QUASIVARIETIES AND VARIETIES OF ORDERED ALGEBRAS: REGULARITY AND EXACTNESS

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Abstract. We characterise quasivarieties and varieties of ordered algebras categorically in terms of regularity, exactness and the existence of a suitable generator. The notions of regularity and exactness need to be understood in the sense of category theory enriched over posets.

We also prove that finitary varieties of ordered algebras are cocompletions of their theories under sifted colimits (again, in the enriched sense).

1. Introduction

Since the very beginning of the categorical approach to universal algebra, the intrinsic characterisation of varieties and quasivarieties of algebras has become an interesting question. First steps were taken already in John Isbell's paper [15], William Lawvere's seminal PhD thesis [26] and Fred Linton's paper [27]. The compact way of characterising varieties and quasivarieties can be, in modern language, perhaps best stated as follows:

> A category $\mathscr{A}$ is equivalent to a (quasi)variety of algebras iff it is (regular) exact and it possesses a "nice" generator.

For the excellent modern categorical treatment of (quasi)varieties of algebras in the sense of classical universal algebra, see the book by Jiří Adámek, Jiří Rosický and Enrico Vitale [5].

In the current paper, we will give a characterisation of categories of varieties and quasivarieties of *ordered* algebras in essentially the same spirit:

> A category $\mathscr{A}$, *enriched over posets*, is equivalent to a (quasi)variety of *ordered* algebras iff it is (regular) exact and it possesses a "nice" generator.

Above, however, the notions of regularity and exactness need to be reformulated so that the notions suit the realm of categories enriched over posets.

There are at least two approaches to what an ordered algebra can be. Let us briefly comment on both:

The approach of Bloom and Wright [10]**:** An *algebra* for a signature Σ consists of a poset X, together with a monotone map $[\![\sigma]\!] : X^n \longrightarrow X$, for each specified n-ary operation σ, where n is a *set*. A *homomorphism* is a monotone map, preserving the operations on the nose.

Such a concept is a direct generalisation of the classical notion of an algebra [12].

The approach of Kelly and Power [20]**:** An *algebra* for a signature Σ consists of a poset X, together with a monotone map $[\![\sigma]\!] : X^n \longrightarrow X$, for each specified n-ary operation σ, where n is a *poset*. Here, X^n denotes the poset of all *monotone* maps from n to X. A *homomorphism* is a monotone map, preserving the operations on the nose.

This concept stems from the theory of enriched monads. It allows for operations that are defined only *partially*. As we will see later, such an approach is also quite natural and handy in practice.

We will choose the first concept as the object of our study. For technical reasons, we will also allow the collection Σn of all n-ary operations to be a poset. Then, for every algebra for Σ on a poset X, the inequality $[\![\sigma]\!] \leq [\![\tau]\!]$ is required to hold in the poset of monotone functions from X^n to X, whenever $\sigma \leq \tau$ holds in the poset Σn of all n-ary operations.

Varieties and quasivarieties in the first sense were studied by Stephen Bloom and Jesse Wright in [9] and [10]. In [9], a Birkhoff-style characterisation of classes of algebras is given:

(1) Varieties are *defined* as classes of algebras satisfying *formal inequalities* of the form

$$t' \sqsubseteq t$$

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where t' and t are Σ-terms. Varieties can be *characterised* as precisely the HSP-classes of Σ-algebras.

(2) Quasivarieties are *defined* as classes of algebras that satisfy *formal implications* of the form

$$(\bigwedge_{i\in I} s'_i \sqsubseteq s_i) \Rightarrow t' \sqsubseteq t$$

where I is a set, s'_i, s_i, t' and t are Σ-terms. Quasivarieties can be *characterised* as precisely the SP-classes of Σ-algebras.

One has to be precise, however, in saying what the closure operators H and S mean. As it turns out, when choosing monotone surjections as the notion of a homomorphic image, then the proper concept of a subalgebra is that of a monotone homomorphism that reflects the orders.

Example 1.1. Sets and mappings form a quasivariety $\mathscr{A}$ of ordered algebras. More precisely:

(1) Let Σ be a signature with no specified operation. Hence Σ-algebras are exactly the posets and Σ-homomorphisms are the monotone maps.

(2) Let the objects of $\mathscr{A}$ be Σ-algebras, subject to the implication

$$x \sqsubseteq y \Rightarrow y \sqsubseteq x$$

Clearly, any object $\mathscr{A}$ can be identified with a set and Σ-homomorphisms in $\mathscr{A}$ can be identified with mappings.

It is easy to see that $\mathscr{A}$ is an SP-class in the category of all Σ-algebras. But it is not an HSP-class: consider the identity-on-objects monotone mapping $e : 2 \longrightarrow \mathbb{2}$, where 2 is the discrete poset on two elements and $\mathbb{2}$ is the two-element chain. Then 2 is an object of $\mathscr{A}$, while $\mathbb{2}$ is not.

This example also shows the distinction between two possible approaches to universal algebra over posets. Namely: the obvious discrete-poset functor $U : \mathsf{Set} \longrightarrow \mathsf{Pos}$ is easily seen to be monadic. Hence Set appears as a "variety" in the world where arities as posets are allowed. More precisely: consider the signature Γ, where $\Gamma\mathbb{2} = \mathbb{2}$ and $\Gamma n = \emptyset$ otherwise. Then the set of equations

$$\sigma_0(x, y) = y, \quad \sigma_1(x, y) = x$$

defines Set over Pos equationally, where $\sigma_0 \leq \sigma_1$ are the only elements of $\Gamma\mathbb{2}$. See [20] for more details on presenting monads by operations and equations.

The system (monotone surjective maps, monotone maps reflecting orders) is a factorisation system in the category Pos of posets and monotone maps. One can therefore ask whether this system can play the rôle of the (regular epi,mono) factorisation system on the category of sets that is so vital in giving intrinsic categorical characterisations of varieties and quasivarieties in classical universal algebra. We prove that this is the case, if we pass from the world of categories to the world of categories *enriched* in posets. Namely:

(1) We give the definition of regularity and exactness of a category enriched in posets. We show that Pos is an exact category.

(2) We give intrinsic characterisations of both varieties and quasivarieties of ordered algebras, see Theorems 5.7 and 5.11 below. Our main results then have the same phrasing as in the classical case, the only difference is that all the notions have their meaning in category theory enriched in posets.

Related work. The notion of regularity and exactness for 2-categories goes back to Ross Street [33], we were much inspired by its polished version of Mike Shulman [31] and the recent PhD thesis of John Bourke [11]. Bourke studies exactness for a different factorisation system, though. Varieties and quasivarieties from the current text were named P-varieties and P-quasivarieties by Stephen Bloom and Jesse Wright in [10]. The authors did not use the standard terminology and they only worked with effective congruences, hence they missed the notion of exactness. However, they give an "almost intrinsic" characterisation of varieties and quasivarieties that we found extremely useful.

Organisation of the text. The necessary notions of enriched category theory are recalled in Section 2. Regularity and exactness are defined in Section 3. Section 4 contains the technicalities that we need in order to prove our main characterisation results in Section 5. We prove in Section 6 that finitary varieties of ordered algebras can be characterised as algebras for a special class of monads — the strongly finitary ones. In Section 7 we indicate directions for future work.

2. Preliminaries

We briefly recall the basic notions of enriched category that we will use later on. For more details, see Max Kelly's book [17].

We will work with categories enriched in the cartesian closed category $(\mathsf{Pos}, \times, \mathbb{1})$ of posets and monotone maps. We will omit the prefix Pos- when speaking of Pos-categories, Pos-functors, etc. Thus, in what follows:

1. A *category* $\mathscr{X}$ is given by objects X, Y, ... such that every hom-object $\mathscr{X}(X,Y)$ is a poset. The partial order on $\mathscr{X}(X,Y)$ is denoted by $\leq$. We require the composition to preserve the order in both arguments: $(g' \cdot f') \leq (g \cdot f)$ holds, whenever $g' \leq g$ and $f' \leq f$.
2. A *functor* $F : \mathscr{A} \longrightarrow \mathscr{B}$ is given by the functorial object-assignment that is locally monotone, i.e., $Ff \leq Fg$ holds, whenever $f \leq g$.

When we want to speak of non-enriched categories, functors, etc., we will call them *ordinary*.

In diagrams, we will denote, for parallel morphisms f, g, the fact $f \leq g$ by an arrow between morphisms and we will speak of a *2-cell*:

$$\begin{tikzcd} X \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\uparrow"] & Y \end{tikzcd}$$

This notation complies with the fact that categories enriched in posets are (rather special) 2-categories.

The category of functors from $\mathscr{A}$ to $\mathscr{B}$ and natural transformations between them is denoted by $[\mathscr{A}, \mathscr{B}]$. The opposite category $\mathscr{X}^{op}$ of $\mathscr{X}$ has just the sense of morphisms reversed, the order on hom-posets remains unchanged.

The proper concept of a limit and a colimit in enriched category theory is that of a weighted (co)limit. More in detail, for every *diagram* $D : \mathscr{D} \longrightarrow \mathscr{X}$, $\mathscr{D}$ small, we define its *tilde-conjugate*

$$\widetilde{D} : \mathscr{X} \longrightarrow [\mathscr{D}^{op}, \mathsf{Pos}], \quad X \mapsto \mathscr{X}(D-, X)$$

and its *hat-conjugate*

$$\widehat{D} : \mathscr{X} \longrightarrow [\mathscr{D}, \mathsf{Pos}]^{op}, \quad X \mapsto \mathscr{X}(X, D-)$$

Then a *colimit* of D weighted by $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$ is an object $W * D$, together with an isomorphism

$$\mathscr{X}(W * D, X) \cong [\mathscr{D}^{op}, \mathsf{Pos}](W, \widetilde{D}X)$$

of posets, natural in X. A *limit* of D weighted by $W : \mathscr{D} \longrightarrow \mathsf{Pos}$ is an object $\{W, D\}$, together with an isomorphism

$$\mathscr{X}(X, \{W, D\}) \cong [\mathscr{D}, \mathsf{Pos}]^{op}(\widehat{D}X, W)$$

of posets, natural in X.

Hence, for a category $\mathscr{X}$ admitting all colimits of the diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$, the assignment $X \mapsto X * D$ is the value of a left adjoint to $\widetilde{D} : \mathscr{X} \longrightarrow [\mathscr{D}^{op}, \mathsf{Pos}]$. A special instance is the case of a one-morphism category $\mathscr{D}$: the diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$ can be identified with an object D of $\mathscr{X}$, the functor $\widetilde{D}$ is the representable functor $\mathscr{X}(D, -) : \mathscr{X} \longrightarrow \mathsf{Pos}$ and its left adjoint assigns the *tensor* $X \bullet D$ of the object D and the poset X.

Analogously, the assignment $X \mapsto \{X, D\}$ is a right adjoint to $\widehat{D} : \mathscr{X} \longrightarrow [\mathscr{D}, \mathsf{Pos}]^{op}$ in case $\mathscr{X}$ admits all limits of $D : \mathscr{D} \longrightarrow \mathscr{X}$.

Recall from [18] that a (co)limit is finite, if it is weighted by a finite weight. The latter is a functor $W : \mathscr{D} \longrightarrow \mathsf{Pos}$ such that $\mathscr{D}$ has finitely many objects, every $\mathscr{D}(d', d)$ is a finite poset, and every Wd is a finite poset.

We will, besides other finite (co)limits, use *coinserters*. The weight $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$ for coinserters has $\mathscr{D}$ consisting of a parallel pair of morphisms that is sent to the parallel pair

$$\mathbb{1} \overset{1}{\underset{0}{\rightrightarrows}} \mathbb{2}$$

in Pos. In elementary terms, a coinserter in $\mathscr{X}$ of a parallel pair

$$X_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} X_0$$

consists of a morphism $c : X_0 \longrightarrow C$ such that $c \cdot d_0 \leq c \cdot d_1$ holds and such that it satifies the following couniversal property:

1. For any $h : X_0 \longrightarrow D$ such that $h \cdot d_0 \leq h \cdot d_1$ there is a unique $h^\sharp : C \longrightarrow D$ such that $h^\sharp \cdot c = h$.
2. For any pair $k', k : C \longrightarrow D$ that satisfies $k' \cdot c \leq k \cdot c$, the inequality $k' \leq k$ holds.

Thus the couniversal property has two aspects: the 1-dimensional aspect (concerning 1-cells) and the 2-dimensional aspect (concerning the order between 1-cells). This will be always the case for weighted (co)limits that we encounter and it is caused by the fact that we enrich over posets. As such, our (co)limits will be rather special 2-(co)limits. The enrichment in posets will usually simplify substantially the 2-dimensional aspect of 2-(co)limits. See [16] for more details.

Example 2.1 (**Explicit computation of coinserters in** Pos). Suppose that

$$X_1 \overset{d_1}{\underset{d_0}{\rightrightarrows}} X_0$$

is a pair of morphisms in Pos. The coinserter $c : X_0 \longrightarrow C$ of d_0, d_1 can be described as follows:

(1) Define a binary relation R on the set $\mathrm{ob}(X_0)$ of objects of X_0 as follows:
  $x' \; R \; x$ iff there is a finite sequence $f_0, \ldots, f_{n-1}$ of objects in X_1 such that the inequalities
  $$x' \leq d_0(f_0), \quad d_1(f_0) \leq d_0(f_1), \quad d_1(f_1) \leq d_0(f_2), \quad \ldots, \quad d_1(f_{n-1}) \leq x$$
  hold in X_0.
  It is easy to see that R is reflexive and transitive. Put $E = R \cap R^{op}$ to obtain an equivalence relation on the set $\mathrm{ob}(X_0)$.

(2) The poset C has as $\mathrm{ob}(C)$ the quotient set $\mathrm{ob}(X_0)/E$, we put $[x'] \leq [x]$ in C to hold iff $x' \; R \; x$ holds. The monotone mapping $c : X_0 \longrightarrow C$ is the canonical map sending x to $[x]$.

It is now routine to verify that we have defined a coinserter.

3. Regularity and exactness

Regularity and exactness in ordinary category theory [6] is defined relative to a factorisation system. In this section we will introduce the factorisation system

(surjective on objects, representably fully faithful)

on the class of morphisms of a general category $\mathscr{X}$. When $\mathscr{X} = \mathsf{Pos}$, the above system coincides with the factorisation system (monotone surjective maps, monotone maps reflecting orders).

We introduce the factorisation system by starting with its "mono" part. The "strong epi" part of the factorisation system is then derived by the orthogonal property that is appropriate for the enrichment in posets. We then show that, in cases of interest, the "strong epi" part of the factorisation system is given by a suitable generalisation of a coequaliser. This is a gist of the second part of this section: we introduce congruences and their quotients and the corresponding notions of regularity and exactness.

3.A. **The factorisation system.**

Definition 3.1. We say that $m : X \longrightarrow Y$ in $\mathscr{X}$ is *representably fully faithful* (or, that it is an rff-morphism), provided that the monotone map $\mathscr{X}(Z, m) : \mathscr{X}(Z, X) \longrightarrow \mathscr{X}(Z, Y)$ reflects orders (i.e., if it is fully faithful as a functor in Pos), for every Z.

A morphism $e : A \longrightarrow B$ is *surjective on objects* (or, that it is an so-morphism), provided that the square

$$\begin{tikzcd} \mathscr{X}(B,X) \arrow[r, "\mathscr{X}(e,X)"] \arrow[d, "\mathscr{X}(B,m)"'] & \mathscr{X}(A,X) \arrow[d, "\mathscr{X}(A,m)"] \\ \mathscr{X}(B,Y) \arrow[r, "\mathscr{X}(e,Y)"'] & \mathscr{X}(A,Y) \end{tikzcd} \tag{3.1}$$

is a pullback in Pos, for every rff-morphism $m : X \longrightarrow Y$.

We say that $\mathscr{X}$ *has* $(\mathsf{so}, \mathsf{rff})$-*factorisations* if every f can be factored as an so-morphism followed by an rff-morphism.

Example 3.2. In Pos, so-morphisms are exactly the monotone surjections, rff-morphisms are order-reflecting monotone maps. Clearly, Pos has $(\mathsf{so}, \mathsf{rff})$-factorisations.

The description extends to "presheaf" categories $[\mathscr{S}^{op}, \mathsf{Pos}]$, where $\mathscr{S}$ is small, in the usual "pointwise" way.

Remark 3.3. The rff-morphisms are called *P-monos* in [10], and *chronic* in [33]. We choose the acronym rff to remind us of *representably fully faithful*. The so-morphisms are called *surjections* in [10] and *acute* in [33].

Remark 3.4. That the diagram (3.1) is a pullback on the level of sets states the usual "diagonal fill-in" property. Hence classes of so-morphisms and rff-morphisms are mutually *orthogonal*. This means that in every commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dotted, "d"'] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

with e an so-morphism and m an rff-morphism, there is a unique diagonal d as indicated, making both triangles commutative.

That the diagram (3.1) is in fact a pullback on the level of posets describes a finer, *2-dimensional aspect* of orthogonality.

Namely, for two pairs $u_1 \leq u_2 : A \longrightarrow X$, $v_1 \leq v_2 : B \longrightarrow Y$ such that both squares

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u_1"'] & B \arrow[d, "v_1"] \\ X \arrow[r, "m"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u_2"'] & B \arrow[d, "v_2"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

commute, we have an inequality $d_1 \leq d_2$ for the respective diagonals.

3.B. **Congruences and their quotients.** We will define congruences and their quotients.[1] Since the general poset-enriched concept of a congruence is rather technical, we start with the following intuition for equivalence relations on sets:

> An equivalence relation E on a set X is a "recipe" how to glue elements of X together. That is: E imposes *new equations* on the set X, besides those already valid.

A congruence E on a poset X should impose *new inequalities* besides those already valid. Moreover, E should be a poset again.

Hence an "element" of a congruence E should be a formal "broken" arrow $x' \relbar\!\!\!+\!\!\!\rightarrow x$ that specifies the formal inequality x' *is smaller than* x. The formal arrows should interact nicely with the actual arrows (representing already valid inequalities in X), i.e., both $x'' \longrightarrow x' \relbar\!\!\!+\!\!\!\rightarrow x$ and $x' \relbar\!\!\!+\!\!\!\rightarrow x \longrightarrow x''$ should have an unambiguous meaning (and both should compose to a "broken" arrow). Furthermore, "broken" arrows should compose (imposing inequations is reflexive and transitive).

The above can be stated more formally: a congruence is a category object, whose domain-codomain span is a two-sided discrete fibration of a certain kind. Before giving the precise definition (Definition 3.7 below), let us see an example of a congruence in Pos:

Example 3.5 (Kernel congruences in posets). Every monotone map $f : A_0 \longrightarrow B$ gives rise to a *kernel congruence* $\ker(f)$ on A_0 as follows:

(1) Form a comma object

$$\begin{tikzcd} A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\ A_0 \arrow[r, "f"'] & B \end{tikzcd}$$

That is: objects of A_1 are pairs (a, b) such that $fa \leq fb$ holds in B. The pair (a, b) should be thought of as a new inequality that we want to impose. We denote such a formal inequality by $a \relbar\!\!\!+\!\!\!\rightarrow b$.

The pairs (a, b) in A_1 inherit the order from the product $A_0 \times A_0$. In other words: the map $\langle d_0^1, d_1^1 \rangle : A_1 \longrightarrow A_0 \times A_0$ is an rff-morphism.

It will be useful to denote the inequality $(a, b) \leq (a', b')$ in A_1 by a formal square

$$\begin{tikzcd} a \arrow[r, "|" marking] \arrow[d] & b \arrow[d] \\ a' \arrow[r, "|" marking] & b' \end{tikzcd}$$

Observe that there is an associative and unital way of *vertical composition* of formal squares by pasting one on top of another.

[1] The standard teminology of 2-category theory for quotients is *codescent*, see [23] or [11]. We prefer to use the term quotient to comply with the intuitions of classical universal algebra.

It is well-known (see, e.g., [32]) that the span (d_0^1, A_1, d_1^1) is a (two-sided) *discrete fibration*. This means that for every pair

$$
\begin{tikzcd}
a \arrow[d] & \\
a' \arrow[r, "|" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "|" marking] & b \arrow[d] \\
 & b'
\end{tikzcd}
$$

of "niches" there are "unique fill-ins" of the form

$$
\begin{tikzcd}
a \arrow[r, "|" marking] \arrow[d] & b \arrow[d, equal] \\
a' \arrow[r, "|" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "|" marking] \arrow[d, equal] & b \arrow[d] \\
a \arrow[r, "|" marking] & b'
\end{tikzcd}
$$

and that every formal square

$$
\begin{tikzcd}
a \arrow[r, "|" marking] \arrow[d] & b \arrow[d] \\
a' \arrow[r, "|" marking] & b'
\end{tikzcd}
$$

can be written uniquely as a vertical composite

$$
\begin{tikzcd}
a \arrow[r, "|" marking] \arrow[d, equal] & b \arrow[d] \\
a \arrow[r, "|" marking] \arrow[d] & b' \arrow[d, equal] \\
a' \arrow[r, "|" marking] & b'
\end{tikzcd}
$$

of such fillings.

(2) Besides pasting the formal squares vertically, we show how to paste them horizontally as in

$$
\begin{tikzcd}
a \arrow[r, "|" marking] \arrow[d] & b \arrow[d] \\
a' \arrow[r, "|" marking] & b'
\end{tikzcd}
\quad
\begin{tikzcd}
b \arrow[r, "|" marking] \arrow[d] & c \arrow[d] \\
b' \arrow[r, "|" marking] & c'
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
a \arrow[r, "|" marking] \arrow[d] & c \arrow[d] \\
a' \arrow[r, "|" marking] & c'
\end{tikzcd}
$$

To allow for the *horizontal composition* of the squares, form a pullback

$$
\begin{tikzcd}
A_2 \arrow[r, "d_2^2"] \arrow[d, "d_0^2"'] & A_1 \arrow[d, "d_0^1"] \\
A_1 \arrow[r, "d_1^1"'] & A_0
\end{tikzcd}
$$

It is straightforward to see that the elements of A_2 are triples (a'', a', a) satisfying $fa'' \leq fa' \leq fa$. The triples are ordered pointwise. Every such triple (a'', a', a) can be drawn as a "composable pair" $a'' \mathrel{\mapstochar\rightarrow} a' \mathrel{\mapstochar\rightarrow} a$. of "broken" arrows. We now define two monotone maps

$$d_1^2 : A_2 \longrightarrow A_1, \quad d_0^0 : A_0 \longrightarrow A_1$$

with the intention that d_1^2 (the composition map) sends $a'' \mathrel{\mapstochar\rightarrow} a' \mathrel{\mapstochar\rightarrow} a$ to $a'' \mathrel{\mapstochar\rightarrow} a$ and $d_0^0 : A_0 \longrightarrow A_1$ (the identity map) that produces the "identity broken arrow" $a \mathrel{\mapstochar\rightarrow} a$ for each a in A_0.

One can use the universal property of the comma square to define $d_1^2 : A_2 \longrightarrow A_1$ as the unique map such that the equality

$$
\begin{tikzcd}
A_2 \arrow[dr, "d_1^2"] & & \\
 & A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\
 & A_0 \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_2 \arrow[r, "d_2^2"] \arrow[d, "d_0^2"'] & A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\
A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[r, "f"'] \arrow[d, "f"] & B \arrow[d, equal] \\
A_0 \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

holds. It is clear that d_1^2 sends (a'', a', a) to (a'', a).

Analogously, one can define $i_0^0 : A_0 \longrightarrow A_1$ as the unique map such that the equality

$$\begin{tikzcd} A_0 \arrow[dr, "i_0^0"] & & \\ & A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\ & A_0 \arrow[r, "f"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A_0 \arrow[dr, "f"] & \\ & B \end{tikzcd}$$

holds. Explicitly: i_0^0 sends a to the pair (a, a).

To summarise: the above constructions yield a category object

$$\ker(f) \equiv A_2 \begin{array}{c} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{array} A_1 \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} A_0$$

in Pos such that $\langle d_0^1, d_1^1 \rangle$ is an rff-morphism and the span (d_0^1, A_1, d_1^1) is a two-sided discrete fibration.

Remark 3.6. Clearly, the steps of the above construction of $\ker(f)$ can be performed in *any* category $\mathscr{X}$ admitting finite limits. In fact, the resulting category object will have the two additional properties as well, since:

(1) A span (d_0^1, A_1, d_1^1) in a general category $\mathscr{X}$ is *defined* to be a two-sided discrete fibration if it is representably so. This means that the span $(\mathscr{X}(X, d_0^1), \mathscr{X}(X, A_1), \mathscr{X}(X, d_1^1))$ of monotone maps is a two-sided discrete fibration in Pos, for every X.

(2) The morphism $\langle d_0^1, d_1^1 \rangle : A_1 \longrightarrow A_0 \times A_0$ is easily proved to be an rff-morphism in a general category $\mathscr{X}$ iff the morphism $\langle \mathscr{X}(X, d_0^1), \mathscr{X}(X, d_1^1) \rangle : \mathscr{X}(X, A_1) \longrightarrow \mathscr{X}(X, A_0) \times \mathscr{X}(X, A_0)$ is an rff-morphism in Pos, for every X.

The above considerations lead us to the following definition:

Definition 3.7 ([33], [31])**.** Suppose A_0 is an object of $\mathscr{X}$. We say that a category object

$$\sim \equiv A_2 \begin{array}{c} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{array} A_1 \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} A_0$$

in $\mathscr{X}$, where the span (d_0^1, A_1, d_1^1) is a (two-sided) discrete fibration and $\langle d_0^1, d_1^1 \rangle : A_1 \longrightarrow A_0 \times A_0$ is an rff-morphism, is a *congruence* on A_0.

Remark 3.8. For a congruence $\sim$ as above, think of A_0 as the object of objects, A_1 as the object of morphisms, $i_0^0 : A_0 \longrightarrow A_1$ picks up the identity morphisms, $d_0^1 : A_1 \longrightarrow A_0$ is the domain map, $d_1^1 : A_1 \longrightarrow A_0$ is the codomain map, A_2 is the object of "composable pairs of morphisms" (since A_2 is the vertex of a pullback of d_0^1 and d_1^1), in a composable pair, $d_0^2 : A_2 \longrightarrow A_1$ picks the "morphism on the left", $d_2^2 : A_2 \longrightarrow A_1$ picks the "morphism on the right", and $d_1^2 : A_2 \longrightarrow A_1$ is the composition.

To treat congruences (and their quotients) conceptually, let us introduce the following notation:

Notation 3.9 ([11])**.** Let $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$ denote the chains on one, two, three elements, respectively. We denote by Δ_2^- the simplicial category truncated at stage two and with the morphisms between stage three and stage two omitted. More precisely: the category Δ_2^- is given by the graph

$$\Delta_2^- \equiv \mathbb{1} \begin{array}{c} \xrightarrow{\delta_1^1} \\ \xleftarrow{\iota_0^0} \\ \xrightarrow[\delta_0^1]{} \end{array} \mathbb{2} \begin{array}{c} \xrightarrow{\delta_0^2} \\ \xrightarrow{\delta_1^2} \\ \xrightarrow[\delta_2^2]{} \end{array} \mathbb{3}$$

subject to equalities

$$\iota_0^0 \cdot \delta_0^1 = 1, \quad \iota_0^0 \cdot \delta_1^1 = 1, \quad \delta_0^2 \cdot \delta_1^1 = \delta_2^2 \cdot \delta_0^1, \quad \delta_1^2 \cdot \delta_0^1 = \delta_0^2 \cdot \delta_0^1, \quad \delta_1^2 \cdot \delta_1^1 = \delta_2^2 \cdot \delta_1^1.$$

We denote by $J^- : \Delta_2^- \longrightarrow \mathsf{Pos}$ the inclusion.

Definition 3.10 ([23])**.** A diagram $D : {\Delta_2^-}^{op} \longrightarrow \mathscr{X}$ is called a *coherence datum* in $\mathscr{X}$. The colimit $J^- * D$ is called a *quotient* of D.

Remark 3.11. The colimit $J^- * D$ of a coherence datum is called a *codescent* of D in [23]. In our context, we prefer to call the colimit $J^- * D$ a *quotient* of D rather than a codescent of D.

Since every congruence is a coherence datum, the above definition can be applied to congruences. Thus

Definition 3.12. The *quotient of a congruence* is the quotient of the underlying coherence datum.

Remark 3.13. Due to enrichment in posets, the *computation* of quotients of general coherence data reduces to the computation of coinserters of $D\delta_0^1$, $D\delta_1^1$. This follows from the general coherence conditions for a quotients (see [23], where quotients are called codescents), specialised to the case of enrichment over posets.

Although the *computation* of quotients of congruences can be simplified, the *definition* of a congruence cannot be simplified. Observe that we need the full strength of the definition of a congruence in the proof of exactness of Pos, see Proposition 3.19. More in detail: congruences should be "transitive" and this is exactly what the object A_2 and the morphism $d_1^2 : A_2 \longrightarrow A_1$ are responsible for.

Definition 3.14. We say that a morphism is *effective* if it is a coinserter of some pair.

Remark 3.15. Effective morphisms are called *P-regular* in [10].

Lemma 3.16. *Any effective morphism is an* so*-morphism.*

Proof. Easy: use couniversality of a coinserter. The 1-dimensional aspect yields the required diagonal and the 2-dimensional aspect yields the 2-dimensional aspect of orthogonality. □

The above result establishes that "every reg-epi is strong epi" for our factorisation system of so-morphisms and rff-morphisms. The gist of the definition of regularity is the converse of this statement. The gist of the definition of exactness is that "congruences are precisely the kernel congruences".

Definition 3.17. A category $\mathscr{X}$ is called *regular*, provided that the following four properties are satisfied:

(R1) $\mathscr{X}$ has finite limits.
(R2) $\mathscr{X}$ is an $(\mathsf{so}, \mathsf{rff})$-category.
(R3) so-morphisms are stable under pullbacks.
(R4) so-morphisms are exactly the effective morphisms.

If, in addition, $\mathscr{X}$ verifies the following condition

(Ex) Every congruence in $\mathscr{X}$ is effective, i.e., it is of the form $\ker(f)$.

then $\mathscr{X}$ is called *exact*.

Remark 3.18. Let us stress our convention: when we say a category, we mean a category enriched in posets. Categories that are not enriched, are called ordinary.

In Example 3.20 below we show that the *enriched* category Set is regular but not exact in the enriched sense, although the *ordinary* category Set is exact in the ordinary sense (see [6]).

Proposition 3.19 (Exactness of presheaf categories). *Every category* $[\mathscr{S}^{op}, \mathsf{Pos}]$*,* $\mathscr{S}$ *small, is an exact category.*

Proof. We prove exactness of Pos, exactness of $[\mathscr{S}^{op}, \mathsf{Pos}]$ follows by reasoning pointwise.

The only non-trivial condition to verify is (Ex). Suppose therefore that

$$\sim\ \equiv\ \begin{tikzcd} A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "i_0^0" description] \end{tikzcd}$$

is a congruence on A_0. Form its quotient $q : A_0 \longrightarrow Q$ as in Example 2.1 and consider the kernel

$$\ker(q)\ \equiv\ \begin{tikzcd} P_2 \arrow[r, shift left=2, "p_2^2"] \arrow[r, "p_1^2" description] \arrow[r, shift right=2, "p_0^2"'] & q/q \arrow[r, shift left=2, "p_1^1"] \arrow[r, shift right=2, "p_0^1"'] & A_0 \arrow[l, "p_0^0" description] \end{tikzcd}$$

We claim that $\ker(q) = \sim$.

Denote by $z : A_1 \longrightarrow q/q$ the unique morphism such that the equality

$$\begin{tikzcd} A_1 \arrow[dr, "z"] & & \\ & q/q \arrow[r, "p^1_1"] \arrow[d, "p^1_0"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "q"] \\ & A_0 \arrow[r, "q"'] & Q \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[r, "d^1_1"] \arrow[d, "d^1_0"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "q"] \\ A_0 \arrow[r, "q"'] & Q \end{tikzcd}$$

holds, where the lax square on the left is a comma object.

In particular, the diagram

$$\begin{tikzcd} A_1 \arrow[r, "z"] \arrow[dr, "\langle d^1_0, d^1_1 \rangle"'] & q/q \arrow[d, "\langle p^1_0, p^1_1 \rangle"] \\ & A_0 \times A_0 \end{tikzcd}$$

commutes. It follows that z reflects order, since $\langle d^1_0, d^1_1\rangle$ does ($\sim$ is a congruence). We need to prove that z is surjective. To that end, consider an object of q/q, i.e., a pair (a', a) such that $qa' \leq qa$. Use now the description of inequality in a quotient of Example 2.1 to find a finite sequence $f_0, \dots, f_{n-1}$ of objects in A_1 such that the inequalities

$$a' \leq d^1_0(f_0), \quad d^1_1(f_0) \leq d^1_0(f_1), \quad d^1_1(f_1) \leq d^1_0(f_2), \quad \dots, \quad d^1_1(f_{n-1}) \leq a$$

hold in A_0.

Using the fact that the span (d^1_0, A_0, d^1_1) is a two-sided discrete fibration, one can find a sequence $f^*_0, \dots, f^*_{n-1}$ of elements of A_1 such that the equalities

$$a' = d^1_0(f^*_0), \quad d^1_1(f^*_0) = d^1_0(f^*_1), \quad d^1_1(f^*_1) = d^1_0(f^*_2), \quad \dots, \quad d^1_1(f^*_{n-1}) = a$$

hold in A_0. Since $\sim$ is a category object, the sequence $f^*_0, \dots, f^*_{n-1}$ composes (using d^2_1) to an element f^* of A_1 such that $a' = d^1_0(f^*)$ and $d^1_1(f^*) = a$. Hence $z(f^*) = (a', a)$, and we proved that $z : A_1 \longrightarrow q/q$ is surjective.

Thus $q/q = A_1$, hence $A_2 = P_2$ by uniqueness of pullbacks. It remains to be proved that $d^2_1 = p^2_1$. But this follows easily.

We proved that $\ker(q) = \sim$, the proof of exactness of Pos is finished. □

Example 3.20 (The category Set (having discrete orders on hom-sets) is regular but not exact). Regularity of Set is easy: observe that the effective morphisms are precisely the epis (and these are precisely the surjective mappings).

We exhibit a congruence that is not effective. Consider the *truncated nerve*

$$\mathsf{nerve}(2) \equiv \begin{tikzcd} A_2 \arrow[r, shift left=2, "d^2_2"] \arrow[r, "d^2_1" description] \arrow[r, shift right=2, "d^2_0"'] & A_1 \arrow[r, shift left=2, "d^1_1"] \arrow[r, shift right=2, "d^1_0"'] & A_0 \arrow[l, "i^1_0" description] \end{tikzcd}$$

of the two-element chain 2.

More in detail: A_0 is the two-element set $\{0, 1\}$, the set A_1 has as elements the pairs (i, j) with $i \leq j$ in 2, the set A_2 has as elements the triples (i, j, k) with $i \leq j \leq k$ in 2. All the connecting morphisms are defined in the obvious way.

It is easy to see that $\mathsf{nerve}(2)$ is a congruence. Yet there is no mapping $f : A_0 \longrightarrow X$ such that $\ker(f)$ would be $\mathsf{nerve}(2)$.

4. Some technical results

In this section we gather some auxilliary results that we will use in Section 5:

(1) We prove that the category $\mathsf{Cong}(\mathscr{X})$ of all congruences on an exact category $\mathscr{X}$ has all limits that $\mathscr{X}$ has.
(2) We summarise properties of an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ in case the counit $\varepsilon_A : FUA \longrightarrow A$ is an effective morphism (i.e., when it is a coinserter of some pair).
(3) We prove that the category $\mathscr{X}^{\mathbb{T}}$ of Eilenberg-Moore algebras for a monad $\mathbb{T}$ is regular, whenever $\mathscr{X}$ is regular and the functor of the monad $\mathbb{T}$ preserves so-morphisms.

4.A. **Limits of congruences.** We denote by $\mathsf{Cong}(\mathscr{X})$ the full subcategory of $[\Delta_2^{-op}, \mathscr{X}]$ spanned by congruences in $\mathscr{X}$. To be more specific: given coherence data

$$
\mathbb{X} \equiv X_2 \;\substack{\xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{}}\; X_1 \;\substack{\xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{}}\; X_0 \quad \text{and} \quad \mathbb{Y} \equiv Y_2 \;\substack{\xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{}}\; Y_1 \;\substack{\xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{}}\; Y_0
$$

then a *morphism* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is a triple $f_0 : X_0 \longrightarrow Y_0$, $f_1 : X_1 \longrightarrow Y_1$, $f_2 : X_2 \longrightarrow Y_2$ of morphisms in $\mathscr{X}$ making all the relevant squares commutative. Given morphisms $f, g : \mathbb{X} \longrightarrow \mathbb{Y}$, we put $f \leq g$ iff $f_i \leq g_i$ for all $i = 0, 1, 2$.

Lemma 4.1. *Suppose $\mathscr{X}$ is exact. Then the category $\mathsf{Cong}(\mathscr{X})$ is reflective in $[\Delta_2^{-op}, \mathscr{X}]$. In particular, $\mathsf{Cong}(\mathscr{X})$ is closed in $[\Delta_2^{-op}, \mathscr{X}]$ under limits.*

Proof. Suppose

$$
\mathbb{X} \equiv X_2 \;\substack{\xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{}}\; X_1 \;\substack{\xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{}}\; X_0
$$

is a coherence datum. Define the congruence

$$
\mathbb{X}^* \equiv X_2^* \;\substack{\xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{}}\; X_1^* \;\substack{\xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{}}\; X_0^*
$$

as $\ker(q)$, where $q : X_0 \longrightarrow Q$ is the quotient of $\mathbb{X}$.

We claim that there is a morphism $e : \mathbb{X} \longrightarrow \mathbb{X}^*$ that is universal.

(1) Definition of e.

The morphism e has to be a natural transformation. Thus we define morphisms $e_0 : X_0 \longrightarrow X_0^*$, $e_1 : X_1 \longrightarrow X_1^*$, $e_2 : X_2 \longrightarrow X_2^*$, and prove that all the naturality squares commute.

We put $e_0 = 1_{X_0}$, morphisms e_1, e_2 are defined using univesal properties:

$$
\begin{tikzcd}
 & & X_0^* \arrow[dr, "q"] \arrow[dd, phantom, "\uparrow"] & \\
X_1 \arrow[r, "e_1"] & X_1^* \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & Q \\
 & & X_0^* \arrow[ur, "q"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & X_0 \arrow[dr, "q"] \arrow[dd, phantom, "\uparrow"] & \\
X_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & Q \\
 & X_0 \arrow[ur, "q"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & & X_0^* \arrow[dr, "d_0^1"] & \\
X_2 \arrow[r, "e_2"] & X_2^* \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & X_0^* \\
 & & X_1^* \arrow[ur, "d_1^1"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & X_1 \arrow[r, "e_1"] & X_1^* \arrow[dr, "d_0^1"] & \\
X_2 \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & & X_0^* \\
 & X_1 \arrow[r, "e_1"'] & X_1^* \arrow[ur, "d_1^1"'] &
\end{tikzcd}
$$

where we use the universal property of a comma square and a pullback, respectively.

That $e : \mathbb{X} \longrightarrow \mathbb{X}^*$ is natural follows by straightforward computations.

(2) Universality of e.

Given $f : \mathbb{X} \longrightarrow \mathbb{Y}$ where $\mathbb{Y}$ is a congruence, we define a unique $f^\sharp : \mathbb{X}^* \longrightarrow \mathbb{Y}$ extending f along e.

Since $\mathscr{X}$ is exact, there is $z : Y_0 \longrightarrow K$ such that $\mathbb{Y} = \ker(z)$. Further, the existence of f yields $z^\sharp : Q \longrightarrow K$ such that the square

$$
\begin{tikzcd}
X_0 \arrow[r, "q"] \arrow[d, "f_0"'] & Q \arrow[d, "z^\sharp"] \\
Y_0 \arrow[r, "z"'] & K
\end{tikzcd}
$$

commutes.

We put $f_0^\sharp = f_0$, and $f_1^\sharp$, $f_2^\sharp$ are defined by universal properties:

$$\begin{tikzcd}[column sep=small]
 & & Y_0^* \arrow[dr, "z"] & & & & X_0^* \arrow[dr, "q"] & & \\
X_1^* \arrow[r, "f_1^\sharp"] & Y_1^* \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & K & = & X_1^* \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & Q \arrow[r, "z^\sharp"] & K \\
 & & Y_0^* \arrow[ur, "z"'] & & & & X_0^* \arrow[ur, "q"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
 & & Y_1^* \arrow[dr, "d_0^1"] & & & & X_1^* \arrow[dr, "d_0^1"] & & \\
X_2^* \arrow[r, "f_2^\sharp"] & Y_2^* \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & Y_0^* & = & X_2^* \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & X_0^* \arrow[r, "f_1^\sharp"] & Y_1^* \\
 & & Y_1^* \arrow[ur, "d_1^1"'] & & & & X_1^* \arrow[ur, "d_1^1"'] & &
\end{tikzcd}$$

where we have used the universal property of a comma square and a pullback, respectively.

The 2-dimensional aspect of universality of e is verified analogously, using 2-dimensional aspects of universality of comma squares and pullbacks.

□

Remark 4.2. Lemma 4.1 is the generalisation of the case of classical universal algebra: congruences form a complete lattice; meet of congruences is the intersection of the underlying relations; join of congruences is the congruence generated by the union of the underlying relations.

Indeed: $\mathsf{Cong}(\mathscr{X})$ is as (co)complete as $\mathscr{X}$. Reflectivity states that limits in $\mathsf{Cong}(\mathscr{X})$ are formed on the level of $[\Delta_2^{-\,op}, \mathscr{X}]$; whereas colimits in $\mathsf{Cong}(\mathscr{X})$ are the reflections of colimits in $[\Delta_2^{-\,op}, \mathscr{X}]$.

4.B. **Properties of $F \dashv U$ with an effective counit.** In Proposition 4.7 below we show that, when the counit of $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is a coinserter of some pair, then the underlying functor U has nice properties. The properties resemble the properties of adjunctions of descent type in ordinary category theory. In proving these results we were much inspired by arguments given by John Duskin in [13] for the case of monadicity over set-like ordinary categories.

We first prove an easy result on the interaction of U with rff-morphisms:

Lemma 4.3. *Suppose that $\mathscr{A}$ has finite limits and $U : \mathscr{A} \longrightarrow \mathscr{B}$ preserves them. Then U preserves rff-morphisms. If, moreover, U is conservative (i.e., if U reflects isomorphisms), then U reflects rff-morphisms.*

Proof. It is easy to see that $m : X \longrightarrow Y$ is an rff-morphism in $\mathscr{A}$ iff the canonical map $c(m) : 1_X/1_X \longrightarrow m/m$ between the comma objects is an isomorphism. Hence U preserves rff-morphisms if U preserves comma objects.

If, moreover, U reflects isomorphisms, then U reflects rff-morphisms, by the same argument. □

For the proof of Proposition 4.7 we will need the following "dual" of rff-morphisms.

Definition 4.4. We say that $e : A \longrightarrow B$ is a *co-rff-morphism* if it is rff in $\mathscr{X}^{op}$, or, equivalently, if $\mathscr{X}(e, Z) : \mathscr{X}(B, Z) \longrightarrow \mathscr{X}(A, Z)$ is order-reflecting, for every Z.

Remark 4.5. Co-rff-morphisms are called P-epis in [10], or absolutely dense in [7].

Lemma 4.6. *Suppose $\mathscr{X}$ has finite (enriched) limits. Then every so-morphism is a co-rff-morphism.*

Proof. Let $e : A \longrightarrow B$ be an so-morphism. Consider $u \cdot e \leq v \cdot e$ and form the inserter i of u and v. Consider the unique mediating map $k : A \longrightarrow E$ such that $i \cdot k = e$. Then the square

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[d, "1_B"] \\
E \arrow[r, "i"'] & B
\end{tikzcd}$$

commutes. Since i is in rff by its universal property, we can infer that i is a split epi, hence an isomorphism. Thus $u \leq v$ and we proved that e is a co-rff-morphism. □

Proposition 4.7. *Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is an adjunction, such that every component ε_A of the counit is effective. Then the following hold:*

(1) *U is locally order-reflecting. That is, the monotone action $U_{A',A} : \mathscr{A}(A', A) \longrightarrow \mathscr{X}(UA', UA)$ of the functor U is order-reflecting, for every A', A.*
(2) *U preserves and reflects congruences.*

(3) *U preserves and reflects limits.*
(4) *The comparison functor $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is fully faithful.*
(5) *If, moreover, $\mathscr{A}$ is regular, then U reflects effective morphisms.*

Proof. One at a time:

(1) Every effective morphism is a co-rff-morphism (use couniversal property of coinserters for that).
Hence every ε_A is a co-rff-morphism. Since the diagram

$$\begin{tikzcd} \mathscr{A}(A',A) \arrow[r, "U_{A',A}"'] \arrow[rr, bend left, "\mathscr{A}(\varepsilon_{A'},A)"] & \mathsf{Pos}(UA',UA) \arrow[r, "\cong"'] & \mathscr{A}(FUA',A) \end{tikzcd}$$

commutes, the proof is finished.

(2) Since U is a right adjoint, it preserves congruences. Indeed: suppose

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "d_0^0" description] \end{tikzcd}$$

is a congruence in $\mathscr{A}$.

Since U preserves (finite) limits, it preserves category objects. Thus

$$\begin{tikzcd} UA_2 \arrow[r, shift left=2, "Ud_2^2"] \arrow[r, "Ud_1^2" description] \arrow[r, shift right=2, "Ud_0^2"'] & UA_1 \arrow[r, shift left=2, "Ud_1^1"] \arrow[r, shift right=2, "Ud_0^1"'] & UA_0 \arrow[l, "Ud_0^0" description] \end{tikzcd}$$

is a category object in $\mathscr{X}$.

By the same argument $U\langle d_0^1, d_1^1\rangle \cong \langle Ud_0^1, Ud_1^1\rangle$. Since U preserves rff-morphisms (being a right adjoint, see Lemma 4.3), we proved that $\langle Ud_0^1, Ud_1^1\rangle$ is an rff-morphism.

Since being a two-sided discrete fibration is a representable notion, see Remark 3.6, the isomorphisms

$$\mathscr{X}(X, \langle Ud_0^1, Ud_1^1\rangle) \cong \mathscr{X}(X, U\langle d_0^1, d_1^1\rangle) \cong \mathscr{X}(FX, \langle d_0^1, d_1^1\rangle)$$

prove that the span $(\mathscr{X}(X, Ud_0^1)\mathscr{X}(X, UA_1), \mathscr{X}(X, Ud_1^1))$ is a two-sided discrete fibration, for any X. Hence the span (Ud_0^1, UA_1, Ud_1^1) is a two-sided discrete fibration.

For the reflection of congruences, consider a coherence datum $\mathbb{D} : \Delta_2^{-\,op} \longrightarrow \mathscr{A}$ such that the composite $U\mathbb{D} : \Delta_2^{-\,op} \longrightarrow \mathscr{X}$ is a congruence. To prove that $\mathbb{D}$ is a congruence, we need to prove that the composite $\mathscr{A}(A, \mathbb{D}) : \Delta_2^{-\,op} \longrightarrow \mathsf{Pos}$ is a congruence, for every A. Observe that every $\mathscr{A}(FX, \mathbb{D})$ is a congruence, since $\mathscr{A}(FX, \mathbb{D}) \cong \mathscr{X}(X, U\mathbb{D})$ holds.

Thus it suffices to present $\mathscr{A}(A, \mathbb{D})$ as a limit of congruences in Pos and then use Lemma 4.1.

Since ε_A is assumed to be effective, there is a a coinserter of the form

$$\begin{tikzcd} & FUA \arrow[dr, "\varepsilon_A"] \arrow[dd, phantom, "\uparrow"] & \\ A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & A \\ & FUA \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

We claim that the pasting

$$\begin{tikzcd} & & FUA \arrow[dr, "\varepsilon_A"] \arrow[dd, phantom, "\uparrow"] & \\ FUA_1 \arrow[r, "\varepsilon_{A_1}"] & A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & A \\ & & FUA \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

is a coinserter diagram. That is easy: ε_{A_1} is a co-rff-morphism, hence coinserter "cocones" for d_0^1, d_1^1 coincide with coinserter "cocones" for $d_0^1 \cdot \varepsilon_{A_1}$, $d_1^1 \cdot \varepsilon_{A_1}$.

Therefore we have an *inserter* diagram

$$\begin{tikzcd}
& \mathscr{A}(FUA,\mathbb{D}) \arrow[dr, "\mathscr{A}(d_1^1\varepsilon_{A_1},\mathbb{D})"] & \\
\mathscr{A}(A,\mathbb{D}) \arrow[ur, "\mathscr{A}(\varepsilon_A,\mathbb{D})"] \arrow[dr, "\mathscr{A}(\varepsilon_A,\mathbb{D})"'] & \uparrow & \mathscr{A}(FUA_1,\mathbb{D}) \\
& \mathscr{A}(FUA,\mathbb{D}) \arrow[ur, "\mathscr{A}(d_0^1\varepsilon_{A_1},\mathbb{D})"'] &
\end{tikzcd}$$

in $[\Delta_2^{-op},\mathsf{Pos}]$. But both $\mathscr{A}(FUA_1,\mathbb{D})$ and $\mathscr{A}(FUA,\mathbb{D})$ are congruences. By Lemma 4.1, $\mathscr{A}(A,\mathbb{D})$ is a congruence.

(3) U preserves limits since it is a right adjoint.

For reflecting limits, consider a diagram $D:\mathscr{D}\longrightarrow\mathscr{A}$ and a weight $W:\mathscr{D}\longrightarrow\mathsf{Pos}$. Suppose $\gamma : W\longrightarrow\mathscr{A}(A,D-)$ is a cylinder such that the composite

$$\overline{\gamma}\equiv W\xrightarrow{\gamma}\mathscr{A}(A,D-)\xrightarrow{U_{A,D-}}\mathscr{X}(UA,UD-)$$

is a limit cylinder in $\mathscr{X}$. This means that the monotone map

$$\overline{\varphi}_X:\mathscr{X}(X,UA)\longrightarrow[\mathscr{D},\mathsf{Pos}](W,\mathscr{X}(X,UD-)),\quad f\mapsto\mathscr{X}(f,-)\cdot\overline{\gamma}$$

is an isomorphism, naturally in X.

We need to prove that the monotone map

$$\varphi_{A'}:\mathscr{A}(A',A)\longrightarrow[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(A',D-)),\quad f\mapsto\mathscr{A}(f,-)\cdot\gamma$$

is an isomorphism, naturally in A'.

We will use a similar trick to (2) above. For observe that φ_{FX} is an isomorphism for every X: this follows from the commutative square

$$\begin{tikzcd}
\mathscr{A}(FX,A) \arrow[r, "\varphi_{FX}"] \arrow[d, "\cong"'] & {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FX,D-))} \arrow[d, "\cong"] \\
\mathscr{X}(X,UA) \arrow[r, "\overline{\varphi}_X"'] & {[\mathscr{D},\mathsf{Pos}](W,\mathscr{X}(X,UD-))}
\end{tikzcd}$$

where the vertical maps are given by the adjunction bijections.

Expressing $\varepsilon_{A'}$ as a coinserter

$$\begin{tikzcd}
& FUA' \arrow[dr, "\varepsilon_{A'}"] & \\
FUA'_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & A' \\
& FUA' \arrow[ur, "\varepsilon_{A'}"'] &
\end{tikzcd}$$

in the same way as in (2) above, we see that both

$$\begin{tikzcd}
& \mathscr{A}(FUA',A) \arrow[dr, "\mathscr{A}(d_1^1,A)"] & \\
\mathscr{A}(A',A) \arrow[ur, "\mathscr{A}(\varepsilon_{A'},A)"] \arrow[dr, "\mathscr{A}(\varepsilon_{A'},A)"'] & \uparrow & \mathscr{A}(FUA'_1,A) \\
& \mathscr{A}(FUA',A) \arrow[ur, "\mathscr{A}(d_0^1,A)"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FUA',D-)} \arrow[dr, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(d_1^1,D-))}"] & \\
{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(A',D-)} \arrow[ur, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(\varepsilon_{A'},D-))}"] \arrow[dr, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(\varepsilon_{A'},D-))}"'] & \uparrow & {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FUA'_1,D-)} \\
& {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FUA',D-)} \arrow[ur, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(d_0^1,D-))}"'] &
\end{tikzcd}$$

are inserters of isomorphic diagrams. Thus φ_A is an isomorphism by the essential uniqueness of inserters.

(4) Since $U = U^{\mathbb{T}} \cdot K$, the functor K is order-reflecting. In particular, K is faithful.

We prove that the functor K is full. To that end, consider $f : KA \longrightarrow KB$. Thus suppose the square

$$\begin{tikzcd} UFUA \arrow[r, "UFf"] \arrow[d, "U\varepsilon_A"'] & UFUB \arrow[d, "U\varepsilon_B"] \\ UA \arrow[r, "f"'] & UB \end{tikzcd}$$

commutes.

Since $\varepsilon_A : FUA \longrightarrow A$ is effective, there is a coinserter

$$\begin{tikzcd} & FUA \arrow[dr, "\varepsilon_A"] \arrow[dd, phantom, "\uparrow"] & \\ A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & A \\ & FUA \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

To prove that

$$\begin{tikzcd} & FUA \arrow[dr, "\varepsilon_B \cdot Ff"] \arrow[dd, phantom, "\uparrow"] & \\ A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & B \\ & FUA \arrow[ur, "\varepsilon_B \cdot Ff"'] & \end{tikzcd}$$

consider first the pasting

$$\begin{tikzcd} & UFUA \arrow[r, "UFf"] \arrow[dr, "U\varepsilon_A"] \arrow[dd, phantom, "\uparrow"] & UFUB \arrow[dr, "U\varepsilon_B"] & \\ UA_1 \arrow[ur, "Ud_1^1"] \arrow[dr, "Ud_0^1"'] & & UA \arrow[r, "f"] & UB \\ & UFUA \arrow[r, "UFf"'] \arrow[ur, "U\varepsilon_A"'] & UFUB \arrow[ur, "U\varepsilon_B"'] & \end{tikzcd}$$

and then use that U is locally order-reflecting.

By the universal property of coinserters there is a unique $h : A \longrightarrow B$ such that the square

$$\begin{tikzcd} FUA \arrow[r, "Ff"] \arrow[d, "\varepsilon_A"'] & FUB \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

commutes. Therefore $Uh \cdot U\varepsilon_A = f \cdot U\varepsilon_A$ holds (both are equal to $U\varepsilon_B \cdot UFf$). Since $U\varepsilon_A$ is epi, $Uh = f$ follows. Hence K is full.

(5) To prove the last assertion, suppose $e : A \longrightarrow B$ is such that Ue is effective in $\mathscr{X}$.

Then FUe is effective. Thus in the naturality square

$$\begin{tikzcd} FUA \arrow[r, "FUe"] \arrow[d, "\varepsilon_A"'] & FUB \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "e"'] & B \end{tikzcd}$$

the passage first-right-then-down is an so-morphism (use that every effective morphism is an so-morphism in $\mathscr{A}$). Therefore e is an so-morphism, hence effective.

The proof is finished. $\square$

4.C. **Regularity of $\mathscr{X}^{\mathbb{T}}$.** The proof of regularity of $\mathscr{X}^{\mathbb{T}}$ for a class of monads $\mathbb{T}$ that preserve so-morphisms is rather standard, see [6] for the classical case. We include the proof for the sake of self-containedness.

Definition 4.8. Say that a monad $\mathbb{T} = (T, \eta, \mu)$ is an so-*monad*, if T preserves so-morphisms.

Example 4.9 ([10], Section 8, Example 5)**.** Consider the adjunction $- \bullet \mathbb{2} \dashv [\mathbb{2}, -] : \mathsf{Pos} \longrightarrow \mathsf{Pos}$. The resulting monad $\mathbb{T} = (T, \eta, \mu)$ on Pos is not an so-monad: the so-morphism $e : 2 \longrightarrow \mathbb{2}$ is not preserved.

We will need the following technical notion.

Definition 4.10 ([22])**.** Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ is any functor. We say that $f : A \longrightarrow B$ is U-*final* if the following commutative diagram

$$\begin{tikzcd}
\mathscr{A}(B,B') \arrow[r, "\mathscr{A}(f,B')"] \arrow[d, "U_{B,B'}"'] & \mathscr{A}(A,B') \arrow[d, "U_{A,B'}"] \\
\mathscr{X}(UB,UB') \arrow[r, "\mathscr{X}(Uf,UB')"'] & \mathscr{X}(UA,UB')
\end{tikzcd}$$

is a pullback, for every B'.

Remark 4.11. Thus, as expected, U-finality has two aspects:

(1) For every $g : UB \longrightarrow UB'$, if $g \cdot Uf$ is of the form Uh, then there is a unique $g' : B \longrightarrow B'$ such that $Ug' = g$.
(2) If $g_1 \leq g_2 : UB \longrightarrow UB'$ and if $g_1 \cdot Uf \leq g_2 \cdot Uf$ has the form $Uh_1 \leq Uh_2$, then $g_1' \leq g_2'$.

Lemma 4.12. *Suppose $\mathscr{X}$ has finite limits and $\mathbb{T}$ is an so-monad on $\mathscr{X}$. If $U^{\mathbb{T}}e : A \longrightarrow B$ is an so-morphism in $\mathscr{X}$, then $e : (A,a) \longrightarrow (B,b)$ is $U^{\mathbb{T}}$-final.*

Proof. Consider $f : B \longrightarrow U^{\mathbb{T}}(C,c)$, such that the diagram

$$\begin{tikzcd}
TA \arrow[r, "Te"] \arrow[d, "a"'] & TB \arrow[r, "Tf"] \arrow[d, "b"'] & TC \arrow[d, "c"] \\
A \arrow[r, "e"'] & B \arrow[r, "f"'] & C
\end{tikzcd}$$

commutes. The morphism $Te : TA \longrightarrow TB$ is in so, hence epi by Lemma 4.6. Thus $f : (B,b) \longrightarrow (C,c)$ is a $\mathbb{T}$-algebra morphism.

The 2-dimensional aspect of finality follows analogously, using the fact that Te is a co-rff-morphism by Lemma 4.6. ☐

Proposition 4.13. *Suppose $\mathscr{X}$ has finite (enriched) limits and let $\mathbb{T}$ be an so-monad. Then $U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$ reflects so-morphisms. If $\mathscr{X}$ has $(\mathsf{so}, \mathsf{rff})$-factorisations, $U^{\mathbb{T}}$ preserves so-morphims.*

Proof. Suppose $e : (A,a) \longrightarrow (B,b)$ is a $\mathbb{T}$-algebra morphism such that $U^{\mathbb{T}}e = e : A \longrightarrow B$ is an so-morphism. Consider a commutative square

$$\begin{tikzcd}
(A,a) \arrow[r, "e"] \arrow[d, "u"'] & (B,b) \arrow[d, "v"] \\
(X,x) \arrow[r, "m"'] & (Y,y)
\end{tikzcd}$$

with m an rff-morphism in $\mathscr{X}^{\mathbb{T}}$. Since $U^{\mathbb{T}}$ preserves and reflects rff-morphisms by Lemma 4.3, the square

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\
X \arrow[r, "m"'] & Y
\end{tikzcd}$$

has a unique diagonal fill-in $d : B \longrightarrow X$ that is a $\mathbb{T}$-algebra morphism by $U^{\mathbb{T}}$-finality. This proves that $U^{\mathbb{T}}$ reflects so-morphisms.

The preservation: consider an so-morphism $e : (A, a) \longrightarrow (B, b)$. Form the $(\mathsf{so}, \mathsf{rff})$-factorisation $m \cdot e'$ of $U^{\mathbb{T}}e$. Then the diagram

$$\begin{tikzcd} TA \arrow[r, "Te'"'] \arrow[d, "a"'] \arrow[rr, bend left, "Te"] & TA' \arrow[r, "Tm"'] \arrow[d, dotted, "a'"] & TB \arrow[d, "b"] \\ A \arrow[r, "e'"] \arrow[rr, bend right, "e"'] & A' \arrow[r, "m"] & B \end{tikzcd}$$

commutes and there is a diagonal fill-in $a' : TA' \longrightarrow A'$ as indicated, since Te' is an so-morphism. The pair (A', a') is a $\mathbb{T}$-algebra, since m is a monomorphism. Thus we have $e = m \cdot e'$ in $\mathscr{X}^{\mathbb{T}}$. But e is in so and m is in rff (by Lemma 4.3). Therefore m is an isomorphism and we have proved that $e = e'$. Thus $U^{\mathbb{T}}$ reflects so-morphisms. □

Corollary 4.14. *Suppose that $\mathscr{X}$ has finite limits and $(\mathsf{so}, \mathsf{rff})$-factorisations. Suppose further that $\mathbb{T}$ is any monad on $\mathscr{X}$. Then the following are equivalent:*

(1) *$\mathbb{T}$ is an so-monad.*
(2) *$U^{\mathbb{T}}$ preserves so-morphisms.*

Proof. By Proposition 4.13 it suffices to prove that (2) implies (1).

Suppose that $e : A \longrightarrow B$ is an so-morphism. We prove that $Te : (TA, \mu_A) \longrightarrow (TB, \mu_B)$ is an so-morphism in $\mathscr{X}^{\mathbb{T}}$. To that end, consider the square

$$\begin{tikzcd} (TA, \mu_A) \arrow[r, "Te"] \arrow[d, "u"'] & (TB, \mu_B) \arrow[d, "v"] \\ (X, x) \arrow[r, "m"'] & (Y, y) \end{tikzcd}$$

with m an rff-morphism in $\mathscr{X}^{\mathbb{T}}$.

Then the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \arrow[ddl, dotted, "d"] \\ TA \arrow[d, "u"'] & TB \arrow[d, "v"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

commutes in $\mathscr{X}$ and the transpose $d^\sharp : (TB, \mu_B) \longrightarrow (X, x)$ under $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ of the unique diagonal d proves the 1-dimensional aspect of Te being an so-morphism. The 2-dimensional aspect is proved analogously.

Since $Te : (TA, \mu_B) \longrightarrow (TB, \mu_B)$ is an so-morphism in $\mathscr{X}^{\mathbb{T}}$, so is $U^{\mathbb{T}}Te = Te : TA \longrightarrow TB$. □

Corollary 4.15. *Suppose $\mathscr{X}$ is regular and $\mathbb{T}$ is an so-monad. Then $\mathscr{X}^{\mathbb{T}}$ is regular.*

Proof. $\mathscr{X}^{\mathbb{T}}$ has finite limits since $\mathscr{X}$ has them and $U^{\mathbb{T}}$ creates limits. Proposition 4.13 and Lemma 4.3 prove that $(\mathsf{so}, \mathsf{rff})$-factorisations exist in $\mathscr{X}^{\mathbb{T}}$. Moreover, so-morphisms in $\mathscr{X}^{\mathbb{T}}$ are pullback stable, since $U^{\mathbb{T}}$ preserves pullbacks, and preserves and reflects so-morphisms.

It remains to be proved that so-morphisms of $\mathscr{X}^{\mathbb{T}}$ are exactly the quotients of congruences in $\mathscr{X}^{\mathbb{T}}$. By Lemma 3.16 it suffices to prove that every so-morphism in $\mathscr{X}^{\mathbb{T}}$ is effective.

Consider an so-morphism $e : (A, a) \longrightarrow (B, b)$ and form its kernel congruence $\ker(e)$. By Proposition 4.7 $U^{\mathbb{T}} \ker(e)$ is a congruence and it is easy to see that $U^{\mathbb{T}} \ker(e) = \ker(U^{\mathbb{T}}e)$. Hence $U^{\mathbb{T}}e$ is a quotient of $U^{\mathbb{T}} \ker(e)$, since $\mathscr{X}$ is regular. Now use $U^{\mathbb{T}}$-finality of $e : (A, a) \longrightarrow (B, b)$ to conclude that e is a quotient of $\ker(e)$. □

5. Quasivarieties and varieties

In this section we prove our main results (Theorems 5.7 and 5.11 below) that characterise varieties and quasivarieties of ordered algebras for signatures in the sense of Stephen Bloom and Jesse Wright [10].

We start with precise definitions of signatures and their algebras.

Definition 5.1. Let λ be a regular cardinal. Denote by $|\mathsf{Set}_\lambda|$ the discrete category having λ-small sets as objects. A *λ-ary signature* Σ is a functor $\Sigma : |\mathsf{Set}_\lambda| \longrightarrow \mathsf{Pos}$. The signature Σ is called *bounded* if it is λ-ary for some regular cardinal λ.

Thus, a signature is a collection $(\Sigma n)_n$ of posets, indexed by sets of cardinalities smaller than λ. The elements of the poset Σn are called *n-ary operations*.

Definition 5.2. Given a λ-ary signature Σ, we denote by $H_\Sigma : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ the corresponding *polynomial functor*, defined by

$$H_\Sigma X = \coprod_n X^n \bullet \Sigma n$$

where the coproduct ranges over λ-small sets. A category Pos^{H_Σ} of *Σ-algebras and their homomorphisms* is the category of algebras for the functor H_Σ and algebra homomorphisms.

It is often convenient to think of a Σ-algebra as of a pair $(X, [\![-]\!])$ consisting of a poset X and monotone maps $[\![\sigma]\!] : X^n \longrightarrow X$ for every σ in Σn. If $\sigma \leq \tau$ in Σn, then there is an inequality $[\![\sigma]\!] \leq [\![\tau]\!]$ in the poset $\mathsf{Pos}(X^n, X)$.

Definition 5.3. Suppose that Σ is a bounded signature. We say that

1. $\mathscr{A}$ is a *quasivariety* if $\mathscr{A}$ is equivalent to a full subcategory of Pos^{H_Σ}, defined by implications of the form
$$\bigwedge_{i \in I} (s'_i(x_{ij}) \sqsubseteq s_i(x_{ij})) \Rightarrow t'(x_k) \sqsubseteq t(x_k)$$
where I is a set.
2. $\mathscr{A}$ is a *variety* if $\mathscr{A}$ is equivalent to a full subcategory of Pos^{H_Σ}, defined by inequations of the form
$$t'(x_k) \sqsubseteq t(x_k)$$

Remark 5.4. Since we are dealing with λ-ary signatures, one expects that λ-*filtered* colimits will play a prominent rôle. This is indeed the case: we only stress that all the notions concerning λ-filtered colimits are those that are appropriate for category theory enriched in posets.

We briefly recall the basic notions of the theory of λ-filtered colimits (and specialise them for the enrichment in posets). For details, see Max Kelly's paper [18]. Notice that the phrasing and results are the same as in the case of ordinary categories, see [14] or [3].

1. By a λ-*filtered* colimit in $\mathscr{X}$ we mean a *conical* colimit of an ordinary functor $D : \mathscr{D} \longrightarrow \mathscr{X}_o$, where $\mathscr{D}$ is a λ-filtered ordinary category and $\mathscr{X}_o$ denotes the underlying ordinary category of $\mathscr{X}$.
   Here, by a *conical* colimit of $D : \mathscr{D} \longrightarrow \mathscr{X}_o$ we understand a colimit weighted by the functor that is constantly the one-element poset.
2. A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is called *λ-accessible* if $\mathscr{A}$ has λ-filtered colimits and F preserves them.
3. An object X is called *λ-presentable* if the hom-functor $\mathscr{X}(X, -) : \mathscr{X} \longrightarrow \mathsf{Pos}$ is λ-accessible.
4. A category $\mathscr{X}$ is called *locally λ-presentable* if $\mathscr{X}$ is cocomplete and there is a small full dense subcategory $E : \mathscr{X}_\lambda \longrightarrow \mathscr{X}$ representing all λ-presentable objects of $\mathscr{X}$.

As examples of locally λ-presentable categories serve: the category Pos, every category of the form $[\mathscr{A}, \mathsf{Pos}]$ where $\mathscr{A}$ is small, every category of the form $\mathscr{X}^{\mathbb{T}}$ where $\mathscr{X}$ is locally λ-presentable and $\mathbb{T}$ is a λ-accessible monad (i.e., one, whose underlying functor T is λ-accessible). See [18] and [8].

Every (quasi)variety $\mathscr{A}$ is equipped by a functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ that arises as the composite of the fully faithful functor $K : \mathscr{A} \longrightarrow \mathsf{Pos}^{H_\Sigma}$ and the λ-accessible monadic functor $U^\Sigma : \mathsf{Pos}^{H_\Sigma} \longrightarrow \mathsf{Pos}$.

Lemma 5.5. *Let $\mathscr{A}$ be a (quasi)variety for a λ-ary signature. Then $\mathscr{A}$ has λ-filtered colimits and the full inclusion $K : \mathscr{A} \longrightarrow \mathsf{Pos}^{H_\Sigma}$ preserves them.*

Proof. It suffices to prove that if $(C, [\![-]\!])$ is a conical λ-filtered colimit of Σ-algebras $(Dd, [\![-]\!]^d)$ satisfying an implication

$$\bigwedge_{i \in I} (s'_i(x_{ij}) \sqsubseteq s_i(x_{ij})) \Rightarrow t'(x_k) \sqsubseteq t(x_k)$$

then $(C, [\![-]\!])$ satisfies this implication. Suppose therefore that $[\![(s'_i(x_{ij})]\!] \leq [\![s_i(x_{ij})]\!]$ holds in $(C, [\![-]\!])$, for all i. Since the diagram of $(Dd, [\![-]\!]^d)$'s is λ-filtered and since Σ is a λ-ary signature, there is d_0 such that $[\![(s'_i(x_{ij})]\!]^{d_0} \leq [\![s_i(x_{ij})]\!]^{d_0}$ holds in $(Dd_0, [\![-]\!]^{d_0})$. Therefore $t'(x_k) \sqsubseteq t(x_k)$ holds in $(Dd_0, [\![-]\!]^{d_0})$. Using monotonicity of the colimit injections, $t'(x_k) \sqsubseteq t(x_k)$ holds in $(C, [\![-]\!])$. □

Thus, we can work with (quasi)varieties as categories equipped with an accessible functor into Pos. Using this observation, we can reformulate the main result of [10] as follows:

Theorem 5.6 (**The main theorem of** [10]). *Suppose $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ is an accessible functor. Then U exhibits $\mathscr{A}$ as a quasivariety for a bounded signature iff the following conditions*

(Q1) *$\mathscr{A}$ has coinserters.*
(Q2) *The action of U on hom-posets is order reflecting.*
(Q3) *U has a left adjoint F.*
(Q4) *U preserves and reflects effective morphisms.*
(Q5) *U reflects isomorphisms.*

are satisfied.

The functor U exhibits $\mathscr{A}$ as a variety for a bounded signature iff, in addition, the condition

(V) *U reflects effective congruences.*

holds.

Our first intrinsic characterisation concerns varieties of ordered algebras. Compare the phrasing with Corollary 5.13 of [13] and Proposition 3.2 of [34].

Theorem 5.7 (**Intrinsic characterisation of λ-ary P-varieties**). *For $\mathscr{A}$, the following are equivalent:*

(1) *There is a λ-accessible functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$, exhibiting $(\mathscr{A}, U)$ as a λ-ary P-variety.*
(2) *$\mathscr{A}$ is exact and there is an equivalence $\mathscr{A} \simeq \mathsf{Pos}^{\mathbb{T}}$, for a λ-accessible monad $\mathbb{T}$ on Pos.*
(3) *$\mathscr{A}$ is exact, has coinserters, and there is an object P such that:*
 (a) *Tensors $X \bullet P$ exist for every poset X.*
 (b) *P is a λ-presentable object.*
 (c) *P is projective w.r.t. so-morphisms.*
 (d) *P is an so-generator, i.e., the canonical $\varepsilon_A : \mathscr{A}(P, A) \bullet P \longrightarrow A$ is an so-morphism.*

Proof. (1) implies (2). By [10, Section 6, Lemma 4], $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ is a λ-accessible monadic functor. Hence $\mathscr{A} \simeq \mathsf{Pos}^{\mathbb{T}}$ for the λ-accessible monad $\mathbb{T}$ given by U. Since U preserves so-morphisms, the category $\mathscr{A}$ is regular by Corollary 4.15. Since U reflects effective congruences, $\mathscr{A}$ is exact.

(2) implies (3). Assume $\mathscr{A} = \mathsf{Pos}^{\mathbb{T}}$. Then $\mathscr{A}$ is a locally λ-presentable category by [8, Theorem 6.9]. Thus $\mathscr{A}$ has coinserters.

To conclude the proof, put P to be the free algebra $F\mathbb{1}$ on the one-element poset.

(a) The tensor $X \bullet P$ is isomorphic to FX.
(b) The functor $U^{\mathbb{T}} \cong \mathscr{A}(P, -)$ is λ-accessible, hence P is λ-presentable.
(c) Since $U^{\mathbb{T}} \cong \mathscr{A}(P, -)$ holds, $\mathscr{A}(P, -)$ preserves so-morphisms by Corollary 4.14. This means precisely that P is so-projective.
(d) We only need to show that the counit ε_A of $F \dashv U$ is an so-morphism. But this is trivial: $U\varepsilon_A$ is a split epimorphism, hence an so-morphism in Pos. The monadic functor $U^{\mathbb{T}} : \mathsf{Pos}^{\mathbb{T}} \longrightarrow \mathsf{Pos}$ reflects so-morphisms, since $\mathbb{T}$ preserves so-morphisms by Proposition 4.13.

(3) implies (1). Define $U = \mathscr{A}(P, -)$. Then U is λ-accessible, since P is λ-presentable. We verify conditions (Q1)–(Q5) and (V) for the pair $(\mathscr{A}, U)$.

(Q1) $\mathscr{A}$ has coinserters.
 Trivial.
(Q3) U has a left adjoint.
 Easy: $F \cong - \bullet P$.
(Q2) U is locally order-reflecting.
 Since P is an so-generator, the counit ε_A of $F \dashv U$ is an so-morphism. Since $\mathscr{A}$ has finite limits (being locally presentable), every so-morphism is a co-rff-morphism, see Lemma 4.6.
 Thus every $\mathscr{A}(\varepsilon_{A'}, A) \cong U_{A',A}$ is order-reflecting.
(Q4) U preserves and reflects effective morphisms.
 Every effective morphism in $\mathscr{A}$ is an so-morphism. But U preserves so-morphisms, since P is so-projective. And every so-morphism in Pos is effective.
 U reflects effective morphisms by Proposition 4.7.
(Q5) U reflects isomorphisms.

Suppose $f : A \longrightarrow B$ is such that Uf is an isomorphism. Since $\varepsilon_A : FUA \longrightarrow A$ and $\varepsilon_B : FUB \longrightarrow B$ are so-morphisms, the naturality square

$$\begin{tikzcd} FUA \arrow[r, "FUf"] \arrow[d, "\varepsilon_A"'] & FUB \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

tells us that f is an so-morphism.

We prove that f is an rff-morphism. To that end, consider an inequality $f \cdot u \leq f \cdot u$. Since Uf is an isomorphism, $Uu \leq Uv$ holds. And $u \leq v$ holds by (Q2).

(V) U reflects effective congruences.
Use Proposition 4.7 and the fact that $\mathscr{A}$ is exact.

□

Example 5.8 (The category Set is not a variety of ordered algebras). Recall that from Example 1.1 the finitary monadic discrete-poset functor $U : \mathsf{Set} \longrightarrow \mathsf{Pos}$. Hence $\mathsf{Set} \simeq \mathsf{Pos}^{\mathbb{T}}$ for a finitary monad $\mathbb{T}$. By Example 3.20, the category Set is not exact (in the enriched sense). Hence Set is not equivalent to *any* variety of ordered algebras by Theorem 5.7. Of course, Set is a quasivariety of ordered algebras, see Example 1.1.

Remark 5.9. The equivalence of conditions of Theorem 5.7 can be easily extended to the "many-sorted" case. More in detail: for a category $\mathscr{A}$, the following conditions are equivalent:

(1) $\mathscr{A}$ is an S-sorted variety of ordered Σ-algebras for some set S and some λ-ary signature Σ of S-sorted operations.
(2) $\mathscr{A}$ is exact and there is an equivalence $\mathscr{A} \simeq [S, \mathsf{Pos}]^{\mathbb{T}}$, for a λ-accessible monad $\mathbb{T}$ on $[S, \mathsf{Pos}]$, where S is a set, considered as a discrete category.
(3) $\mathscr{A}$ is exact, has coinserters, and there is a set S and a functor $P : S^{op} \longrightarrow \mathscr{A}$ such that:
   (a) Colimits $X * P$ exist for every functor $X : S \longrightarrow \mathsf{Pos}$.
   (b) P is a λ-presentable object in $[S, \mathsf{Pos}]$.
   (c) P is projective w.r.t. so-morphisms in $[S, \mathsf{Pos}]$.
   (d) P is an so-generator, i.e., the canonical $\varepsilon_A : \mathscr{A}(P-, A) * P \longrightarrow A$ is an so-morphism.

Above, we need to be careful in what we mean by a many-sorted variety. The proper definition is as follows: $\mathscr{A}$ is an S-sorted variety if it is an HSP class of algebras for an S-sorted signature Σ and $\mathscr{A}$ is closed under λ-filtered colimits in all Σ-algebras.

Example 5.8 exhibited a finitary monad $\mathbb{T}$ on the category Pos such that $\mathsf{Pos}^{\mathbb{T}}$ is not a variety of ordered algebras. Next example shows that a category of the form $\mathsf{Pos}^{\mathbb{T}}$, $\mathbb{T}$ a finitary monad, need not even be a quasivariety of ordered algebras.

Example 5.10 (Category of the form $\mathsf{Pos}^{\mathbb{T}}$ that is not a quasivariety). Let $\mathbb{T}$ be the monad of the adjunction $F \dashv U : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ with $UX = [2, X]$ and $FX = X \bullet 2$. The adjunction $F \dashv U$ is *not monadic*, since 2 is not projective w.r.t. so-morphisms. This result is in contrast with the case of ordinary categories. See, e.g. [29] for discussion of monadicity of functors of the form $[S, -] : \mathscr{C} \longrightarrow \mathscr{C}$ in regular ordinary cartesian closed categories $\mathscr{C}$.

Moreover, the monad $\mathbb{T}$ of $F \dashv U$ is not an so-monad, see Example 4.9. Therefore the category $\mathsf{Pos}^{\mathbb{T}}$ is not a quasivariety by [10, Section 7, Proposition 2].

The difference between quasivarieties and varieties of ordered algebras is essentially the difference between regularity and exactness, as the next result shows.

Theorem 5.11 (Intrinsic characterisation of P-quasivarieties). *For $\mathscr{A}$, the following are equivalent:*

(1) *There is a λ-accessible functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ such that $(\mathscr{A}, U)$ is a λ-ary P-quasivariety.*
(2) *$\mathscr{A}$ is regular, has coinserters, and there exists an object P, such that:*
   (a) *Tensors $X \bullet P$ exist for every poset X.*
   (b) *P is a λ-presentable object.*
   (c) *P is projective w.r.t. so-morphisms.*
   (d) *P is an so-generator, i.e., the canonical $\varepsilon_A : \mathscr{A}(P, A) \bullet P \longrightarrow A$ is an so-morphism.*

Proof. (1) implies (2). By assumption, there is an adjunction $F \dashv U$. Define P as $F\mathbb{1}$. Then U is necessarily isomorphic to $\mathscr{A}(P, -)$ and F is isomorphic to $- \bullet P$.

We need to prove that $\mathscr{A}$ is regular. Observe first that the counit ε_A of $F \dashv U$ is effective. This follows from the fact that $U\varepsilon_A$ is effective in Pos (being a split epi) and U is assumed to reflect effective morphisms. Hence Proposition 4.7(4) can be applied: the comparison functor $K : \mathscr{A} \longrightarrow \mathsf{Pos}^{\mathbb{T}}$ is fully faithful. Moreover, $\mathsf{Pos}^{\mathbb{T}}$ is a regular category by Corollary 4.15 and it is a quasivariety by [10, Section 7, Proposition 2].

(R1) $\mathscr{A}$ has finite limits.
This follows from [10, Section 4, Corrollary 1].

(R2) $\mathscr{A}$ has $(\mathsf{so}, \mathsf{rff})$-factorisations.
First of all, U preserves and reflects rff-morphisms by Lemma 4.3. Moreover, $\mathscr{A}$ clearly has (effective,rff)-factorisations. Furthermore K preserves effective morphisms, since U preserves them and $U^{\mathbb{T}}$ reflects them (since $\mathsf{Pos}^{\mathbb{T}}$ is a quasivariety).
Therefore K preserves (effective, rff)-factorisations, these being $(\mathsf{so}, \mathsf{rff})$-factorisations in the quasivariety $\mathsf{Pos}^{\mathbb{T}}$. Since K is fully faithful, K reflects so-morphisms and therefore $\mathscr{A}$ has $(\mathsf{so}, \mathsf{rff})$-factorisations.

(R3) so-morphisms are stable under pullbacks.
This follows from the fact that K is fully faithful, preserves limits, and $\mathsf{Pos}^{\mathbb{T}}$ is regular.

(R4) so-morphisms coincide with the effective morphisms.
This follows from the above.

We proved that $\mathscr{A}$ is regular.

(a) Tensors $X \bullet P$ exist for every poset X.
This is clear: $X \bullet P \cong FX$.

(b) P is a λ-presentable object.
Clear: $U = \mathscr{A}(P, -)$ is λ-accessible.

(c) P is projective w.r.t. so-morphisms.
Clear: $U = \mathscr{A}(P, -)$ is assumed to preserve so-morphisms.

(d) P is an so-generator, i.e., the canonical $\varepsilon_A : \mathscr{A}(P, A) \bullet P \longrightarrow A$ is an so-morphism.
This was proved already.

(2) implies (1). Define $U = \mathscr{A}(P, -)$. Then U is λ-accessible and Conditions (Q1)–(Q5) for $(\mathscr{A}, U)$ are verified in the same way as in the proof of Theorem 5.7. ☐

Remark 5.12. Theorems 5.7 and 5.11 above were stated for an abstract category $\mathscr{A}$. Similar results can be stated for a pair $(\mathscr{A}, U)$ consisting of a category $\mathscr{A}$ and a functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$, since the properties of the object P from the above statements reflect the properties of U. More precisely, P is the representing object of U.

6. Finitary varieties and strongly finitary monads

In case when the signature Σ is *finitary*, i.e., when $\Sigma : |\mathsf{Set}_{fp}| \longrightarrow \mathsf{Pos}$, one can give yet other characterisations of varieties of Σ-algebras.

(1) The first characterisation involves the notion of *strongly finitary functors* introduced by Max Kelly and Steve Lack in [19].
We prove in Theorem 6.8 below that finitary varieties over Pos are precisely the strongly finitary monadic categories over Pos.

(2) The notion of strongly finitary functors is closely related to a certain class of weighted colimits, called *sifted*, see [11].
We prove in Theorem 6.10 that finitary varieties are precisely free cocompletions of their theories under sifted colimits.

Definition 6.1 ([19])**.** A functor $H : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is *strongly finitary* if it is a left Kan extension of its restriction along the discrete-poset functor $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$.

A monad $\mathbb{T}$ on Pos is *strongly finitary* if its functor is strongly finitary.

Remark 6.2. By definition, a functor $H : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is strongly finitary iff it has a coend expansion

$$HX = \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, X) \bullet Hn$$

for every poset X.

Since every Dn is a finitely presentable object in Pos, every strongly finitary functor H is *a fortiori* finitary.

Lemma 6.3. *Every strongly finitary functor $H : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ preserves* so*-morphisms.*

Proof. Consider an so-morphism $e : A \longrightarrow B$. Then $He : HA \longrightarrow HB$ has a coend expansion

$$\int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, e) \bullet Hn : \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, A) \bullet Hn \longrightarrow \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, B) \bullet Hn$$

Since every $\mathsf{Pos}(Dn, e) : \mathsf{Pos}(Dn, A) \longrightarrow \mathsf{Pos}(Dn, B)$ is surjective, so is every

$$\mathsf{Pos}(Dn, e) \bullet Hm : \mathsf{Pos}(Dn, A) \bullet Hm \longrightarrow \mathsf{Pos}(Dn, B) \bullet Hm$$

Thus He is surjective as a colimit of surjections. □

Example 6.4. None of the implications *strongly finitary ⇒ finitary and* so*-preserving ⇒ finitary* can be reversed.

(1) The functor $T : X \mapsto [\mathbb{2}, X \bullet \mathbb{2}]$ is finitary but it does not preserve so-morphisms. This follows from Example 4.9.
(2) Consider the connected-component functor $\pi_0 : \mathsf{Pos} \longrightarrow \mathsf{Pos}$. It preserves so-morphisms and it is finitary. The functor π_0 is, however, not strongly finitary. Suppose it were, then

$$\pi_0(X) = \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, X) \bullet \pi_0 n = \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, X) \bullet n = X$$

would hold for every poset X: use that $\pi_0 n = n$ for every discrete poset and, for the last equality, use that the inclusion $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$ is dense. But $\pi_0(\mathbb{2}) = \mathbb{1} \not\cong \mathbb{2}$.

Incidentally, both T and π_0 have the structure of a monad: for T, consider the adjunction $- \bullet \mathbb{2} \dashv [\mathbb{2}, -]$, and for π_0, consider the (monadic) adjunction $C \dashv U : \mathsf{Set} \longrightarrow \mathsf{Pos}$, where U is the discrete-poset functor and C assigns the set of components to a poset.

It can be proved that $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$ exhibits Pos as a *free cocompletion* of Set_{fp} w.r.t. a certain class of colimits that include filtered colimits and an enriched analogue of reflexive coequalisers, namely *quotients of reflexive coherence data* (see below). This follows by a modification of arguments given in [11, Section 8.4].

Definition 6.5 ([11]). Denote by Δ_2 the full simplicial category truncated at stage two. That is, Δ_2 is given by the graph

$$\begin{tikzcd} \Delta_2 \equiv \mathbb{1} \arrow[r, shift left=3, "\delta^1_1"] \arrow[r, shift right=3, "\delta^1_0"'] & \mathbb{2} \arrow[l, "\iota^0_0" description] \arrow[r, shift left=6, "\delta^2_0"] \arrow[r, "\delta^2_1" description] \arrow[r, shift right=6, "\delta^2_2"'] & \mathbb{3} \arrow[l, shift right=3, "\iota^1_1" description] \arrow[l, shift left=3, "\iota^1_0" description] \end{tikzcd}$$

subject to simplicial equalities. See, e.g., [28].

A *reflexive coherence datum* in $\mathscr{X}$ is a diagram $R : {\Delta_2}^{op} \longrightarrow \mathscr{X}$. A *quotient* of a reflexive coherence datum $R : {\Delta_2}^{op} \longrightarrow \mathscr{X}$ is a colimit $J * R$, where $J : \Delta_2 \longrightarrow \mathsf{Pos}$ denotes the full inclusion.

Filtered colimits and quotients of reflexive coherence data form a density presentation in the sense of [17] of the fully faithful dense functor $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$.

The *saturation* (the *closure*, in the terminology of [2]) of the class of filtered colimits and quotients of reflexive coherence data is the class of weights, called *sifted*. This is in analogy to the case of ordinary sifted colimits introduced by Christian Lair [25]. More in detail: a weight $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$ is called *sifted*, if the n-fold product functor $\Pi_n : [n, \mathsf{Pos}] \longrightarrow \mathsf{Pos}$ preserves W-colimits, for every finite discrete poset n.

Example 6.6. Every filtered colimit and every quotient of a reflexive coherence datum is an example of a sifted colimit. Every reflexive coequaliser is a sifted colimit.

Using various types of sifted colimits, we can give a characterisation of functors preserving sifted colimits. We formulate the result for functors preserving finite limits between exact categories, since this is how we will need it.

Proposition 6.7. *Suppose $H : \mathscr{K} \longrightarrow \mathscr{L}$ preserves finite limits and suppose $\mathscr{K}$ and $\mathscr{L}$ are cocomplete exact categories. Then the following are equivalent:*

(1) *H preserves sifted colimits.*
(2) *H preserves filtered colimits and quotients of reflexive coherence data.*
(3) *H preserves filtered colimits and quotients of congruences.*

Proof. Clearly, (1) is equivalent to (2). That (2) implies (3) follows from the fact that every congruence is a reflexive coherence datum. For (3) implies (2) it suffices to prove that H preserves quotients of reflexive coherence data. Consider a reflexive coherence datum

$$\mathbb{D} \equiv X_2 \begin{array}{c} \xrightarrow{d_2^2} \\ \xleftarrow{i_0^1} \\ \xrightarrow{d_1^2} \\ \xleftarrow{i_1^1} \\ \xrightarrow[d_0^2]{} \end{array} X_1 \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} X_0$$

and observe that, for the quotient $q : X_0 \longrightarrow X$ of $\mathbb{D}$, the congruence $\ker(q)$ has the same cocones as $\mathbb{D}$. ☐

We can now formulate the first characterisation of finitary varieties.

Theorem 6.8. *For a category $\mathscr{A}$, the following conditions are equivalent:*
1. *$\mathscr{A}$ is equivalent to a variety of algebras for a finitary signature.*
2. *$\mathscr{A}$ is equivalent to $\mathsf{Pos}^{\mathbb{T}}$ for a strongly finitary monad $\mathbb{T}$ on Pos.*

Proof. (1) implies (2). By Theorem 5.7 we know that $\mathscr{A}$ is an exact category and that $\mathscr{A}$ is equivalent to $\mathsf{Pos}^{\mathbb{T}}$ for a finitary monad $\mathbb{T}$ on Pos. Moreover, the monad $\mathbb{T}$ is given by the adjunction $- \bullet P \dashv \mathscr{A}(P, -)$, where P is a free algebra on $\mathbb{1}$.

To prove that the monad $\mathbb{T}$ is strongly finitary, by Proposition 6.7 it therefore suffices to prove that its functor $X \mapsto \mathscr{A}(P, X \bullet P)$ preserves quotients of congruences in Pos. The left adjoint $X \mapsto X \bullet P$ preserves all colimits. And $\mathscr{A}(P, -)$ does preserve quotients of congruences, since $\mathscr{A}$ is a variety.

(2) implies (1). We only need to prove that $\mathsf{Pos}^{\mathbb{T}}$ is an exact category. Since $\mathbb{T}$ is an so-monad by Lemma 6.3, the category $\mathsf{Pos}^{\mathbb{T}}$ is regular by Corollary 4.15. Thus it remains to be proved that congruences are effective in $\mathsf{Pos}^{\mathbb{T}}$. To that end, consider a congruence

$$\sim \equiv (X_2, a_2) \begin{array}{c} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{array} (X_1, a_1) \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} (X_0, a_0)$$

in $\mathsf{Pos}^{\mathbb{T}}$. Then there is $f : X_0 \longrightarrow X$ in Pos such that $U^{\mathbb{T}}(\sim) = \ker(f)$. Since T preserves quotients of congruences, we can form

$$TU^{\mathbb{T}}(\sim) \equiv TX_2 \begin{array}{c} \xrightarrow{Td_2^2} \\ \xrightarrow{Td_1^2} \\ \xrightarrow[Td_0^2]{} \end{array} TX_1 \begin{array}{c} \xrightarrow{Td_1^1} \\ \xleftarrow{Ti_0^0} \\ \xrightarrow[Td_0^1]{} \end{array} TX_0$$

having $Tf : TX_0 \longrightarrow TX$ as its quotient. Define $a : TX \longrightarrow X$ as the unique mediating map:

$$\begin{array}{ccccccc}
TX_2 & \begin{array}{c} \xrightarrow{Td_2^2} \\ \xrightarrow{Td_1^2} \\ \xrightarrow[Td_0^2]{} \end{array} & TX_1 & \begin{array}{c} \xrightarrow{Td_1^1} \\ \xleftarrow{Ti_0^0} \\ \xrightarrow[Td_0^1]{} \end{array} & TX_0 & \xrightarrow{Tf} & TX \\
{\scriptstyle a_2}\downarrow & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_0} & & \vdots{\scriptstyle a} \\
X_2 & \begin{array}{c} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{array} & X_1 & \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} & X_0 & \xrightarrow[f]{} & X
\end{array}$$

It is then easy to see that (X, a) is a $\mathbb{T}$-algebra and f is a $\mathbb{T}$-algebra homomorphism. Moreover, $\sim\, = \ker(f)$ in $\mathsf{Pos}^{\mathbb{T}}$. ☐

We prove now that finitary varieties of ordered algebras are free cocompletions of certain small categories under sifted colimits.

Definition 6.9. Suppose $\mathbb{T} = (T, \eta, \mu)$ is a strongly finitary monad on Pos. By $\mathsf{Th}(\mathbb{T})$ we denote the full subcategory of $\mathsf{Pos}^{\mathbb{T}}$ spanned by free $\mathbb{T}$-algebras on objects of Set_{fp}. The category $\mathsf{Th}(\mathbb{T})$ is called the *theory* of $\mathbb{T}$.

The following result states that the category of algebras for $\mathbb{T}$ is the free cocompletion of $\mathsf{Th}(\mathbb{T})$ under sifted colimits. This is the enriched analogue of the classical result. See, e.g., Theorem 4.13 of [5].

Theorem 6.10. *Let $\mathbb{T} = (T, \eta, \mu)$ be a strongly finitary monad on* Pos*. Then the embedding $E : \mathsf{Th}(\mathbb{T}) \longrightarrow \mathsf{Pos}^{\mathbb{T}}$ exhibits $\mathsf{Pos}^{\mathbb{T}}$ as a free cocompletion of $\mathsf{Th}(\mathbb{T})$ under sifted colimits.*

Proof. We will use Proposition 4.2 of [21]. Since E is fully faithful and $\mathsf{Pos}^{\mathbb{T}}$ cocomplete, we only need to prove that $\mathsf{Pos}^{\mathbb{T}}$ is the closure of $\mathsf{Th}(\mathbb{T})$ under sifted colimits and that every functor $\mathsf{Pos}^{\mathbb{T}}((Tn, \mu_n), -) : \mathsf{Pos}^{\mathbb{T}} \longrightarrow \mathsf{Pos}$, where n is discrete and finite poset, preserves sifted colimits.

(1) We prove that every $\mathbb{T}$-algebra is an iterated sifted colimit of $\mathbb{T}$-algebras free on discrete posets. This is done in three steps:
  (a) Using quotients of truncated nerves that are reflexive coherence data, one can exhibit every algebra free on a finite poset.
  More in detail: given a finite poset P, exhibit it as a quotient $q : P_0 \longrightarrow P$ of its truncated nerve

$$\mathsf{nerve}(P) \equiv P_2 \;\begin{smallmatrix} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{smallmatrix}\; P_1 \;\begin{smallmatrix} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{smallmatrix}\; P_0$$

  in an analogous way as it was done for $\mathbb{2}$ in Example 3.20. Since $\mathsf{nerve}(P)$ can clearly be augmented to form a reflexive coherence datum, we proved that $F^{\mathbb{T}}P$ arises as a sifted colimit of free algebras on finite discrete posets.
  (b) Further, using filtered colimits, one can exhibit every algebra free on a poset.
  More in detail: suppose X is any poset. Then X can be written as a filtered colimit of finite posets. Hence $F^{\mathbb{T}}X$ is a filtered (hence, sifted) colimit of algebras of the form $F^{\mathbb{T}}P$, where P is a finite poset.
  (c) Finally, using canonical presentations that are reflexive coequalisers, one can exhibit every $\mathbb{T}$-algebra.
  More in detail, given a $\mathbb{T}$-algebra (X, a), consider the diagram

$$(TTX, \mu_{TX}) \overset{Ta}{\underset{\mu_X}{\rightrightarrows}} (TX, \mu_X) \xrightarrow{a} (X, a)$$

  that is a reflexive coequaliser in $\mathsf{Pos}^{\mathbb{T}}$. Hence (TX, a) is a sifted colimit of free algebras.

(2) The functor $\mathsf{Pos}^{\mathbb{T}}((Tn, \mu_n), -) \cong \mathsf{Pos}(n, U^{\mathbb{T}}-) = \mathsf{Pos}(n, -) \cdot U^{\mathbb{T}}$, preserves sifted colimits, since every $\mathsf{Pos}(n, -)$ does and $U^{\mathbb{T}}$ preserves filtered colimits and quotients of congruences. Hence, by Proposition 6.7, $U^{\mathbb{T}}$ preserves sifted colimits.

This concludes the proof. □

7. Conclusions and future work

We gave intrinsic characterisations of categories equivalent to (quasi)varieties of ordered algebras in the sense of Stephen Bloom and Jesse Wright. Namely, we showed that, for the notion of an ordered algebra as a poset equipped with monotone operations of discrete arities, such characterisation theorems are very similar to the classical case of unordered algebras [5]. The only difference to the classical case is the ubiquitous need for the use of 2-dimensional notions. Hence one can say that ordered universal algebra in the sense of Stephen Bloom and Jesse Wright is the "poset-version" of the classical set-based universal algebra.

We believe that our work is only an opening study in the direction of understanding ordered universal algebra using categorical methods. In fact, much of the results surveyed in [5] need to be investigated. Let us mention just a few:

(1) The rôle of sifted colimits in the enriched sense in the study of generalised varieties, see [4] for the classical case. Also, it is not clear how the non-existence of λ-sifted colimits, λ-uncountable, in the set-based case (see [1]) transfers to the enriched setting.
(2) The connection of (quasi)varieties and regular and exact completions of categories enriched over posets. See, e.g., the paper [34] for the ordinary case.
(3) The Morita-type theorems concerning Morita equivalence of ordered theories.

We also believe that the categorical theory of ordered algebras will lead to a better understanding of order-algebraizable logics in the sense of James Raftery [30].

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