FREE HEYTING ALGEBRAS: REVISITED

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1. Introduction

There are at least two different methods for describing finitely generated free Heyting algebras. One uses a description of the points of finite depth of the dual frame of the free Heyting algebra. For the details of this construction we refer to [6, Section 8.7] and [3, Section 3.2]. The other one, observed by Ghilardi [7], builds the free Heyting algebra on a distributive lattice step by step by freely adding to the original lattice the implications of degree n, for each $n \in \omega$. Ghilardi [7] used this technique to show that every finitely generated free Heyting algebra is a bi-Heyting algebra. A more detailed account of Ghilardi's construction can be found in [5] and [9]. Ghilardi and Zawadowski [9], based on this method, derive a model-theoretic proof of Pitts' uniform interpolation theorem. In [2] a similar construction is used to describe free linear Heyting algebras over a finite distributive lattice and [11] uses the same method to construct high order cylindric Heyting algebras. This construction can also be extended to the modal case [8, 1, 4]. In this note we approach the Ghilardi construction from a coalgebraic perspective. We split the construction into two steps. We first construct free weak Heyting algebras. Weak Heyting algebras are axiomatized by equations of rank 1. This allows a straightforward application of coalgebraic techniques. After that we build free Heyting algebras on top of free weak Heyting algebras. We show that the rooted admissible sets used by Ghilardi [7] can be obtained using this approach in a simple and systematic way. We also give an example of a formula of intuitionistic logic of rank 1 that can not be derived from other formulas of rank 0-1.

2. Discrete duality for distributive lattices

We recall that an element a of a distributive lattice D is called *join-irreducible* if for every $b, c \in D$ we have that $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. For every distributive lattice D let $J(D)$ denote the set of all join-irreducible elements of D. Let also $\leq$ be the restriction of the order of D to $J(D)$. Then $(J(D), \leq)$ is a poset. Recall also that for every poset X a subset $U \subseteq X$ is called a *downset* if $x \in U$ and $y \leq x$ imply $y \in U$. For every poset X we denote by $D(X)$ the distributive lattice $(D(X), \cap, \cup, \emptyset, X)$ of all downsets of X. Then every finite distributive lattice D is isomorphic to the lattice of all downsets of $(J(D), \leq)$ and vice versa, every poset X is isomorphic to a poset of join-irreducible elements of $D(X)$. We call $(J(D), \leq)$ the *dual poset* of D and we call $D(X)$ the *dual lattice* of X.

This duality can be extended to the duality of the category $\mathbf{DL}_{fin}$ of finite distributive lattices and lattice morphisms and the category $\mathbf{Pos}_{fin}$ of finite posets and order-preserving maps. In fact, if $h : D \to D'$ is a lattice morphism, then the restriction of h to $J(D)$ is an order-preserving map between $(J(D), \leq)$ and $(J(D'), \leq')$, and if $f : X \to X'$ is an order-preserving map between two posets X and X', then $f^{-1} : D(X') \to D(X)$ is a lattice morphism. Moreover, injective lattice morphisms (embeddings) correspond to surjective order-preserving maps and surjective lattice morphisms (homomorphic images) correspond injective order-preserving maps which are in one-to-one correspondence with subsets of the corresponding poset.

We also recall that an element a of a distributive lattice D is called *meet-irreducible* if for every $b, c \in D$ we have that $b \vee c \leq a$ implies $b \leq a$ or $c \leq a$. We let $M(D)$ denote the set of all meet-irreducible elements of D.

Proposition 2.1. *Let D be a finite distributive lattice. Then*

(1) *For every $p \in J(D)$, there exists $\kappa(p) \in M(D)$ such that for every $a \in D$ we have*
$$p \leq a \quad \text{or} \quad a \leq \kappa(p).$$

(2) *For every $m \in M(D)$, there exists $\delta(m) \in J(D)$ such that for every $a \in D$ we have*
$$\delta(m) \leq a \quad \text{or} \quad a \leq m.$$

3. Freely adding weak implications

Definition 3.1. *A distributive lattice $(A, \vee, \wedge, 0, 1)$ is called a* weak Heyting algebra *if there is a binary operation $\rightarrow$ on A such that for every $a, b, c \in A$:*

(1) $a \rightarrow a = 1$,
(2) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.
(3) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$.
(4) $(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c$.

We call $\rightarrow$ a *weak implication.*

Let D and D' be distributive lattices. We let $\rightarrow(D \times D')$ denote the set $\{a \rightarrow b : a \in D$ and $b \in D'\}$. For every distributive lattice D we also let $F_{DL}(\rightarrow(D \times D))$ denote the free distributive lattice over $\rightarrow(D \times D)$. Moreover, we let
$$H(D) = F_{DL}(\rightarrow(D \times D))/\approx$$
where $\approx$ is the DL congruence generated by the axioms (1)–(4).

Theorem 3.2. *Let D be a finite distributive lattice and $X = (J(D), \leq)$ its dual poset. Then*

(1) $J(H(D)) = \{\bigwedge q_i \rightarrow \kappa(q_i) : q_i \in J(D)\}$.
(2) *The poset $(J(H(D)), \leq)$ is isomorphic to the poset $(\mathcal{P}(X), \subseteq)$ of all subsets of X ordered by inclusion.*

4. Freely adding Heyting implications

Definition 4.1. *A weak Heyting algebra A is called a Heyting algebra if for every $a, b \in A$:*

(1) $b \leq a \rightarrow b$,
(2) $a \wedge (a \rightarrow b) \leq b$.

Since both D and $H(D)$ are embedded in $D + H(D)$ (where $+$ is the coproduct in the category of distributive lattices) we will not distinguish between the elements of D and $H(D)$ and their images in $D + H(D)$. Let $\approx$ be a distributive lattice (DL for short) congruence generated by the axioms (1)–(2). We denote $(D + H(D))/\approx$ by $V(D)$. Now we spell out the connection between our construction and the one of Ghilardi. For every finite poset X a set $\{p\} \cup \{q_1, \ldots, q_n\}$, for $p, q_i \in X$, is called *rooted* if $q_i < p$.

Theorem 4.2. *Let D be a distributive lattice and $X = (J(D), \leq)$ its dual poset. Then*

(1) $J(V(D)) = \{p \wedge \bigwedge q_i \rightarrow \kappa(q_i) : p \in J(D), q_i \in J(D)$ *and* $q_i < p\}$.
(2) *The poset $(J(V(D)), \leq)$ is isomorphic to the poset $(X^S, \subseteq)$ of all rooted subsets of X ordered by inclusion.*

Example 4.3. Let D be a finite distributive lattice and let $H'(D)$ denote $F_{DL}(\rightarrow(D \times D))$ modulo axioms (1),(2) of Definition 3.1. We also let $V'(D)$ denote $D + H'(D)$ modulo axioms (1),(2) of Definition 4.1. Then we can show that in general, $V'(D)$ is not isomorphic to $V(D)$. In fact, the inequality $(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c$ will not be valid on $V'(D)$, whereas on $V(D)$ it is valid by definition. We recall that axioms (1),(2) of Definition 3.1 and (1),(2) of Definition 4.1 are sufficient to axiomatize Heyting algebras; see e.g., [10, Lemma 1.10] or [3, Theorem 2.2.6]. In logical terms the above observation means that the inequality $(a \to b) \wedge (b \to c) \leq (a \to c)$ is an example of a valid rank 1 inequality of the theory of Heyting algebras (intuitionistic logic) that can not be derived from other valid equations of rank 0-1.

5. Free weak Heyting algebras and free Heyting algebras

Let D be the free distributive lattice over n generators. Since the variety of DLs is locally finite, D is finite. We put $D_0 = D$ and $D_{k+1} = H(D_k)$ and we let i_k be the embedding of D_k into D_{k+1}. We also let $D'_0 = D$ and $D'_{k+1} = V(D_k)$ modulo the equations $i_k(a \to_{k-1} b) = i_{k-1}a \to_k i_{k-1}b$, for each $a, b \in D'_{k-1}$ We let i'_k denote the restriction of i_k to D'_k.

Theorem 5.1.

(1) *The algebra* $(D_\omega, \to_\omega)$ *is the free* n*-generated weak Heyting algebra, where* D_ω *is the direct limit of* $\{D_k\}_{k\in\omega}$ *with the maps* $i_k : D_k \to D_{k+1}$ *in the categoty* **DL** *of distributive lattices, and* $a \to_\omega b = a \to_k b$*, for* $a, b \in D_k$*.*

(2) *The algebra* $(D'_\omega, \to'_\omega)$ *is the free* n*-generated Heyting algebra, where* D'_ω *is the direct limit of* $\{D'_k\}_{k\in\omega}$ *with the maps* $i'_k : D'_k \to D'_{k+1}$ *in the category* **DL** *of distributive lattices, and* $a \to'_\omega b = a \to'_k b$*, for* $a, b \in D'_k$*.*

We finish the abstract by reformulating Theorem 5.2 in dual terms. By doing so we obtain Ghilardi's representation of the dual posets of the D'_ks. Let f_{k-1} be a map from X_k onto X_{k-1}. We call a rooted subset $S \subseteq X_k$, f_{k-1}-*admissible* if for any $x, s \in X_k$ such that $s \in S$ and $x \leq s$ there exists $s' \leq x$ with $f_{k-1}(s) = f_{k-1}(x)$. Let X_0 be a poset. Let X_1 be the poset of all rooted subsets of X_0 ordered by inclusion. We also let f_1 be a map that maps every rooted subset to its root. Now for every $k \in \omega$ we let X_{k+1} be the poset of f_{k-1}-admissible subsets of X_k ordered by inclusion. Then we have the following theorem.

Theorem 5.2. *The inverse limit of the sequence* $\{X_k\}_{k\in\omega}$ *(where* X_0 *is the dual poset to* D_0*) with the maps* $f_k : X_{k+1} \to X_k$ *in the category of Priestley spaces is dual to the free Heyting algebra over* D_0*.*

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