cap B) \in \beta_2^\mathcal{Q} \), and so \((\tau_1^Q, \tau_2^Q) \in \mathcal{EB}_X \).

Now suppose that \((\tau_1', \tau_2') \in \mathcal{EB}_X \). By Lemma 6.17, \( Q_{(\tau_1', \tau_2')} \) is a Priestley quasi-order on \( X \) extending \( \tau \). We show that the lattice of clopen \( Q_{(\tau_1', \tau_2')} \)-upsets of \((X, \tau, \leq)\) is closed under \( \to \). Let \( A \) and \( B \) be clopen \( Q_{(\tau_1', \tau_2')} \)-upsets of \((X, \tau, \leq)\). Then \( A \in \beta_1' \) and \( B \in \beta_2' \). Therefore, \( \text{Cl}(A \cap B') \in \beta_2' \), and so \( \text{Cl}(A \cap B') \) is a clopen \( Q_{(\tau_1', \tau_2')} \)-downset of \((X, \tau, \leq)\). By Corollary 6.6, \( \text{Cl}(A \cap B') = \downarrow(A \cap B') \). Consequently, \( \downarrow(A \cap B') \) is a clopen \( Q_{(\tau_1', \tau_2')} \)-downset of \((X, \tau, \leq)\), so \( A \to B = [\downarrow(A \cap B')]^c \) is a clopen \( Q_{(\tau_1', \tau_2')} \)-upset of \((X, \tau, \leq)\), and so the lattice of clopen \( Q_{(\tau_1', \tau_2')} \)-upsets of \((X, \tau, \leq)\) is closed under \( \to \). This implies that the lattice of clopen \( Q_{(\tau_1', \tau_2')} \)-upsets of \((X, \tau, \leq)\) is a Heyting subalgebra of the Heyting algebra of all clopen upsets of \((X, \tau, \leq)\), which, by Theorem 7.16, gives us that \( Q_{(\tau_1', \tau_2')} \in \mathcal{EQ}_X \).

Let \((X, \tau)\) be a spectral Esakia space. We call a topology \( \tau' \) on \( X \) a spectral Esakia topology if \( \tau' \) is strongly coherent and \( A \in \mathcal{E}(X, \tau') \), \( B \in \Delta(X, \tau') \) imply \( \text{Cl}(A \cap B) \in \Delta(X, \tau') \). For a spectral Esakia space \((X, \tau)\), let \((\mathcal{SE}_X, \subseteq)\) denote the poset of spectral Esakia topologies on \( X \) coarser than \( \tau \).

**Lemma 7.18.** Let \((X, \tau_1, \tau_2)\) be a bitopological Esakia space and \((X, \tau_1)\) be the corresponding spectral Esakia space. Then \((\mathcal{EB}_X, \subseteq)\) is isomorphic to \((\mathcal{SE}_X, \subseteq)\).

**Proof.** In view of Lemma 6.19, we only need to show that if \((\tau_1', \tau_2') \in \mathcal{EB}_X \), then \( \tau_1' \in \mathcal{SE}_X \), and that if \( \tau_1' \in \mathcal{SE}_X \), then \( \tau_1' \in \mathcal{EB}_X \). Let \( \tau_1' \in \mathcal{SE}_X \). By Lemma 6.19, \( \tau_1' \) is a strongly coherent topology coarser than \( \tau_1 \). Moreover, since \( \beta_1' = \mathcal{E}(X, \tau_1') \) and \( \beta_2' = \Delta(X, \tau_1') \), for \( A \in \mathcal{E}(X, \tau_1') \) and \( B \in \Delta(X, \tau_1') \), we have \( A \in \beta_1' \) and \( B \in \beta_2' \), so \( \text{Cl}(A \cap B) \in \beta_2' \), and so \( \text{Cl}(A \cap B) \in \Delta(X, \tau_1') \). Therefore, \( \tau_1' \in \mathcal{SE}_X \). Now let \( \tau_1' \in \mathcal{SE}_X \). By Lemma 6.19, \( \tau_1' \) is a zero-dimensional bitopology coarser than \( (\tau_1, \tau_2) \). Moreover, since \( \mathcal{E}(X, \tau_1') = \beta_1' \) and \( \Delta(X, \tau_1') = \beta_2' \), for \( A \in \beta_1' \) and \( B \in \beta_2' \), we have \( A \in \mathcal{E}(X, \tau_1') \) and \( B \in \Delta(X, \tau_1') \), so \( \text{Cl}(A \cap B) \in \Delta(X, \tau_1') \), and so \( \text{Cl}(A \cap B) \in \beta_2' \). Thus, \( (\tau_1, \tau_2) \in \mathcal{EB}_X \).
Putting Lemmas 7.17 and 7.18 together, we obtain the following dual description of Heyting subalgebras of a Heyting algebra.

**Corollary 7.19.** Let $A$ be a Heyting algebra, $(X, \tau, \leq)$ be the Esakia space of $A$, $(X, \tau_1, \tau_2)$ be the bitopological Esakia space of $A$, and $(X, \tau_1)$ be the spectral Esakia space of $A$. Then $(HS_A, \subseteq)$ is dually isomorphic to $(EQ_X, \subseteq)$, and is isomorphic to $(EB_X, \subseteq)$ and $(SE_X, \subseteq)$.

In Table 2 we gather together the dual descriptions of different algebraic concepts for Heyting algebras by means of their Esakia spaces, bitopological Esakia spaces, and spectral Esakia spaces obtained in this section. This can be thought of as a dictionary of duality theory for Heyting algebras.

We conclude by mentioning that two more natural subclasses of the class of distributive lattices that play an important role in the study of non-classical logics are the classes of co-Heyting algebras and bi-Heyting algebras, respectively. We recall that a *co-Heyting algebra* is a bounded distributive lattice $A$ with a binary operation $\leftarrow: A^2 \to A$ such that for all $a, b, c \in A$ we have:

$$c \geq a \iff b \lor c \geq a.$$

We also recall that $(A, \to, \leftarrow)$ is a *bi-Heyting algebra* if $(A, \to)$ is a Heyting algebra and $(A, \leftarrow)$ is a co-Heyting algebra. The first duality for co-Heyting algebras and bi-Heyting algebras was developed by Esakia [9]. It is a restricted version of Priestley’s duality, and is a modified version of Esakia’s duality for Heyting algebras [8]. The bitopological and spectral dualities for co-Heyting and bi-Heyting algebras can be developed by an obvious modification of the bitopological and spectral dualities for Heyting algebras developed in this section. We skip the details, which can be recovered by an appropriate modification of the proofs given above, and only mention that there is a dictionary of duality theory for co-Heyting algebras and bi-Heyting algebras similar to the one for Heyting algebras given in Table 2.

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