Extendable formulas in two variables in intuitionistic logic

Nick Bezhanishvili and Dick de Jongh

1 Introduction

In this paper we discuss projective, exact and extendable formulas of the intuitionistic propositional calculus **IPC**. Exact formulas were introduced in [6] as the formulas that axiomatize the theories of substitutions. In an attempt to find a semantic characterization of exact formulas, de Jongh and Visser [7] defined the notion of extendable formulas and proved that every exact formula is extendable. Later Ghilardi [4], motivated by finding the most general unifiers for intuitionistic formulas, introduced projective formulas and proved that every extendable formula is projective and that every projective formula is exact. Thus all these three notions are equivalent. In this paper we give an alternative proof of Ghilardi's theorem for formulas in two variables. We also give a full description of projective, exact and extendable formulas in two variables. Our main tools are the n-universal models — the general frames that are dual to Lindenbaum-Tarski algebras of **IPC** in n-variables.

2 Exact, extendable and projective formulas

For the definition and basic properties of the intuitionistic propositional calculus **IPC** and the n-universal models of **IPC** we refer to [3, Section 8.7] and [2, Section 3.2].

Definition 2.1 (de Jongh [6])**.** *A formula φ is called an* exact formula *if there is a substitution σ such that*

1. $\mathbf{IPC} \vdash \sigma(\varphi)$,
2. *For any formula ψ, if $\mathbf{IPC} \vdash \sigma(\psi)$, then $\varphi \vdash \psi$.*

It is well known that every substitution σ can be seen as a homomorphims between Lindenbaum-Tarski algebras. Let $\sigma : F(n) \to F(m)$ be a substitution, where $F(n)$ and $F(m)$ are Lindenbaum-Tarski algebras of **IPC** in n and m variables, respectively. The *theory of* σ is the filter $\sigma^{-1}(\top)$. The theory of σ is finitely axiomatizable if $\sigma^{-1}(\top)$ is a principal filter; that is, if there exists a formula φ such that $\sigma^{-1}(\top) = [\varphi)$. The next proposition shows that exact formulas are exactly those formulas that axiomatize the theories of substitutions.

Proposition 2.2. *A formula φ is exact iff there is a substitution $\sigma : F(n) \to F(m)$, such that φ axiomatizes the theory of σ.*

Next we discuss the question whether the theory of every substitution is finitely axiomatizable. In fact, its positive answer is a direct consequence of Pitts' Uniform Interpolation Theorem. We formulate this result in its more general form; see e.g., [8, 5].

Theorem 2.3 (Pitts [8])**.** *Every substitution $\sigma : F(n) \to F(m)$ possesses right and left adjoints $\exists_\sigma, \forall_\sigma : F(m) \to F(n)$. That is, for any formula $\varphi(p_1, \ldots, p_m)$, there are formulas $\exists_\sigma\varphi$ and $\forall_\sigma\varphi$ in n variables such that for any $\psi(p_1, \ldots, p_n)$:* (1) $\varphi \vdash \sigma(\psi)$ *iff* $\exists_\sigma\varphi \vdash \psi$ *and* (2) $\sigma(\psi) \vdash \varphi$ *iff* $\psi \vdash \forall_\sigma\varphi$.

Pitts' theorem immediately implies the following result.

Corollary 2.4. *1. For every substitution $\sigma : F(n) \to F(m)$, the theory of σ is finitely axiomatizable by the formula $\exists_\sigma \top$.*

2. A formula φ is exact iff there is a substitution $\sigma : F(n) \to F(m)$ such that φ is equivalent to $\exists_\sigma \top$.

Recall that a subset U of the n-universal model $\mathcal{U}(n) = (U(n), R, V)$ is called *definable* if there is a formula φ such that $U = V(\varphi)$. A p-morphism between universal models is called *definable* if the inverse image of every definable subset is again definable. Since the algebra of all definable upsets of the n-universal model $\mathcal{U}(n)$ is dual to the Lindenbaum-Tarski algebra $F(n)$ (see, e.g., [2, 3.2.2]), the above result gives us the following characterization of exact formulas in terms of universal models.

Corollary 2.5. *A formula φ is exact iff there exists a definable p-morphism $f : \mathcal{U}(m) \to \mathcal{U}(n)$ such that $V(\varphi) = f(U(m))$.*

Next we discuss extendable formulas. Extendable formulas were introduced in an attempt to find a semantic characterization of exact formulas.

Definition 2.6 (de Jongh, Visser [7])**.** *A formula φ is called* extendable *if any disjoint union of finite rooted Kripke models validating φ can be extended to a Kripke model validating φ by adding a new root to this disjoint union of Kripke models.*

Example 2.7. *For every formula φ, the formula $\varphi \to p$ is extendable.*

Next we will characterize extendable formulas in terms of universal models. We write $w \prec \Delta$ if Δ is the set of all immediate successors of w.

Definition 2.8. *An upset U of the n-universal model $\mathcal{U}(n)$ is called* extendable *if for every antichain $\Delta \subseteq U$, there exists a point $w \in U$ such that $w \prec \Delta$.*

In fact, there is a one-to-one correspondence between definable extendable upsets of the n-universal model and extendable formulas in n-variables. For the detailed proof of the next theorem see [1, Theorem 12].

Theorem 2.9. *A formula $\varphi(p_1, \ldots, p_n)$ is extendable iff $V(\varphi)$ is an extendable subset of the n-universal model $\mathcal{U}(n)$.*

Now we are ready to give an alternative proof of Visser's theorem that every exact formula is extendable.

Theorem 2.10 (de Jongh and Visser [7])**.** *If a formula φ is exact, then it is extendable.*

Proof. Let φ be an exact formula. Then by Corollary 2.4 there is a definable p-morphism $f : U(m) \to U(n)$ such that $V(\varphi) = f(U(m))$. We show that $V(\varphi)$ is an extendable upset. Let $\Delta \subseteq V(\varphi)$ be any finite antichain. Since f is order-preserving $f^{-1}(\Delta)$ is an antichain of $\mathcal{U}(n)$. Let $x \in U(m)$ be such that $x \prec f^{-1}(\Delta)$ (by the construction of $\mathcal{U}(n)$ we know that such a point always exists). Then $f(x) \in V(\varphi)$ and it is easy to see that $f(x) \prec \Delta$. Therefore, $V(\varphi)$ is an extendable upset of $\mathcal{U}(m)$ and, by Theorem 2.9, φ is an extendable formula. □

In the remainder of this section we characterize extendable upsets of n-universal models.

Definition 2.11. *For every upset U of the n-universal model, we call a point $x \in U(n)$ a* U-border point *if $x \notin U$, but for every proper successor y of x we have $y \in U$. For every upset U let $B(U)$ denote the set of all U-border points.*

Let φ be any formula. The $V(\varphi)$-border points we simply call *φ-border points*. The set of all φ-border points we denote by $B(\varphi)$. The points that belong to $V(\varphi)$ will be called φ-points. Let $x \in U(n)$ be totally covered by an antichain Δ; that is, $x \prec \Delta$. We call a point $y \in U(n)$ a *sister* of x if $x \neq y$ and $y \prec \Delta$. The next theorem characterizes extendable upsets of $\mathcal{U}(n)$.

Theorem 2.12. *Let U be an upset of the n-universal model $\mathcal{U}(n)$. Then U is extendable iff $U = U(n) \setminus R^{-1}(S)$, where $S \subseteq \bigcup_{i=1}^{n} V(p_i) \cup \bigcup_{i=1}^{n} B(p_i)$, and if $x \in S$, then x has a sister $y \notin S$.*

Next we recall the definition of projective formulas.

Definition 2.13 (Ghilardi [4])**.** *A formula φ is called* projective *if there is a substitution σ such that*

1. **IPC** $\vdash \sigma(\varphi)$,
2. *For any formula ψ, we have $\varphi \vdash \psi \leftrightarrow \sigma(\psi)$.*

For the connection of projective formulas with the theory of unification and with finitely generated projective Heyting algebras we refer to [4] and [1], respectively. Now we will explore the connection of projective formulas with the exact and extendable formulas. We first note the following simple fact.

Theorem 2.14. *Every projective formula is exact.*

Proof. In fact the substitution σ that makes φ projective will also make it exact. For this all one needs to observe is that if $\vdash \sigma\psi$, then $\varphi \vdash \psi \leftrightarrow \sigma\psi$ implies $\varphi \vdash \psi$. □

Next we give a characterization of projective formulas in terms of the universal models.

Theorem 2.15. *A formula φ is projective iff there exists a definable p-morphism $f : \mathcal{U}(n) \to \mathcal{U}(n)$ such that $V(\varphi) = f(U(n))$ and $f(x) = x$, for every $x \in V(\varphi)$.*

This theorem together with Theorem 2.12 has a consequence that every projective formula is exact. Moreover, Theorem 2.15 implies that $V(\varphi)$ is a retract of $\mathcal{U}(n)$.

3 Extendable formulas in two variables

We will use the propositional letters p and q instead of p_1 and p_2. We call the sets $F \subseteq B(p)$ and $G \subseteq B(q)$ cofinite, if the sets $B(p) \setminus F$ and $B(q) \setminus G$ are finite.

Theorem 3.1. *Let U be an upset of the 2-universal model $\mathcal{U}(2)$. Then U is definable and extendable iff $U = U(2) \setminus R^{-1}(S)$, where $S \subseteq V(p) \cup V(q)$, the sets $S \cap B(p)$ and $S \cap B(q)$ are finite or cofinite subsets of $B(p)$ and $B(q)$, respectively, and if $x \in S$, then there exists a sister y of x such that $y \notin S$.*

Now we are ready to give an alternative proof of a theorem of Ghilardi that (for the formulas in two variables) exact, extendable and projective formulas are the same.[1]

[1] In fact Ghilardi [4] proved the theorem for all formulas, not only in two variables.

Theorem 3.2 (Ghilardi [4]). *Let φ be a formula in 2 variables. The following three conditions are equivalent.*

1. *φ is projective,*
2. *φ is exact,*
3. *φ is extendable.*

Proof. (Sketch) (1) $\Rightarrow$ (2) is Theorem 2.14. and (2) $\Rightarrow$ (3) is Theorem 2.10. Now we prove that (3) implies (1). In fact, this proof will also be a direct proof of (3) $\Rightarrow$ (2). Let φ be an extendable formula. Then $V(\varphi)$ is an extendable upset of the 2-universal model. By Theorem 2.12, we have that $V(\varphi) = U(2) \setminus R^{-1}(S)$, where S contains only p-points, q-points p-border points and q-border points. We will construct a definable p-morphism $f : U(2) \to U(2)$, such that $f(U(2)) = V(\varphi)$ and $f(x) = x$, for every $x \in V(\varphi)$. By Theorem 2.15 this will imply that φ is a projective formula. Note that by Theorem 3.1, for every x is S, x's sister belongs to $V(\varphi)$. Intuitively speaking, f will be the the least definable p-morphism that maps the points in S to their sisters. We skip the technical details. □

Next we give a syntactic description of the extendable formulas in two variables.

Definition 3.3. *The* Rieger-Nishimura polynomials *are given by the following recursive definition:* $g_0(p) = p$, $g_1(p) = \neg p$, $f_1(p) = p \vee \neg p$, $g_2(p) = \neg\neg p$, $g_3(p) = \neg\neg p \to p$, $g_{n+4}(p) = g_{n+3}(p) \to (g_n(p) \vee g_{n+1}(p))$, $f_{n+2}(p) = g_{n+2}(p) \vee g_{n+1}(p)$.

Exact formulas in one variable were characterized in [6].

Theorem 3.4 (de Jongh [6]). *The only projective formulas in one variable are the formulas* $p \to p$, p, $\neg p$, $\neg\neg p$, *and* $\neg\neg p \to p$.

Now we move to the two-variable case.

Definition 3.5. *For every $k \in \omega$ we let*

1. $r_k(p, q) = p \to g_{k+1}(q)$.
2. $h_k(p, q) = ((p \vee (p \to g_{k+1}(q))) \to (p \wedge g_k(q)) \to p$.
3. $j_k(p, q) = ((p \vee (p \to (g_{k+1}(q)) \vee (p \to g_{k+2}(q))) \to ((p \wedge f_{k+2}(q)))) \to p$.
4. $a_k(p, q) = g_k(q) \to p$.
5. $b_k(p, q) = (g_k(q) \to p) \to p$.

Finally, we arrive at the following characterization of the extendable formulas in two variables.

Theorem 3.6. *If $\varphi(p, q)$ is an extendable formula in two variables, then it is equivalent to a formula* $\varphi_1 \wedge \cdots \wedge \varphi_k$, $k \geq 1$, *where each φ_i belongs to* $\{r_n(p, q), r_n(q, p), h_n(p, q), h_n(q, p), j_n(p, q), j_n(q, p), a_n(p, q), a_n(q, p), b_n(p, q), b_n(q, p)\colon n \in \omega\}$, *and for each $n \in \omega$, the formulas* $r_n(p, q) \wedge j_n(p, q)$, $r_n(q, p) \wedge j_n(q, p)$, $r_n(p, q) \wedge b_n(p, q)$ *and* $r_n(q, p) \wedge b_n(q, p)$ *are not part of* $\varphi_1 \wedge \cdots \wedge \varphi_k$.

One can make many restrictions on the forms that occur. For example if $r_n(p, q)$ occurs as a φ_i, then no $r_m(p, q)$ and $h_m(p, q)$ need to occur for any $m > n + 1$. Finding a projective substitution for the formulas occurring in Theorem 3.6 would provide an alternative proof to Theorem 3.2.

References

[1] N. Arevadze, *Finite projective formulas*, Master's thesis, ILLC, University of Amsterdam, 2001.

[2] N. Bezhanishvili, *Lattices of intermediate and cylindric modal logics*, Ph.D. thesis, ILLC, University of Amsterdam, 2006.

[3] A. Chagrov and M. Zakharyaschev, *Modal logic*, Oxford University Press, 1997.

[4] S. Ghilardi, *Unification in intuitionistic logic*, Journal of Symbolic Logic **64** (1999), 859–880.

[5] S. Ghilardi and M. Zawadowski, *A sheaf representation and duality for finitely presented Heyting algebras*, Journal of Symbolic Logic **60** (1995), 911–939.

[6] D. de Jongh, *Formulas in one propositional variable in intuitionistic arithmetic*, The L.E.J Brouwer Centenary Symposium (A. Troelstra and D. van Dalen, eds.), North Holland Publishing Company, 1982.

[7] D. de Jongh and A. Visser, *Embeddings of Heyting algebras*, Logic: from foundations to applications (W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, eds.), Oxford University Press, 1996.

[8] A.M. Pitts, *On an interpretation of second order quantification in firs order intuitionistic propositional logic*, Journal of Symbolic Logic **57** (1992), 33–52.