Intuitionistic Logic

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Contents

1 Introduction

In this course we give an introduction to intuitionistic logic. We concentrate on the propositional calculus mostly, make some minor excursions to the predicate calculus and to the use of intuitionistic logic in intuitionistic formal systems, in particular Heyting Arithmetic. We have chosen a selection of topics that show various sides of intuitionistic logic. In no way we strive for a complete overview in this short course. Even though we approach the subject for the most part only formally, it is good to have a general introduction to intuitionism. This we give in section 2 in which also natural deduction is introduced. For more extensive introductions see [35],[17].

After this introduction we start with other proof systems and the Kripke models that are used for intuitionistic logic. Completeness with respect to Kripke frames is proved. Metatheorems, mostly in the form of disjunction properties and admissible rules, are explained. We then move to show how classical logic can be interpreted in intuitionistic logic by Gödels's negative translation and how in its turn intuitionistic logic can be interpreted by another translation due to Gödel into the modal logic **S4** and several other modal logics. Finally we introduce the infinite fragment of intutionistic logic of 1 propositional variable. The Kripke model called the Rieger-Nishimura Ladder that comes up while studying this fragment will play a role again later in the course. The next subject is a short subsection in which the complexity of the intuitionistic propositional calculus is shown to be **PSPACE**-complete. We end up with a discussion of a recently developed game for intuitionistic propositional logic [25].

In the next section Heyting algebras are discussed. We show the connections with intuitionistic logic itself and also with Kripke frames and topology. Completeness of **IPC** with respect to Heyting algebras is shown. Unlike in the case of Kripke models this can straightforwardly be generalized to extensions of **IPC**, the so-called intermediate logics. The topological connection leads also to closure algebras that again give a relation to the modal logic **S4** and its extension **Grz**.

The final section is centered around the usage of Jankov formulas in intuitionistic logic and intermediate logics. The Jankov formula of a (finite rooted) frame $\mathfrak{F}$ axiomatizes the least logic that does not have $\mathfrak{F}$ as a frame. We approach this type of formulas via n-universal models. These are minimal infinite models in which all distinctions regarding formulas in n propositional variables can be made. Jankov formulas correspond to point generated upsets of n-universal models. We show how some well-known logics can be easily axiomatized by Jankov formulas and how this fails in other cases. We also show how Jankov formulas are used to prove that there are uncountably many intermediate logics.

In the final subsection we are concerned with the logic of the Rieger-Nishimura Ladder. Many of the interesting properties of this logic can be approached with the aid of Jankov formulas.

2 Intuitionism

Intuitionism is one of the main points of view in the philosophy of mathematics, nowadays usually set opposite *formalism* and *Platonism*. As such intuitionism is best seen as a particular manner of implementing the idea of *constructivism* in mathematics, a manner due to the Dutch mathematician Brouwer and his pupil Heyting. Constructivism is the point of view that mathematical objects exist only in so far they have been constructed and that proofs derive their validity from constructions; more in particular, existential assertions should be backed up by *effective* constructions of objects. Mathematical truths are rather seen as being created than discovered. Intuitionism fits into *idealistic* trends in philosophy: the mathematical objects constructed are to be thought of as idealized objects created by an idealized mathematician (IM), sometimes called *the creating* or *the creative subject*. Often in its point of view intuitionism skirts the edges of *solipsism* when the idealized mathematician and the proponent of intuitionism seem to fuse.

Much more than formalism and Platonism, intuitionism is in principle *normative*. Formalism and Platonism may propose a foundation for existing mathematics, a reduction to logic (or set theory) in the case of Platonism, or a consistency proof in the case of formalism. Intuitionism in its stricter form leads to a reconstruction of mathematics: mathematics as it is, is in most cases not acceptable from an intuitionistic point of view and it should be attempted to rebuild it according to principles that are constructively acceptable. Typically it is not acceptable to prove $\exists x\, \phi(x)$ (for some x, $\phi(x)$ holds) by deriving a contradiction from the assumption that $\forall x \neg \phi(x)$ (for each x, $\phi(x)$ does not hold): *reasoning by contradiction*. Such a proof does not create the object that is supposed to exist.

Actually, in practice the intuitionistic point of view hasn't lead to a large scale and continuous rebuilding of mathematics. For what has been done in this respect, see e.g. [4]. In fact, there is less of this kind of work going on now even than before. On the other hand, one might say that intuitionism describes a particular portion of mathematics, the constructive part, and that it has been described very adequately by now what the meaning of that constructive part is. This is connected with the fact that the intuitionistic point of view has been very fruitful in *metamathematics*, the construction and study of systems in which parts of mathematics are formalized. After Heyting this has been pursued by Kleene, Kreisel and Troelstra (see for this, and an extensive treatment of most other subjects discussed here, and many other ones [35]). Heyting's [17] will always remain a quickly readable but deep introduction to the intuitionistic ideas. In theoretical computer science many of the formal systems that are of foundational importance are formulated on the basis of intuitionistic logic.

L.E.J. Brouwer first defended his constructivist ideas in his dissertation of 1907 ([8]). There were predecessors defending constructivist positions: mathematicians like Kronecker, Poincaré, Borel. Kronecker and Borel were prompted by the increasingly abstract character of concepts and proofs in the mathematics of the end of the 19th century, and Poincaré couldn't accept the formalist or

Platonist ideas proposed by Frege, Russell and Hilbert. In particular, Poincaré maintained in opposition to the formalists and Platonists that mathematical induction (over the natural numbers) cannot be reduced to a more primitive idea. However, from the start Brouwer was more radical, consistent and encompassing than his predecessors. The most distinctive features of intuitionism are:

1. The use of a distinctive logic: *intuitionistic logic*. (Ordinary logic is then called *classical* logic.)
2. Its construction of the *continuum*, the totality of the real numbers, by means of *choice sequences*.

Intuitionistic logic was introduced and axiomatized by A. Heyting, Brouwer's main follower. The use of intuitionistic logic has most often been accepted by other proponents of constructive methods, but the construction of the continuum much less so. The particular construction of the continuum by means of choice sequences involves principles that contradict classical mathematics. Constructivists of other persuasion like the school of Bishop often satisfy themselves in trying to constructively prove theorems that have been proved in a classical manner, and shrink back from actually contradicting ordinary mathematics.

Intuitionistic logic. We will indicate the formal system of intuitionistic propositional logic by **IPC** and intuitionistic predicate logic by **IQC**; the corresponding classical systems will be named **CPC** and **CQC**. Formally the best way to characterize intuitionistic logic is by a *natural deduction system* à la Gentzen. (For an extensive treatment of natural deduction and sequent systems, see [34].) In fact, natural deduction is more natural for intuitionistic logic than for classical logic. A natural deduction system has *introduction* rules and *elimination* rules for the logical connectives $\wedge$ (and), $\vee$ (or) and $\rightarrow$ (if ..., then) and *quantifiers* $\forall$ (for all) and $\exists$ (for at least one). The rules for $\wedge$, $\vee$ and $\rightarrow$ are:

- $I\wedge$: From ϕ and ψ conclude $\phi\wedge\psi$,
- $E\wedge$: From $\phi\wedge\psi$ conclude ϕ and conclude ψ,
- $E\rightarrow$: From ϕ and $\phi\rightarrow\psi$ conclude ψ,
- $I\rightarrow$: If one has a derivation of ψ from premise ϕ, then one may conclude to $\phi\rightarrow\psi$ (simultaneously dropping assumption ϕ),
- $I\vee$: From ϕ conclude to $\phi\vee\psi$, and from ψ conclude to $\phi\vee\psi$,
- $E\vee$: If one has a derivation of χ from premise ϕ and a derivation of χ from premise ψ, then one is allowed to conclude χ from premise $\phi\vee\psi$ (simultaneously dropping assumptions ϕ and ψ),
- $I\forall$: If one has a derivation of $\phi(x)$ in which x is not free in any premise, then one may conclude $\forall x\phi(x)$,

- $E\forall$: If one has a derivation of $\forall x\phi(x)$, then on may conclude $\phi(t)$ for any term t,
- $I\exists$: From $\phi(t)$ for any term t one may conclude $\exists x\phi(x)$,
- $E\exists$: If one has a derivation of ψ from $\phi(x)$ in which x is not free in in ψ itself or in any premise other than $\phi(x)$, then one may conclude ψ from premise $\exists x\phi(x)$, dropping the assumption $\phi(x)$ simultaneously.

One usually takes negation $\neg$ (not) of a formula ϕ to be defined as ϕ implying a contradiction ($\bot$). One adds then the *ex falso sequitur quodlibet rule* that

- anything can be derived from $\bot$.

If one wants to get classical propositional or predicate logic one adds the rule that

- if $\bot$ is derived from $\neg\phi$, then one can conclude to ϕ, simultaneously dropping the assumption $\neg\phi$.

Note that this is not a straightforward introduction or elimination rule as the other rules.

The natural deduction rules are strongly connected with the so-called BHK-interpretation (named after Brouwer, Heyting and Kolmogorov) of the connectives and quantifiers. This interpretation gives a very clear foundation of intuitionistically acceptable principles and makes intuitionistic logic one of the very few non-classical logics in which reasoning is clear, unambiguous and all encompassing but nevertheless very different from reasoning in classical logic.

In classical logic the meaning of the *connectives*, i.e. the meaning of complex statements involving the connectives, is given by supplying the *truth conditions* for complex statements that involve the informal meaning of the same connectives. For example:

- $\phi \wedge \psi$ is true if and only if ϕ is true *and* ψ is true,
- $\phi \vee \psi$ is true if and only if ϕ is true *or* ψ is true,
- $\neg\phi$ is true iff ϕ is *not* true

The BHK-interpretation of intuitionistic logic is based on the notion of *proof* instead of truth. (N.B! *Not* formal proof, or derivation, as in natural deduction or Hilbert type axiomatic systems, but intuitive (informal) proof, i.e. convincing mathematical argument.) The meaning of the connectives and quantifiers is then just as in classical logic explained by the informal meaning of their intuitive counterparts:

- A proof of $\phi \wedge \psi$ consists of a proof of ϕ *and* a proof of ψ plus the conclusion $\phi \wedge \psi$,

- A proof of $\phi \vee \psi$ consists of a proof of ϕ *or* a proof of ψ plus a conclusion $\phi \vee \psi$,
- A proof of $\phi \to \psi$ consists of a *method of converting* any proof of ϕ into a proof of ψ,
- *No* proof of $\perp$ exists,
- A proof of $\exists x\, \phi(x)$ consists of a name d of *an* object constructed in the intended domain of discourse plus a proof of $\phi(d)$ and the conclusion $\exists x\, \phi(x)$,
- A proof of $\forall x\, \phi(x)$ consists of a method that *for any* object d constructed in the intended domain of discourse produces a proof of $\phi(d)$.

For negations this then means that a proof of $\neg\, \phi$ is a method of converting any supposed proof of ϕ into a proof of a contradiction. That $\perp \to \phi$ has a proof for any ϕ is based on the intuitive counterpart of the ex falso principle. This may seem somewhat less natural then the other ideas, and Kolmogorov did not include it in his proposed rules.

Together with the fact that statements containing negations seem less contentful constructively this has lead Griss to consider doing completely without negation. Since however it is often possible to prove such more negative statements without being able to prove more positive counterparts this is not very attractive. Moreover, one can do without the formal introduction of $\perp$ in natural mathematical systems, because a statement like $1=0$ can be seen to satisfy the desired properties of $\perp$ without making any ex falso like assumptions. More precisely, not only statements for which this is obvious like $3=2$, but all statements in those intuitionistic theories are derivable from $1=0$ without the use of the rules concerning $\perp$. If one nevertheless objects to the ex falso rule, one can use the logic that arises without it, called *minimal logic.*

The intuitionistic meaning of a disjunction is only superficially close to the classical meaning. To prove a disjunction one has to be able to prove one of its members. This makes it immediately clear that there is no general support for $\phi \vee \neg\, \phi$: there is no way to invariably guarantee a proof of ϕ or a proof of $\neg\, \phi$. However, many of the laws of classical logic remain valid under the BHK-interpretation. Various *decision methods* for **IPC** are known, but it is often easy to decide intuitively:

- A disjunction is hard to prove: for example, of the four directions of the two *de Morgan laws* only $\neg\,(\phi \wedge \psi) \to \neg\phi \vee \neg\psi$ is not valid, other examples of such invalid formulas are
  - $\phi \vee \neg\phi$ (the law of the *excluded middle*)
  - $(\phi \to \psi) \to \neg\phi \vee \psi$
  - $(\phi \to \psi \vee \chi) \to (\phi \to \psi) \vee (\phi \to \chi)$
  - $((\phi \to \psi) \to \psi) \to (\phi \vee \psi)$

- An existential statement is hard to prove: for example, of the four directions of the classically valid interactions between negations and quantifiers only $\neg\forall x\,\phi \to \exists x \neg\,\phi$ is not valid,
- statements directly based on the two-valuednes of truth values are not valid, e.g. $\neg\neg\,\phi \to \phi$ or $((\phi \to \psi) \to \phi) \to \phi$ (*Peirce's law*), and *contraposition* in the form $(\neg\psi \to \neg\phi) \to \phi \to \psi)$,
- On the other hand, many basic laws naturally remain valid, commutativity and associativity of conjunction and disjunction, both distributivity laws, and
  - $(\phi \to \psi \wedge \chi) \leftrightarrow (\phi \to \psi) \wedge (\phi \to \chi)$,
  - $(\phi \to \chi) \wedge (\psi \to \chi) \leftrightarrow ((\phi \vee \psi) \to \chi))$,
  - $(\phi \to (\psi \to \chi)) \leftrightarrow (\phi \wedge \psi) \to \chi$.
  - $(\phi \vee \psi) \wedge \neg\phi \to \psi)$ (needs *ex falso*!),
  - $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$,
  - $(\phi \to \psi) \to (\neg\psi \to \neg\phi)$ (the *converse* form of contraposition),
  - $\phi \to \neg\neg\phi$,
  - $\neg\neg\neg\phi \leftrightarrow \neg\phi$ (triple negations are not needed).

Slightly less obvious is that *double negation shift* is valid for $\wedge$ and $\to$ but not for $\forall$, at least in one direction. Valid are:

- $\neg\neg(\phi \wedge \psi) \leftrightarrow \neg\neg\phi \wedge \neg\neg\psi$,
- $\neg\neg(\phi \to \psi) \leftrightarrow \neg\neg\phi \to \neg\neg\psi$,
- $\neg\neg\forall x\phi(x) \to \forall x\neg\neg\phi(x)$ (but not its converse).

The BHK-interpretation was independently given by Kolmogorov and Heyting, with Kolmogorov's formulation in terms of the solution of problems rather than in terms of executing proofs. Of course, both extracted the idea from Brouwer's work. In any case, it is clear from the above that, if a logical schema is (formally) provable in **IPC** (say, by natural deduction), then any instance of the scheme will have an informal proof following the BHK-interpretation.

Clearly, in the most direct sense intuitionistic logic is weaker than classical logic. However, from a different point of view the opposite is true. By Gödel's so-called *negative translation* classical logic can be translated into intuitionistic logic. To translate a classical statement one puts $\neg\neg$ in front of all atomic formulas and then replaces each subformula of the form $\phi \vee \psi$ by $\neg(\neg\,\phi \wedge \neg\,\psi)$ and each subformula of the form $\exists x\,\phi(x)$ by $\neg\forall x\neg\,\phi(x)$ in a recursive manner. The formula obtained is provable in intuitionistic logic exactly when the original one is provable in classical logic. Some examples are:

- $p \vee \neg p$ becomes in translation $\neg(\neg\neg p \wedge \neg\neg\neg p)$,

- $(\neg q \to \neg p) \to (p \to q)$ becomes $(\neg\neg\neg q \to \neg\neg\neg p) \to (\neg\neg p \to \neg\neg q)$,
- $\neg\forall x\, Ax \to \exists x \neg Ax$ becomes $\neg\forall x \neg\neg Ax \to \neg\forall x \neg\neg Ax$

Thus, one may say that intuitionistic logic accepts classical reasoning in a particular form and is therefore richer than classical logic.

3 Kripke models, Proof systems and Metatheorems

3.1 Other proof systems

We start this section with a Hilbert type system for intuitionistic logic. We will call the intuitionistic propositional calculus **IPC** and the intuitionistic predicate calculus **IQC** in contrast to the classical systems **CPC** and **CQC**. For extensive information on the topics treated in this section, see [34].

Axioms for a Hilbert type system for IPC.

1. $\phi \to (\psi \to \phi)$.
2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$.
3. $\phi \wedge \psi \to \phi$.
4. $\phi \wedge \psi \to \psi$.
5. $\phi \to \phi \vee \psi$.
6. $\psi \to \phi \vee \psi$.
7. $(\phi \to \chi) \to ((\psi \to \chi) \to (\phi \vee \psi \to \chi))$.
8. $\bot \to \phi$.

with the rule of *modus ponens*: from ϕ and $\phi \to \psi$ conclude ψ.

This system is closely related to the natural deduction system. The first two axiom schemes are exactly sufficient to prove the deduction theorem, which mirrors the introduction rule for implication.

Theorem 1. (*Deduction Theorem*) *If* $\Gamma, \phi \vdash_{\mathbf{IPC}} \psi$, *then* $\Gamma \vdash_{\mathbf{IPC}} \phi \to \psi$.

Proof. By induction on the length of the derivation, using the fact that $\chi \to \chi$ is derivable. □

Exercise 2. Show that $\vdash_{\mathbf{IPC}} \chi \to \chi$.

Gentzen sequent calculus for IPC.

- Structural rules (like weakening),
- Axioms: $\Gamma, \phi, \Delta \Longrightarrow \Theta, \phi, \Lambda$ and $\Gamma, \bot, \Delta \Longrightarrow \Theta$,
- $L\wedge$
$$\frac{\Gamma, \phi, \psi, \Delta \Longrightarrow \Theta}{\Gamma, \phi \wedge \psi, \Delta \Longrightarrow \Theta},$$
- $R\wedge$:
$$\frac{\Gamma \Longrightarrow \Delta, \phi, \Theta \text{ and } \Gamma \Longrightarrow \Delta, \psi, \Theta}{\Gamma \Longrightarrow \Delta, \phi \wedge \psi, \Theta},$$
- $L\vee$:
$$\frac{\Gamma, \phi, \Delta \Longrightarrow \Theta \,\text{and}\, \Gamma, \psi, \Delta \Longrightarrow \Theta}{\Gamma, \phi \vee \psi, \Delta \Longrightarrow \Theta},$$
- $R\vee$:
$$\frac{\Gamma \Longrightarrow \Delta, \phi, \psi, \Theta}{\Gamma \Longrightarrow \Delta, \phi \vee \psi, \Theta},$$
- $R\rightarrow$:
$$\frac{\Gamma, \phi, \Delta \Longrightarrow \psi}{\Gamma, \Delta \Longrightarrow \Theta, \phi \rightarrow \psi, \Lambda},$$
- $L\rightarrow$:
$$\frac{\Gamma, \psi, \Delta \Longrightarrow \Theta \text{ and } \Gamma, \phi \rightarrow \psi, \Delta \Longrightarrow \phi, \Theta}{\Gamma, \phi \rightarrow \psi, \Delta \Longrightarrow \Theta},$$
- Cut
$$\frac{\Gamma \Longrightarrow \Delta, \phi \text{ and } \phi, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta.}$$

This, not very common, sequent calculus system for **IPC** has the advantage that, read from bottom to top these rules are rules for a *semantic tableau* system for **IPC**. A more standard sequent calculus system for **IPC** is obtained by restricting in a sequent calculus for classical logic **CPC** the sequence of formulas on the right to one formula (or none). For example, $R\vee$ becomes:

$$\frac{\Gamma \Longrightarrow \phi}{\Gamma \Longrightarrow \phi \vee \psi}$$

plus the same for ψ instead of ϕ. In both systems Cut can be eliminated. This means that there is a way of transforming a derivation with cuts into a derivation without cut. A similar theorem applies to natural deduction. A derivation natural deduction can be transformed into a *normal* derivation, i.e. a derivation in which formulas are not first introduced and then eliminated.

Predicate calculus.
We give just the axioms for the Hilbert type system:

1. $\forall x\, \phi(x) \to \phi(t)$, with t not containing variables that become bound in $\phi(t)$.
2. $\phi(t) \to \exists x\, \phi(x)$, with t not containing x or variables that become bound in $\phi(t)$.

   and the rules:

3. From $\phi \to \psi(x)$ conclude $\phi \to \forall x\, \psi(x)$, if x not free in ϕ.
4. From $\phi(x) \to \psi$ conclude $\exists x\, \phi(x) \to \psi$, if x not free in ψ.

3.2 Arithmetic and analysis

Classical arithmetic of the natural numbers is formalized in **PA** by the so-called *Peano axioms* (the idea of which is originally due to Dedekind). The axioms for intuitionistic arithmetic (or *Heyting arithmetic*) **HA** are the same:
These axioms can simply be added to the Hilbert system for the predicate calculus, or, for that matter to a natural deduction or sequent system. **HA**-models are simply predicate logic models for the language of **HA** in which the **HA**-axioms are verified at each node.
Arithmetic. Classical arithmetic of the natural numbers is formalized in **PA** by the so-called *Peano axioms* (the idea of which is originally due to Dedekind). These axioms

- $x + 1 \neq 0$,
- $x + 1 = y + 1 \to x = y$,
- $x + 0 = x$,
- $x + (y + 1) = (x + y) + 1$,
- $x \,.\, 0 = 0$,
- $x \,.\, (y + 1) = x \,.\, y + x$,

and the *induction* scheme

- For each $\phi(x)$, $\phi(0) \wedge \forall x(\phi(x) \to \phi(x + 1)) \to \forall x \phi(x)$.

can simply be added to the Hilbert system for the predicate calculus, or, for that matter to a natural deduction or sequent system. Of course an intuitionist does not simply accept these axioms face value but checks their (intuitive) provability from the basic idea of what natural numbers are (Brouwer in his inaugural address: "... This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers,

inasmuch as one of the elements of the two-oneness maybe thought of as a new two-oneness, which process may be repeated indefinitely ...").
Worth while noting is that the scheme

- For each $\phi(x)$, $\exists x\phi(x) \rightarrow \exists x(\phi(x) \wedge \forall y < x \neg\phi(y))$

is classically but not intuitionistically equivalent to the induction scheme. (Here $y < x$ is defined as $\exists z(y + (z+1) = x)$.)

Gödels' negative translation is applicable to **HA**/**PA**. Of course, also Gödel's incompleteness theorem applies to **HA**: there exists a ϕ such that neither $\vdash_{\mathbf{HA}} \phi$, nor $\vdash_{\mathbf{HA}} \neg\phi$, and this ϕ can be taken to have the form $\forall x\psi(x)$ for some $\psi(x)$ such that, for each n, $\vdash_{\mathbf{HA}} \psi(\bar{n})$. (Here $\bar{n}$ stands for $1 + \ldots + 1$ with n ones, a term with the value n.)

Free choice sequences. A great difficulty in setting up constructive versions of mathematics is the continuum. It is not difficult to reason about individual real numbers via for example *Cauchy sequences*, but one loses that way the intuition of the totality of all real numbers which does seem to be a primary intuition. Brouwer based the continuum on the idea of choice sequences. For example, a choice sequence α of natural numbers is viewed as an ever unfinished, ongoing process of choosing natural number values $\alpha(0), \alpha(1), \alpha(2), \cdots$ by the ideal mathematician IM. At any stage of IM's activity only finitely many values have been determined by IM, plus possibly some restrictions on future choices. This straightforwardly leads to the idea that a function f giving values to all choice sequences can do so only by having the value $f(\alpha)$ for any particular choice sequence α determined by a finite initial segment $\alpha(0), \ldots, \alpha(m)$ of that choice sequence, in the sense that all choice sequences β starting with the same initial segment $\alpha(0), \ldots, \alpha(m)$ have to get the same value under the function: $f(\beta) = f(\alpha)$. This idea will lead us to Brouwer's theorem that every real function on a bounded closed interval is necessarily uniformly continuous. Of course, this is in clear contradiction with classical mathematics.

Before we get to a characteristic example of a less severe distinction between classical and intuitionistic mathematics, the *intermediate value theorem*, let us discuss the fact that counterexamples to classical theorems in logic or mathematics can be given as *weak* counterexamples or *strong* counterexamples. A weak counterexample to a statement just shows that one cannot hope to prove that statement, a strong counterexample really derives a contradiction from the general application of the statement. For example, to give a weak counterexample to $p \vee \neg p$ it suffices to give a statement ϕ that has not been proved or refuted, especially a statement of a kind that can always be reproduced if the original problems is solved after all. A strong counterexample to $\phi \vee \neg\phi$ cannot consist of proving $\neg(\phi \vee \neg\phi)$ for some particular ϕ, since $\neg(\phi \vee \neg\phi)$ is even in intuitionistic logic contradictory (it is directly equivalent to $\neg\phi \wedge \neg\neg\phi$). But a predicate $\phi(x)$ in intuitionistic analysis can be found such that $\neg\forall x\,(\phi(x) \vee \neg\phi(x))$ can be proved, which can reasonably be called a strong counterexample.

For weak counterexamples Brouwer often used the decimal expansion of

π. For example consider the number $a = 0, a_0 a_1 a_2 \ldots$ for which the decimal expansion[1] defined as follows:

As long as no sequence 1234567890 has occurred in the decimal expansion of π, a_n is defined to be 3. If a sequence 1234567890 has occcurred in the decimal expansion of π starting at some m with $m \leqslant n$, then, if the first such m is even a_n is 0 for all $n \geqslant m$, if it is odd, $a_m = 4$ and $a_n = 0$ for all $n > m$. As long as the problem has not been solved whether such a sequence exists it is not known whether $a < \frac{1}{3}$ or $a = \frac{1}{3}$ or $a > \frac{1}{3}$. That this is time bound is shown by the fact that in the meantime this particular problem has been solved, m does exist and is even, so $a < \frac{1}{3}$ [7]. But that does not matter, such problems can, of course, be multiplied endlessly, and (even though we don't take the trouble to change our example) this shows that it is hopeless to try to prove that, for any a, $a < \frac{1}{3} \vee a = \frac{1}{3} \vee a > \frac{1}{3}$. Note that, also a cannot be shown to be rational, because for that, p and q should be given such that $a = \frac{p}{q}$, which clearly cannot be done without solving the problem. On the other hand, obviously, $\neg\neg(a < \frac{1}{3} \vee a = \frac{1}{3} \vee a > \frac{1}{3})$ does hold, a is not not rational. In any case, weak counterexamples are not mathematical theorems, but they do show which statements one should not try to prove. Later on, Brouwer used unsolved problems to provide weak and strong counterexamples in a stronger way by making the decimal expansion of a dependent on the creating subjects' insight whether he had solved a particular unsolved problem at the moment of the construction of the decimal in question. Attempts to formalize these so-called *creative subject arguments* have lead to great controversy and sometimes paradoxical consequences. For a reconstruction more congenial to Brouwer's ideas that avoids such problematical consequences, see [26].

Let us now move to using a weak counterexample to show that one cannot hope to prove the so-called *intermediate value theorem.* A continuous function f that has value -1 at 0 and value 1 at 1 reaches the value 0 for some value between 0 and 1 according to classical mathematics. This does not hold in the constructive case: a function f that moves linearly from value -1 at 0 to value $a - \frac{1}{3}$ at $\frac{1}{3}$, stays at value $a - \frac{1}{3}$ until $\frac{2}{3}$ and then moves linearly to 1 cannot be said to reach the value 0 at a particular place if one does not know whether $a > \frac{1}{3}$, $a = \frac{1}{3}$ or $a < \frac{1}{3}$. Since there is no method to settle the latter problem in general, one cannot determine a value x where $f(x) = 0$. (See Figure 1.)

Constructivists of the Russian school did not accept the intuitionistic construction of the continuum, but neither did they shrink from results contradicting classical mathematics. They obtained such results in a different manner however, by assuming that effective constructions are *recursive* constructions, and thus in particular when one restricts functions to effective functions that all functions are recursive functions. Thus, in opposition to the situation in classical mathematics, accepting the so-called *Church-Turing thesis* that all ef-

[1] To make arguments easier to follow, we discuss these problems regarding real numbers with arguments pertaining to their decimal expansions. This was not Brouwer's habit, he even showed with a weak counteraxmple that not all real numbers have a decimal expansion (how to start the decimal expansion of a if one does not know whether it is smaller than, equal to, or greater than 0?).

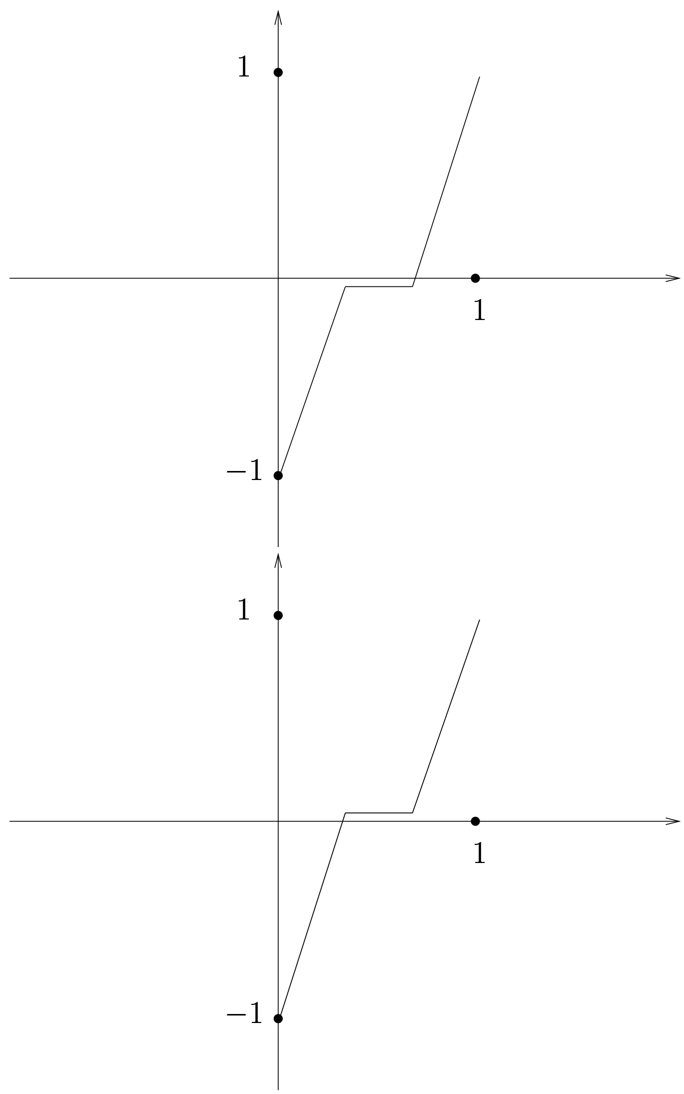


Figure 1: Counter-example to the intermediate value theorem

fective functions are recursive does influence the validity of mathematical results directly.

Let us remark finally that, no matter what ones standpoint is, the resulting formalized intuitionistic analysis has a more complicated relationship to classical analysis than the one between **HA** and **PA**, the negative translation does no longer apply.

Realizability. Kleene used recursive functions in a different manner than the Russian constructivists. Starting in the 1940's he attempted to give a faithful interpretation of intuitionistic logic and (formalized) mathematics by means of recursive functions. To understand this, we need to know two basic facts. The first is that there is a recursive way of coding pairs of natural numbers by a single one, j is a bijection from $\mathbb{N}^2$ to $\mathbb{N}$: $j(m,n)$ codes the pair (m,n) as a single natural number. Decoding is done by the functions $(\,)_0$ and $(\,)_1$: if $j(m.n)=p$, then $(p)_0=m$ and $(p)_1=n$. The second insight is that all recursive functions, or easier to think about, all the Turing machines that calculate them can be coded by natural numbers as well. If e codes a Turing machine, then $\{e\}$ is the function that is calculated by it, i.e. for each natural number n, $\{e\}(n)$ has a certain value if on input n the Turing machine coded by e delivers that value. Now Kleene defines how a natural number realizes an arithmetic statement (in the language of **HA**):

- Any $n\in\mathbb{N}$ realizes an atomic sentence iff the statement is true,
- n realizes $\phi\wedge\psi$ iff $(n)_0$ realizes ϕ and $(n)_1$ realizes ψ,
- n realizes $\phi\vee\psi$ iff $(n)_0=0$ and $(n)_1$ realizes ϕ, or $(n)_0=1$ and $(n)_1$ realizes ψ,
- n realizes $\phi\rightarrow\psi$ iff, for any $m\in\mathbb{N}$ that realizes ϕ, $\{n\}(m)$ has a value that realizes ψ,
- n realizes $\forall x\phi(x)$ iff, for each $m\in\mathbb{N}$, $\{n\}(m)$ has a value that realizes $\phi(\overline{m})$,
- n realizes $\exists x\phi(x)$ iff, $(n)_1$ realizes $\phi(\overline{(n)_0})$.

One cannot say that realizability is a faithful interpretation of intuitionism, as Kleene later realized very well. For example, it turns out that at least from the classical point of view there exist in **IPC** unprovable formulas all of whose arithmetic instances are realizible. But realizability has been an enormously successful concept that has multiplied into countless variants. One important fact Kleene was immediately able to produce by means of realizability is that, if **HA** proves a statement of the form $\forall x\exists y\phi(x,y)$, then ϕ is satisfied by a recursive function $\{e\}$, and even, for each $n\in\mathbb{N}$, **HA** proves $\phi(\overline{n},\overline{\{e\}(n)})$. For more on realizability, see e.g. [33].

Intuitionistic logic in intuitionistic formal systems. Intuitionistic logic, in the form of propositional logic or predicate logic satisfies the so-called *disjunction property*: if $\phi \vee \psi$ is derivable, then ϕ is derivable or ψ. This is very characteristic for intuitionistic logic: for classical logic $p \vee \neg p$ is an immediate counterexample to this assertion. The property also transfers to the usual formal systems for arithmetic and analysis. Of course, this is in harmony with the intuitionistic philosophy. If $\phi \vee \psi$ is formally provable, then if things are right it is informallly provable as well. But then, according to the BHK-interpretation, ϕ or ψ should be provable informally as well. It would at least be nice if the formal system were complete enough to provide this formal proof, and in the usual case it does. For existential statements something similar holds, an *existence property*, if $\exists x\, \phi(x)$ is derivable in Heyting's arithmetic, then $\phi(\bar{n})$ is derivable for some $\bar{n}$. Statements of the form $\forall y \exists x\, \phi(y,x)$ express the existence of functions, and, for example for Heyting's arithmetic, the existence property then transforms in: if such a statements is derivable, then also some instantiation of it as a recursive function as was stated above already. In classical Peano arithmetic such properties only hold for particularly simple, e.g. quantifier-free, ϕ. In fact, with regard to the latter statements, classical and intuitionistic arithmetic are of the same strength.

Some formal systems may be decidable (e.g. some theories of order) and then one will have classical logic in most cases. However, in Heyting's arithmetic one has de Jongh's *arithmetic completeness theorem* stating that its logic is exactly the intuitionistic one: if a formula is not derivable in intuitionistic logic an arithmetic substitution instance can be found that is not derivable in Heyting's arithmetic (see e.g. [21], [32]). For the particular case of $p \vee \neg p$ this is easy to see, it follows immediately from Gödel's incompleteness theorem and the disjunction property: by Gödel a sentence ϕ exists which **HA** can neither prove nor refute, by the disjunction property **HA** will then not be able to prove $\phi \vee \neg\phi$ either.

3.3 Kripke models

Definition 3. *A Kripke* frame $\mathfrak{K} = (K, R)$ *for* **IPC** *has a reflexive partial order* R*. A Kripke* model (K, R, V) *for* **IPC** *on such a frame is* persistent*, in the sense that, if* wRw' *and* $w \in V(p)$*, then* $w' \in V(p)$*.*

The rules for *forcing* of the formulas are:

1. $w \vDash p$ iff $w \in V(p)$,
2. $w \vDash \phi \wedge \psi$ iff $w \vDash \phi$ and $w \vDash \psi$,
3. $w \vDash \phi \vee \psi$ iff $w \vDash \phi$ or $w \vDash \psi$,
4. $w \vDash \phi \to \psi$ iff, for all w' such that wRw', if $w' \vDash \phi$, then $w' \vDash \psi$,
5. $w \nvDash \bot$.

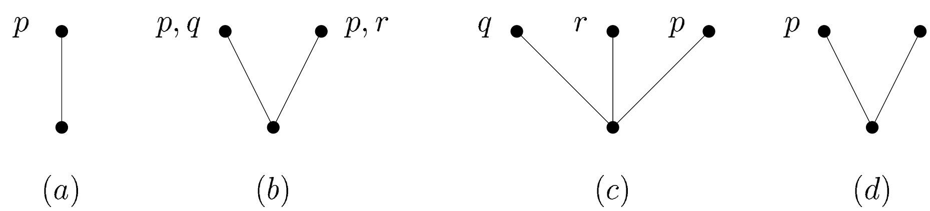


Figure 2: Counter-models for the propositional formulas

Most of our Kripke models will be *rooted* models, they have a least node (often w_0), a *root*. For the predicate calculus each node w of a model is equipped with a domain D_w in such a way that, if wRw', then $D_w \subseteq D_{w'}$. Persistency comes in this case down to the fact that D_w is a submodel of $D_{w'}$ in the normal sense of the word. The clauses for the quantifiers are (adding names for the elements of the domain to the language):

1. $w \vDash \exists x \phi(x)$ iff, for some $d \in D_w$, $w \vDash \phi(d)$.
2. $w \vDash \forall x \phi(x)$ iff, for each w' with wRw' and all $d \in D_{w'}$, $w' \vDash \phi(d)$.

HA-models are simply predicate-logic models for the language of **HA** in which the **HA**-axioms are verified at each node.
Persistency transfers to formulas: if wRw' and $w \vDash \phi$, then $w' \vDash \phi$.

Exercise 4. Prove that persistency transfers to formulas.

It is helpful to note that $w \vDash \neg\neg\phi$ iff, for each w' such that wRw', there exists w'' with $w'Rw''$ with $w'' \vDash \phi$. For finite models this simplifies to $w \vDash \neg\neg\phi$ iff for all maximal nodes w' above w, $w' \vDash \phi$.

Theorem 5. (Glivenko) $\vdash_{\mathbf{CPC}} \phi$ *iff* $\vdash_{\mathbf{IPC}} \neg\neg\phi$.

Exercise 6. Show Glivenko's Theorem in two ways. First, by using one of the proof systems. Secondly, assuming the completeness theorem with respect to finite Kripke models.

We will see shortly that this does not extend to the predicate calculus or arithmetic.

The following models invalidate respectively $p \vee \neg p$, $\neg\neg p \to p$ (both Figure 2a), $(\neg\neg p \to p) \to p \vee \neg p)$ (Figure 2d), $(p \to q \vee r) \to (p \to q) \vee (p \to r)$ (Figure 2b), $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$ (Figure 2c), $\neg\neg\forall x(Ax \vee \neg Ax)$ (Figure 3a, constant domain $\mathbb{N}$), $\forall x(A \vee Bx) \to A \vee \forall x Bx$ (Figure 3b).

Exercise 7. Show that Glivenko's Theorem does not extend to predicate logic.

The usual constructions with Kripke models and frames are applicable and have the usual properties. Three that we will use are the following.

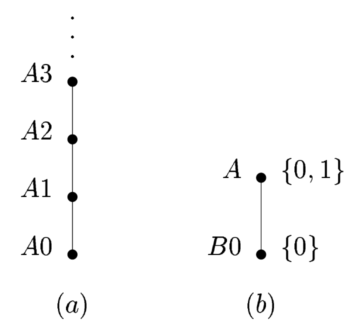


Figure 3: Counter-models for the predicate formulas

Definition 8.

1. *If* $\mathfrak{F} = (W, R)$ *is a frame and* $w \in W$*, then the* generated subframe $\mathfrak{F}_w$ *is* $(R(w), R')$*, where* $R(w) = \{w' \in W \mid wRw'\}$ *and* R' *the restriction of* R *to* $R(w)$*. If* $\mathfrak{K}$ *is a model on* $\mathfrak{F}$*, then the* generated submodel $\mathfrak{K}_w$ *is obtained by restricting the forcing on* $\mathfrak{F}_w$ *to* $R(w)$*.*

2. (a) *If* $\mathfrak{F} = (W, R)$ *and* $\mathfrak{F}' = (W', R')$ *are frames, then* $f\colon W \to W'$ *is a* p-morphism (*also* bounded morphism) *from* $\mathfrak{F}$ *to* $\mathfrak{F}'$ *iff*
   i. *for each* $w, w' \in W$*, if* wRw'*, then* $f(w)Rf(w')$*,*
   ii. *for each* $w \in W$*,* $w' \in W'$*, if* $f(w)Rw'$*, then there exists* $w'' \in W$*,* wRw'' *and* $f(w'') = w'$*.*

   (b) *If* $\mathfrak{K} = (W, R, V)$ *and* $\mathfrak{K}' = (W', R', V')$ *are models, then* $f\colon W \to W'$ *is a* p-morhism *from* $\mathfrak{K}$ *to* $\mathfrak{K}'$ *iff* f *is a p-morhism of the respective frames and, for all* $w \in W$*,* $w \in V(p)$ *iff* $f(w) \in V'(p)$*.*

3. *If* $\mathfrak{F}_1 = (W_1, R_1)$ *and* $\mathfrak{F}_2 = (W_2, R_2)$*, then their* disjoint union $\mathfrak{F}_1 \uplus \mathfrak{F}_2$ *has as its set of worlds the disjoint union of* W_1 *and* W_2*, and* R *is* $R_1 \cup R_2$*. To obtain the disjoint union of two models the union of the two valuations is added.*

Theorem 9.

1. *If* w' *is a node in the generated submodel* $\mathfrak{M}_w$*, then, for each* ϕ*,* $w' \vDash \phi$ *in* $\mathfrak{M}$ *iff* $w' \vDash \phi$ *in* $\mathfrak{M}_w$*.*

2. *If* f *is a p-morphism from* $\mathfrak{M}$ *to* $\mathfrak{M}'$ *and* $w \in W$*, then, for each* ϕ*,* $w \vDash \phi$ *iff* $f(w) \vDash \phi$*.*

3. *If* $w \in W_1$*, then* $w \vDash \phi$ *in* $\mathfrak{M}_1 \uplus \mathfrak{M}_2$ *iff* $w \vDash \phi$ *in* $\mathfrak{M}_1$*, etc.*

The first part of this theorem means among many other things that when we have a formula falsified in a world in which some other formulas are true, we may w.l.o.g. assume that this situation occurs in the root of the model.

The method of canonical models used in modal logic can be adapted to the case of intuitionistic logic. Instead of considering maximal consistent sets of formulas we consider theories with the disjunction property.

Definition 10. *A* theory *is a set of formulas that is closed under* **IPC**-*consequence. A set of formulas* Γ *has the* disjunction property *if* $\phi \vee \psi \in \Gamma$ *implies* $\phi \in \Gamma$ *or* $\psi \in \Gamma$.

The Lindenbaum type lemma that is then needed is the following.

Lemma 11. *If* $\Gamma \nvdash_{\mathbf{IPC}} \psi \to \chi$*, then a theory with the disjunction property* Δ *that includes* Γ *exists such that* $\psi \in \Delta$ *and* $\chi \notin \Delta$.

Proof. Enumerate all formulas: $\phi_0, \phi_1, \cdots$ and ddefine

- $\Delta_0 = \Gamma \cup \{\psi\}$,
- $\Delta_{n+1} = \Delta_n \cup \{\phi_n\}$ if this does not prove χ,
- $\Delta_{n+1} = \Delta_n$ otherwise.

Take Δ to be the union of all Δ_n. As in the usual Lindenbaum construction Δ is a theory and none of the Δ_n or Δ itself prove χ. χ simply takes the place that $\bot$ has in classical proofs. Claim is that Δ also has the the disjunction property and therefore satisfies all the desired properties. Assume that $\phi \vee \psi \in \Delta$, and $\phi \notin \Delta$, $\psi \notin \Delta$. Let $\phi = \phi_m$ and $\psi = \phi_n$ and w.l.o.g. let n be the larger of m, n. Then χ is provable in both $\Delta_n \cup \{\phi\}$ and $\Delta_n \cup \{\psi\}$ and thus in $\Delta_n \cup \{\phi \vee \psi\}$ as well. But that cannot be true since $\Delta_n \cup \{\phi \vee \psi\} \subseteq \Delta$ and $\Delta \nvdash_{\mathbf{IPC}} \chi$. □

Definition 12. (**Canonical model**) *The canonical model of* **IPC** *is a Kripke model based on a frame* $\mathfrak{F}^C = (W^C, R^C)$*, where* W^C *is the set of all consistent theories with the disjunction property and* R^C *is the inclusion. The canonical valuation on* $\mathfrak{F}^C$ *is defined by putting:* $\Gamma \vDash p$ *if* $p \in \Gamma$.

Theorem 13. (**Completeness theorem for IQC, IPC**) $\vdash_{\mathbf{IQC},\ \mathbf{IPC}} \phi$ *iff* ϕ *is valid in all Kripke models for* **IQC, IPC** (*for* **IPC** *the finite models are sufficient*).

Proof. We give the proof for **IPC** and make some comments about **IQC**.

As in modal logic the proof proceeds by showing by induction on the length of ϕ that $\Gamma \vDash \phi$ iff $\phi \in \Gamma$. The only interesting case is the step of showing that, if $\psi \to \chi \notin \Gamma$, then a theory with the disjunction property Δ that includes Γ exists such that $\psi \in \Delta$ and $\chi \notin \Delta$. This is the content of Lemma 11.

Finally, assume $\Gamma \nvdash_{\mathbf{IPC}} \chi$. Then $\Gamma \nvdash_{\mathbf{IPC}} \top \to \chi$, so, again applying the Lindenbaum Lemma an extension Δ of Γ with the disjunction property, not containing χ , exists. In the canonical model, $\Delta \nvDash \chi$.

The *finite model property* for **IPC** (i.e. if $\nvdash_{\mathbf{IPC}} \phi$, then there exists a finite model on which ϕ is falsified) can be obtained by restricting the whole proof to a finite so-called *adequate* set, a set closed under taking subformulas, that contains all relevant formulas. Another way of doing this is using *filtration*. This works exactly as in modal logic.

The Henkin type proof for **IQC** is only slightly more complicated than a combination of the above proof and the proof for the classical predicate calculus. Of course, one needs the theories to have not only the disjunction property but also the existence property: if $\exists x\phi(x) \in \Gamma$ then $\phi(d) \in \Gamma$ for some d in the language of Γ. Since one needs growing domains one needs theories in different languages. Let $C_0, C_1, C_2, \cdots$ be a sequence of disjoint countably infinite sets of new constants. It suffices to consider theories in the languages obtained by adding $C_0 \cup C_1 \cdots \cup C_n$ to the original language. For students who know the classical proof this turns then into a larger **exercise** (see [35], Volume 1). $\square$

Remark 14. If we restrict the propositional language to only finitely many variables, we obtain finite variable canonical models. These models provide completeness for finite variable fragments of **IPC** and will come up later on in the study of n-universal models.

Exercise 15. Prove, using an adequate set, the finite model property for **IPC**.

When one adds schemes to the Hilbert type system of **IPC** one obtains so-called *intermediate* (or *superintuitionistic*) logics. For example adding $\phi \vee \neg\phi$, or $\neg\neg\phi \to \phi$ or $((\phi \to \psi) \to \phi) \to \phi$ (*Peirce's law*) one obtains classical logic **CPC**. Other well-known intermediate logics are:

- **LC** (*Dummett's logic*) axiomatized by $(\phi \to \psi) \vee (\psi \to \phi)$. **LC** characterizes the linear frames and is complete with regard to the finite ones. Equivalent axiomatizations are $(\phi \to \psi \vee \chi) \to (\phi \to \psi) \vee (\phi \to \chi)$ or $((\phi \to \psi) \to \psi) \to \phi \vee \psi$.
- **KC** (*logic of the weak excluded middle*), axiomatized by $\neg\phi \vee \neg\neg\phi$, complete with regard to the finite frames with a largest element.
- $((\chi \to (((\phi \to \psi) \to \phi) \to \phi)) \to \chi) \to \chi$ (3-Peirce) characterizes the frames with depth 2 and is complete with regard to the finite ones.
- $\forall x(\phi \vee \psi x) \to \phi \vee \forall x\psi x$ is a predicate intermediate logic sound and complete for the frames with constant domains.

Information on propositional intermediate logics can be found in [11].

Exercise 16. 1. Show the different axiomatizations of **LC** to be equivalent.

2. Show that in **KC** it is sufficient to assume the axioms for atomic formulas.
3. Give a counterexample to "3-Peirce" on the linear frame of 3 elements. Formulate a conjecture for the logic that is complete with regard to frames of depth n.
4. Show that $\forall x(\phi \vee \psi x) \to \phi \vee \forall x\psi x$ is valid on frames with a constant domain.

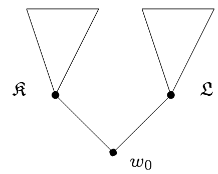


Figure 4: Proving the disjunction property

3.4 The Disjunction Property, Admissible Rules

Theorem 17. $\vdash_{\mathbf{IPC}} \phi \vee \psi$ *iff* $\vdash_{\mathbf{IPC}} \phi$ *or* $\vdash_{\mathbf{IPC}} \psi$. (*This extends to the predicate calculus and arithmetic.*)
Proof. The idea of the nontrivial direction of the proof for **IPC** is to equip two supposed counter-models $\mathfrak{K}$ and $\mathfrak{L}$ for ϕ and ψ respectively, with a new root w_0. In w_0, $\phi \vee \psi$ is falsified (see Figure 4). It is in the present case not relevant how the forcing of the new root is defined as long as it is in line with persistency. In the case of arithmetic this method works also, but only for some models at the new root and it is difficult to prove, except when one uses for the new root the standard model $\mathbb{N}$. That the latter is always possible is known as (*Smoryński's trick*). We will return to it presently.□

We call **HA**-models simply predicate logic models for the language of **HA** in which the **HA**-axioms are verified at each node. By the (strong) completeness theorem the sentences true on all these models are the ones derivable from **HA**.

Lemma 18. *In each node of each* **HA**-*model there exists in the domain* D_w *of each world* w *a unique sequence of distinct elements that are the interpretations of the* numerals $\overline{0}, \overline{1}, \cdots, \overline{n}, \cdots$, *where* $\overline{n} = \underbrace{S \cdots S}_{n\ times} 0$
Proof. Straightforward from the axioms. □

Theorem 19. (Smoryński's trick) *If* Σ *is a set of* **HA**-*models, then a new* **HA**-*model is obtained by taking the disjoint union of* Σ *adding a new root* w_0 *below it and taking* $\mathbb{N}$ *as its domain* D_{w_0}.
Proof. The only thing to show is that the **HA**-axioms hold at w_0. This is obvious for the simple universal axioms. Remains to prove it for the induction axioms. Assume $w_0 \vDash \phi(\overline{0})$ and $w_0 \vDash \forall x(\phi(x) \to \phi(Sx))$. By using an (intuitive) induction one sees that, for each $n \in \mathbb{N}$, $w_0 \vDash \phi(\overline{n})$. $w_0 \vDash \forall x \phi(x)$ immediately follows because no problems can arise at nodes other than w_0. □

Corollary 20. $\vdash_{\mathbf{HA}} \phi \vee \psi$ *iff* $\vdash_{\mathbf{HA}} \phi$ *or* $\vdash_{\mathbf{HA}} \psi$.

An easy syntactic way to prove the disjunction property for intuitionistic systems was invented by Kleene when he introduced the notion of *slash* [22]. Nowadays mostly the variant introduced by Aczel is used.

Definition 21. (**Aczel slash**)

1. $\Gamma \,|\, p$ *iff* $\Gamma \vdash p$,
2. $\Gamma \,|\, \phi \wedge \psi$ *iff* $\Gamma \,|\, \phi$ *and* $\Gamma \,|\, \psi$,
3. $\Gamma \,|\, \phi \vee \psi$ *iff* $\Gamma \,|\, \phi$ *or* $\Gamma \,|\, \psi$,
4. $\Gamma \,|\, \phi \to \psi$ *iff* $\Gamma \vdash \phi \to \psi$ *and* (*not* $\Gamma \,|\, \phi$ *or* $\Gamma \,|\, \psi$) .

Can be extended to predicate calculus and arithmetic.

Theorem 22. *If* $\Gamma \vdash \phi$ *and* $\Gamma \,|\, \chi$ *for each* $\chi \in \Gamma$, *then* $\Gamma \,|\, \phi$.

This theorem is proved by induction on the length of the derivation in one of the proof systems.

Corollary 23.

1. $\chi \,|\, \chi$ *iff, for all* ϕ, ψ, *if* $\vdash_{\mathbf{IPC}} \chi \to \phi \vee \psi$, *then* $\vdash_{\mathbf{IPC}} \chi \to \phi$ *or* $\vdash_{\mathbf{IPC}} \chi \to \psi$.
2. *If* $\vdash_{\mathbf{IPC}} \neg\chi \to \phi \vee \psi$, *then* $\vdash_{\mathbf{IPC}} \neg\chi \to \phi$ *or* $\vdash_{\mathbf{IPC}} \neg\chi \to \psi$.
3. $\chi \,|\, \chi$ *iff, for all rooted* $\mathfrak{M}, M'$, *if* $\mathfrak{M} \models \chi$ *and* $\mathfrak{M}' \models \chi$, *then* $\mathfrak{N} \models \chi$ *exists such that* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *are generated subframes of* $\mathfrak{N}$.

This theorem can be read as a rule $\neg\chi \to \phi \vee \psi / \neg(\chi \to \phi) \vee (\neg\chi \to \psi)$ that can be applied to **IPC** even though the rule does not follow directly from **IPC**. That such rules exist is very characteristic for intuitionistic systems.

Definition 24. *An* admissible rule *is a schematic rule of the form:* $\phi(\chi_1, \ldots, \chi_k)/\psi(\chi_1, \ldots, \chi_k)$ *with* ϕ *and* ψ *particular formulas and the property that, for all* **IPC***-formulas* $\chi_1, \ldots, \chi_k$, *if* $\vdash_{\mathbf{IPC}} \phi(\chi_1, \ldots, \chi_k)$, *then* $\vdash_{\mathbf{IPC}} \psi(\chi_1, \ldots, \chi_k)$.

Example 25. *Admissible rules that do not correspond to derivable formulas of* **IPC** *are for example:*

1. $\neg\chi \to \phi \vee \psi / (\neg\chi \to \phi) \vee (\neg\chi \to \psi)$,
2. $g_n(\phi) / \neg\neg\phi \vee (\neg\neg\phi \to \phi)$.

The second of these rules will occur in section 3.6.

Theorem 26. (R. Iemhoff) *All admissible rules can be obtained using only derivability in* **IPC** *from the rules*

$$\eta \to \phi \vee \psi / (\eta \to \chi_1) \vee \cdots \vee (\eta \to \chi_k) \vee (\eta \to \phi) \vee (\eta \to \psi),$$

where $\eta = (\chi_1 \to \delta_1) \wedge \cdots \wedge (\chi_k \to \delta_k)$.

Exercise 27. Show that the rules used in Iemhoff's theorem are admissible in two ways: semantically and using the Aczel slash.

We conclude this section with another application of Smoryński's trick: proving arithmetic completeness (de Jongh's theorem).

Theorem 28. **(Arithmetic Completeness)** $\vdash_{\mathbf{IPC}} \phi(p_1, \ldots, p_m)$ *iff, for all arithmetic sentences* $\psi_1, \ldots, \psi_m$*,* $\vdash_{\mathbf{HA}} \phi(\psi_1, \ldots, \psi_m)$.

Proof. (sketch). Assume $\nvdash_{\mathbf{IPC}} \phi(p_1, \cdots, p_m)$ (the other direction being trivial). A finite Kripke model on a frame $\mathfrak{F}$ exists in which $\phi(p_1, \cdots, p_k)$ is falsified in the root. By a standard procedure we can also assume that the frame is ordered as a tree, which at each node (except the maximal ones) has at least a binary split. The purpose of the latter is to ensure that each node is uniquely characterized by the maximal elements above it. Assume that $w_1, \cdots, w_k$ are the maximal nodes of the tree. Using arithmetic considerations one can construct arithmetic sentences $\alpha_1, \cdots, \alpha_k$ and **PA**-models $\mathfrak{M}_1, \cdots, \mathfrak{M}_k$ such that $\mathfrak{M}_i \vDash \alpha_j$ iff $i = j$. Noting that the one-node models $\mathfrak{M}_i$ are immediately **HA**-models as well one now applies Smoryński's trick repeatedly to fill out the model by assigning $\mathbb{N}$ to each node. One so obtains an **HA**-model on $\mathfrak{F}$. Next one notes that for each node w, the sentence $\psi_w = \neg\neg(\alpha_{i_1} \vee \cdots \vee \alpha_{i_m})$, where $w_{i_1}, \cdots, w_{i_m}$ are the maximal elements that are successors of w is forced at w and its successors and nowhere else. Finally taking each ψ_i to be the disjunction of those ψ_w where p_i is forced one sees that the ψ_i behave in the **HA**-model exactly like the p_i in the original Kripke model and thus one gets that $\phi(\psi_1, \cdots, \psi_m)$ is falsified in the **HA**-model and hence cannot be a theorem of **HA**. □

For a full version of this proof and more information on the application of Kripke models to arithmetical systems, see Sm73.

3.5 Translations

First we give Gödel's so-called *negative translation* of classical logic into intuitionistic logic.

Definition 29.

1. $p^n = \neg\neg p$,
2. $(\phi \wedge \psi)^n = \phi^n \wedge \psi^n$,
3. $(\phi \vee \psi)^n = \neg\neg(\phi^n \vee \psi^n)$,
4. $(\phi \to \psi)^n = \phi^n \to \psi^n$,
5. $\bot^n = \bot$.

There are many variants of this definition that give the same result.

Theorem 30. $\vdash_{\mathbf{CPC}} \phi$ *iff* $\vdash_{\mathbf{IPC}} \phi^n$. (*This extends to the predicate calculus and arithmetic.*)

Proof. for the propositional calculus.
$\Longleftarrow$: Of course, $\vdash_{\mathbf{CPC}} \psi \leftrightarrow \psi^n$. Also, if $\vdash_{\mathbf{IPC}} \phi$, then $\vdash_{\mathbf{CPC}} \phi$. Thus, this direction follows.
$\Longrightarrow$: One first proves, by induction on the length of ϕ, that $\vdash_{\mathbf{IPC}} \phi^n \leftrightarrow \neg\neg\phi^n$ (ϕ^n is *negative*). This is straightforward; for the case of implication one uses that $\vdash_{\mathbf{IPC}} \neg\neg(\phi \to \psi) \leftrightarrow (\neg\neg\phi \to \neg\neg\psi)$, and for conjunction the analogous fact for $\wedge$. Then, one proves, by induction on the length of the proof in the Hilbert type system that, if $\vdash_{\mathbf{CPC}} \phi$, then $\vdash_{\mathbf{IPC}} \phi^n$. In some cases one needs the fact first proved that χ^n is a negative formula, e.g. in the axiom $\neg\neg\phi \to \phi$ that is added to **IPC** to obtain **CPC**. $\square$

If one uses in the above proof the natural deduction system or a Gentzen system one automatically gets the slightly stronger result that $\Gamma \vdash_{\mathbf{CPC}} \phi$ iff $\Gamma^n \vdash_{\mathbf{IPC}} \phi^n$.

Exercise 31. Give a translation satisfying Theorem 30 that uses $\wedge$ and $\neg$ only.

Exercise 32. Prove Glivenko's theorem using the Gödel translation.

The propositional modal-logical systems **S4**, **Grz** and **GL** are obtained by adding to the axiom $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ of the modal logic **K**, the axioms $\Box\phi \to \phi$, $\Box\phi \to \Box\Box\phi$ for **S4**, in addition to this Grzegorczyk's axiom $\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi$ for **Grz**, and $\Box(\Box\phi \to \phi) \to \Box\phi$ for **GL**. Completeness holds for **S4** with respect to the finite reflexive, transitive Kripke models, for **Grz** with respect to the finite partial orders (reflexive, transitive, anti-symmetric), and for **GL** with respect to the finite transitive, conversely well-founded (i.e. irreflexive) Kripke models.

Of course, one may note the closeness of **IPC** and **Grz** or **S4** when one thinks of intuitionistic implication as necessary ('strict') implication and notices the resemblance of the models. Gödel saw the connection long before the existence of Kripke models by noting that interpreting $\Box$ as the intuitive notion of provability the **S4**-axioms $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, $\Box\phi \to \phi$, $\Box\phi \to \Box\Box\phi$ as well as its rule of necessitation $\phi/\Box\phi$ become plausible. He constructed the following translation from **IPC** into **S4**.

Definition 33. Gödel translation

1. $p^\Box = \Box p$,
2. $(\phi \wedge \psi)^\Box = \phi^\Box \wedge \psi^\Box$,
3. $(\phi \vee \psi)^\Box = \phi^\Box \vee \psi^\Box$,
4. $(\phi \to \psi)^\Box = \Box(\phi^\Box \to \psi^\Box)$,

Theorem 34. $\vdash_{\mathbf{IPC}} \phi$ *iff* $\vdash_{\mathbf{S4}} \phi^\Box$ *iff* $\vdash_{\mathbf{Grz}} \phi^\Box$.

Proof. $\Longrightarrow$: Trivial from **S4** to **Grz**. From **IPC** to **S4** it is simply a matter of using one of the proof systems of **IPC** and to find the needed proofs in **S4**. Using natural deduction or sequents one finds the obvious slight strengthening.

$\Longleftarrow$: It is sufficient to note that it is easily provable by induction on the length of the formula ϕ that for any world w in a Kripke model with a persistent valuation $w \vDash \phi$ iff $w \vDash \phi^{\Box}$ (where on the left the forcing is interpreted in the intuitionistic manner and on the right in the modal manner). This means that if $\nvdash_{\mathbf{IPC}} \phi$ one can interpret the finite **IPC**-counter model to ϕ provided by the completeness theorem immediately as a finite **Grz**-counter model to $\phi^{\Box}$. $\Box$

A natural adaptation of Gödel's translation can be given from **IPC** into provability logic **GL** when one notes that **Grz**-models and **GL**-models only differ in the fact that **GL** has irreflexive instead of reflexive models.

Definition 35.

1. $p^{\boxdot} = \Box p \wedge p$,
2. $\bot^{\boxdot} = \bot$,
3. $(\phi \wedge \psi)^{\boxdot} = \phi^{\boxdot} \wedge \psi^{\boxdot}$,
4. $(\phi \vee \psi)^{\boxdot} = \phi^{\boxdot} \vee \psi^{\boxdot}$,
5. $(\phi \to \psi)^{\boxdot} = \Box(\phi^{\boxdot} \to \psi^{\boxdot}) \wedge (\phi^{\boxdot} \to \psi^{\boxdot})$,

Theorem 36. $\vdash_{\mathbf{IPC}} \phi$ *iff* $\vdash_{\mathbf{GL}} \phi^{\boxdot}$.

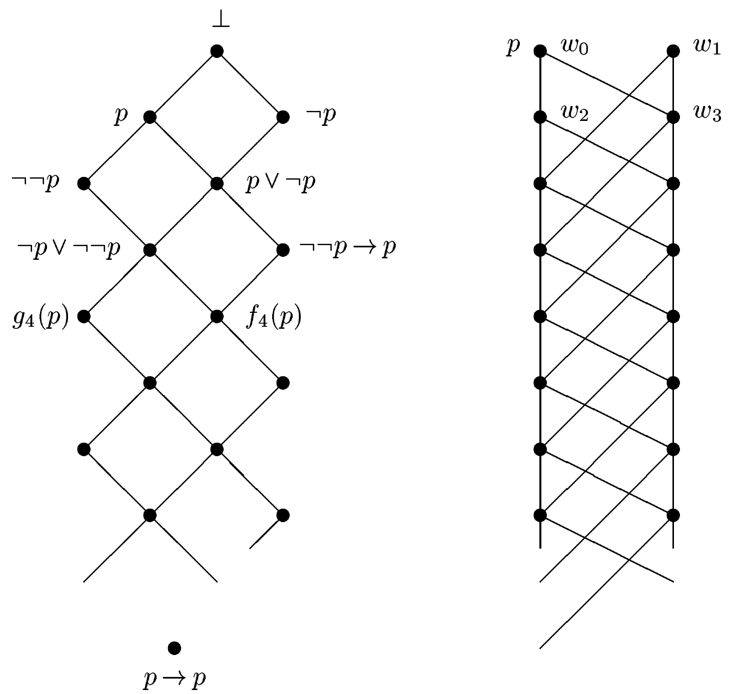
3.6 The Rieger-Nishimura Lattice and Ladder

This section will introduce the fragment of **IPC** of one propositional variable. We will see later that this is essentially the same as the free Heyting algebra on one generator. It is known under the name of the people who discovered it: Rieger [29] and Nishimura [27]. It is accompanied by a universal Kripke model: the Rieger-Nishimura Ladder.

Definition 37. (**Rieger-Nishimura Lattice**)

1. $g_0(\phi) = f_0(\phi) =_{def} \phi$,
2. $g_1(\phi) = f_1(\phi) =_{def} \neg\phi$,
3. $g_2(\phi) =_{def} \neg\neg\phi$,
4. $g_3(\phi) =_{def} \neg\neg\phi \to \phi$,
5. $g_{n+4}(\phi) =_{def} g_{n+3}(\phi) \to g_n(\phi) \vee g_{n+1}(\phi)$,
6. $f_{n+2}(\phi) =_{def} g_n(\phi) \vee g_{n+1}(\phi)$.

Theorem 38. *Each formula $\phi(p)$ with only the propositional variable p is* **IPC**-*equivalent to a formula $f_n(p)$ $(n \geqslant 2)$ or $g_n(p)$ $(n \geqslant 0)$ or $\top$ or $\bot$. All formulas $f_n(p)$ $(n \geqslant 2)$ and $g_n(p)$ $(n \geqslant 0)$ are nonequivalent in* **IPC**. *In fact, in the Rieger-Nishimura Ladder w_i validates $g_n(p)$ for $i \leqslant n$ only.*



Rieger-Nishimura Lattice (left) and Rieger-Nishimura Ladder (right)

In the Rieger-Nishimura lattice a formula $\phi(p)$ implies $\psi(p)$ in **IPC** iff $\psi(p)$ can be reached from $\phi(p)$ by a downward going line.

Exercise 39. Check that in the Rieger-Nishimura Ladder w_i validates $g_n(p)$ for $i \geqslant n$ only.

Theorem 40. *([21]) If* $\vdash_{\mathbf{IPC}} g_n(\phi)$ *for any* $n \in \mathbb{N}$*, then* $\vdash_{\mathbf{IPC}} \neg\neg\phi$ *or* $\vdash_{\mathbf{IPC}} \neg\neg\phi \to \phi$*. (Can be extended to arithmetic.)*

Proof. (Harder) **Exercise**. □

3.7 Complexity of IPC

It was first proved by R. Statman that **IPC** is **PSPACE**-complete. The idea of the following proof showing its hardness is due to V. Švejdar. A very basic acquaintance with the notions of complexity theory is presupposed (see e.g. [28]). Validity in **IPC** is **PSPACE**. The Gödel translation shows this by providing a **PTIME** reduction to **S4** the validity of which is well-known to be **PSPACE**. One could see this directly by carefully considering a tableaux method to satisfy a formula.

We reduce quantified Boolean propositional logic **QBF**, known to be **PSPACE**-complete, to **IPC**. **QBF** is an extension of **CPC** in which the propositional quantifiers can be thought of as ranging over the set of truth values $\{0,1\}$. Consider the **QBF**-formula

$$A = Q_m p_m \cdots Q_1 p_1 B(p_1, \cdots, p_m)$$

with B quantifier-free. We write

- $\overrightarrow{p}$ for $p_1, \cdots, p_m$
- $\overrightarrow{q}$ for $q_1, \cdots, q_m$
- $_j\overrightarrow{p}$ for $p_1, \cdots, p_{j-1}$
- $_j\overrightarrow{q}$ for $q_1, \cdots, q_{j-1}$
- $\overrightarrow{p_j}$ for $p_{j+1}, \cdots, p_m$,
- $\overrightarrow{q_j}$ for $q_{j+1}, \cdots, q_m$,
- A_j for $\overrightarrow{Q}\, p_j, \overrightarrow{p_j}(p_1, \cdots, p_m)$

We construct $\overline{A}_j$ (linearly in A) by recursion on j:

$$\overline{A}_0(\overrightarrow{p}) = \neg B(\overrightarrow{p}).$$

$$\overline{A}_j({}_j\overrightarrow{p}, p_j, \overrightarrow{p_j}, {}_j\overrightarrow{q}) = (p_j \vee \neg p_j) \to \overline{A}_{j-1}({}_j\overrightarrow{p}, p_j, \overrightarrow{p_j}, {}_j\overrightarrow{q})$$

if Q_j is $\exists$, and

$$\overline{A}_j({}_j\overrightarrow{p}, p_j, \overrightarrow{p_j}, q_j, {}_j\overrightarrow{q}) = (\overline{A}_{j-1} \to q_j) \to (p_j \to q_j) \vee (\neg p_j \to q_j)$$

if Q_j is $\forall$.

Claim For any valuation v on $\overrightarrow{p}$:

$$v \vDash A_j(\overrightarrow{p_j}) \iff \exists \mathfrak{M}\, \exists w \in \mathfrak{M}(w \nvDash \overline{A}_j({}_j\overrightarrow{p}, p_j, \overrightarrow{p_j}, \overrightarrow{q}) \,\&\, \mathfrak{M}(\overrightarrow{p_j}) = v(\overrightarrow{p_j})),$$

where $\mathfrak{M}(\overrightarrow{p_j}) = v(\overrightarrow{p_j})$ means that all nodes of $\mathfrak{M}$ valuate $\overrightarrow{p_j}$ as v does.

Proof of Claim. By induction on j
$j=0$. $\Longrightarrow$: Take w to be the unique node of the Kripke model corresponding to the valuation v.
$\Longleftarrow$: If v is a valuation that agrees with all nodes of $\mathfrak{M}$, then $\mathfrak{M}$ has a constant valuation and if it doesn't verify $\neg B$ at a world it verifies $B = A_0$ at that world.

$Q_j = \exists \implies$: Assume $v \vDash \exists p_j A_{j-1}(p_j, \vec{p_j})$. Then, for some v' obtained from v by adding a value for p_j, $v' \vDash A_{j-1}(p_j, \vec{p_j})$. By the induction hypothesis, w' exists in some Kripke model $\mathfrak{M}$ with $\mathfrak{M}(p_j, \vec{p_j}) = v'(p_j, \vec{p_j})$ and $w' \nvDash \overline{A}_{j-1}(\vec{{}_jp}, p_j, \vec{p_j}, \vec{{}_jq})$. Since $w' \vDash p_j \vee \neg p_j$, $w' \nvDash p_j \vee \neg p_j \to \overline{A}_{j-1} = \overline{A}_j$, and so $\mathfrak{M}, w'$ satisfy the requirements for w.
$\Longleftarrow$: Assume for $w, \mathfrak{M}$ with $\mathfrak{M}(\vec{p_j}) = v(\vec{p_j})$,

$$w \nvDash ((p_j \vee \neg p_j) \to \overline{A}_{j-1}(\vec{{}_jp}, p_j, \vec{p_j}, \vec{q})).$$

Since the valuation of the $\vec{p_j}$ is constant we can assume w.l.o.g. that $w \vDash p_j \vee \neg p_j$ and $w \nvDash \overline{A}_{j-1}(\vec{{}_jp}, p_j, \vec{p_j}, \vec{q}))$. Add the valuation of p_j in w to v to obtain v'. We then have $\mathfrak{M}_w(p_j \vec{p_j}) = v'(p_j, \vec{p_j})$. By the induction hypothesis for all valuations, so also for v', $v' \vDash A_{j-1}(p_j, \vec{p_j})$. Hence, $v \vDash \exists p_j A_{j-1}(p_j, \vec{p_j})$.

$Q_j = \forall \implies$: Assume $v \vDash \forall p_j A_{j-1}(p_j, \vec{p_j})$. Then, for both ways, v_0 and v_1, of extending v, $v_i \vDash A_{j-1}(p_j, \vec{p_j})$. By the induction hypothesis, we can find models $\mathfrak{M}_i$, with respective roots w_i and $\mathfrak{M}_i(p_j, \vec{p_j}) = v_i(p_j, \vec{p_j})$, such that $w_i \nvDash \overline{A}_{j-1}(\vec{{}_jp}, p_j, \vec{p_j}, \vec{{}_jq}))$. Note that $w_0 \vDash p_j, w_1 \vDash \neg p_j$. Add below the disjoint union of those two models a new root w. The $\vec{p_j}$ is valued on w as on w_1 and w_2, the $\vec{p} p_j, \vec{{}_jq}$ in conformity with persistency; q_j is forced precisely where $\overline{A}_{j-1}$ is forced. Thus $\overline{A}_{j-1} \to q_j$ is forced in $\mathfrak{M}$, and $p_j \to q_j$ and $\neg p_j \to q_j$ are not: $w \nvDash \overline{A}_j$.
$\Longleftarrow$: Assume $\mathfrak{M}$ and its root w are such that $\mathfrak{M}(\vec{p_j}) = v(\vec{p_j})$ and

$$w \nvDash (\overline{A}_{j-1} \to q_j) \to (p_j \to q_j) \vee (\neg p_j \to q_j).$$

There exist w_0, $w_1 \geqslant w$ such that

- $w_0 \vDash \neg p_j$,
- $w_0 \nvDash \overline{A}_{j-1}(\vec{{}_jp}, p_j, \vec{p_j}, \vec{{}_jq})$,
- $w_1 \vDash p_j$,
- $w_1 \nvDash \overline{A}_{j-1}(\vec{{}_jp}, p_j, \vec{p_j}, \vec{{}_jq})$,

Let v_0, v_1 be the extensions of v with v_0 satisfying $\neg p_j$ and v_1 satisfying p_j. Clearly $\mathfrak{M}_{w_i}(p_j, \vec{p_j}) = v_i(p_j, \vec{p_j})$ in both cases. So, by the induction hypothesis, in both cases, $v_i \vDash A_{j-1}(p_j, \vec{p_j})$. Hence, $v \vDash \forall p_j A_{j-1}(p_j, \vec{p_j})$.

The final conclusion is that the mapping from A to $\overline{A}_m$ is the desired reduction. For the universal case $(p \to \overline{A}_{j-1}) \vee (\neg p \to \overline{A}_{j-1})$ would have worked as well in the proof above, but that would not have given us a **PTIME** transformation.

3.8 Mezhirov's game for IPC

. We like to end up with something that has recently been developed: a game that is sound and complete for intuitionistic propositional logic announced in [25]. The games played are ϕ*-games* with ϕ being a formula of the propositional calculus. The game has two players P (*proponent*) and O (*opponent*). The playing field is the set of subformulas of ϕ. A *move* of a player is *marking* a formula that has not been marked before. Only O is allowed to mark atoms. The first move is made by P, and consists of marking ϕ. Players do not move in turn; whose move it is is determined by the *state of the game*. The player who has to move in a state where no move is available *loses*. The state of the game is determined by the markings and by a classical valuation Val that is developed along with the markings. The rules for this valuation are at each stage

- for atoms that $Val(p) = 1$ iff p is marked,
- for complex formulas $\psi \circ \chi$ that, if $\psi \circ \chi$ is unmarked, $Val(\psi \circ \chi) = 0$, and if $\psi \circ \chi$ is marked, $Val(\psi \circ \chi) = Val(\psi) \circ_B Val(\chi)$ where $\circ_B$ is the Boolean function associated with $\circ$.

If a player has marked a formula that gets the valuation 0, then that is considered to be a *fault* by that player. If P has a fault and O doesn't then P moves, in all other cases (i.e. if O has fault and P does or doesn't, or if neither player has a fault) O moves. The completeness theorem can be stated as follows.

Theorem 41. *$\vdash_{\mathbf{IPC}} \phi$ iff P has a winning strategy in the ϕ-game.*

We will first prove

Theorem 42. *If $\nvdash_{\mathbf{IPC}} \phi$, then O has a winning strategy in the ϕ-game.*

Proof. We write the sequences of formulas marked by O and P respectively as $\mathbf{O}$ and $\mathbf{P}$. O keeps in mind a *minimal* counter-model for ϕ, i.e., in the root w_0, ϕ is not satisfied, but in all other nodes of the model ϕ is satisfied. The strategy of O is as follows. As long as P does not choose formulas false in nodes higher up in the model O just picks formulas that are true in w_0. As soon as P does choose a formula ψ that is falsified at a higher up in the model, O keeps in mind the submodel generated by a maximal node w that falsifies ψ. O keeps repeating the same tactic with respect to the node where the game has lead the players. It is sufficient to prove the following:

Claim. If there are no formulas left for O to choose when following this strategy, i.e. all formulas that are true in the w that is fixed in O's mind have been marked, then it is P's move.

This is sufficient because it means that in such a situation P can only move onwards in the model, or, in case w is a maximal node, P loses.

Proof of Claim. We write $|\theta|_w$ for the truth value of θ in w. As we will see it is sufficient to show that, if the situation in the game is as in the assumptions of the claim, then $|\theta|_w = Val(\theta)$ for all θ. We prove, by induction on θ, $|\theta|_w = 1 \iff Val(\theta) = 1$.

- If θ is atomic, then O has marked all the atoms that are forced in w and no other, so those have become true and no other.
- Induction step $\Rightarrow$: Assume $|\theta \circ \psi|_w = 1$. Then $\theta \circ \psi$ is marked, because otherwise O could do so, contrary to assumption. We have $|\theta|_w \circ_B |\psi|_w = 1$. By IH, $Val(\theta) \circ_B Val(\psi) = 1$, so $Val(\theta \circ \psi) = 1$.
- Induction step $\Leftarrow$:
- $Val(\theta \wedge \psi) = 1 \Rightarrow Val(\theta) = 1$ and $Val(\psi) = 1 \Rightarrow_{IH}$ $|\theta|_w = 1$ and $|\psi|_w = 1 \Rightarrow |\theta \wedge \psi|_w = 1$.
- $\vee$ is same as $\wedge$.
- $Val(\theta \to \psi) = 1 \Rightarrow Val(\theta) = 0$ or $Val(\psi) = 1$, and thus by IH, $|\theta|_w = 0$ or $|\psi|_w = 1$. Also, $\theta \to \psi$ is marked, hence in $\mathbf{O}$ or $\mathbf{P}$. In the first case $|\theta \to \psi|_w = 1$ immediate, in the second, $|\theta \to \psi|_s = 1$ for all $s > w$ (P has marked no formulas false higher up, otherwise O would have shifted attention another node) and hence $|\theta|_s = 0$ or $|\psi|_s = 1$ for all $s > w$. Indeed, $|\theta \to \psi|_w = 1$.

We are now faced with the fact that O has only chosen formulas true in the world in O's mind and those stay true higher up in the model. So, $Val(\theta) = 1$ for all $\theta \in \mathbf{O}$. On the other hand, P has at least one fault, the formula ξ chosen by P that landed the game in w in the first place: $Val(\xi) = 0$. Indeed, it is P's move.□

We now turn to the second half:

Theorem 43. *If $\vdash_{\mathbf{IPC}} \phi$, then P has a winning strategy in the ϕ-game.*
Proof. P's strategy is to choose only formulas that are provable from $\mathbf{O}$. Note that P's first forced choice of ϕ is in line with this strategy. For this case it is sufficient to prove the following claim.

Claim If all formulas that are provable from $\mathbf{O}$ are marked, then it is O's move.

This is sufficient because it means that in such a situation O can only mark a completely new formula, and when there are no such formulas left loses.

Proof of Claim. Create a model in the following manner. Assume $\chi_1, \ldots, \chi_k$ are the formulas unprovable from $\mathbf{O}$ and hence the unmarked ones. By the completeness of **IPC** there are k models making $\mathbf{O}$ true and falsifying respectively $\chi_1, \ldots, \chi_k$ in their respective roots. Adjoin to these models a new root r verifying exactly the $\mathbf{O}$-atoms (this obeys persistency). As in the other direction we will prove: $|\theta|_r = Val(\theta)$ for all θ, or, equivalently, $|\theta|_r = 1 \iff Val(\theta) = 1$.

- Atoms are forced in r iff marked by O and then have Val 1, otherwise 0.
- Induction step $\Rightarrow$: Assume $|\theta \circ \psi|_r = 1$. Then $\theta \circ \psi$ is marked, because if it wasn't it would be one of the χ_i, falsifying persistency. We can reason on as in the other direction.
- Induction step $\Leftarrow$:
- $\vee$ and $\wedge$ as in the other direction.
- $Val(\theta \to \psi) = 1 \Rightarrow Val(\theta) = 0$ or $Val(\psi) = 1$, and thus by IH, $|\theta|_r = 0$ or $|\psi|_r = 1$. Also, $\theta \to \psi$ is marked, hence in **O** or **P**, and so $|\theta \to \psi|_s = 1$ and hence $|\theta|_s = 0$ or $|\psi|_s = 1$ for all $s > r$. Indeed, $|\theta \to \psi|_r = 1$.

We are now faced with the fact that P has only marked formulas provable from **O** and those will remain provable from **O**. So, $Val(\theta) = 1$ for all $\theta \in \mathbf{P}$. So, P has no fault. By the rules of the game it is O's move. □

4 Heyting algebras

4.1 Lattices, distributive lattices and Heyting algebras

We begin by introducing some basic notions. A partially ordered set $(A, \leq)$ is called a *lattice* if every two element subset of A has a least upper and greatest lower bound. Let $(A, \leq)$ be a lattice. For $a, b \in A$ let $a \vee b := sup\{a, b\}$ and $a \wedge b := inf\{a, b\}$. We assume that every lattice is bounded, i.e., it has a least and a greatest element denoted by $\bot$ and $\top$ respectively. The next proposition shows that lattices can also be defined axiomatically.

Proposition 44. *A structure* $(A, \vee, \wedge, \bot, \top)$ *is a lattice iff for every* $a, b, c \in A$ *the following holds:*

1. $a \vee a = a$, $a \wedge a = a$;
2. $a \vee b = b \vee a$, $a \wedge b = b \wedge a$;
3. $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
4. $a \vee \bot = a$, $a \wedge \top = a$;
5. $a \vee (b \wedge a) = a$, $a \wedge (b \vee a) = a$.

Proof. It is a matter of routine checking that every lattice satisfies the axioms 1–5. Now suppose $(A, \vee, \wedge, \bot, \top)$ satisfies the axioms 1–5. We say that $a \leq b$ if $a \vee b = b$ or equivalently if $a \wedge b = a$. It is left to the reader to check that $(A, \leq)$ is a lattice. □

From now on we will denote lattices by $(A, \vee, \wedge, \bot, \top)$.

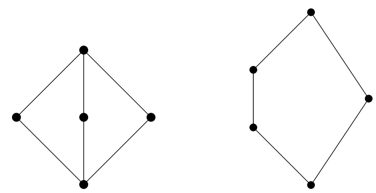


Figure 5: Non-distributive lattices M_5 and N_5

Definition 45. *A lattice* $(A, \vee, \wedge, \bot, \top)$ *is called* distributive *if it satisfies the* distributivity laws*:*

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Exercise 46. Show that the lattices shown in Figure 5 are not distributive.

The next theorem shows that, in fact, the lattices in Figure 5 are typical examples of non-distributive lattices. For the proof the reader is referred to Balbes and Dwinger [1].

Theorem 47. *A lattice* L *is distributive iff* N_5 *and* M_5 *are not sublattices of* L*.*

We are ready to define the main notion of this section.

Definition 48. *A distributive lattice* $(A, \wedge, \vee, \bot, \top)$ *is said to be a* Heyting algebra *if for every* $a, b \in A$ *there exists an element* $a \to b$ *such that for every* $c \in A$ *we have:*

$$c \leq a \to b \;\; \textit{iff} \;\; a \wedge c \leq b.$$

We call $\to$ a *Heyting implication* or simply an *implication.* For every element a of a Heyting algebra, let $\neg a := a \to 0$.

Remark 49. It is easy to see that if $\mathfrak{A}$ is a Heyting algebra, then $\to$ is a binary operation on $\mathfrak{A}$, as follows from Proposition 51(1). Therefore, we should add $\to$ to the signature of Heyting algebras. Note also that $0 \to 0 = 1$. Hence, we can exclude 1 from the signature of Heyting algebras. From now on we will let $(A, \vee, \wedge, \to, 0)$ denote a Heyting algebra.

As in the case of lattices, Heyting algebras can be defined in a purely axiomatic way; see, e.g., [20, Lemma 1.10].

Theorem 50. *A distributive lattice*[2] $\mathfrak{A} = (A, \vee, \wedge, 0, 1)$ *is a Heyting algebra iff there is a binary operation* $\to$ *on* A *such that for every* $a, b, c \in A$*:*

[2] In fact, it is not necessary to state that $\mathfrak{A}$ is distributive. Every lattice satisfying conditions 1–4 of Theorem 50 is automatically distributive [20, Lemma 1.11(i)].

1. $a \to a = 1$,
2. $a \wedge (a \to b) = a \wedge b$,
3. $b \wedge (a \to b) = b$,
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

Proof. Suppose $\mathfrak{A}$ satisfies the conditions 1–4. Assume $c \leq a \to b$. Then by (2), $c \wedge a \leq (a \to b) \wedge a = a \wedge b \leq b$. For the other direction we first show that for every $a \in A$ the map $(a \to \cdot)$ is monotone, i.e., if $b_1 \leq b_2$ then $a \to b_1 \leq a \to b_2$. Indeed, since $b_1 \leq b_2$ we have $b_1 \wedge b_2 = b_1$. Hence, by (4), $(a \to b_1) \wedge (a \to b_2) = a \to (b_1 \wedge b_2) = a \to b_1$. Thus, $a \to b_1 \leq a \to b_2$. Now suppose $c \wedge a \leq b$. By (3), $c = c \wedge (a \to c) \leq 1 \wedge (a \to c)$. By (1) and (4), $1 \wedge (a \to c) = (a \to a) \wedge (a \to c) = a \to (a \wedge c)$. Finally, since $(a \to \cdot)$ is monotone, we obtain that $a \to (a \wedge c) \leq a \to b$ and therefore $c \leq a \to b$.

It is easy to check that $\to$ from Definition 48 satisfies the conditions 1–4. We skip the proof. □

We say that a lattice $(A, \wedge, \vee)$ is *complete* if for every subset $X \subset A$ there exist $\bigwedge X = sup(X)$ and $\bigvee X = inf(X)$. For the next proposition consult [20, Theorem 4.2].

Proposition 51.

1. *In every Heyting algebra* $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ *we have that for every* $a, b \in A$*:*
$$a \to b = \bigvee\{c \in A : a \wedge c \leq b\}.$$
2. *A complete distributive lattice* $(A, \wedge, \vee, 0, 1)$ *is a Heyting algebra iff it satisfies the infinite distributive law*
$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$
for every $a, b_i \in A$, $i \in I$.

Proof. (1) Clearly $a \to b \leq a \to b$. Hence, $a \wedge (a \to b) \leq b$. So, $a \to b \leq \bigvee\{c \in A : a \wedge c \leq b\}$. On the other hand, if c is such that $c \wedge a \leq b$, then $c \leq a \to b$. Therefore, $\bigvee\{c \in A : a \wedge c \leq b\} \leq a \to b$.

(2) Suppose $\mathfrak{A}$ is a Heyting algebra. For every $i \in I$ we have that $a \wedge b_i \leq a \wedge \bigvee_{i \in I} b_i$. Hence, $\bigvee_{i \in I}(a \wedge b_i) \leq a \wedge \bigvee_{i \in I} b_i$. Now let $c \in A$ be such that $\bigvee_{i \in I}(a \wedge b_i) \leq c$. Then $a \wedge b_i \leq c$ for every $i \in I$. Therefore, $b_i \leq a \to c$ for every $i \in I$. This implies that $\bigvee_{i \in I} b_i \leq a \to c$, which gives us that $a \wedge \bigvee_{i \in I} b_i \leq c$. Thus, taking $\bigvee_{i \in I}(a \wedge b_i)$ as c we obtain $a \wedge \bigvee_{i \in I} b_i \leq \bigvee_{i \in I}(a \wedge b_i)$.

Conversely, suppose that a complete distributive lattice satisfies the infinite distributive law. Then we put $a \to b = \bigvee\{c \in A : a \wedge c \leq b\}$. It is now easy to see that $\to$ is a Heyting implication. □

Next we will give a few examples of Heyting algebras.

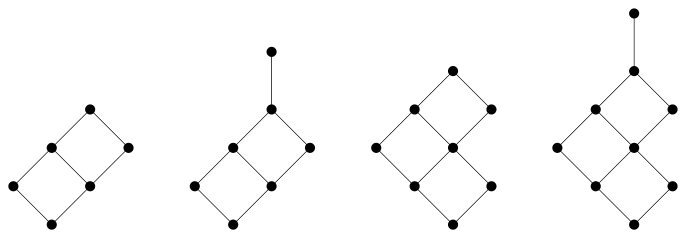


Figure 6: The algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{B}_1$, $\mathfrak{B}_2$

Example 52.

1. Every finite distributive lattice is a Heyting algebra. This immediately follows from Proposition 51(2), since every finite distributive lattice is complete and satisfies the infinite distributive law.

2. Every chain $\mathfrak{C}$ with a least and greatest element is a Heyting algebra and for every $a, b \in \mathfrak{C}$ we have

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

3. Every Boolean algebra $\mathfrak{B}$ is a Heyting algebra, where for every $a, b \in \mathfrak{B}$ we have

$$a \to b = \neg a \vee b$$

Exercise 53. Give an example of a non-complete Heyting algebra.

The next proposition characterizes those Heyting algebras that are Boolean algebras. For the proof see, e.g., [20, Lemma 1.11(ii)].

Proposition 54. *Let $\mathfrak{A} = (A, \vee, \wedge, \to, 0)$ be a Heyting algebra. Then the following three conditions are equivalent:*

1. *$\mathfrak{A}$ is a Boolean algebra,*

2. *$a \vee \neg a = 1$ for every $a \in A$,*

3. *$\neg\neg a = a$ for every $a \in A$.*

4.2 The connection of Heyting algebras with Kripke frames and topologies

Next we will spell out in detail the connection between Kripke frames and Heyting algebras. Let $\mathfrak{F} = (W, R)$ be a partially ordered set (i.e., an intuitionistic Kripke frame). For every $w \in W$ and $U \subseteq W$ let

$R(w) = \{v \in W : wRv\}$,

$R^{-1}(w) = \{v \in W : vRw\}$,

$R(U) = \bigcup_{w \in U} R(w)$,

$R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w)$.

A subset $U \subseteq W$ is called an *upset* if $w \in U$ and wRv implies $v \in U$. Let $Up(\mathfrak{F})$ be the set of all upsets of $\mathfrak{F}$. Then $(Up(\mathfrak{F}), \cap, \cup, \to, \emptyset)$ forms a Heyting algebra, where $U \to V = \{w \in W :$ for every $v \in W$ with wRv if $v \in U$ then $w \in V\} = W \setminus R^{-1}(U \setminus V)$.

Exercise 55.

1. Verify this claim. That is, show that for every Kripke frame $\mathfrak{F} = (W, R)$, the algebra $(Up(\mathfrak{F}), \cap, \cup, \to, \emptyset)$ is a Heyting algebra.
2. Draw a Heyting algebra corresponding to the 2-fork frame (W, R), where $W = \{w, v, u\}$ and $R = \{(w, w), (v, v), (u, u), (w, v), (w, u)\}$.
3. Draw a Heyting algebra corresponding to the frame (W, R), where $W = \{w, v, u, z\}$ and $R = \{(w, w), (v, v), (u, u), (z, z)(w, v), (w, u), (w, z), (v, z)\}$.
4. Show that if a frame $\mathfrak{F}$ is rooted, then the corresponding Heyting algebra has a second greatest element.

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. Let $\mathcal{A}$ be a set of upsets of $\mathfrak{F}$ closed under $\cap, \cup, \to$ and containing $\emptyset$. Then $\mathcal{A}$ is a Heyting algebra. A triple $(W, R, \mathcal{A})$ is called a *general frame*.

We will next discuss the connection of Heyting algebras with topology.

Definition 56. *A pair $\mathcal{X} = (X, \mathcal{O})$ is called a topological space if $X \neq \emptyset$ and $\mathcal{O}$ is a set of subsets of X such that*

- $X, \emptyset \in \mathcal{O}$
- *If $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$*
- *If $U_i \in \mathcal{O}$, for every $i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{O}$*

For $Y \subseteq X$, the *interior* of Y is the set $\mathbf{I}(Y) = \bigcup\{U \in \mathcal{O} : U \subseteq Y\}$. Let $\mathcal{X} = (X, \mathcal{O})$ be a topological space. Then the algebra $(\mathcal{O}, \cup, \cap, \to, \emptyset)$ forms a Heyting algebra, where $U \to V = \mathbf{I}((X \setminus U) \cup V)$ for every $U, V \in \mathcal{O}$.

Exercise 57. Verify this claim. That is, show that for every topological space $\mathcal{X} = (X, \mathcal{O})$, the algebra $(\mathcal{O}, \cup, \cap, \to, \emptyset)$ is a Heyting algebra.

We already saw how to obtain a Heyting algebra from a Kripke frame. Now we will show how to construct a Kripke frame from a Heyting algebra. The construction of a topological space from a Heyting algebra is more sophisticated. We will not discuss it here. The interested reader is referred to [20, §1.3].

Let $\mathfrak{A} = (A, \wedge, \vee, \to, \bot)$ be a Heyting algebra. $F \subseteq A$ is called a *filter* if

- $a, b \in F$ implies $a \wedge b \in F$
- $a \in F$ and $a \leq b$ imply $b \in F$

A filter F is called *prime* if

- $a \vee b \in F$ implies $a \in F$ or $b \in F$

In a Boolean algebra every prime filter is maximal. However, this is not the case for Heyting algebras.

Exercise 58. Give an example of a Heyting algebra $\mathfrak{A}$ and a filter F of $\mathfrak{A}$ such that F is prime, but not maximal.

Now let $W := \{F : F \text{ is a prime filter of } \mathfrak{A}\}$. For $F, F' \in W$ we put FRF' if $F \subseteq F'$. It is clear that R is a partial order and hence (W, R) is an intuitionistic Kripke frame.

Exercise 59. Draw Kripke frames corresponding to:

1. The two and four element Boolean algebras.
2. A finite chain consisting of n elements for $n \in \omega$.
3. The Heyting algebras drawn in Figure 6.
4. Show that if a Heyting algebra $\mathfrak{A}$ has a second greatest element, then the Kripke frame corresponding to $\mathfrak{A}$ is rooted.

Let $\mathfrak{A} = (A, \wedge, \vee, \to, \bot)$ and $\mathfrak{A}' = (A', \wedge', \vee', \to', \bot')$ be Heyting algebras. A map $h : A \to A'$ is called a *Heyting homomorphism* if

- $h(a \wedge b) = h(a) \wedge' h(b)$
- $h(a \vee b) = h(a) \vee' h(b)$
- $h(a \to b) = h(a) \to' h(b)$
- $h(\bot) = \bot'$

An algebra $\mathfrak{A}'$ is called a *homomorphic* image of $\mathfrak{A}$ if there exists a homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two Heyting algebras. We say that an algebra $\mathfrak{A}'$ is a *subalgebra* of $\mathfrak{A}$ if $A' \subseteq A$ and for every $a, b \in A'$ $a \wedge b, a \vee b, a \to b, \bot \in A'$.

A *product* $\mathfrak{A} \times \mathfrak{A}'$ of $\mathfrak{A}$ and $\mathfrak{A}'$ is the algebra $(A \times A', \wedge, \vee, \to, \bot)$, where

- $(a, a') \wedge (b, b') := (a \wedge b, a' \wedge' b')$
- $(a, a') \vee (b, b') := (a \vee b, a' \vee' b')$
- $(a, a') \to (b, b') := (a \to b, a' \to' b')$

- $\bot := (\bot, \bot')$

Let **Heyt** be a category whose objects are Heyting algebras and whose morphisms Heyting homomorphisms.[3] Let **Kripke** denote the category of intuitionistic Kripke frames and p-morphisms.

Next we define contravariant functors Φ : **Heyt** $\to$ **Kripke** and Ψ : **Kripke** $\to$ **Heyt**. For every Heyting algebra $\mathfrak{A}$ let $\Phi(\mathfrak{A})$ be the Kripke frame described above. For a homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ define $\Phi(h) : \Phi(\mathfrak{A}') \to \Phi(\mathfrak{A})$ by putting $\Phi(h) = h^{-1}$, that is, for every element $F \in \Phi(\mathfrak{A}')$ (a prime filter of $\mathfrak{A}'$) we let $\Phi(h)(F) = h^{-1}(F)$.

Exercise 60. 1. Show that $\Phi(h)$ is a well-defined p-morphism.

2. Prove that Φ is a contravariant functor.

We now define a functor Ψ : **Kripke** $\to$ **Heyt**. For every Kripke frame $\mathfrak{F}$ let $\Psi(\mathfrak{F}) = (Up(\mathfrak{F}), \cap, \cup, \to, \emptyset)$. If $f : \mathfrak{F} \to \mathfrak{F}'$ is a p-morphism, then define $\Psi(f) : \Psi(\mathfrak{F}') \to \Psi(\mathfrak{F})$ by putting $\Psi(f) = f^{-1}$, that is, for every element of $U \in \Psi(\mathfrak{F}')$ (an upset of $\mathfrak{F}'$) we let $\Psi(f)(U) = f^{-1}(U)$.

Exercise 61. 1. Show that $\Psi(f)$ is a well-defined Heyting homomorphism.

2. Prove that Ψ is a contravariant functor.

The next theorem spells out the connection between homomorphisms, subalgebras and products with generated subframes, p-morphisms and disjoint unions. The proofs are not very difficult; the reader is referred to any of the following textbooks in modal logic [11], [5], [23].

Theorem 62. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras and $\mathfrak{F}$ and $\mathfrak{G}$ Kripke frames.*

1.
   - *If $\mathfrak{A}$ is a homomorphic image of $\mathfrak{B}$, then $\Phi(\mathfrak{A})$ is isomorphic to a generated subframe of $\Phi(\mathfrak{B})$.*
   - *If $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, then $\Phi(\mathfrak{A})$ is isomorphic to a p-morphic image of $\Phi(\mathfrak{B})$.*
   - *If $\mathfrak{A} \times \mathfrak{B}$ is a product of $\mathfrak{A}$ and $\mathfrak{B}$, then $\Phi(\mathfrak{A} \times \mathfrak{B})$ is isomorphic to the disjoint union $\Phi(\mathfrak{A}) \uplus \Phi(\mathfrak{B})$.*

2.
   - *If $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$, then $\Psi(\mathfrak{F})$ is a homomorphic image of $\Psi(\mathfrak{G})$.*
   - *If $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$, then $\Psi(\mathfrak{F})$ is isomorphic to a subalgebra of $\Psi(\mathfrak{G})$.*
   - *If $\mathfrak{F} \uplus \mathfrak{G}$ is a disjoint union of $\mathfrak{F}$ and $\mathfrak{G}$, then $\Psi(\mathfrak{F} \uplus \mathfrak{G})$ is isomorphic to the product $\Psi(\mathfrak{F}) \times \Psi(\mathfrak{G})$.*

[3] We assume that the reader is familiar with the very basic notions of category theory, such as a category and (covariant and contravariant) functor. For an extensive study of category theory the reader is referred to [24].

Let **Top** be the category of topological spaces and open maps. Then as for Kripke frames, one can define a contravariant functor from **Top** to **Heyt**.

Exercise 63. Define a contravariant functor from **Top** to **Heyt**.

To define a contravariant functor from **Heyt** to **Top** is a bit trickier, in fact it is impossible. We will not discuss it here. The interested reader is referred to [20].

Now the question is whether $\Phi(\mathbf{Heyt}) \simeq \mathbf{Kripke}$, $\Psi(\mathbf{Kripke}) \simeq \mathbf{Heyt}$ and if not how to characterize $\Phi(\mathbf{Heyt})$ and $\Psi(\mathbf{Kripke})$.

Next we will characterize the Heyting algebras that are isomorphic to Φ-images of Kripke frames. As a result we obtain that there exist Heyting algebras that are not isomorphic to Φ-images of Kripke frames.

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. The lattice $Up(\mathfrak{F})$ is complete. To see this, first observe that arbitrary unions and intersections of upsets are upsets again. Now it is a routine to check that for every $\{U_i\}_{i\in I} \subseteq Up(\mathfrak{F})$, we have that $\bigwedge_{i\in I} U_i = \bigcap_{i\in I} U_i$ and $\bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i$.

Remark 64. Note that despite the fact that an intersection of open sets in general is not open, for every topological space $\mathcal{X} = (X, \mathcal{O})$, the Heyting algebra $\mathcal{O}$ of all open sets of $\mathcal{X}$ is a complete Heyting algebra. For every $\{U_i\}_{i\in I} \subseteq \mathcal{O}$, we have that $\bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i$ and $\bigwedge_{i\in I} U_i = \mathbf{I}(\bigcap_{i\in I} U_i)$. We leave it as an exercise to the reader to check that so defined $\bigvee$ and $\bigwedge$ are indeed the infinite join and meet operations in $\mathcal{O}$.

An element $a \in A$ of a complete lattice $(\mathfrak{A}, \wedge, \vee, \to \bot)$ is called *completely join prime* if $a \leq \bigvee X$ implies there is $x \in X$ such that $a \leq x$. The sets of the form $R(w) = \{v \in W : wRv\}$ are the only completely join-prime elements of $Up(\mathfrak{F})$. Moreover, for every upset $U \subseteq W$ we have that $U = \bigcup\{R(w) : w \in U\}$. Therefore, every element of $Up(\mathfrak{F})$ is a join of completely join-prime elements. Thus, we just showed that for every Kripke frame $\mathfrak{F}$, the Heyting algebra of all upsets of $\mathfrak{F}$ is complete and every element is a join of completely join-prime elements. In fact, the converse of this statement is also true. That is, the following theorem holds.

Theorem 65. *A Heyting algebra $\mathfrak{A}$ is isomorphic to $Up(\mathfrak{F})$ for some Kripke frame $\mathfrak{F}$ iff $\mathfrak{A}$ is complete and every element of $\mathfrak{A}$ is a join of completely join-prime elements of $\mathfrak{A}$.*

Exercise 66. Give an example of a Heyting algebra $\mathfrak{A}$ such that $\mathfrak{A}$ is not isomorphic to $\Phi(\mathfrak{F})$ for any Kripke frame $\mathfrak{F}$. [Hint: observe that it is sufficient to construct a non-complete Heyting algebra].

On the other hand, to characterize those Kripke frames which are Ψ-images of Heyting algebras is much more complicated. It is still an open question to find a decent characterization of such Kripke frames.

Open Question 67. *Characterize Kripke frames in* $\Phi(\mathbf{Heyt})$.

However, if we restrict our attention to the finite case then the correspondence between Heyting algebras and Kripke frames becomes one-to-one. In fact, we have the following theorem.

Theorem 68. *For every finite Heyting algebra* $\mathfrak{A}$ *there exists a Kripke frame* $\mathfrak{F}$ *such that* $\mathfrak{A}$ *is isomorphic to* $Up(\mathfrak{F})$.

To make the correspondence between Heyting algebras and Kripke frames one-to-one we have to generalize the notion of Kripke frames to *descriptive frames*, which are a special kind of general frames. Even though descriptive frames play an important role in the investigation of intuitionistic logic and Heyting algebras we will not discuss them in this course.

4.3 Algebraic completeness of IPC and its extensions

In this section we will discuss the connection between intuitionistic logic and Heyting algebras. We first recall the definition of a variety.

Let K be a class of algebras of the same signature. We say that K is a *variety* if K is closed under homomorphic images, subalgebras and products. It can be shown that K is a variety iff $\mathsf{K} = \mathbf{HSP}(\mathsf{K})$, where $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ are respectively the operations of taking homomorphic images, subalgebras and products. The next, classical, theorem gives another characterization of varieties. For the proof we refer to any of the text books in universal algebra (see e.g., Burris and Sankappanavar [10] or Grätzer [14])

Theorem 69. (*Birkhoff*) *A class of algebras forms a variety iff it is equationally defined.*

Corollary 70. Heyt *is a variety.*

We are now ready to spell out the connection of Heyting algebras and intuitionistic logic and obtain an algebraic completeness result for **IPC**.

Let $\mathcal{P}$ be the (finite or infinite) set of propositional variables and *Form* the set of all formulas in this language. Let $\mathfrak{A} = (A, \wedge, \vee, \rightarrow, \bot)$ be a Heyting algebra. A function $v : \mathcal{P} \rightarrow A$ is called a *valuation* into the Heyting algebra $\mathfrak{A}$.

We extend the valuation from $\mathcal{P}$ to the whole of *Form* by putting:

- $v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$
- $v(\phi \vee \psi) = v(\phi) \vee v(\psi)$
- $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$
- $v(\bot) = \bot$

A formula ϕ is *true* in $\mathfrak{A}$ under v if $v(\phi) = \top$. ϕ is *valid* in $\mathfrak{A}$ if ϕ is true for every valuation in $\mathfrak{A}$.

Proposition 71. (*Soundness*) $\mathbf{IPC} \vdash \phi$ *implies that* ϕ *is valid in every Heyting algebra.*

We define an equivalence relation $\equiv$ on $Form$ by putting

$$\phi \equiv \psi \quad \text{iff} \quad \vdash_{\mathbf{IPC}} \phi \leftrightarrow \psi.$$

Let $[\phi]$ denote the $\equiv$-equivalence class containing ϕ. $Form/_{\equiv} := \{[\phi] : \phi \in Form\}$. Define the operations on $Form/_{\equiv}$ by letting:

- $[\phi] \wedge [\psi] = [\phi \wedge \psi]$
- $[\phi] \vee [\psi] = [\phi \vee \psi]$
- $[\phi] \to [\psi] = [\phi \to \psi]$

Exercise 72. Show that the operations on $Form/_{\equiv}$ are well-defined. That is, show that if $\phi' \equiv \phi''$ and $\psi' \equiv \psi''$, then $\phi' \circ \psi' \equiv \phi'' \circ \psi''$, for $\circ \in \{\vee, \wedge, \to\}$.

The algebra $(Form/_{\equiv}, \wedge, \vee, \to, \bot)$ we denote by $F(\omega)$ (by $F(n)$ in case $\mathcal{P}$ is finite and consists of n-many propositional variables). We call $F(\omega)$ $(F(n))$ the Lindenbaum-Tarski algebra of **IPC** or the ω-generated (n-generated) free Heyting algebra.

Theorem 73. *1. $F(\alpha)$, for $\alpha \leq \omega$ is a Heyting algebra.*

2. **IPC** $\vdash \phi$ *iff ϕ is valid in $F(\omega)$.*

3. **IPC** $\vdash \phi$ *iff ϕ is valid in $F(n)$, for any formula ϕ in n variables.*

Proof. The proof is very similar to the proof of Theorem 13 and is left to the reader. □

Corollary 74. **IPC** *is sound and complete with respect to algebraic semantics.*

Exercise 75. Show that the canonical frame $\mathfrak{F}$ of **IPC** is isomorphic to $\Phi(F(\omega))$.

The analogue of Exercise 75 for the functor Ψ does not hold. To see this, one has to observe that $F(\omega)$ is not a complete Heyting algebra. In order to obtain from the canonical model an algebra that is isomorphic to the Lindenbaum-Tarski algebra of **IPC** we have to restrict ourselves to so-called definable subsets, see Theorem 92 below.

We can extend the algebraic semantics of **IPC** to all the intermediate logics. With every intermediate logic $L \supseteq$ **IPC** we associate the class $\mathbf{V}_L$ of Heyting algebras in which all the theorems of L are valid. It follows from Theorem 69 that $\mathbf{V}_L$ is a variety. For example $\mathbf{V}_{\mathbf{IPC}} = \mathbf{Heyt}$ and $\mathbf{V}_{\mathbf{CPC}} = \mathbf{Bool}$, where **CPC** and **Bool** denote the classical propositional calculus and the variety of all Boolean algebras respectively. For every $\mathbf{V} \subseteq \mathbf{Heyt}$ let $L_{\mathbf{V}}$ be the set of all formulas valid in $\mathbf{V}$. Note that $L_{\mathbf{Heyt}} = \mathbf{IPC}$ and $L_{\mathbf{Bool}} = \mathbf{CPC}$.

Theorem 76. *Every extension L of* **IPC** *is sound and complete with respect to algebraic semantics.*

Proof. The proof is similar to the proof of Corollary 74 and uses the Lindenbaum-Tarski construction. □

The connection between varieties of Heyting algebras and intermediate logics which we described above is one-to-one. That is, $L_{\mathbf{V}_L} = L$ and $\mathbf{V}_{L_{\mathbf{V}}} = \mathbf{V}$.

For every variety $\mathbf{V}$ of algebras the set of its subvarieties forms a lattice which we denote by $(\Lambda(\mathbf{V}), \vee, \wedge, \bot, \top)$. The trivial variety generated by the one element algebra is the least element and $\mathbf{V}$ is the greatest element of $\Lambda(\mathbf{V})$. For every $\mathbf{V}_1, \mathbf{V}_2 \subseteq \Lambda(\mathbf{V})$ we have that $\mathbf{V}_1 \wedge \mathbf{V}_2 = \mathbf{V}_1 \cap \mathbf{V}_2$ and $\mathbf{V}_1 \vee \mathbf{V}_2 = \mathbf{HSP}(\mathbf{V}_1 \cup \mathbf{V}_2)$ – i.e., the smallest variety containing both $\mathbf{V}_1$ and $\mathbf{V}_2$.

Similarly to this, for every propositional logic L, the set of extensions of L forms a lattice $(Lat(L), \vee, \wedge, \bot, \top)$.

Exercise 77. Describe the operations on $(Lat(L), \vee, \wedge, \bot, \top)$.

Then we have that for every $L_1, L_2, \supseteq$ **IPC**, $L_1 \subseteq L_2$ iff $\mathbf{V}_{L_1} \supseteq \mathbf{V}_{L_2}$ and moreover the following theorem is true.

Theorem 78. *The lattice of extensions of* **IPC** *is anti-isomorphic to the lattice of subvarieties of* **Heyt**.

4.4 The connection of Heyting and closure algebras

Finally, we briefly mention the connection of Heyting and closure (interior) algebras. We will give some examples and mention a few important results without providing any proofs. All the proofs can be found in [11].

A pair $(B, \Box)$ is called an *interior* (*closure*) algebra if B is a Boolean algebra and $\Box : B \to B$ is such that for every $a, b \in B$:

1. $\Box\top = \top$
2. $\Box a \leq a$
3. $\Box\Box a \geq \Box a$
4. $\Box(a \wedge b) = \Box a \wedge \Box b$

The best known examples of interior (closure) algebras come from topology (this is the reason why such algebras are called 'interior' and 'closure' algebras). Let $\mathcal{X} = (X, \mathcal{O})$ be a topological space, then the interior and closure operations can be seen as operations on a Boolean algebra $P(X)$, the power set of X. Thus, $(P(X), \mathbf{I})$ is a Boolean algebra with an operator. Moreover, $(P(X), \mathbf{I})$ satisfies the axioms 1–4.

Exercise 79. Verify this claim.

We will now consider another natural example of interior and closure algebras. Let (W, R) be a *quasi-order*, i.e., R is reflexive and transitive. Then $(P(W), [R])$ is an interior algebra, where $[R](U) = \{w \in W :$ for every $v \in W$ wRv implies $v \in U\} = W \setminus R^{-1}(W \setminus U)$.

Exercise 80. Verify this claim.

Using the same arguments as in the previous section one can prove the algebraic completeness of the modal logic **S4** with respect to interior algebras.

Theorem 81. *The modal system* **S4** *is sound and complete with respect to interior algebras.*

Moreover, the lattice of extensions of **S4** is anti-isomorphic to the lattice of subvarieties of the variety of all interior algebras.

Note that the fixed points of **I** in the algebra $(P(X), \mathbf{I})$ are open sets and hence form a Heyting algebra. This correspondence can be extended to all interior algebras.

Theorem 82.

1. *For every interior algebra* $\mathcal{B} = (B, \Box)$ *the fixed points of* $\Box$, *that is,* $\{a \in B : \Box a = a\}$ *form a Heyting algebra.*
2. *For every Heyting algebra* $\mathfrak{A}$ *there exists a closure algebra* $\mathcal{B} = (B, \Box)$ *such that* $\mathfrak{A}$ *is the Heyting algebra of the fixed points of* $\Box$.

Using Theorem 82 we can give an alternative, algebraic proof of Theorem 34.

Corollary 83. $\mathbf{IPC} \vdash \phi$ *iff* $\mathbf{S4} \vdash \phi^{\Box}$.

Proof. Suppose $\mathbf{IPC} \not\vdash \phi$. Then there exists a Heyting algebra $\mathfrak{A}$ and a valuation $v : Form \to \mathfrak{A}$ such that $v(\phi) \neq \top$ (one can always take $F(\omega)$ as $\mathfrak{A}$). By Theorem 82(2) there exists an interior algebra $\mathcal{B} = (B, \Box)$ such that $\mathfrak{A}$ is a Heyting algebra of the fixed points of $\Box$. But this means that there exists a valuation v' on $\mathcal{B}$ such that $v'(\phi^{\Box}) \neq \top$ in $\mathcal{B}$. Therefore, $\mathbf{S4} \not\vdash \phi^{\Box}$. The other direction is left as an **exercise** to the reader.

$\Box$

An interior algebra $(B, \Box)$ is said to be a *Grzegorczyk* algebra if the equation

$$\text{(grz)} \qquad \Box(\Box(a \to \Box a) \to a) \to a = \top$$

holds in $(B, \Box)$.

The variety of Grzegorczyk algebras we denote by $\mathbf{V_{Grz}}$. Let **Grz** denote the normal modal logic obtained from **S4** by adding the Grzegorczyk axiom, that is, $\mathbf{Grz} = \mathbf{S4} + (grz)$.

Theorem 84. **Grz** *is complete with respect to the class of finite partially ordered Kripke frames.*

Theorem 85.

1. $\mathbf{IPC} \vdash \phi$ *iff* $\mathbf{Grz} \vdash \phi^{\Box}$.
2. **Grz** *is the greatest extension of* **S4** *for which (1) holds.*

Denote by $\Lambda(\mathbf{Heyt})$ and $\Lambda(\mathbf{V_{Grz}})$ the lattices of subvarieties of all Heyting and Grzegorczyk algebras respectively. Let also $Lat(\mathbf{IPC})$ and $Lat(\mathbf{Grz})$ be the lattices of extensions of **IPC** and **Grz** respectively. We will close this section by the following fundamental result, linking modal and intermediate logics (see [6], [13] and [11, Theorem 9.66]).

Theorem 86. (*Blok-Esakia*)

1. $\Lambda(\mathbf{Heyt})$ *is isomorphic to* $\Lambda(\mathbf{V_{Grz}})$*;*
2. $Lat(\mathbf{IPC})$ *is isomorphic to* $Lat(\mathbf{Grz})$*.*

5 Jankov formulas and intermediate logics

5.1 n-universal models

Fix a propositional language $\mathcal{L}_n$ consisting of finitely many propositional letters $p_1, \dots, p_n$ for $n \in \omega$. Let $\mathfrak{M}$ be an intuitionistic Kripke model. With every point w of $\mathfrak{M}$, we associate a sequence $i_1 \dots i_n$ such that for $k = 1, \dots, n$:

$$i_k = \begin{cases} 1 & \text{if } w \models p_k, \\ 0 & \text{if } w \not\models p_k \end{cases}$$

We call the sequence $i_1 \dots i_n$ associated with w the *color* of w and denote it by $col(w)$.

Definition 87. *We order colors according to the relation* $\leq$ *such that* $i_1 \dots i_n \leq i'_1 \dots i'_n$ *if for every* $k = 1, \dots, n$*, we have that* $i_k \leq i'_k$*. We write* $i_1 \dots i_n < i'_1 \dots i'_n$ *if* $i_1 \dots i_n \leq i'_1 \dots i'_n$ *and* $i_1 \dots i_n \neq i'_1 \dots i'_n$*.*

Thus, the set of colors of length n ordered by $\leq$ forms a 2^n-element Boolean algebra. For a frame $\mathfrak{F} = (W, R)$ and $w, v \in W$, we say that a point w is an *immediate successor* of a point v if there are not intervening points, i.e., for every $u \in W$such that vRu and uRv we have $u = v$ or $u = w$. We say that a set A *totally covers* a point v and write $v \prec A$ if A is the set of all immediate successors of v. Note that $\prec$ is a relation relating points and sets. We will use the shorthand $v \prec w$ for $v \prec \{w\}$. Thus, $v \prec w$ means not only that w is an immediate successor of v, but that w is the only immediate successor of w. It is easy to see that if every point of W has only finitely many successors, then R is the reflexive and transitive closure of the immediate successor relation. Therefore, if (W, R) is such that every point of W has only finitely many successors, then R is uniquely defined by the immediate successor relation and vice versa. Thus, to define such a frame (W, R), it is sufficient to define the relation $\prec$. A set $A \subseteq W$ is called an *anti-chain* if $|A| > 1$ and for every $w, v \in A$, $w \neq v$ implies $\neg(wRv)$ and $\neg(vRw)$.

Now we are ready to construct the n-universal model of **IPC** for such $n \in \omega$. As we mentioned above, to define $\mathcal{U}(n) = (U(n), R, V)$, it is sufficient to define the set $U(n)$, relation $\prec$ relating points and sets, and valuation V on $U(n)$.

Definition 88. *The* n-universal model $\mathcal{U}(n)$ *is the model satisfying the following three conditions:*

1. $max(\mathcal{U}(n))$ *consists of* 2^n *points of distinct colors.*

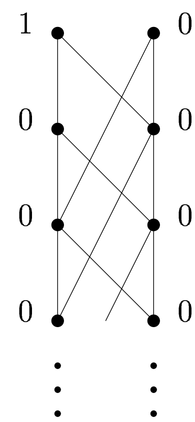


Figure 7: The 1-universal model

2. *For every $w \in U(n)$ and every color $i_1 \dots i_n < col(w)$, there exists a unique $v \in U(n)$ such that $v \prec w$ and $col(v) = i_1 \dots i_n$.*

3. *For every finite anti-chain A in $U(n)$ and every color $i_1 \dots i_n$ such that $i_1 \dots i_n \leq col(u)$ for all $u \in A$, there exists a unique $v \in U(n)$ such that $v \prec A$ and $col(v) = i_1 \dots i_n$.*

First we show that Definition 88 uniquely defines the n-universal model.

Proposition 89. *The n-universal model $\mathcal{U}(n)$ is unique up to isomorphism.*

Proof. Let $\mathcal{W}(n)$ be a model satisfying conditions (1)-(3) of Definition 88. Then it follows that every point $\mathcal{W}(n)$ has finite depth. Now we will prove by an easy induction on the number of layers of $\mathcal{W}(n)$ that $\mathcal{U}(n)$ and $\mathcal{W}(n)$ are isomorphic. By (1) of Definition 88, $max(\mathcal{U}(n))$ and $max(\mathcal{W}(n))$ are isomorphic. Now assume that first m layers of $\mathcal{U}(n)$ and $\mathcal{W}(n)$ are isomorphic. Then by (2) and (3) of Definition 88 it follows immediately that the $m+1$ layers of $\mathcal{U}(n)$ and $\mathcal{W}(n)$ are also isomorphic, which finishes the proof of the proposition. □

The 1-universal model is shown in Figure 7. The 1-universal model is often called the *Rieger-Nishimura ladder.*

In the remainder of this section, we state the main properties of the n-universal model without proof. All the proofs can be found in [11, Sections 8.6 and 8.7], [15], [2], [31] or [30].

Theorem 90. *1. For every Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$, there exists a Kripke model $\mathfrak{M}' = (\mathfrak{F}', V')$ such that $\mathfrak{M}'$ is a generated submodel of $\mathcal{U}(n)$ and $\mathfrak{M}'$ is a p-morphic image of $\mathfrak{M}$.*

2. *For every finite Kripke frame $\mathfrak{F}$, there exists a valuation V, and $n \leq |\mathfrak{F}|$ such that $\mathfrak{M} = (\mathfrak{F}, V)$ is a generated submodel of $\mathcal{U}(n)$.*

Theorem 90(1) immediately implies the following corollary.

Corollary 91. *For every formula ϕ in the language $\mathcal{L}_n$, we have that*

$$\vdash_{\mathbf{IPC}} \phi \quad \textit{iff} \quad \mathcal{U}(n) \models \phi.$$

There is a close connection between n-universal models and free Heyting algebras. Call a set $V \subseteq U(n)$ *definable* if there is a formula ϕ such that $V = \{w \in U(n) : w \models \phi\}$. It can be shown that not every upset of the n-universal model is definable (for $n > 1$). It is easy to see that definable subsets of $\mathcal{U}(n)$ form a subalgebra of $Up(\mathcal{U}(n))$. We have the following theorem.

Theorem 92. *The Heyting algebra of all definable subsets of the n-universal model is isomorphic to the free n-generated Heyting algebra.*

We say that a frame $\mathfrak{F} = (W, R)$ is of *depth* $n < \omega$, and write $d(\mathfrak{F}) = n$ if there is a chain of n points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than n points. If for every $n \in \omega$, $\mathfrak{F}$ contains a chain consisting of n points, then $\mathfrak{F}$ is said to be of *infinite depth*. The *depth* of a point $w \in W$ is the depth of $\mathfrak{F}_w$, i.e., the depth of the subframe of $\mathfrak{F}$ based on the set $R(w)$. The depth of w we denote by $d(w)$.

Remark 93. The n-universal models are closely related to finite variable canonical models. That is, canonical models in the language $\mathcal{L}_n$ (see Remark 14). Let $\mathfrak{M}(n)$ be the canonical model in the language $\mathcal{L}_n$, which is also called the Henkin model. Then, the generated submodel of $\mathfrak{M}(n)$ consisting of all the points of finite depth is (isomorphic to) $\mathcal{U}(n)$. Therefore, $\mathfrak{M}(n)$ can be represented as a union $\mathfrak{M}(n) = \mathcal{U}(n) \cup \mathcal{T}(n)$, where $\mathcal{T}(n)$ is a submodel of $\mathfrak{M}(n)$ consisting of all the points of infinite depth. Moreover, it can be shown that for every point w in $\mathcal{T}(n)$, there exists a point $v \in U(n)$ such that wRv. In other words, universal models are "upper parts" of canonical models.

5.2 Formulas characterizing point generated subsets

In this section, we will introduce the so-called De Jongh formulas of **IPC** and prove that they define point generated submodels of universal models. We will also show that they do the same job as Jankov's characteristic formulas for **IPC**. For more details on this topic, we refer to [16, §2.5].

Let w be a point in the n-universal model. Recall that $R(w) = \{v \in U(n) : wRv\}$ and $R^{-1}(w) = \{v \in U(n) : vRw\}$. Now we define formulas ϕ_w and ψ_w inductively. If $d(w) = 1$ then let

$$\phi_w := \bigwedge\{p_k : w \models p_k\} \wedge \bigwedge\{\neg p_j : w \not\models p_j\} \text{ for each } k, j = 1, \ldots, n$$

and

$$\psi_w = \neg\phi_w.$$

If $d(w) > 1$, then let $\{w_1, \ldots, w_m\}$ be the set of all immediate successors of w. Let

$$prop(w) := \{p_k : w \models p_k\}$$

and

$$newprop(w) := \{p_k : w \not\models p_k \text{ and for all } i \text{ such that } 1 \leq i \leq m,\ w_i \models p_k\}.$$

Let

$$\phi_w := \bigwedge prop(w) \wedge \left((\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}) \to \bigvee_{i=1}^{m} \phi_{w_i} \right)$$

and

$$\psi_w := \phi_w \to \bigvee_{i=1}^{m} \phi_{w_i}$$

We call ϕ_w and ψ_w the *de Jongh formulas*.

Theorem 94. *For every $w \in U(n)$ we have that:*

- *$R(w) = \{v \in U(n) : v \models \phi_w\}$, i.e., ϕ_w defines $R(w)$.*
- *$U(n) \setminus R^{-1}(w) = \{v \in U(n) : v \models \psi_w\}$, i.e., ψ_w defines $U(n) \setminus R^{-1}(w)$.*

Proof. We prove the theorem by induction on the depth of w. Let the depth of w be 1. This means, that w belongs to the maximum of $\mathcal{U}(n)$. By Definition 88(1) for every $v \in max(\mathcal{U}(n))$ such that $w \neq v$ we have $col(v) \neq col(w)$ and thus $v \not\models \phi_w$. Therefore, if $u \in U(n)$ is such that uRv for some maximal point v of $\mathcal{U}(n)$ distinct from w, then $u \not\models \phi_w$. Finally, assume that vRw and v is not related to any other maximal point. By Definition 88(2) and (3), this implies that $col(v) < col(w)$. Therefore, $v \not\models \phi_w$, and so $v \models \phi_w$ iff $v = w$. Thus, $V(\phi_w) = \{w\}$. Consequently, by the definition of the intuitionistic negation, we have that $V(\psi_w) = V(\neg\phi_w) = U(n) \setminus R^{-1}(V(\phi_w)) = U(n) \setminus R^{-1}(w)$.

Now suppose the depth of w is greater than 1 and the theorem holds for the points with depth strictly less than $d(w)$. This means that the theorem holds for every immediate successor w_i of w, i.e., for each $i = 1, \ldots, m$ we have $V(\phi_{w_i}) = R(w_i)$ and $V(\psi_{w_i}) = U(n) \setminus R^{-1}(w_i)$.

First note that, by the induction hypothesis, $w \not\models \bigvee_{i=1}^{m} \psi_{w_i}$; hence, by the definition of $newprop(w)$, we have $w \not\models \bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}$. Therefore, $w \models \phi_w$, and so, by the persistence of intuitionistic valuations, $v \models \phi_w$ for every $v \in R(w)$.

Now let $v \notin R(w)$. If $v \not\models \bigwedge prop(w)$, then $v \not\models \phi_w$. So suppose $v \models \bigwedge prop(w)$. This means that $col(v) \geq col(w)$. Then two cases are possible:

Case 1. $v \in \bigcup_{i=1}^{m} U(n) \setminus R^{-1}(w_i)$. Then by the induction hypothesis, $v \models \bigvee_{i=1}^{m} \psi_{w_i}$ and since $v \notin R(w)$, we have $v \not\models \bigvee_{i=1}^{m} \phi_{w_i}$. Therefore, $v \not\models \phi_w$.

Case 2. $v \notin \bigcup_{i=1}^{m} U(n) \setminus R^{-1}(w_i)$. Then vRw_i for every $i = 1, \ldots, m$. If vRv' and $v' \in \bigcup_{i=1}^{m} U(n) \setminus R^{-1}(w_i)$, then, by Case 1, $v' \not\models \phi_w$, and so $v \not\models \phi_w$. Now assume that for every $v' \in U(n)$, vRv' implies $v' \notin \bigcup_{i=1}^{m} U(n) \setminus R^{-1}(w_i)$. By the construction of $\mathcal{U}(n)$ (see Definition 88(3)), there exists a point $u \in U(n)$ such that $u \prec \{w_1, \ldots, w_m\}$ and vRu. We again specify two cases.

Case 2.1. $u = w$. Then there exists $t \in U(n)$ such that $t \prec w$ and vRt. So, $col(v) \leq col(t)$ and by Definition 88(2), $col(t) < col(w)$, which is a contradiction.

Case 2.2. $u \neq w$. Since vRu and $col(v) \geq col(w)$, we have $col(u) \geq col(v) \geq col(w)$. If $col(u) > col(w)$, then there exists p_j, for some $j = 1, \ldots, n$, such that $u \models p_j$ and $w \not\models p_j$. Then $w_i \models p_j$, for every $i = 1, \ldots, m$, and hence $p_j \in newprop(w)$. Therefore, $u \models \bigvee newprop(w)$ and $u \not\models \bigvee_{i=1}^{m} \phi_{w_i}$. Thus, $u \not\models \phi_w$ and so $v \not\models \phi_w$. Now suppose $col(u) = col(w)$. Then by Definition 88(3), $u = w$ which is a contradiction.

Therefore, for every point v of $U(n)$ we have:

$$v \models \phi_w \text{ iff } wRv.$$

Now we show that ψ_w defines $U(n) \setminus R^{-1}(w)$. For every $v \in U(n)$, $v \not\models \psi_w$ iff there exists $u \in U(n)$ such that vRu and $u \models \phi_w$ and $u \not\models \bigvee_{i=1}^{m} \phi_{w_i}$, which holds iff $u \in R(w)$ and $u \notin \bigcup_{i=1}^{m} R(w_i)$, which, in turn, holds iff $u = w$. Hence, $v \not\models \psi_w$ iff $v \in R^{-1}(w)$. This finishes the proof of the theorem. □

5.3 The Jankov formulas

In this subsection we show that the de Jongh formulas do the same job as Jankov's characteristic formulas for **IPC**. We first state the Jankov-de Jongh theorem. Note that Jankov's original result was formulated in terms of Heyting algebras. We will formulate it in logical terms. Most of the results in this and subsequent sections have their natural algebraic counterparts but we will not discuss them here.

Theorem 95. (*see Jankov* [18], [11, §9.4] *and de Jongh* [12]) *For every finite rooted frame* $\mathfrak{F}$ *there exists a formula* $\chi(\mathfrak{F})$ *such that for every frame* $\mathfrak{G}$

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \quad \textit{iff } \mathfrak{F} \textit{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

For the proof of Theorem 95 using the so-called *Jankov formulas* the reader is refered to [11, §9.4]. We will give an alternative proof of Theorem 95 using the de Jongh formulas. For this we will need one additional lemma.

Lemma 96. *A frame* $\mathfrak{F}$ *is a p-morphic image of a generated subframe of* $\mathfrak{G}$ *iff* $\mathfrak{F}$ *is a generated subframe of a p-morphic image of* $\mathfrak{G}$.

Proof. The proof follows from Theorem 62 and the general universal algebraic result which says that in every variety $\mathbf{V}$ with the congruence extension property, for every algebra $\mathfrak{A} \in \mathbf{V}$, we have that $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$. □

Proof of Theorem 95:

Proof. By Theorem 90(2) there exists $n \in \omega$ such that $\mathfrak{F}$ is (isomorphic to) a generated subframe of $\mathcal{U}(n)$. Let $w \in U(n)$ be the root of $\mathfrak{F}$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_w$. By Lemma 96, for proving Theorem 95 it is sufficient to show that for every frame $\mathfrak{G}$:

$$\mathfrak{G} \not\models \psi_w \text{ iff } \mathfrak{F}_w \text{ is a generated subframe of a } p\text{-morphic image of } \mathfrak{G}.$$

Suppose $\mathfrak{F}_w$ is a generated subframe of a p-morphic image of $\mathfrak{G}$. Clearly, $w \not\models \psi_w$; hence $\mathfrak{F}_w \not\models \psi_w$, and since p-morphisms preserve the validity of formulas $\mathfrak{G} \not\models \psi_w$.

Now suppose $\mathfrak{G} \not\models \psi_w$. Then, there exists a model $\mathfrak{M} = (\mathfrak{G}, V)$ such that $\mathfrak{M} \not\models \psi_w$. By Theorem 90(1), there exists a submodel $\mathfrak{M}' = (\mathfrak{G}', V')$ of $\mathcal{U}(n)$ such that $\mathfrak{M}'$ is a p-morphic image of $\mathfrak{M}$. This implies that $\mathfrak{M}' \not\models \psi_w$. Now, $\mathfrak{M}' \not\models \psi_w$ iff there exists v in $\mathfrak{G}'$ such that vRw, which holds iff w belongs to $\mathfrak{G}'$. Therefore, w is in $\mathfrak{G}'$, and $\mathfrak{F}_w$ is a generated subframe of $\mathfrak{G}'$. Thus, $\mathfrak{F}_w$ is a generated subframe of a p-morphic image of $\mathfrak{G}$. □

Remark 97. Note that there is one essential difference between the Jankov and de Jongh formulas: the number of propositional variables used in the Jankov formula depends on (is equal to) the cardinality of $\mathfrak{F}$, whereas the number of variables in the de Jongh formula depends on which $\mathcal{U}(n)$ contains $\mathfrak{F}$ as a generated submodel. Therefore, in general, the de Jongh formula contains fewer variables than the Jankov formula. From now on we will use the general term "Jankov formula" to refer to the formulas having the property formulated in Theorem 95 and denote them by $\chi(\mathfrak{F})$.

5.4 Applications of Jankov formulas

Here we will give some illustrations of the use of Jankov formulas. First we show that there are continuum many intermediate logics. Let $\mathfrak{F}$ and $\mathfrak{G}$ be two Kripke frames. We say that

$$\mathfrak{F} \leq \mathfrak{G} \quad \text{if } \mathfrak{F} \text{ is a } p\text{-morphic image of a generated subframe of } \mathfrak{G}.$$ [4]

Exercise 98.

1. Show that $\leq$ is reflexive and transitive.
2. Show that if we restrict ourselves to only finite Kripke frames, then $\leq$ is a partial order.
3. Show that f is a p-morphism from linear ordering $\mathfrak{F}$ to $\mathfrak{G}$ iff f is an order-preserving function.
4. Show that in the infinite case $\leq$, in general, is not antisymmetric [Hint: consider the previous item of this exercise and look for example at the real numbers].

[4] By Lemma 96 this is equivalent to saying that $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{G}$.

5. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two finite rooted frames. Show that if $\mathfrak{F} \leq \mathfrak{F}'$, then for every frame $\mathfrak{G}$ we have that $\mathfrak{G} \models \chi(\mathfrak{F})$ implies $\mathfrak{G} \models \chi(\mathfrak{F}')$.

$\mathfrak{F} \leq \mathfrak{G}$ cannot be replaced by: $\mathfrak{F}$ is a generated subframe or a p-morhic image of $\mathfrak{G}$, even in the finite case. We will show that in a minute, but we first have to develop a few facts about p-morhisms.

Lemma 99. *Let $\mathfrak{F} = (W, R)$ be a Kripke frame.*

1. *Assume that u and v in $\mathfrak{F}$ are such that $R(u)\backslash\{u\} = R(v)\backslash\{v\}$. Define $W' = W\backslash\{u\}$, $R' = R\upharpoonright W' \cup \{(x, v) | (x, u) \in R\}$, $h(u) = v$, $h(x) = x$ if $x \neq u$. Then h is a p-morhism from $\mathfrak{F}$ onto $\mathfrak{F}' = (W', R')$. A function like h is called a β-*reduction.

2. *Assume that u and v in $\mathfrak{F}$ are such that $R(u) = R(v) \cup \{u\}$ (i.e. v is the sole immediate successor of u) . Define $W' = W\backslash\{u\}$, $R' = R\upharpoonright W'$, $h(u) = v$, $h(x) = x$ if $x \neq u$. Then h is a p-morhism from $\mathfrak{F}$ onto $\mathfrak{F}' = (W', R')$. A function like h is called an α-*reduction.

Proof. Easy **Exercise**. □

Proposition 100. *If f is a proper p-morhism from finite $\mathfrak{F}$ onto $\mathfrak{G}$, then there exists a sequence $f_1, \cdots, f_n$ of α- and β-reductions such that $f = f_1 \circ \cdots \circ f_n$.*

Proof. Let f be a proper p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$. Let w be a maximal point of $\mathfrak{G}$ that is the image under f of at least two distinct points of $\mathfrak{F}$. We consider a number of possibilities:

Case 1. $max(f^{-1}(w))$ contains more than one element and u and v are two such elements. Then, by the conditions on a p-morphism, the sets of successors of u and v in $\mathfrak{F}$, disregarding u and v themselves, are the same. There exists a β-reduction h with $g(u) = v$ of $\mathfrak{F}$ onto $\mathfrak{H} = (W\backslash\{u\}, R'')$. It suffices to construct a p-morphism g from $\mathfrak{H}$ onto $\mathfrak{G}$ such that $g \circ h = f$ (and apply induction on the number of points that are identified by f). We can just take g to be the restriction of f to $W\backslash\{u\}$. Checking the clauses of p-morphism is trivial.

Case 2. $max(f^{-1}(w))$ contains exactly one element. Consider the immediate predecessors of that element in $max(f^{-1}(w))$. If there is more than one such element we can proceed as in the first case. Otherwise, there exist two elements u and v such that u is the unique immediate successor of v and v is the unique immediate predecessor of u. This case is left as an **Exercise** to the reader.

□

This proposition will enable us to give an example of a generated subframe of a finite frame that is not a p-morphic image of the frame. Let us start by noting a simple corrollary:

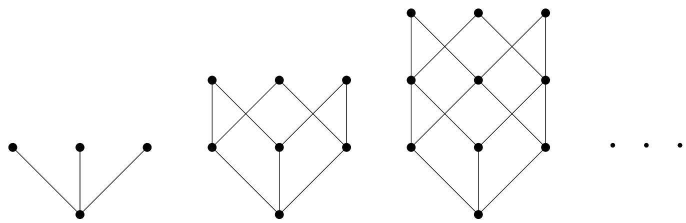


Figure 8: The sequence Δ

Lemma 101. *No p-morphism of a finite frame can increase the number of end nodes.*

Proof. Just note that by an α- or β-reduction the number of end nodes cannot increase. $\square$

Exercise 102. *Show that by p-morhisms of infinte models the number of end nodes can increase.*

Example 103. In Figure $\mathfrak{G}_{\mathfrak{w}}$ is not a p-morhic image of $\mathfrak{G}$.

It is easy to see that $\mathfrak{G}_{\mathfrak{w}}$ is not a p-morhic image of $\mathfrak{G}$, since the only nodes that can be identified by an α- or β-reduction are two of the three end nodes, but that would reduce the number of end nodes.

Exercise 104. Show that if $\mathfrak{G}$ is a tree and $\mathfrak{F}$ is a rooted generated subframe of $\mathfrak{G}$, then $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$.

Example 105. In Figure $\mathfrak{G} \leqslant \mathfrak{F}$, but $\mathfrak{G}$ is not a p-morhic image of $\mathfrak{F}$ and neither is it isomorphic to a generated subframe of $\mathfrak{F}$.

Consider the sequence Δ of finite Kripke frames shown in Figure 8.

Lemma 106. Δ *forms a* $\leq$*-antichain.*

Proof. The proof is left as an **Exercise** to the reader. Hint: Use Lemma 5.4. $\square$

For every set Γ of Kripke frames. Let $Log(\Gamma)$ be the logic of Γ, that is, $Log(\Gamma) = \{\phi : \mathfrak{F} \models \phi \text{ for every } \mathfrak{F} \in \Gamma\}$.

Theorem 107. *For every* $\Gamma_1, \Gamma_2 \subseteq \Delta$*, if* $\Gamma_1 \neq \Gamma_2$*, then* $Log(\Gamma_1) \neq Log(\Gamma_2)$*.*

Proof. Without loss of generality assume that $\Gamma_1 \not\subseteq \Gamma_2$. This means that there is $\mathfrak{F} \in \Gamma_1$ such that $\mathfrak{F} \notin \Gamma_2$. Consider the Jankov formula $\chi(\mathfrak{F})$. By Theorem 95 we have that $\mathfrak{F} \not\models \chi(\mathfrak{F})$. Hence, $\Gamma_1 \not\models \chi(\mathfrak{F})$ and $\chi(\mathfrak{F}) \notin Log(\Gamma_1)$. Now we show that $\chi(\mathfrak{F}) \in Log(\Gamma_2)$. Suppose $\chi(\mathfrak{F}) \notin Log(\Gamma_2)$. Then there is $\mathfrak{G} \in \Gamma_2$ such that $\mathfrak{G} \not\models \chi(\mathfrak{F})$. By Theorem 95 this means that $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Thus, $\mathfrak{F} \leq \mathfrak{G}$ which contradicts the fact that Δ forms a $\leq$-antichain. Therefore, $\chi(\mathfrak{F}) \notin Log(\Gamma_1)$ and $\chi(\mathfrak{F}) \in Log(\Gamma_2)$. Thus, $Log(\Gamma_1) \neq Log(\Gamma_2)$.

$\square$

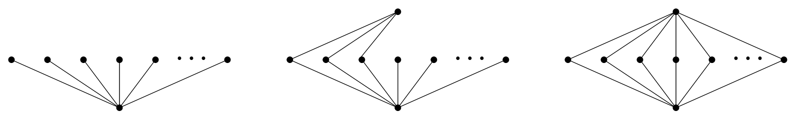


Figure 9: The k, n-kite $\mathfrak{F}_n^k$

We have the following corollary of Theorem 107 first observed by Jankov in [19].

Corollary 108. *There are continuum many intermediate logics.*

Proof. The proof follows immediately from Theorem 107. □

Next we will give a few applications of Jankov formulas in axiomatizations of intermediate logics. Intuitively speaking the Jankov formula of a frame $\mathfrak{F}$ axiomatizes the least logic that does not have $\mathfrak{F}$ as its Kripke frame.

Recall that a frame $\mathfrak{F} = (W, R)$ is of *depth* $\leqslant n < \omega$, if there is a chain of n points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than n points. An intermediate logic $L \supseteq \mathbf{IPC}$ has *depth* $n \in \omega$ if every L-frame (that is, frame validating all the formulas in L) $\mathfrak{F}$ has depth $\leq n$. We say that a finite frame $\mathfrak{F}$ has *branching* $\leq n$ if every point in $\mathfrak{F}$ has at most n distinct immediate successors. $L \supseteq \mathbf{IPC}$ has *branching* $\leq n$ if every L-frame has branching $\leq n$.

We will axiomatize logics of finite depth and branching by Jankov formulas. Let $\mathfrak{C}_n$ denote a transitive chain of length n and let $\chi(\mathfrak{C}_n)$ be a Jankov formula of $\mathfrak{C}_n$.

Theorem 109. *1. A frame $\mathfrak{F}$ has depth $\leq n$ iff $\mathfrak{C}_{n+1} \not\leq \mathfrak{F}$.*

*2. A logic $L \supseteq$ **IPC** has depth $\leqslant n$ iff $\chi(\mathfrak{C}_n) \in L$.*

Proof. (1) If $\mathfrak{C}_{n+1} \leq \mathfrak{F}$, then obviously the depth of $\mathfrak{F}$ is $\geq n+1$. Now suppose the depth of $\mathfrak{F}$ is $> n$. Then there are distinct points $w_0, \dots, w_n$ such that $w_i R w_j$ if $i \leq j$ for every $i, j = 0, \dots, n$. Let $\mathfrak{F}_{w_0}$ be a generated subframe of $\mathfrak{F}$ generated by the point w_0. Define a map f from $\mathfrak{F}_{w_0}$ to $\mathfrak{F}_{w_0}$ by putting:

$$f(w) = w_i \text{ for every } w \in R^{-1}(w_i) \setminus R^{-1}(w_{i-1}) \text{ and } i = 0, \dots, n.$$

We leave it to the reader to verify that f is a p-morphism and the f-image of $\mathfrak{F}_{w_0}$ is isomorphic to $\mathfrak{C}_{n+1}$. Therefore, $\mathfrak{C}_{n+1}$ is a p-morphic image of a generated subframe of $\mathfrak{F}$.

(2) The result follows from (1) and is left as an **exercise** to the reader. □

Now we axiomatize by Jankov formulas the logics of branching $\leq n$. Call a frame shown in Figure 9 a k, n-*kite* if the number of the points of the depth 2 is equal to n and the number of points of depth 2 that see the top point (if it exists) is equal to $k \leq n$. (Figure 9 shows the frames $\mathfrak{F}_n^0$, $\mathfrak{F}_n^3$ and $\mathfrak{F}_n^n$.)

Lemma 110. *If $n' \geq n > 0$ and $k' \geq k > 0$, then $\mathfrak{F}_n^k$ is a p-morphic image of $\mathfrak{F}_{n'}^{k'}$.*

Proof. The proof is left as an **exercise** to the reader. □

Theorem 111. *1. A finite frame $\mathfrak{F}$ has branching $\leq n$ iff $\mathfrak{F}^k_{n+1} \not\leq \mathfrak{F}$, for every $k \leq n$.*

*2. A logic $L \supseteq$ **IPC** has branching $\leqslant n$ iff $\chi(\mathfrak{F}^k_n) \in L$ for every $k \leq n$.*

Proof. (1) It is easy to see that if $\mathfrak{F}^k_n \leq \mathfrak{F}$, for some finite $\mathfrak{F}$ and $k \leq n$, then $\mathfrak{F}$ has branching $\geq n+1$. Now suppose there is a point w in $\mathfrak{F}$ which has m successors, for $m > n$. Let $\mathfrak{F}_w$ denote the rooted subframe of $\mathfrak{F}$ generated by the point w. Let $\{w_1, \ldots, w_m\}$ be the set of all immediate successors of w. We call a point w_i for $i \leq m$ a *hat point* if there exists w_j for $j \neq i$ such that $R(w_i) \cap R(w_j) \neq \emptyset$. Let $k \leq m$ be the number of hat point successors of w. Without loss of generality we can assume that the points $w_1, \ldots, w_k$ are the hat points. Let $\mathfrak{F}^k_m$ be a k, m-kite. Denote by r the root of $\mathfrak{F}^k_m$ and by t the top point (if it exists, in case $k = 0$ the top point does not exist). Let also $x_1 \ldots, x_k$ be the points of $\mathfrak{F}^k_m$ of depth 2 that are related to t and $y_{k+1}, \ldots, y_n$ the points of depth 2 that are not related to t. Define a map $f : \mathfrak{F}_w \to \mathfrak{F}^k_m$ by putting for $i = 1, \ldots, m$:

$$f(x) = \begin{cases} r & \text{if} \quad x = w \\ y_i & \text{if} \quad x \in R(w_i) \text{ and } w_i \text{ is not a hat point} \\ t & \text{if} \quad x \in R(w_i) \setminus \{w_i\} \text{ and } w_i \text{ is a hat point} \\ x_i & \text{if} \quad x = w_i \text{ and } w_i \text{ is a hat point} \end{cases}$$

We leave it as an **exercise** to the reader to check that f is a p-morphism. Thus, $\mathfrak{F}^k_m$ is a p-morphic image of a generated subframe of $\mathfrak{F}$. Finally, note that by Lemma 110 $\mathfrak{F}^k_n$ is a p-morphic image of $\mathfrak{F}^k_m$. Thus, $\mathfrak{F}^k_n \leq \mathfrak{F}$, for some $k \leq n$.

(2) The proof is left to the reader. □

Call a logic L *tabular* if $L = Log(\mathfrak{F})$ for some finite (not necessarily rooted) frame $\mathfrak{F}$.

Theorem 112. *Every tabular logic is finitely axiomatizable.*

Proof. Let $\mathbf{Fr} = \{\mathfrak{G} : \mathfrak{G} \not\leq \mathfrak{F}\}$. It is easy to see that $(\mathbf{Fr}, \leq)$ has finitely many minimal elements (verify this). Let the minimal elements of $(\mathbf{Fr}, \leq)$ be $\mathfrak{G}_1, \ldots, \mathfrak{G}_k$. Then, $L(\mathfrak{F}) = \mathbf{IPC} + \chi(\mathfrak{G}_1) + \ldots + \chi(\mathfrak{G}_k)$. Let the depth of $\mathfrak{F}$ be n. Then one of the $\mathfrak{G}_i$s will be isomorphic to $\mathfrak{C}_{n+1}$ (verify this). Thus, $\chi(\mathfrak{C}_{n+1}) \in \mathbf{IPC} + \chi(\mathfrak{G}_1) + \ldots + \chi(\mathfrak{G}_k)$ and by Theorem 109 the logic $\mathbf{IPC} + \chi(\mathfrak{G}_1) + \ldots + \chi(\mathfrak{G}_k)$ has a finite depth. Every logic of finite depth has the finite model property (see [11]). Therefore, both $L(\mathfrak{F})$ and $\mathbf{IPC} + \chi(\mathfrak{G}_1) + \ldots + \chi(\mathfrak{G}_k)$ have the finite model property. Note that by the definition of $\mathfrak{G}_1, \ldots, \mathfrak{G}_k$ the finite frames of $L(\mathfrak{F})$ and $\mathbf{IPC} + \chi(\mathfrak{G}_1) + \ldots + \chi(\mathfrak{G}_k)$ coincide (verify this). This implies that these two logics are equal and we obtain that $L(\mathfrak{F})$ is finitely axiomatizable by Jankov formulas.

□

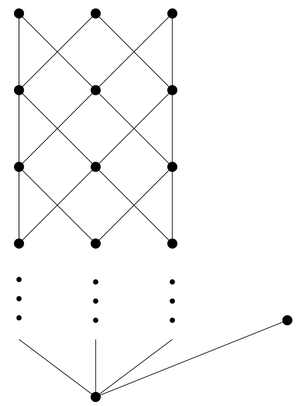


Figure 10: The frame $\mathfrak{G}$

However, by no means every intermediate logic is axiomatized by Jankov formulas. We say that a frame $\mathfrak{F}$ has *width* n if it contains an antichain of n points and there is no antichain of greater cardinality. The width of $\mathfrak{F}$ we denote by $w(\mathfrak{F})$. Let L_3 be the least logic of width 3, that is, $L_3 = Log(\Gamma_3)$, where $\Gamma_3 = \{\mathfrak{F} : w(\mathfrak{F}) \leq 3\}$.
Now we will give a sketch of the proof that L_3 is not axiomatized by Jankov formulas. For the details we refer to [11, Proposition 9.50].

Theorem 113. *L_3 is not axiomatizable by only Jankov formulas.*

Proof. Suppose $L_3 = \mathbf{IPC} + \{\chi(\mathfrak{F}_i) : i \in I\}$. Note that the width of every $\mathfrak{F}_i$ should be greater than 3, otherwise they will be L_3-frames. Consider the frame $\mathfrak{G}$ shown in Figure 10. It is obvious that $\mathfrak{G}$ has width 4 and hence is not an L_3-frame. Thus, there exists $i \in I$ such that $\mathfrak{G} \not\models \chi(\mathfrak{F}_i)$. This means that $\mathfrak{F}_i$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.

Now we leave it as a (nontrivial) **exercise** for the reader to verify that no finite rooted frame of width > 3 can be a p-morphic image of a generated subframe of $\mathfrak{G}$. □

To be able to axiomatize all intermediate logics by 'frame' formulas one has to generalize the Jankov formulas. Zakharyaschev's canonical formulas are extensions of Jankov formulas and provide complete axiomatizations of all intermediate logics. We do not discuss canonical formulas in this course. For a systematic study of canonical formulas the reader is referred to [11, §9].

5.5 The logic of the Rieger-Nishimura ladder

In this last section of the course notes we will discuss the logic of one particular infinite frame. We will study the logic of the Rieger-Nishimura ladder. (We came across this structure a few times before.) This logic is interesting on its own, since it is the greatest 1-conservative extension of **IPC**. However, the main reason of including this material is to give an example how to investigate the logic of one (infinite) frame. We will show how Jankov formulas can help in describing finite rooted frames of such logics, and how can they be used in establishing a lack of the finite model property. Let **RN** denote the logic of $\mathcal{RN}$.

Definition 114. *Suppose L and S are intermediate logics. We say that S is an n-conservative extension of L if $L \subseteq S$ and for every formula $\phi(p_1, \ldots, p_n)$ in n-variables we have $\phi \in L$ iff $\phi \in S$.*

Theorem 115. *$L(\mathcal{U}(n))$ is the greatest n-conservative extension of* **IPC**.

Proof. By Corollary 91, for each formula ϕ in n-variables, we have $\mathcal{U}(n) \models \phi$ iff $\phi \in$ **IPC**. Therefore, $L(\mathcal{U}(n))$ is n-conservative over **IPC**.

Let L be an n-conservative extension of **IPC**. If $L \not\subseteq L(\mathcal{U}(n))$, then there exists a formula ϕ such that $\phi \in L$ and $\phi \notin L(\mathcal{U}(n))$. Therefore, there exists $x \in \mathcal{U}(n)$ such that $x \not\models \phi$. Let $\mathfrak{F}_x$ be the rooted upset of $\mathcal{U}(n)$ generated by x. Then $\mathfrak{F}_x$ is finite and $\mathfrak{F}_x \not\models \phi$. Let also $\chi(\mathfrak{F}_x)$ denote the de Jongh formula of $\mathfrak{F}_x$.[5] By the definition of the de Jongh formula (see §5.2) $\chi(\mathfrak{F}_x)$ is in n variables. If $\chi(\mathfrak{F}_x) \notin L$, then $\mathfrak{F}_x$ is an L-frame refuting ϕ, which contradicts $\phi \in L$. Therefore, $\chi(\mathfrak{F}_x) \in L$. But then $\chi(\mathfrak{F}_x) \in$ **IPC** as L is n-conservative over **IPC**, which is obviously false. Thus, $L \subseteq L(\mathcal{U}(n))$ and $L(\mathcal{U}(n))$ is the greatest n-conservative extension of **IPC**. □

Corollary 116. **RN** *is the greatest* 1*-conservative extension of* **IPC**.

Next, using the technique of Jankov formulas, we will describe the finite rooted frames of **RN**.

Theorem 117. *A finite rooted frame $\mathfrak{F}$ is an* **RN***-frame iff it is a p-morphic image of a generated subframe of $\mathcal{RN}$.*

Proof. Obviously, if $\mathfrak{F}$ is p-morphic image of a generated subframe of $\mathcal{RN}$, then $\mathfrak{F}$ is an **RN**-frame (p-morphisms and generated subframes preserve the validity of formulas). Now suppose $\mathfrak{F}$ is an **RN**-frame and is not a p-morphic image of a generated subframe of $\mathcal{RN}$. Then, by Theorem 95, $\mathcal{RN} \models \chi(\mathfrak{F})$. Therefore, since $\mathfrak{F}$ is an **RN**-frame, we have that $\mathfrak{F} \models \chi(\mathfrak{F})$, which is a contradiction. □

Thus, for characterizing the finite rooted **RN**-frames all we need to do is to characterize those finite rooted frames that are p-morphic images of generated subframes of $\mathcal{RN}$. For this purpose we will need the following definition.

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be two Kripk frames. We define the sum $\mathfrak{F} \oplus \mathfrak{F}'$ of $\mathfrak{F}$ and $\mathfrak{F}'$ as the frame $(W \uplus W', S)$ where

- xSy if $x, y \in W$ and xRy.

[5] Note that in this case it is essential that we take the de Jongh formula. This ensures us that this formula is in n propositional variables.

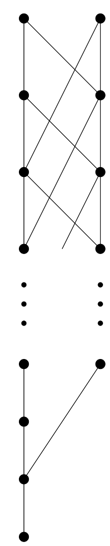


Figure 11: The frame $\mathfrak{H}$

- xSy if $x, y \in W'$ and $xR'y$.
- xSy if $x \in W$ and $y \in W'$.

In other words we put $\mathfrak{F}'$ on the top of $\mathfrak{F}$. Now we are ready to describe all the finite rooted frames of **RN**. The proof is somewhat long, so we will skip it here. Let $\Phi(\mathbf{2})$ denote the frame consisting of one reflexive point. Let also $\Phi(\mathbf{4})$ denote the disjoin union of two reflexive points. The reason we use this notation is the connection of these frames with two and four element Boolean algebras (see Exercise 59).

Theorem 118. *A finite rooted frame $\mathfrak{F}$ is an* **RN***-frame iff $\mathfrak{F}$ is isomorphic to $\mathcal{RN}_k \oplus (\bigoplus_{i=1}^{n} \mathfrak{F}_i)$, where k is even, and each $\mathfrak{F}_i$ is isomorphic to $\Phi(\mathbf{2})$ or $\Phi(\mathbf{4})$.*

Proof. See [3, Corollary 4.2.10]. □

The logic of the Rieger-Nishimura ladder has a very specific and rare property. We will again skip the proof. For the complete proof we refer to [3, Theorem 4.4.13].

Theorem 119. *Every extension of* **RN** *has the finite model property.*

We will close this section by giving an example of a logic of one infinite frame of width 2 that contains **RN** as an extension and does not have the finite model property. We will again use the Jankov formulas for this purpose.

Let $\mathfrak{H}$ be isomorphic to the frame $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \mathcal{RN}$ and let $L = Log(\mathfrak{H})$.

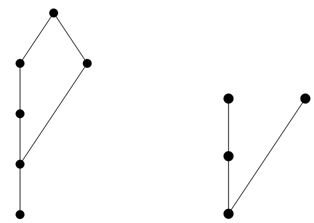


Figure 12: The frames $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \Phi(\mathbf{2})$ and $\mathcal{RN}_4$

Theorem 120. *A finite rooted frame $\mathfrak{F}$ is an L-frame iff $\mathfrak{F}$ is an* **RN***-frame or is isomorphic to $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \bigoplus_{i=1}^{n} \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is isomorphic to $\Phi(\mathbf{2})$ or $\Phi(\mathbf{4})$.*

Proof. Follows from Theorem 118. □

Theorem 121. *L does not have the finite model property.*

Proof. We will construct a formula which is refuted in $\mathfrak{H}$ and is valid on every finite L-frame. Consider the formula $\phi = \chi(\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \Phi(\mathbf{2})) \vee \chi(\mathcal{RN}_4)$ with disjoint variables. The frames $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \Phi(\mathbf{2})$ and $\mathcal{RN}_4$ are shown in Figure 12. Then $\mathfrak{H} \not\models \phi$. To see this, observe that $\mathcal{RN}_4$ is a generated subframe of $\mathfrak{G}$ and $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \Phi(\mathbf{2})$ is a p-morphic image of $\mathfrak{H}$. We leave it to the reader to verify that if a finite rooted L-frame, as described in Theorem 120, contains $\mathcal{RN}_4$ as a generated subframe, then $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \Phi(\mathbf{2})$ cannot be its p-morphic image, and vice versa if there is a p-morphism from a finite L-frame $\mathfrak{F}$ onto $\Phi(\mathbf{2}) \oplus \mathcal{RN}_4 \oplus \Phi(\mathbf{2})$, then $\mathcal{RN}_4$ cannot be a generated subframe of $\mathfrak{F}$. Therefore, no finite rooted L frame refutes ϕ. Thus, L does not have the finite model property. □

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