Frame based formulas for intermediate logics

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Abstract

In this paper we define a new notion of frame based formulas. We show that the well-known examples of formulas arising from a finite frame, such as the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas, are all particular cases of the frame based formulas. We give a criterion for an intermediate logic to be axiomatizable by the frame based formulas and use this criterion to obtain a simple proof that every locally tabular intermediate logic is axiomatizable by the Jankov-de Jongh formulas. We also show that not every intermediate logic is axiomatizable by the frame based formulas.

1 Introduction

Intermediate logics are the logics in between the classical propositional calculus **CPC** and the intuitionistic propositional calculus **IPC**. One of the main tools for studying intermediate logics is formulas that arise from finite frames. The first such formulas were constructed by Jankov [11] and de Jongh [14]. Although syntactically different, both types of formulas have the same semantic properties. We call them the *Jankov-de Jongh formulas*. The Jankov-de Jongh formula of a finite rooted frame $\mathfrak{F}$ is valid on a frame $\mathfrak{G}$ if and only if $\mathfrak{G}$ does not contain $\mathfrak{F}$ as a p-morphic image of a generated subframe. Jankov [12] used these formulas to prove that there are continuum many intermediate logics, and de Jongh [14] applied them to show that intuitionistic logic is the only intermediate logic that satisfies a syntactic condition formulated in terms of the Kleene slash. In modal logic analogues of Jankov-de Jongh formulas were introduced by Fine [9]. They are called *Jankov-Fine formulas* [5]. Jankov [12] also used his formulas to construct intermediate logics without the finite model property and to introduce the so-called splitting technique for lattices of intermediate logics. In fact, using these formulas Jankov [11] proved that every finite subdirectly irreducible Heyting algebra generates a splitting variety. McKenzie [19] developed splitting techniques for varieties of lattices. Blok [6] used the splitting technique to obtain powerful results on the degree of incompleteness of modal logics. Further investigations of splittings in modal and intuitionistic logics were carried out by Rautenberg [20, 21, 22], Kracht [15, 16, 17] and Wolter [25, 27]. Wolter [26] also investigated splittings for temporal logics.

The main application of Jankov-de Jongh formulas is that they provide a useful tool for axiomatizing intermediate logics. Large classes of intermediate logics are axiomatizable by these formulas. However, it turns out that there are logics that cannot be axiomatized by the Jankov-de Jongh formulas. Fine [10] and Zakharyaschev [30] defined another type of frame based formulas, the *subframe formulas*, originally for modal logic. Zakharyaschev [28, 29, 31] defined an analogues notion for intermediate logics. Zakharyaschev [28, 29, 31, 33] also defined the *cofinal subframe formulas* for intermediate and modal logics. Subframe formulas and cofinal subframe formulas are quite closely related to the Jankov-de Jongh formulas and are also used to axiomatize logics. There are intermediate logics that cannot be axiomatized by the Jankov-de Jongh formulas, but are axiomatizable by subframe and cofinal subframe formulas and vice versa. Finally, Zakharyaschev [29, 30, 32] defined the so-called *canonical formulas* and proved that every intermediate and modal logic above **K4** is axiomatizable by the canonical formulas. Canonical formulas, however, are not particular cases of the frame based formulas (as we will see below). They have two parameters, the finite rooted frame and a set of its antichains, whereas the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas have only one parameter—a finite rooted frame.

In this paper we define a new notion of a *frame based formula* which generalizes the notions of the Jankov-de Jongh, subframe and cofinal subframe formulas. We define frame based formulas for every order $\trianglelefteq$ on the class of descriptive frames. We call $\trianglelefteq$ a *frame order*. The Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas become examples of the frame based formulas for particular frame orders. We also give a general criterion when a given logic is axiomatizable by frame-based formulas. This gives us as a corollary criteria for a logic to be axiomatizable by the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas. As a result, we derive simple proofs that every locally tabular logic is axiomatizable by the Jankov-de Jongh formulas, and that every tabular logic is finitely axiomatizable by the Jankov-de Jongh formulas. At the end of the paper we prove that there exist logics that are not axiomatizable by the frame based formulas. This also explains that in order to axiomatize all the intermediate logics by the formulas arising from finite frames we need to consider frame based formulas with an additional parameter, as in Zakharyaschev's canonical formulas. We note that the results presented in this paper are formulated for intermediate logics, but they can be generalized to transitive modal logics.

2 Descriptive frames for intuitionistic logic

For the basic facts about intuitionistic propositional calculus **IPC** including its Kripke and algebraic semantics we refer to [7], [8] and [4]. Let $\mathfrak{F} = (W, R)$ be a partially ordered set (i.e., an intuitionistic Kripke frame). For every $w \in W$ and $U \subseteq W$ let

$$R(w) = \{v \in W : wRv\},$$

$$R^{-1}(w) = \{v \in W : vRw\},$$

$$R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w).$$

Recall also that a subset $U \subseteq W$ of a Kripke frame $\mathfrak{F} = (W, R)$ is an *upset* if $w \in U$ and wRv imply $v \in U$. Next we recall from [7, §8.1 and 8.4] the definitions of general frames and descriptive frames.

Definition 2.1. *An* intuitionistic general frame *or simply a* general frame *is a triple* $\mathfrak{F} = (W, R, \mathcal{P})$*, where* (W, R) *is an intuitionistic Kripke frame and* $\mathcal{P}$ *is a set of upsets such that* $\emptyset$ *and* W *belong to* $\mathcal{P}$*, and* $\mathcal{P}$ *is closed under* $\cup$, $\cap$ *and* $\to$ *defined by*

$$U_1 \to U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W \setminus R^{-1}(U_1 \setminus U_2).$$

Note that every Kripke frame can be seen as a general frame, where $\mathcal{P}$ is the set of all upsets of $\mathfrak{F}$.

Definition 2.2. *Let* $\mathfrak{F} = (W, R, \mathcal{P})$ *be a general frame.*

1. $\mathfrak{F}$ *is called refined if for every* $w, v \in W$*:* $\neg(wRv)$ *implies that there is* $U \in \mathcal{P}$ *such that* $w \in U$ *and* $v \notin U$*.*

2. $\mathfrak{F}$ *is called compact if for every* $\mathcal{X} \subseteq \mathcal{P}$ *and* $\mathcal{Y} \subseteq \{W \setminus U : U \in \mathcal{P}\}$*, if* $\mathcal{X} \cup \mathcal{Y}$ *has the finite intersection property (that is, every intersection of finitely many elements of* $\mathcal{X} \cup \mathcal{Y}$ *is nonempty) then* $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$*.*

3. $\mathfrak{F}$ *is called descriptive if it is refined and compact.*

We call the elements of $\mathcal{P}$ *admissible sets.*

Definition 2.3. *A descriptive frame* $\mathfrak{F} = (W, R, \mathcal{P})$ *is called* rooted *if there exists* $w \in W$ *such that* $R(w) = W$ *and* $W \setminus \{w\} \in \mathcal{P}$*.*

We recall that a descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *finitely generated* if the (Heyting) algebra $(\mathcal{P}, \cup, \cap, \to, \emptyset)$ is finitely generated. The detailed description of the structure of finitely generated descriptive frames can be found in e.g. [7, Section 8] or [4, Section 3.2]. The main property of finitely generated descriptive frames is that every intermediate logic is complete with respect to them; see, e.g. [7, Theorem 8.36].

Theorem 2.4. *Every intermediate logic* L *is complete with respect to its finitely generated rooted descriptive frames.*

Definition 2.5. *Let* L *be an intermediate logic.*

1. *A descriptive frame* $\mathfrak{F}$ *is called* an L-frame *if* $\mathfrak{F}$ *validates all the theorems of* L*.*

2. *Let* $\mathbb{FG}(L)$ *denote the set of all finitely generated rooted descriptive* L*-frames modulo isomorphism.*

3. *Let* $\mathbf{F}_L$ *denote the set of all finite rooted* L*-frames modulo isomorphism.*

Then $\mathbf{F}_{\mathbf{IPC}}$ is the set of all finite rooted frames modulo isomorphism and every logic L is complete with respect to $\mathbb{FG}(L)$.

For the definition of p-morphisms, subframes, generated subframes and cofinal subframes of descriptive frames we refer to e.g. [7] or [4]. Next we recall the Jankov-de Jongh theorem. We refer to e.g. [7, Proposition 9.41] and [4, Theorem 3.3.3] for two different proofs of this theorem.

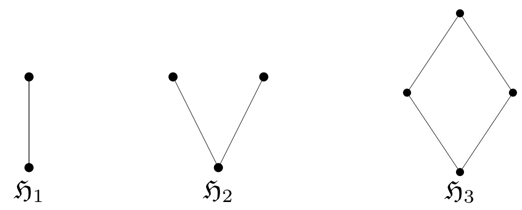

Figure 1: The frames $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$

Theorem 2.6. *For every finite rooted frame $\mathfrak{F}$ there exists a formula $\chi(\mathfrak{F})$, which we call* the Jankov-de Jongh formula of $\mathfrak{F}$,[1] *such that for every descriptive frame $\mathfrak{G}$:*

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \quad \textit{iff } \mathfrak{F} \textit{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Example 2.7. We recall few examples of intermediate logics axiomatizable by the Jankov-de Jongh formulas, for the proofs we refer to e.g., [7, Section 9.4]. Recall that **KC** is an intermediate logics of all directed frames and **LC** is the logic of all linear frames. For every intermediate logic L, we let $L + \varphi$ denote the smallest intermediate logic containing $L \cup \{\varphi\}$. Then

$$\mathbf{CPC} = \mathbf{IPC} + \chi(\mathfrak{H}_1), \quad \mathbf{KC} = \mathbf{IPC} + \chi(\mathfrak{H}_2), \quad \mathbf{LC} = \mathbf{IPC} + \chi(\mathfrak{H}_2) + \chi(\mathfrak{H}_3),$$

where $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ are the frames shown in Figure 1 and **CPC** is the classical propositional calculus. For more examples of intermediate logics axiomatized by the Jankov-de Jongh formulas we refer to [7, Section 9.4] (see also Section 3.2 below).

Next we discuss subframe and cofinal subframe formulas. For two different proofs of the next theorem we refer to [7, Section 9.4] and [4, Section 3.3.3].

Theorem 2.8. *Let $\mathfrak{G} = (W', R', \mathcal{P}')$ be a descriptive frame and let $\mathfrak{F} = (W, R)$ be a finite rooted frame. Then*

1. $\mathfrak{G} \not\models \beta(\mathfrak{F})$ *iff $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{G}$.*
2. $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ *iff $\mathfrak{F}$ is a p-morphic image of a cofinal subframe of $\mathfrak{G}$.*

The formulas $\beta(\mathfrak{F})$ and $\gamma(\mathfrak{F})$ are called the *subframe formula of $\mathfrak{F}$ and cofinal subframe formula of $\mathfrak{F}$*, respectively. For an overview on subframe and cofinal subframe formulas we refer to [7, §9.4]. In [4, Section 3.3.3] the subframe and cofinal subframe formulas are defined differently from [7] and are connected to the NNIL formulas of [23], i.e., the formulas that are preserved under submodels. For an algebraic approach to subframe formulas we refer to [2].

Example 2.9. The logic **LC** of all linear frames can be axiomatized by adding to **IPC** a single subframe formula. In fact, $\mathbf{LC} = \mathbf{IPC} + \beta(\mathfrak{H}_2)$, where $\mathfrak{H}_2$ is the frame shown in

[1] In fact, the Jankov formulas and de Jongh formulas have a different syntactic shape. For the similarities and differences between Jankov and de Jongh formulas we refer to [4, Remark 3.3.5].

Figure 1. (Note that in order to axiomatize **LC** by Jankov-de Jongh formulas we need to use two Jankov-de Jongh formula, see Example 2.7). Moreover, for every $n \in \omega$ the logic of all frames of depth n and the logic of all frames of width n are axiomatized by subframe formulas; see, e.g., [7, Section 9.4] for details. We also note that the logic **KC** of all directed frames cannot be axiomatized by subframe formulas.

Example 2.10. The logic **KC** of all directed frames is an example of a logic axiomatized by a single cofinal subframe formula. In fact,

$$\mathbf{KC} = \mathbf{IPC} + \beta'(\mathfrak{H}_2),$$

where $\mathfrak{H}_2$ is the frame shown in Figure 1. Moreover, there are continuum many logics axiomatized by cofinal subframe formulas that cannot be axiomatized by subframe formulas; see, e.g., [7, Corollary 11.23] for details.

3 Frame based formulas

This section is the main part of the paper. In it we will treat the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas in a uniform framework. We give a definition of frame based formulas and show that these three types of formulas are particular cases of frame based formulas. We prove a criterion for recognizing whether an intermediate logic is axiomatizable by frame based formulas. Using this criterion we show that every locally tabular intermediate logic is axiomatizable by the Jankov-de Jongh formulas. We also give a simple proof of a well-known result that every tabular logic is finitely axiomatizable by these formulas. We also recall the definitions of subframe logics and cofinal subframe logics and as a corollary of the main criterion obtain that a logic is axiomatizable by subframe formulas iff it is a subframe logic and that a logic is axiomatizable by cofinal subframe formulas iff it is a cofinal subframe logic. At the end of the section we show that there are intermediate logics that are not axiomatizable by frame based formulas.

3.1 Axiomatizations

We define three relations on descriptive frames.

Definition 3.1. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be descriptive frames. We say that*

1. $\mathfrak{F} \leq \mathfrak{G}$ *iff* $\mathfrak{F}$ *is a p-morphic image of a generated subframe of* $\mathfrak{G}$.
2. $\mathfrak{F} \preccurlyeq \mathfrak{G}$ *iff* $\mathfrak{F}$ *is a p-morphic image of a subframe of* $\mathfrak{G}$.
3. $\mathfrak{F} \preccurlyeq' \mathfrak{G}$ *iff* $\mathfrak{F}$ *is a p-morphic image of a cofinal subframe of* $\mathfrak{G}$.

We write $\mathfrak{F} < \mathfrak{G}$, $\mathfrak{F} \prec \mathfrak{G}$ and $\mathfrak{F} \prec' \mathfrak{G}$ if $\mathfrak{F} \leq \mathfrak{G}$, $\mathfrak{F} \preccurlyeq \mathfrak{G}$ and $\mathfrak{F} \preccurlyeq' \mathfrak{G}$, respectively, and $\mathfrak{F}$ is not isomorphic to $\mathfrak{G}$. The next proposition discusses some basic properties of $\leq$, $\preccurlyeq$ and $\preccurlyeq'$. The proof is simple and we will skip it.

Proposition 3.2.

1. *Each of $\leq$, $\preccurlyeq$ and $\preccurlyeq'$ is reflexive and transitive.*

2. *If we restrict ourselves to finite frames, then each of $\leq$, $\preccurlyeq$ and $\preccurlyeq'$ is a partial order.*

3. *In the infinite case none of $\leq$, $\preccurlyeq$, $\preccurlyeq'$ is in general anti-symmetric.*

4. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two finite rooted frames. Let $\mathfrak{G}$ be an arbitrary descriptive frame. Then*

   (a) $\mathfrak{F} \leq \mathfrak{F}'$ *and* $\mathfrak{G} \models \chi(\mathfrak{F})$ *imply* $\mathfrak{G} \models \chi(\mathfrak{F}')$.

   (b) $\mathfrak{F} \preccurlyeq \mathfrak{F}'$ *and* $\mathfrak{G} \models \beta(\mathfrak{F})$ *imply* $\mathfrak{G} \models \beta(\mathfrak{F}')$.

   (c) $\mathfrak{F} \preccurlyeq' \mathfrak{F}'$ *and* $\mathfrak{G} \models \gamma(\mathfrak{F})$ *imply* $\mathfrak{G} \models \gamma(\mathfrak{F}')$.

Note that Theorems 2.6 and 2.8 can be formulated in terms of the relations $\leq$, $\preccurlyeq$ and $\preccurlyeq'$ as follows:

Theorem 3.3. *For every finite rooted frame $\mathfrak{F}$ there exist formulas $\chi(\mathfrak{F})$, $\beta(\mathfrak{F})$ and $\gamma(\mathfrak{F})$ such that for every descriptive frame $\mathfrak{G}$:*

1. $\mathfrak{G} \not\models \chi(\mathfrak{F})$ *iff* $\mathfrak{F} \leq \mathfrak{G}$.

2. $\mathfrak{G} \not\models \beta(\mathfrak{F})$ *iff* $\mathfrak{F} \preccurlyeq \mathfrak{G}$.

3. $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ *iff* $\mathfrak{F} \preccurlyeq' \mathfrak{G}$.

Proposition 3.2 and Theorem 3.3 clearly indicate that these three types of formulas can be treated in a uniform framework. Next we give a general definition of frame based formulas and show that the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas are particular cases of frame based formulas. Let $\trianglelefteq$ be a relation on $\mathbb{FG}(L)$. We write $\mathfrak{F} \triangleleft \mathfrak{G}$ if $\mathfrak{F} \trianglelefteq \mathfrak{G}$ and $\mathfrak{F}$ and $\mathfrak{G}$ are not isomorphic.

Definition 3.4. *We call a reflexive and transitive relation $\trianglelefteq$ on $\mathbb{FG}(\mathbf{IPC})$ a* frame order *if the following two conditions are satisfied:*

1. *For every $\mathfrak{F}, \mathfrak{G} \in \mathbb{FG}(L)$, $\mathfrak{G} \in \mathbf{F_{IPC}}$ and $\mathfrak{F} \triangleleft \mathfrak{G}$ imply $|\mathfrak{F}| < |\mathfrak{G}|$.*

2. *For every finite rooted frame $\mathfrak{F}$ there exists a formula $\alpha(\mathfrak{F})$ such that for every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$*

$$\mathfrak{G} \not\models \alpha(\mathfrak{F}) \quad \textit{iff} \quad \mathfrak{F} \trianglelefteq \mathfrak{G}.$$

We call the formula $\alpha(\mathfrak{F})$ the frame based formula for $\trianglelefteq$ *of $\mathfrak{F}$.*

Obviously, the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas are frame based formulas for $\leq$, $\preccurlyeq$ and $\preccurlyeq'$, respectively. Next we prove some auxiliary lemmas.

Lemma 3.5.

1. *The restriction of $\trianglelefteq$ to $\mathbf{F_{IPC}}$ is a partial order.*

2. $\mathbf{F_{IPC}}$ *is a* $\trianglelefteq$*-downset, i.e.,* $\mathfrak{F} \in \mathbf{F_{IPC}}$ *and* $\mathfrak{F}' \trianglelefteq \mathfrak{F}$ *imply* $\mathfrak{F}' \in \mathbf{F_{IPC}}$.

Proof. The relation $\trianglelefteq$ is reflexive and transitive by definition. That the restriction of $\trianglelefteq$ is anti-symmetric on finite frames follows from Definition 3.4(1). That $\mathbf{F_{IPC}}$ is a $\trianglelefteq$-downset, also follows immediately from Definition 3.4(1). □

Lemma 3.6. *Let* $\mathfrak{F}$ *and* $\mathfrak{F}'$ *be finite rooted frames.*

$$\text{If } \mathfrak{F} \trianglelefteq \mathfrak{F}', \text{ then } \mathbf{IPC} + \alpha(\mathfrak{F}) \vdash \alpha(\mathfrak{F}').$$

Proof. Let $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$ and $\mathfrak{G} \not\models \alpha(\mathfrak{F}')$, then $\mathfrak{F}' \trianglelefteq \mathfrak{G}$. By the transitivity of $\trianglelefteq$ we then have that $\mathfrak{F} \trianglelefteq \mathfrak{G}$ and $\mathfrak{G} \not\models \alpha(\mathfrak{F})$. By Corollary 2.4 we get that $\mathbf{IPC} + \alpha(\mathfrak{F}) \vdash \alpha(\mathfrak{F}')$. □

Definition 3.7. *Let L be an intermediate logic and let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. We say that L is* axiomatizable *by frame based formulas for $\trianglelefteq$ if there exists a family $\{\mathfrak{F}_i\}_{i \in I}$ of finite rooted frames such that $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}_i) : i \in I\}$.*

Let $\mathfrak{F} = (W, R)$ be a (descriptive) frame. We call a point w of $\mathfrak{F}$ *maximal* (*minimal*) if for every $v \in W$ we have that wRv (vRw) implies $w = v$. For every frame $\mathfrak{F}$ we let $max(\mathfrak{F})$ and $min(\mathfrak{F})$ denote the sets of all maximal and minimal points of $\mathfrak{F}$, respectively. For every subset U of $\mathbb{FG}(L)$ we let $min_{\trianglelefteq}(U)$ denote the set of the $\trianglelefteq$-minimal elements of U.

Definition 3.8. *Let L be an intermediate logic. We let*

$$\mathbf{M}(L, \trianglelefteq) := min_{\trianglelefteq}(\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L))$$

We give a criterion recognizing whether an intermediate logic is axiomatized by frame based formulas.

Theorem 3.9. *Let L be an intermediate logic and let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. Then L is axiomatizable by frame based formulas for $\trianglelefteq$ iff the following two conditions are satisfied.*

1. *$\mathbb{FG}(L)$ is a $\trianglelefteq$-downset. That is, for every $\mathfrak{F}, \mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$, if $\mathfrak{G} \in \mathbb{FG}(L)$ and $\mathfrak{F} \trianglelefteq \mathfrak{G}$, then $\mathfrak{F} \in \mathbb{FG}(L)$.*

2. *For every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{F} \trianglelefteq \mathfrak{G}$.*

Moreover, if L is axiomatizable by frame-based formulas for $\trianglelefteq$, then $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$.

Proof. Suppose L is axiomatizable by frame based formulas for $\trianglelefteq$. Then $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}_i) : i \in I\}$, for some family $\{\mathfrak{F}_i\}_{i \in I}$ of finite rooted frames. First we show that $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset. Suppose, for some $\mathfrak{F}, \mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$ we have $\mathfrak{G} \in \mathbb{FG}(L)$ and $\mathfrak{F} \trianglelefteq \mathfrak{G}$. Assume that $\mathfrak{F} \notin \mathbb{FG}(L)$. Then there exists $i \in I$ such that $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i)$. Therefore, by Definition 3.4(2), $\mathfrak{F}_i \trianglelefteq \mathfrak{F}$. By the transitivity of $\trianglelefteq$, we have that $\mathfrak{F}_i \trianglelefteq \mathfrak{G}$, which implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i)$, a contradiction. Thus, $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset.

Suppose there exist $i, j \in I$ such that $i \neq j$ and $\mathfrak{F}_i \trianglelefteq \mathfrak{F}_j$. Then by Lemma 3.6, $\mathbf{IPC} + \alpha(\mathfrak{F}_i) \vdash \alpha(\mathfrak{F}_j)$. Therefore, we can exclude $\alpha(\mathfrak{F}_j)$ from the axiomatization of L. So it is sufficient to consider only $\trianglelefteq$-minimal elements of $\{\mathfrak{F}_i\}_{i\in I}$. (By Definition 3.4(1), the set of $\trianglelefteq$-minimal elements of an infinite set of finite rooted frames is non-empty.) Thus, without loss of generality we may assume that $\neg(\mathfrak{F}_i \trianglelefteq \mathfrak{F}_j)$, for $i \neq j$. To verify the second condition suppose $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. Then $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i)$ for some $i \in I$, which implies $\mathfrak{F}_i \trianglelefteq \mathfrak{G}$. Hence, if we show that $\mathfrak{F}_i \in \mathbf{M}(L, \trianglelefteq)$, then condition (2) of the theorem is satisfied.

We now prove that every $\mathfrak{F}_i$ belongs to $\mathbf{M}(L, \trianglelefteq)$. By the reflexivity of $\trianglelefteq$, we have $\mathfrak{F}_i \not\models \alpha(\mathfrak{F}_i)$ for all $i \in I$. Therefore, $\mathfrak{F}_i \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. Now suppose $\mathfrak{F} \triangleleft \mathfrak{F}_i$. By Definition 3.4(1), $|\mathfrak{F}| < |\mathfrak{F}_i|$ implying that $\mathfrak{F}$ is finite. By Lemma 3.5, $\trianglelefteq$ is anti-symmetric on finite frames, therefore $\neg(\mathfrak{F}_i \trianglelefteq \mathfrak{F})$. If $\mathfrak{F}_j \trianglelefteq \mathfrak{F}$, for some $j \in I$ and $j \neq i$, then by the transitivity of $\trianglelefteq$ we have $\mathfrak{F}_j \trianglelefteq \mathfrak{F}_i$, which is a contradiction. Therefore, $\neg(\mathfrak{F}_j \trianglelefteq \mathfrak{F})$, for all $j \in I$. So $\mathfrak{F} \models \alpha(\mathfrak{F}_j)$, for all $j \in I$, which implies that $\mathfrak{F} \in \mathbb{FG}(L)$ and that $\mathfrak{F}_i$ is a minimal element of $\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. Thus, $\mathfrak{F}_i \in \mathbf{M}(L, \trianglelefteq)$, condition (2) is satisfied.

For the right to left direction, first note that, by our assumption, $\mathbf{M}(L, \trianglelefteq)$ consists of only finite frames. We show that $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. We prove this by showing that the finitely generated rooted descriptive frames of L and of $\mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$ coincide. Let $\mathfrak{G} \in \mathbb{FG}(L)$, then since $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset, for every $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ we have that $\neg(\mathfrak{F} \trianglelefteq \mathfrak{G})$ and hence $\mathfrak{G} \models \alpha(\mathfrak{F})$. On the other hand, if $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, then by our assumption there exists $\mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)$ such that $\mathfrak{F} \trianglelefteq \mathfrak{G}$. Therefore, $\mathfrak{G} \not\models \alpha(\mathfrak{F})$ and $\mathfrak{G}$ is not a frame for $\mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. Since every intermediate logic is complete with respect to its finitely generated rooted descriptive frames (Corollary 2.4), we obtain that $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. This also shows that if L is axiomatizable by frame based formulas for $\trianglelefteq$, then $L = \mathbf{IPC} + \{\alpha(\mathfrak{F}) : \mathfrak{F} \in \mathbf{M}(L, \trianglelefteq)\}$. □

Next we apply this criterion to the Jankov-de Jongh formulas, subframe formulas and cofinal subframe formulas.

Theorem 3.10. *Let L be an intermediate logic. Then*

1. *$\mathbb{FG}(L)$ is a $\leq$-downset.*
2. *For every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \preccurlyeq)$ such that $\mathfrak{F} \preccurlyeq \mathfrak{G}$.*
3. *For every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \preccurlyeq')$ such that $\mathfrak{F} \preccurlyeq' \mathfrak{G}$.*

Proof. (1) is trivial since generated subframes and p-morphisms preserve the validity of formulas. The proofs of (2) and (3) are quite involved, we will skip them here. For the proofs we refer to [7, Theorem 11.15]. □

These results allow us to obtain the following criterion.

Corollary 3.11. *Let L be an intermediate logic.*

1. *L is axiomatized by the Jankov-de Jongh formulas iff for every frame $\mathfrak{G}$ in $\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$.*
2. *L is axiomatizable by subframe formulas iff $\mathbb{FG}(L)$ is a $\preccurlyeq$-downset.*
3. *L is axiomatizable by cofinal subframe formulas iff $\mathbb{FG}(L)$ is a $\preccurlyeq'$-downset.*

Proof. The result is an immediate consequence of Theorems 3.9 and 3.10. □

Definition 3.12. *Let L be an intermediate logic.*

1. *L is called a* subframe logic *if for every L-frame $\mathfrak{G}$, every subframe $\mathfrak{G}'$ of $\mathfrak{G}$ is also an L-frame.*

2. *L is called a* cofinal subframe logic *if for every L-frame $\mathfrak{G}$, every cofinal subframe $\mathfrak{G}'$ of $\mathfrak{G}$ is also an L-frame.*

For the next theorem consult [7, Theorem 11.21].

Corollary 3.13. *Let L be an intermediate logic.*

1. *L is axiomatizable by subframe formulas iff L is a subframe logic.*

2. *L is axiomatizable by cofinal subframe formulas iff L is a cofinal subframe logic.*

Proof. Since every intermediate logic L is complete with respect to $\mathbb{FG}(L)$, it is easy to see that L is a subframe logic iff $\mathbb{FG}(L)$ is a $\preccurlyeq$-downset and L is a cofinal subframe logic iff $\mathbb{FG}(L)$ is a $\preccurlyeq'$-downset. The proof now follows from Corollary 3.11. □

It turns out that not every intermediate logic is axiomatizable by Jankov-de Jongh formulas and (cofinal) subframe formulas. Moreover, there exist logics axiomatizable by (cofinal) subframe formulas that are not axiomatizable by Jankov-de Jongh formulas and vice versa. For an example of an intermediate logic axiomatizable by (cofinal) subframe formulas that are not axiomatizable by Jankov-de Jongh formulas we refer to [7, Proposition 9.50] (see also [4, 3.4.31 and 3.4.32]). In Section 3.2 we will construct a simple example of a logic that is axiomatizable by Jankov-de Jongh formulas but is not axiomatizable by (cofinal) subframe formulas.

Next we discuss a method for constructing continuum many intermediate logics using frame based formulas. Let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. A set of frames Δ is called an $\trianglelefteq$-*antichain* if for every distinct $\mathfrak{F}, \mathfrak{G} \in \Delta$ we have $\neg(\mathfrak{F} \trianglelefteq \mathfrak{G})$ and $\neg(\mathfrak{G} \trianglelefteq \mathfrak{F})$. For every set Γ of frames. Let $Log(\Gamma)$ be the logic of Γ, that is, $Log(\Gamma) = \{\phi : \mathfrak{F} \models \phi$ for every $\mathfrak{F} \in \Gamma\}$. We also write $Log(\mathfrak{F})$ instead of $Log(\{\mathfrak{F}\})$.

Theorem 3.14. *Let $\Delta = \{\mathfrak{F}_i\}_{i \in \omega}$ be an $\trianglelefteq$-antichain of finite rooted frames. For all $\Gamma_1, \Gamma_2 \subseteq \Delta$, if $\Gamma_1 \neq \Gamma_2$, then $Log(\Gamma_1) \neq Log(\Gamma_2)$.*

Proof. Without loss of generality we may assume that $\Gamma_1 \not\subseteq \Gamma_2$. This means that there is $\mathfrak{F} \in \Gamma_1$ such that $\mathfrak{F} \notin \Gamma_2$. Consider the frame based formula $\alpha(\mathfrak{F})$. Then, by the reflexivity of $\trianglelefteq$, we have $\mathfrak{F} \not\models \alpha(\mathfrak{F})$. Hence, $\alpha(\mathfrak{F}) \notin Log(\Gamma_1)$. Now we show that $\alpha(\mathfrak{F}) \in Log(\Gamma_2)$. Suppose $\alpha(\mathfrak{F}) \notin Log(\Gamma_2)$. Then there is $\mathfrak{G} \in \Gamma_2$ such that $\mathfrak{G} \not\models \alpha(\mathfrak{F})$. This means that $\mathfrak{F} \trianglelefteq \mathfrak{G}$, which contradicts the fact that Δ forms an $\trianglelefteq$-antichain. Therefore, $\alpha(\mathfrak{F}) \notin Log(\Gamma_1)$ and $\alpha(\mathfrak{F}) \in Log(\Gamma_2)$. Thus, $Log(\Gamma_1) \neq Log(\Gamma_2)$. □

There are many examples of $\leq$-antichains. Here we give a simple example of a $\leq$-antichain first observed by Jankov [12] and de Jongh [14]. Consider the sequence Δ of finite rooted frames shown in Figure 2. For short direct proof of the next lemma we refer to [4, Theorem 3.4.19].

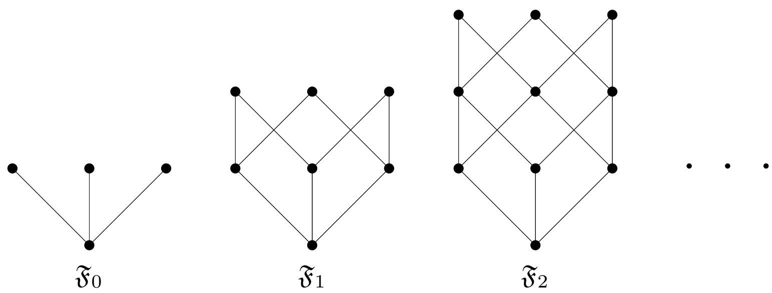

Figure 2: The sequence Δ

Lemma 3.15. Δ *forms an* $\leq$*-antichain.*

As a direct corollary of Theorem 3.14 and Lemma 3.15 we obtain that there are continuum many intermediate logics; a fact first observed by Jankov [12]. For the examples of infinite $\preccurlyeq$ and $\preccurlyeq'$-antichains of finite rooted frames consult [7, Lemma 11.18 and Theorem 11.19].

3.2 Locally tabular and tabular intermediate logics

Next we show that every locally tabular intermediate logic is axiomatizable by the Jankov-de Jongh formulas, and that every tabular logic is finitely axiomatizable by the Jankov-de Jongh formulas. We recall that an intermediate logic L is called *locally tabular* if for every $n \in \omega$ there are only finitely many pairwise non-L-equivalent formulas in n variables. We note that for transitive modal logics the notion of local tabularity coincides with the notion of finite depth; that is, a transitive modal logic is locally tabular iff it is of finite depth; see, e.g., [7, Theorem 12.21]. Moreover, a transitive modal logic is locally tabular iff the 1-generated free algebra of the corresponding variety is finite; see e.g., [7, Corollary 12.22]. Also a normal extension L of the modal logic **S4** is not locally tabular iff the modal logic **Grz.3** of all linear posets contains L; see e.g., [7, Theorem 12.23]. This means that **Grz.3** is the only pre-locally tabular normal extension of **S4**. The notion of local tabularity becomes much more complex for intermediate logics. Although every intermediate logic of finite depth is locally tabular, there exist logics of infinite depth (e.g., the logic **LC** of linear frames) that have infinite depth and still are locally tabular. Mardaev [18] showed that, unlike extensions of **S4**, there are continuum pre-locally tabular intermediate logics. G. Bezhanishvili and Grigolia [3] gave a characterization of locally tabular intermediate logics using coproducts of three element Heyting algebra. They also conjectured that an intermediate logic L is locally tabular iff the 2-generated free algebra in the corresponding variety is finite.

Here we apply our criterion from Corollary 3.11 to prove that every locally tabular intermediate logic is axiomatizable by the Jankov-de Jongh formulas. We will use the following criterion of local tabularity established in [1]. We recall that a descriptive frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *n-generated* if the algebra $(\mathcal{P}, \cup, \cap, \to, \emptyset)$ is an n-generated Heyting algebra.

Theorem 3.16. *A logic L is locally tabular iff the class of rooted finitely generated descriptive L-frames is uniformly locally tabular. That is, for every natural number n there exists a* *natural number $M(n)$ such that for every n-generated rooted descriptive L-frame $\mathfrak{F}$ we have $|\mathfrak{F}| \leq M(n)$.*

We will use the following well-known property of infinite finitely generated descriptive frames. For the proof we refer to e.g. [7, Section 8.7] or [4, Section 3.1].

Lemma 3.17.

1. *For every infinite finitely generated descriptive frame $\mathfrak{G}$, we have*

$$sup\{|\mathfrak{H}| : \mathfrak{H} \textit{ is a finite rooted generated subframe of } \mathfrak{G}\} = \infty.$$

2. *For every n-generated descriptive frame $\mathfrak{G}$, its every generated subframe $\mathfrak{H}$ is also n-generated.*

We will also need the following auxiliary lemma.

Lemma 3.18. *Let L be an intermediate logic. Then*

1. *$(\mathbf{F}_L, \leq)$ is well-founded.*
2. *For every finite rooted frame $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, there exists a finite rooted $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$.*

Proof. (1) The proof follows immediately from the fact that if $\mathfrak{F}, \mathfrak{G} \in \mathbf{F}_L$ then $\mathfrak{F} < \mathfrak{G}$ implies $|\mathfrak{F}| < |\mathfrak{G}|$.

(2) The proof is similar to the proof of (1). □

Theorem 3.19. *Every locally tabular intermediate logic is axiomatized by Jankov-de Jongh formulas.*

Proof. Let L be a locally tabular intermediate logic. By Corollary 3.11(1), we need to show that for every $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$ there exists a finite $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$. Suppose $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$. If $\mathfrak{G}$ is finite, then by Lemma 3.18(2), there exists a finite rooted $\mathfrak{F} \in \mathbf{M}(L, \leq)$ such that $\mathfrak{F} \leq \mathfrak{G}$. Now assume that $\mathfrak{G}$ is infinite. Let $\mathfrak{H}$ be a finite rooted frame such that $\mathfrak{H} < \mathfrak{G}$. If $\mathfrak{H} \in \mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, then by Lemma 3.18(2), there exists $\mathfrak{F} \in \mathbf{M}(L, \leq)$ with $\mathfrak{F} \leq \mathfrak{H}$. Since $\leq$ is transitive, we obtain that $\mathfrak{F} \leq \mathfrak{G}$. Now suppose, for every finite rooted $\mathfrak{H}$ such that $\mathfrak{H} < \mathfrak{G}$ we have $\mathfrak{H} \in \mathbb{FG}(L)$. By Theorem 3.17(1), $sup\{|\mathfrak{H}| : \mathfrak{H}$ is a finite rooted generated subframe of $\mathfrak{G}\} = \infty$. Since $\mathfrak{G}$ is finitely generated, there exists $n \in \omega$ such that $\mathfrak{G}$ is n-generated. Since $\mathfrak{H}$ is a generated subframe of $\mathfrak{G}$, by Theorem 3.17(2), $\mathfrak{H}$ is also n-generated. This means that the set of all rooted finitely generated descriptive L-frames is not uniformly locally finite. By Theorem 3.16, L is not locally tabular, which is a contradiction. Thus, by Corollary 3.11(1) L is axiomatized by the Jankov-de Jongh formulas. □

Remark 3.20. We note that the fact that every locally tabular transitive modal logic is axiomatizable by the Jankov-Fine formulas (modal logic analogues of the Jankov-de Jongh formulas) is much easier to prove. As we mentioned above every locally tabular transitive modal logic has the finite depth and therefore has the finite model property [7, Theorem 8.85]. Therefore, in Corollary 3.11, instead of considering all finitely generated frames, it suffices to consider only finite ones. This immediately implies that the condition of Corollary 3.11 is automatically satisfied. Thus, we derive that all locally tabular transitive modal logics are axiomatizable by Jankov-Fine formulas.

Recall that an intermediate logic L is called *tabular* if there exists a finite (not necessarily rooted) frame $\mathfrak{F}$ such that $L = Log(\mathfrak{F})$. Since every tabular logic is locally tabular, it follows from Theorem 3.19 that every tabular logic is also axiomatizable by the Jankov-de Jongh formulas. Next we show that every tabular logic is in fact finitely axiomatizable by the Jankov-de Jongh formulas. For an alternative proof of the theorem consult [7, Theorem 12.4]. First we prove two auxiliary lemmas.

Lemma 3.21. *For every finite rooted frame $\mathfrak{F}$, consisting of at least two points, there exists a frame $\mathfrak{G}$ and a p-morphism $f : \mathfrak{F} \to \mathfrak{G}$ such that f identifies only two points.*

Proof. If $max(\mathfrak{F})$ contains more than one point, we consider the map that identifies two distinct maximal points of $\mathfrak{F}$. It is easy to check that such a map is a p-morphism. If $max(\mathfrak{F})$ is a singleton set, we consider the second layer of $\mathfrak{F}$. By our assumption the second layer is not empty. If the second layer of $\mathfrak{F}$ consists of one point, then consider the map that identifies the point of the second layer with the maximal point. It is easy to verify that such a map is a p-morphism. If the second layer of $\mathfrak{F}$ consists of at least two points, we consider a map that identifies two points from the second layer. It is a again easy to check that this map is a p-morphism. $\square$

Lemma 3.22. *Let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$. Suppose that $\mathfrak{F}$ is a finite rooted L-frame, where $L = Log(\mathfrak{G})$, for some $\mathfrak{G} \in \mathbb{FG}(\mathbf{IPC})$. Then $\mathfrak{F} \trianglelefteq \mathfrak{G}$.*

Proof. Suppose $\neg(\mathfrak{F} \trianglelefteq \mathfrak{G})$. Then $\mathfrak{G} \models \alpha(\mathfrak{F})$, where $\alpha(\mathfrak{F})$ is the frame based formula for $\trianglelefteq$. Therefore, since $\mathfrak{F}$ is an L-frame, $\mathfrak{F} \models \alpha(\mathfrak{F})$. This is a contradiction since $\trianglelefteq$ is reflexive. $\square$

Theorem 3.23. *Every tabular logic is finitely axiomatizable by Jankov-de Jongh formulas.*

Proof. Let L be tabular. Then $L = Log(\mathfrak{F})$ for some finite frame $\mathfrak{F}$. By Lemma 3.22, for every rooted L-frame $\mathfrak{F}'$ we have $\mathfrak{F}' \leq \mathfrak{F}$. Therefore, if $\mathfrak{F}' \in \mathbf{F}_L$, then $|\mathfrak{F}'| \leq |\mathfrak{F}|$. Hence, every finite rooted L-frame contains at most $|\mathfrak{F}|$ points. We will show that $\mathbf{M}(L, \leq)$ is finite.

Claim 3.24. *For every $\mathfrak{H} \in \mathbf{M}(L, \leq)$ we have $|\mathfrak{H}| \leq |\mathfrak{F}| + 1$.*

Proof. Assume $\mathfrak{H} \in \mathbf{M}(L, \leq)$. If $|\mathfrak{H}| = 1$, then trivially $|\mathfrak{H}| \leq |\mathfrak{F}| + 1$. Now suppose $\mathfrak{H}$ is such that $|\mathfrak{H}| > 1$. Then by Lemma 3.21, there exists a frame $\mathfrak{H}'$ such that $\mathfrak{H}' < \mathfrak{H}$ and $|\mathfrak{H}| = |\mathfrak{H}'| + 1$. If $\mathfrak{H}' \notin \mathbf{F}_L$, then $\mathfrak{H}$ is not a minimal element of $\mathbb{FG}(\mathbf{IPC}) \setminus \mathbb{FG}(L)$, that is, $\mathfrak{H} \notin \mathbf{M}(L, \leq)$, which is a contradiction. Now assume $\mathfrak{H}' \in \mathbf{F}_L$. Then as $\mathfrak{H}'$ is an L-frame, $|\mathfrak{H}'| \leq |\mathfrak{F}|$. Thus, $|\mathfrak{H}| \leq |\mathfrak{F}| + 1$. $\square$

There are only finitely many non-isomorphic frames consisting of m points for $m \in \omega$. Therefore, $\mathbf{M}(L, \leq)$ is finite. Let $\mathbf{M}(L, \leq) = \{\mathfrak{G}_1, \ldots, \mathfrak{G}_k\}$. Then, by Theorem 3.9, we have $L(\mathfrak{F}) = \mathbf{IPC} + \chi(\mathfrak{G}_1) + \cdots + \chi(\mathfrak{G}_k)$. □

Now we are ready to give a simple example of a tabular logic that is not axiomatizable by subframe and cofinal subframe formulas.

Theorem 3.25. *There are intermediate logics that are axiomatizable by Jankov-de Jongh formulas but not axiomatizable by subframe formulas or by cofinal subframe formulas.*

Proof. Let Δ be as in Lemma 3.15. Consider $\mathfrak{F}_i \in \Delta$ such that $i > 0$. Then $L = Log(\mathfrak{F}_i)$ is tabular and by Theorem 3.23, L is finitely axiomatizable by the Jankov-de Jongh formulas. Now we show that L is neither a subframe nor a cofinal subframe logic. It is easy to see that $\mathfrak{F}_0$ is a subframe of $\mathfrak{F}_i$, moreover it is a cofinal subframe. By Lemma 3.22, if $\mathfrak{F}_0$ is an L-frame, then $\mathfrak{F}_0 \leq \mathfrak{F}_i$. This is a contradiction because by Theorem 3.14, Δ is an $\leq$-antichain. Therefore L is neither a subframe nor a cofinal subframe logic and by Corollary 3.13, it is not axiomatizable by subframe formulas and cofinal subframe formulas. □

3.3 Splittings and the finite model property

In this section we overview the connection between frame based formulas, the so-called splittings of lattices of logics and the finite model property. First we recall the definition of a splitting, which was introduced in lattice theory by Whitman [24].

Let A be a lattice and $a, b \in A$. We say that a pair (a, b) *splits* A, if $a \nleq b$ and for every $c \in A$ we have

$$a \leq c \quad \text{or} \quad c \leq b.$$

Then a is called a *splitting element* and b is called a *co-splitting element*.

Theorem 3.26. *Let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$ such that for every intermediate logic L condition (1) of Theorem 3.9 is satisfied. That is, $\mathbb{FG}(L)$ is $\trianglelefteq$-downset for every intermediate logic L. Then for every finite rooted frame $\mathfrak{F}$, the pair $(\mathbf{IPC} + \alpha(\mathfrak{F}), Log(\mathfrak{F}))$ is a splitting pair in the lattice of all intermediate logics.*

Proof. Since $\mathfrak{F} \not\models \alpha(\mathfrak{F})$, we have that $\alpha(\mathfrak{F}) \notin Log(\mathfrak{F})$ and therefore $\mathbf{IPC} + \alpha(\mathfrak{F}) \not\subseteq Log(\mathfrak{F})$. Now let L be an intermediate logic. If $\alpha(\mathfrak{F}) \in L$, then $\mathbf{IPC}+\alpha(\mathfrak{F}) \subseteq L$. So suppose $\alpha(\mathfrak{F}) \notin L$, then there exists a finitely generated rooted L-frame $\mathfrak{G} \in \mathbb{FG}(L)$ such that $\mathfrak{G} \not\models \alpha(\mathfrak{F})$. By Definition 3.4, we obtain that $\mathfrak{F} \trianglelefteq \mathfrak{G}$. By our assumption, $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset, therefore $\mathfrak{F}$ is an L-frame. Thus, $Log(\mathfrak{F}) \supseteq L$. So for every intermediate logic L we have $\mathbf{IPC}+\alpha(\mathfrak{F}) \subseteq L$ or $L \subseteq Log(\mathfrak{F})$. This finishes the proof of the theorem. □

Corollary 3.27. *For every finite rooted frame $\mathfrak{F}$, the pair $(\mathbf{IPC}+\chi(\mathfrak{F}), Log(\mathfrak{F}))$ is a splitting pair in the lattice of all intermediate logics.*

Proof. The result follows immediately from Theorems 3.26 and 3.10(1). □

Remark 3.28. In fact, the converse to Corollary 3.27 also holds; that is, for every splitting pair (L_1, L_2) there exists a finite rooted frame $\mathfrak{F}$ such that $L_1 = \mathbf{IPC}+\chi(\mathfrak{F})$ and $L_2 = Log(\mathfrak{F})$; see, e.g., [7, 10.47(2)]. This result was originally proved by Jankov [11, 12, 13]. McKenzie [19] proved a more general result that if a variety of algebras is congruence distributive and finitely approximable (i.e., generated by its finite members), then every splitting variety in its lattice of subvarieties is generated by a finite subdirectly irreducible algebra. McKenzie [19] also gave an example of a finite subdirectly irreducible lattice that does not generate a splitting variety. Blok [6] obtained the same result for modal algebras. In fact, a rooted finite frame splits a lattice of normal extensions of the basic modal logic $\mathbf{K}$ iff $\mathfrak{F}$ is cycle free. For the details we refer to e.g., [7, Theorem 10.53].

As we saw in Theorem 3.25, there exist intermediate logics not satisfying condition (1) of Theorem 3.9. Therefore, Theorem 3.26, in general, does not apply to logics axiomatizable by a single subframe formula. Moreover, an analogue of Corollary 3.27 does not hold, in general, for subframe logics. The intermediate logic $\mathbf{LC}$ of all linear frames provides a simple counter-example. As follows from Example 2.7, $\mathbf{LC} = \mathbf{IPC} + \beta(\mathfrak{H}_2)$. But by Example 2.9, $\mathbf{LC} = \mathbf{IPC} + \chi(\mathfrak{H}_2) + \chi(\mathfrak{H}_3)$. Therefore, $\mathbf{LC}$ cannot be axiomatized by a single Jankov-de Jongh formula and thus by Remark 3.28, $\mathbf{LC}$ is not a splitting logic. We also recall that an intermediate logic L is called a *union-splitting* if it is a join of splitting logics in the lattice of all intermediate logics. Therefore, an intermediate logic is a union-splitting iff it is axiomatizable by Jankov-de Jongh formulas. Thus, $\mathbf{LC}$ provides an example of a union-splitting that is not a splitting.

Next we discuss the finite model property of the logics axiomatizable by frame based formulas. We recall that an intermediate logic L has *the finite model property*, the *fmp* for short, if every non-L-theorem is refuted in a finite L-frame. Note that in the normal extensions of the modal logic $\mathbf{K}$ each union-splitting has the finite model property; see, e.g., [7, Theorem 10.54]. However, this is not the case for transitive modal logics. For an example of a normal extension of $\mathbf{K4}$ that is a union-splitting but does not have the finite model property we refer to e.g., [7, Example 10.56]. An example of an intermediate logic axiomatizable by Jankov-de Jongh formulas that lacks the finite model property was first constructed by Jankov [12] (see also Kracht [17]). Next we discuss the result that intermediate logics axiomatizable by subframe and cofinal subframe formulas enjoy the finite model property.

Theorem 3.29. *Let $\trianglelefteq$ be a frame order on $\mathbb{FG}(\mathbf{IPC})$ such that for every intermediate logic L condition (2) of Theorem 3.9 is satisfied. Then every intermediate logic axiomatizable by frame based formulas for $\trianglelefteq$ enjoys the finite model property.*

Proof. Let L be an intermediate logic axiomatizable by frame based formulas for $\trianglelefteq$. Suppose $L \not\vdash \phi$. Then there exists $\mathfrak{F} \in \mathbb{FG}(L)$ such that $\mathfrak{F} \not\models \phi$. Consider $L + \phi$. If it is inconsistent, then every finite L-frame refutes ϕ. Thus, assume $L + \phi$ is consistent. Then it is an intermediate logic, and by our assumption, there is $\mathfrak{F}' \in \mathbf{M}(L + \phi, \trianglelefteq)$ such that $\mathfrak{F}' \trianglelefteq \mathfrak{F}$. As L is axiomatizable by frame based formulas for $\trianglelefteq$, by Theorem 3.9, $\mathbb{FG}(L)$ is a $\trianglelefteq$-downset. Therefore, $\mathfrak{F}' \in \mathbf{F}_L$. Since $\mathfrak{F}' \in \mathbf{M}(L + \phi, \trianglelefteq)$ we have $\mathfrak{F}' \not\models \phi$. Thus, L has the fmp. □

For a direct proof of the next corollary we refer to [7, Theorem 11.20]. An algebraic proof of the result can be found in [2].

Corollary 3.30. *All subframe logics and cofinal subframe logics enjoy the finite model property.*

Proof. The result follows immediately from Theorems 3.29, 3.10(2),(3) and Corollary 3.13. □

We note that an analogue of Corollary 3.30 does not hold, in general, for logics axiomatizable by Jankov-de Jongh formulas. As we pointed out above, unlike (cofinal) subframe logics, there are logics that are axiomatizable by Jankov-de Jongh formulas that lack the finite model property.

We will close this paper by showing that for every frame order $\trianglelefteq$ there are intermediate logics that are not axiomatizable[2] by frame based formulas for $\trianglelefteq$. Note that this proof is very non-constructive.

Theorem 3.31. *For every frame order $\trianglelefteq$ on $\mathbb{FG}(\mathbf{IPC})$ there are intermediate logics that are not axiomatizable by frame based formulas for $\trianglelefteq$.*

Proof. We assume that every intermediate logic is axiomatizable by frame based formulas for $\trianglelefteq$ and show that this implies that every intermediate logic has the fmp. This contradicts the fact that there are continuum many intermediate logics without the fmp; see, e.g., [7, Theorem 6.3]. If every intermediate logic is axiomatizable by frame based formulas for $\trianglelefteq$, then by Theorem 3.9, every intermediate logic satisfies condition (2) of Theorem 3.9. Therefore, by Theorem 3.29, every intermediate logic axiomatizable by frame based formulas for $\trianglelefteq$ has the finite model property. By our assumption, every intermediate logic is axiomatizable by frame based formulas for $\trianglelefteq$. Thus, every intermediate logic has the finite model property. This contradiction finishes the proof of the theorem. □

Thus, it is impossible to axiomatize all the intermediate logics by frame based formulas for one given frame order $\trianglelefteq$. This raises the question (which we leave open) whether for every intermediate logic L there exists a frame order $\trianglelefteq_L$ such that L is axiomatizable by frame based formulas for $\trianglelefteq_L$.

In order to axiomatize all intermediate logics by formulas arising from finite frames one has to generalize frame based formulas by introducing a new parameter. Zakharyaschev's canonical formulas are extensions of the Jankov-de Jongh formulas and (cofinal) subframe formulas with a new parameter. Instead of considering just finite rooted frame $\mathfrak{F}$ we need to consider a pair $(\mathfrak{F}, \mathfrak{D})$, where $\mathfrak{D}$ is some set of antichains of $\mathfrak{F}$. We would also need to modify the definition of $\trianglelefteq$ to take this parameter into account. Formulas arising from such pairs are called "canonical formulas". They provide axiomatizations of all intermediate logics. We do not discuss canonical formulas here. For a systematic study of canonical formulas the reader is referred to [7, §9].

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[2]Michael Zakharyaschev pointed out to the author that this result also follows from the fact that intermediate logics do not admit an axiomatic bases.

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