Heyting-valued interpretations for Constructive Set Theory

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Introduction

The theory of locales [23] has a twofold interplay with intuitionistic mathematics: first of all, the internal logic of toposes and intuitionistic set theories provide suitable settings for the development of the theory of locales [24], and secondly, the notion of a locale determines two important forms of toposes and of interpretations for intuitionistic set theories, namely localic toposes [26, Chapter IX] and Heyting-valued interpretations [10]. The combination of these two aspects has led to many proof-theoretic applications [16, 17] and important results in the theory of elementary toposes [25]. The internal logic of toposes with a natural number object [9] and intuitionistic set theories [34] are examples of formal systems that are fully impredicative, in the sense that they have proof-theoretic strength above the one of second-order arithmetic [5].

Formal topology originated by considering whether it was possible to develop pointfree topology in a generalised predicative context [30]. Generalised predicative mathematics is understood here as something more general than the Weyl-Feferman-Schütte notion of predicative mathematics, so as to allow generalised inductive definitions and generalised reflection [15, 29]. For instance, Martin-Löf type theories with well-ordering types and Mahlo universe types are generalised predicative systems, and so is every formal system that is proof-theoretically reducible to them. By virtue of the type-theoretic interpretation [2, 3, 4], the constructive set theories that we consider here are generalised predicative systems.

The development of formal topology shows that it is possible to reconstruct considerable parts of pointfree topology within Martin-Löf type theories [31]. Yet, the second aspect of relationship between locale theory and

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intuitionistic mathematics does not seem to have been explored at the generalised predicative level. Our aim here is to set up an interplay between formal topology and constructive set theories analogous to the one existing between locale theory and intuitionistic set theories. We do so by investigating Heyting-valued interpretations for the Constructive Zermelo-Frankel set theory, CZF [6].

The study of Heyting-valued interpretations reveals many of the differences between intuitionistic and constructive set theories. None of the main choices made to develop Heyting-valued interpretations in the fully impredicative context [10] is suitable for our purposes. First, to model the truth values of the formulas of a constructive set theory, it is appropriate to consider set-generated frames and formal topologies, as defined in Section 2, rather than complete Heyting algebras, as usually defined. The reason for this is that constructive set theories do not have the power-set axiom. Secondly, to define a class of 'Heyting-valued sets', it is preferable to avoid the use of ordinals and instead exploit inductive definitions. This is because there is a well-developed theory of inductive definitions for constructive set theories [6, Chapter 5]. Finally, when it comes to defining the interpretation, it is necessary to pay particular attention to the distinction between arbitrary and restricted formulas that is peculiar to constructive set theories and does not need to be considered in intuitionistic set theories.

Interpretations for constructive set theories in which the truth values are modelled using Grothendieck topologies on posets were studied in [21]. There, it is observed that the validity of the Exponentiation axiom requires an additional hypothesis on the Grothendieck topology. A version of this assumption in the context of formal topology suggests the independence result of Theorem 4.3, and it is used to establish the validity of the Subset Collection axiom, thus strengthening the results of [21]. We also consider the validity of the Strong Collection axiom, that is part of CZF, and was not considered in [21]. The recent work on the analogon of the notion of an elementary topos at the generalised predicative level should also be mentioned as related work [27, 28]. However, the results obtained here are independent of those in [27, 28]. This is because the category of classes of CZF is not an example of the notion of a 'stratified pseudo-topos' axiomatised and studied in [27, 28]. For a discussion of category-theoretic counterparts of CZF, we invite the reader to refer to [20].

Section 1 reviews the aspects of CZF that are most relevant for this paper, and presents some auxiliary results that are needed in the following sections. In Section 2 we take the necessary steps in the development of formal topology in CZF, that allow us to set up and apply Heyting-valued interpretations. Heyting-valued interpretations for CZF are then presented in Section 3. Two kinds of applications of Heyting-valued interpretations are given in Section 4. First, we prove a relative consistency and an independence result concerning the law of restricted excluded middle. Secondly, we transfer at the generalised predicative level a result concerning the relationship between 'internal' and 'external' objects with respect to an Heyting-valued model.

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1 Constructive Set Theory

1.1 Language and Axioms

Constructive set theories will be formulated here in a language $\mathcal{L}$ that extends the language of first-order logic with equality so as to include primitive symbols $(\exists x \in a)$, $(\forall x \in a)$ for restricted quantifiers. The membership relation can be defined in $\mathcal{L}$ by letting $a \in b =_{\text{def}} (\exists x \in b) x = a$. A formula is *restricted* if the only quantifiers contained in it are restricted. We write $\mathcal{L}^{(V)}$ for the extension of $\mathcal{L}$ with constants for sets.

The set of free variables of a formula ϕ is denoted by $\mathrm{FV}\phi$, and, for a formula ϕ with $\mathrm{FV}\phi = \{x_1, \ldots, x_n\}$, we write $\phi[e_1, \ldots, e_n/x_1, \ldots, x_n]$ for the result of simultaneously substituting expressions e_i for the free occurrences of x_i in ϕ for $i = 1, \ldots, n$. In the following, lower-case Greek letters denote formulas: ϕ, ψ, ξ stand for arbitrary formulas, and θ, η stand for restricted formulas. The symbols x, y, z, u, v, w denote variables of $\mathcal{L}$, and lower-case letters that are not used for variables stand for constants for sets.

Constructive Zermelo-Frankel, CZF, is the set theory with usual axioms for first-order intuitionistic logic, standard axioms for restricted quantifiers, and the following set-theoretic axioms: Extensionality, Set Induction, Pairing, Union, Infinity, Restricted Separation, Strong Collection, and Subset Collection. Restricted Separation asserts that, for a set a and a restricted formula θ, the class $\{x \in a \mid \theta\}$ is a set. Full Separation, that is not part of CZF, would allow us to derive the same conclusion with an arbitrary formula in place of the restricted formula θ.

The formulation of Strong Collection and Subset Collection will be recalled in Subsection 1.2. Details on the other axioms can be found in [6, Chapter 2]. The subsystem CZF^- and the extension CZF^+ of CZF will also be considered here: CZF^- is obtained by omitting Subset Collection, and CZF^+ is obtained by adding the Regular Extension axiom [6, Section 5.2].

We will use classes, denoted with upper-case letters $A, B, C, \ldots$, and the notation associated to them. as described in [6, Chapter 3]. It is convenient to introduce some terminology that allows us to treat carefully the crucial distinction between sets and classes. A class P is said to be a *subclass* of a class A if it holds that $P \subseteq A$. When this is the case and P is a set, then P is said to be a *subset* of A. Because of the absence of Full Separation, we may have subclasses of a set. For example, if a is a set and ϕ is a formula, the class $P =_{\mathrm{def}} \{x \in a \mid \phi\}$ is obviously a subclass of a, but without the assumption of Full Separation it is not generally possible to assert that P is a set. The *power class* of a, $Pow(a)$, is defined by letting

$$Pow(a) =_{\mathrm{def}} \{x \mid x \subseteq a\} .$$

Without the assumption of Power Set, this class cannot be asserted to be a set. Observe that the elements of $Pow(a)$ are the subsets, not the subclasses, of a.

1.2 Some consequences of the collection axioms

We prove some consequences of the Strong Collection and Subset Collection axioms of CZF that will be useful in Section 2 and Section 3. Strong Collection is the scheme

$$(\forall x \in a)(\exists y)\phi \rightarrow (\exists u)\, coll(x \in a, y \in u, \phi)$$

where a is a set, ϕ is an arbitrary formula, and we define

$$coll(x \in a, y \in u, \phi) =_{\mathrm{def}} (\forall x \in a)(\exists y \in u)\phi \wedge (\forall y \in u)(\exists x \in a)\phi$$

Note that Strong Collection implies the axiom scheme of Replacement. Subset Collection is the scheme

$$(\exists v)(\forall z)((\forall x \in a)(\exists y \in b)\phi) \rightarrow (\exists u \in v)\, coll(x \in a, y \in v, \phi)$$

where a, b are sets and ϕ is an arbitrary formula.

Proposition 1.1. *Let a be a set, ψ be a formula of $\mathcal{L}^{(V)}$. If*

$$(\forall x \in a)(\exists y)\psi ,$$

then there exists a function g with domain a such that

$$(\forall x \in a)\big((\exists y)(y \in gx) \wedge (\forall y \in gx)\, \psi\big) .$$

Proof. For x, z define $\xi =_{\mathrm{def}} (\exists y)(z = (x, y) \wedge \psi)$. We have $(\forall x \in a)(\exists z)\xi$ by the assumption. By Strong Collection there exists a set u such that

$$coll(x \in a, z \in u, \xi) . \tag{1}$$

Define a function g with domain a by letting, for $x \in a$,

$$gx =_{\text{def}} \{y \mid (x, y) \in u\},$$

and observe that g is a set by Replacement. The required conclusion follows from (1) and the definition of ξ. Discharging the assumption of u, the proof is complete. $\square$

Proposition 1.2. *Let a be a set, ϕ be a formula of $\mathcal{L}^{(V)}$, and Q be a class. If*

$$(\forall x \in a)(\exists y)\big(y \subseteq Q_x \wedge \phi\big) \wedge (\forall x \in a)(\forall y)(\forall z)\big((y \subseteq z \subseteq Q_x \wedge \phi) \rightarrow \phi[z/y]\big)$$

where, for x in a, $Q_x =_{\text{def}} \{y \mid (x, y) \in Q\}$, then there exists a function f with domain a such that

$$(\forall x \in a)(fx \subseteq Q_x \ \wedge \ \phi[fx/y]).$$

Proof. For x, y define $\psi =_{\text{def}} y \subseteq Q_x \wedge \phi$. We have $(\forall x \in a)(\exists y)\psi$ by the assumption. By Proposition 1.1 there is a function g with domain a such that

$$(\forall x \in a)\big((\exists y)(y \in gx) \wedge (\forall y \in gx)\psi\big). \tag{2}$$

Define a function f with domain a by letting, for $x \in a$,

$$fx =_{\text{def}} \bigcup gx,$$

i.e. $(\forall z)(z \in fx \leftrightarrow (\exists y \in gx) z \in y)$, and observe that f is a set by Union and Replacement. For $x \in a$ we now show

$$fx \subseteq Q_x \wedge \phi[fx/y]$$

To prove the first conjunct, let $z \in fx$. There exists $y \in gx$ such that $z \in y$ by the definition of f. We have $y \subseteq Q_x$ by (2) and the definition of ψ, and therefore $z \in Q_x$. Discharging the assumption of y, we have $fx \subseteq Q_x$, as wanted. To prove the second conjunct, observe that there exists $y \in gx$ such that ψ by (2). By the definitions of f and ψ we have

$$y \subseteq fx \subseteq Q_x \wedge \phi.$$

Therefore we get $\phi[fx/y]$, by the assumption in the statement of the proposition. Discharging the assumption of y, we obtain the desired conclusion. The rest of the proof follows easily. $\square$

Proposition 1.3. *Let a be a set, let ϕ be a formula of $\mathcal{L}^{(V)}$ and let P be a class. If*

$$(\forall x \in a)\Big((\exists y)(y \subseteq P \wedge \phi) \wedge (\forall y)(\forall z)\big((y \subseteq z \subseteq P \wedge \phi) \rightarrow \phi[z/y]\big)\Big),$$

then there exists a set b such that $b \subseteq P \wedge (\forall x \in a)\phi[b/y]$.

Proof. Define $Q =_{\text{def}} \{(x,y) \mid x \in a \wedge y \in P\}$. For $x \in A$, we have $Q_x = P$, where Q_x is defined as in Proposition 1.2. By the assumption it follows that

$$(\forall x \in a)(\exists y)\big(y \subseteq Q_x \wedge \phi\big) \wedge (\forall x \in a)(\forall y)(\forall z)\big((y \subseteq z \subseteq Q_x \wedge \phi) \to \phi[z/y]\big)$$

By Proposition 1.2 and the definition of Q, there exists a function g with domain a such that

$$(\forall x \in a)(fx \subseteq P \wedge \phi[fx/y]) \tag{3}$$

Defining $b =_{\text{def}} \bigcup_{x \in a} fx$, we have $b \subseteq P$ by the definition of f and (3). Let $x \in a$, and observe that

$$fx \subseteq b \subseteq A \wedge \phi[fx/y]$$

by the definition of b and (3). Therefore we get $\phi[b/y]$ by the assumption in the statement of the proposition. Universally quantifying over x and discharging the assumption of f, the proof is complete. □

The three propositions we just proved are theorems of CZF$^-$, since Subset Collection has not been applied; the next result instead is proved in CZF. It was first obtained in [1] but we give a proof for completeness.

Proposition 1.4. *Let a and b be sets. Let ϕ be a formula. Then there exists a set c such that*

$$(\forall u \in a)\,(\forall z)\big((\forall x \in u)(\exists y \in b)\phi \to (\exists v \in c)\, \mathit{coll}\,(x \in u, y \in v, \phi)\big)$$

holds.

Proof. For u, w define

$$\psi =_{\text{def}} (\forall z)\big((\forall x \in u)(\exists y \in b)\phi \to (\exists v \in w)\mathit{coll}(x \in u, y \in v, \phi)\big)$$

We have $(\forall u \in a)(\exists w)\psi$ by Subset Collection. By Strong Collection there is d such that

$$\mathit{coll}(u \in a, w \in d, \psi) \tag{4}$$

Define $c =_{\text{def}} \bigcup d$, so that $(\forall v)(v \in c \leftrightarrow (\exists w \in d)v \in w)$. We now show that c satisfies the required conclusion. Let $u \in a$, and let z be a set. Assume

$$(\forall x \in u)(\exists y \in b)\phi$$

We have that there is $w \in d$ such that ψ by (4). Then there is $v \in w$ such that

$$\mathit{coll}(x \in u, y \in v, \phi)$$

by definition of ψ and the assumption. Hence the conclusion, and discharging the assumption of d the proof is complete. □

To simplify some of the applications of Proposition 1.4, we will sometimes use the following pattern of reasoning: given sets u, b, z and a formula ϕ, we will claim that

$$(\forall x \in u)(\exists y \in b)$$

implies the existence of a set c, independent of u, b, z, for which there is $v \in c$ such that

$$\mathit{coll}\,(x \in u, y \in v, \phi)$$

holds. This pattern of reasoning is justified by Proposition 1.4, provided that the sets u are elements of a set a, as we will ensure.

2 Formal spaces

2.1 Set-generated frames and formal topologies

Recall from [6, Chapter 6] that a *poclass* $(A, \leq)$ is a class A equipped with a partial order relation on A, where a relation on A is a subclass of $A \times A$. If $(A, \leq)$ is a poclass in which both A and the partial order relation are sets, we say that it is a *poset*. A morphism of poclasses is a monotone function. The *supremum* of a subclass $P \subseteq A$ is an element $a \in A$ such that

$$(\forall x \in A)\left(a \leq x \leftrightarrow (\forall y \in P) y \leq x\right).$$

holds. We write $\bigvee P$ for the supremum of a subclass P, if it exists. The infimum of a subclass P is defined in a dual way, as usual, and denoted by $\bigwedge P$, if it exists. By a supremum operation we mean an operation that assigns to each subset of A its supremum. Note that a supremum operation is not required to act on subclasses, but only on subsets. The notions of a meet of a pair of elements, and of a top can be defined as usual. From now on, we write $a \wedge b$ for the meet of elements $a, b \in A$, and $\top$ for the top element of A, if they exist.

A *frame* $(A, \leq, \bigvee, \wedge, \top)$ is a poclass $(A, \leq)$ equipped with a supremum operation, a meet operation, and a top element, such that the frame distributivity law,

$$a \wedge \bigvee p = \bigvee \{a \wedge x \mid x \in p\}$$

for all $a \in A$ and all subsets $p \subseteq A$, holds. A *frame morphism* is a poclass morphism that preserves suprema, meets, and top element.

Definition 2.1. A *set-generated frame* $\mathcal{A} = (A, \leq, \bigvee, \wedge, \top, g)$ is given by a frame $(A, \leq, \bigvee, \wedge, \top)$ and a subset $g \subseteq A$, called the *generating set* of $\mathcal{A}$, such that the class $g_a =_{\mathrm{def}} \{x \in g \mid x \leq a\}$ is a set, and $a = \bigvee g_a$ holds, for all $a \in A$.

The properties of the generating set g of a set-generated frame $\mathcal{A}$ allow us to define an infimum operation and a Heyting implication. Given a subset $p \subseteq A$, let $\bigwedge p =_{\text{def}} \bigvee q$, where $q =_{\text{def}} \{x \in g \mid (\forall y \in p) x \leq y\}$. By the assumption that g is a generating set, q is a set and therefore $\bigwedge p$ is well-defined. To define the Heyting implication of $a, b \in A$, let

$$a \to b =_{\text{def}} \bigvee \{x \in g \mid x \wedge a \leq b\}\,.$$

The frame distributivity law implies that $a \to b$ is the Heyting implication of a and b.

Example 2.2. Let $(S, \leq)$ be a poset. For a subclass $P \subseteq S$, let

$$\downarrow P =_{\text{def}} \{x \in S \mid (\exists y \in P) x \leq y\}$$

and observe that $P \subseteq \downarrow P$. We say that P is a *lower class* if it holds that $\downarrow P \subseteq P$, so that $\downarrow P = P$, and say that it is a *lower set* if P is a set. Let $Low(S)$ to be the poclass of lower sets of S, with partial order given by inclusion. A structure of set-generated frame on $Low(S)$ can be given as follows: the supremum operation is union, the meet operation is binary intersection, since the union and intersection of a set of lower sets is a lower set, and S is the top element. The frame distributivity law holds because unions distribute over intersections. A generating set for $Low(S)$ is defined by $g =_{\text{def}} \{\gamma(x) \mid x \in S\}$ where, for a in S, $\gamma(a) =_{\text{def}} \downarrow\{a\}$. The infimum operation in $Low(S)$ is given by intersection.

Example 2.3. Any set S can be seen as a poset by considering the partial order given by equality. Lower sets are just subsets, and so $Pow(S)$ is a set-generated frame, with generating set $g =_{\text{def}} \{\ \{x\} \mid x \in S\}$. The set-generated frame Ω defined by

$$\Omega =_{\text{def}} Pow(1)\,,$$

where $1 =_{\text{def}} \{\emptyset\}$ will be of particular importance here since subclasses and subsets of 1 are in close correspondence with arbitrary and restricted sentences of $\mathcal{L}$ [19, Section 2.3].

The notion of a formal topology that is introduced in the next definition is a slight variation over the one originally presented in [30], as we do not assume a positivity predicate as part of the structure.

Definition 2.4. A *formal topology* $\mathcal{S} = (S, \leq, \lhd)$ is given by a a poset $(S, \leq)$ and a relation $\lhd$ between elements and subsets of S, such that

- if $a \in p$, then $a \lhd p$
- if $a \leq b$ and $b \lhd p$, then $a \lhd p$

- if $a \lhd p$ and $(\forall x \in p)(x \lhd q)$, then $a \lhd q$
- if $a \lhd p$ and $a \lhd q$, then $a \lhd {\downarrow} p \cap {\downarrow} q$

for all $a, b \in S$ and all $p, q \in Pow(S)$.

The notion of a *nucleus* on a frame [35, 23] is very convenient to establish precisely the relationship between set-generated frames and formal topologies. For a formal topology $(S, \leq, \lhd)$, we define a nucleus j on $Low(S)$ by letting, for $p \in Low(S)$,

$$jp =_{\text{def}} \{x \in S \mid x \lhd p\}\,.$$

The properties of a formal topology imply directly that j is a nucleus. In general, for a nucleus j on a set-generated frame $\mathcal{A}$, we define

$$A_j =_{\text{def}} \{x \in A \mid x = jx\}\,.$$

Following the proof of an analogous result for frames [23] it is possible to show that the class A_j is part of the structure of a set-generated frame $\mathcal{A}_j$. Let us recall the definition of the meet, join, and Heyting implication of the set-generated frame $Low(S)_j$. For $p, q \in Low(S)_j$ we have

$$p \wedge q = p \cap q\,, \tag{5}$$

$$p \vee q = j(p \cup q)\,, \tag{6}$$

$$p \rightarrow q = \{x \in S \mid x \in p \rightarrow x \in q\}\,. \tag{7}$$

For a subset $u \subseteq Low(S)_j$, its supremum and infimum in $Low(S)_j$ are given as follows:

$$\bigvee u = j\Big(\bigcup u\Big)\,, \tag{8}$$

$$\bigwedge u = \bigcap u\,. \tag{9}$$

The next proposition is a version of well-known results in formal topology [7, 30], and makes explicit the connection between the notions of set-generated frame and formal topology.

Proposition 2.5. *Let $\mathcal{A} = (A, \leq, \bigvee, \wedge, \top, g)$ be a set-generated frame. There exists a formal topology $(S, \leq, \lhd)$ such that, writing j for the nucleus on $Low(S)$ associated to it, $\mathcal{A}$ and $Low(S)_j$ are isomorphic.*

The nucleus j of Proposition 2.5 can be assumed to extend to an inflationary, monotone and idempotent operator on $Pow(S)$, such that $j({\downarrow} p) = jp$, for all $p \in Pow(S)$. As observed in [13], this extension is not necessarily a nucleus on $Pow(S)$, since j does not need to preserve meets of arbitrary subsets of S. Note that in the characterisation of the subsets $p \subseteq S$ that are in $Low(S)_j$, we do not need to assume that p is a lower subset, since j extends to an operator on $Pow(S)$.

Example 2.6. Theorem 4.3 concerns the *double-negation* formal topology. This is the formal topology on the set $1 = \{\emptyset\}$ that is defined by letting, for $a \in 1$ and a subset $p \subseteq 1$, $a \lhd p =_{\text{def}} \neg\neg\, a \in p$. This formal topology determines a nucleus on the set-generated frame Ω of Example 2.3.

2.2 Extending the formal topology

Given a formal topology $(S, \leq, \lhd)$, let j be the nucleus on $Low(S)$ associated to it. Recall that we can assume that j extends to a closure operator on $Pow(S)$ and that $j(\downarrow p) = jp$, for $p \in Pow(S)$. Under the Heyting-valued interpretation, restricted formulas will be interpreted as elements of $Low(S)_j$, which are subsets $p \subseteq S$ such that $p = jp$. We are then naturally led to consider subclasses of S to interpret arbitrary formulas. To do so correctly, we need to extend the nucleus j to an operator J on lower subclasses of S that coincides with j on lower sets, and that inherits its properties. For a lower subclass $P \subseteq S$ let

$$JP =_{\text{def}} \bigcup \{ jv \mid v \subseteq P \} . \tag{10}$$

The following result, that is proved via direct calculations, shows that J and j coincide on the lower subclasses of S that are sets.

Lemma 2.7. *For all $p \in Low(S)$, it holds that $Jp = jp$.*

The results of Subsection 1.2 are applied to prove that J inherits all the properties of the nucleus j.

Lemma 2.8. *For all lower subclasses $P \subseteq S$, it holds that*

$$(\forall u \subseteq JP)(\exists v)\big(v \subseteq P \wedge u \subseteq jv\big) .$$

Proof. Let u be a subset of JP. For an element $x \in u$ and a subset $v \subseteq P$ define $\phi =_{\text{def}} x \in jv$. By the definition of J and the fact that j is monotone, we have

$$(\forall x \in u)\big((\exists v)(v \subseteq P \wedge \phi) \wedge (\forall v)(\forall w)\big((v \subseteq w \subseteq P \wedge \phi) \rightarrow \phi[w/v]\big)\big) .$$

Proposition 1.3 implies that there is a set v such that $v \subseteq P$ and $(\forall x \in u)\phi$. The desired conclusion follows by the definition of ϕ. $\square$

Proposition 2.9. *Let P, Q be lower subclasses of S. It holds that*

(i) $P \subseteq JP$,

(ii) *if* $P \subseteq Q$ *then* $JP \subseteq JQ$,

(iii) $J(JP) \subseteq JP$,

(iv) $JP \cap JQ \subseteq J(P \cap Q)$.

Proof. Direct calculations suffice to prove (i), (ii) and (iv). Lemma 2.8 implies (iii). □

For an arbitrary subclass $P \subseteq S$, define JP as in (10). Since $JP = J(\downarrow P)$, J extends to a operator on arbitrary subclasses of S. It is a closure operator on subclasses of S, because the properties in (i), (ii) and (iii) of Proposition 2.9 hold without the assumption that P and Q are lower subclasses. In particular, the assumption that P is a lower subclass was never used in the proof of Lemma 2.8. Using the operator J, we can extend the meet, join, and Heyting implication operations to subclasses $P, Q \subseteq S$ such that $P = JP$, $Q = JQ$, by letting

$$P \wedge Q \quad =_{\text{def}} \quad P \cap Q\,, \tag{11}$$

$$P \vee Q \quad =_{\text{def}} \quad J(P \cup Q)\,, \tag{12}$$

$$P \to Q \quad =_{\text{def}} \quad \{x \in S \mid x \in P \to x \in Q\}\,. \tag{13}$$

The definitions in (5) and (11), (6) and (12), (7) and (13), are compatible by Lemma 2.7. The proof of the next lemma follows by direct calculations.

Lemma 2.10. *Let P and Q be subclasses of S such that $P = JP$, $Q = JQ$. The following properties hold:*

(i) *$P \wedge Q$ is a subclass of S such that $J(P \wedge Q) = P \wedge Q$. If R is a subclass of S such that $JR = R$, then $R \subseteq P \wedge Q$ if and only if $R \subseteq P$ and $R \subseteq Q$.*

(ii) *$P \vee Q$ is a subclass of S such that $J(P \vee Q) = P \vee Q$. If R is a subclass of S such that $JR = R$, then $P \vee Q \subseteq R$ if and only if $P \subseteq R$ or $Q \subseteq R$.*

(iii) *$P \to Q$ is a subclass of S such that $J(P \to Q) = P \to Q$. If R is a subclass of S such that $JR = R$, then $R \subseteq P \to Q$ if and only if $R \wedge P \subseteq Q$.*

To interpret correctly unrestricted quantifiers, we need to extend the supremum and infimum operations to *family of subclasses* of S, as defined in [6, Section 3.1]. Let $(P_x)_{x \in U}$ be a family of subclasses of S such that, for all $x \in U$, we have $P_x = J(P_x)$. We define

$$\bigvee_{x \in U} P_x \quad =_{\text{def}} \quad J\Big(\bigcup_{x \in U} P_x\Big)\,, \tag{14}$$

$$\bigwedge_{x \in U} P_x \quad =_{\text{def}} \quad \bigcap_{x \in U} P_x\,. \tag{15}$$

If U is a set and, for all $a \in U$, P_a is a set, then the class $\{P_x \mid x \in U\}$ is a set by Replacement, that is a consequence of Strong Collection. The definitions in (8) and (14), (9) and (15), are therefore compatible by Lemma 2.7. Again, the proof of the next lemma follows by direct calculations.

Lemma 2.11. *Let $(P_x)_{x \in U}$ be a family of subclasses of S such that for all x in U we have $P_x = J(P_x)$. The following hold:*

(i) *$\bigvee_{x \in U} P_x$ is a subclass of S such that $\bigvee_{x \in U} P_x = J\big(\bigvee_{x \in U} P_x\big)$. If R is a subclass of S such that $R = JR$ then $\bigvee_{x \in U} P_x \subseteq R$ if and only if $P_a \subseteq R$ for all $a \in U$.*

(ii) *$\bigwedge_{x \in U} P_x$ is a subclass of S such that $\bigwedge_{x \in U} P_x = J\big(\bigwedge_{x \in U} P_x\big)$. If R is a subclass of S such that $R = JR$ then $R \subseteq \bigvee_{x \in U} P_x$ if and only if $R \subseteq P_a$ for all $a \in U$.*

2.3 Points

A *point* of a set-generated frame $\mathcal{A}$ is a frame morphism from $\mathcal{A}$ to Ω. The next definition, where we use the symbol & to stand for logical conjunction to avoid confusion, and the symbol _ to stand for an anonymous bound variable, presents a variation over the notion of a completely prime filter that is appropriate in our context. Proposition 2.13 gives an alternative characterisation of the points of a set-generated frame. Its proof is essentially straightforward, and therefore is omitted. Details may be found in [19].

Definition 2.12. Let $\mathcal{A} = (A, \leq, \bigvee, \wedge, \top, g)$ be a set-generated frame. We say that a subclass $F \subseteq A$ is a *set-generated completely prime filter* if

- $F \cap g$ is a set,
- F is inhabited, i.e. $(\exists \_ \in F)\top$,
- F is an upper subclass of A, i.e. $(\forall x, y \in A) x \in F \,\&\, x \leq y \to y \in F$
- F is meet-closed, i.e. $(\forall x, y \in A) x \in F \,\&\, y \in F \to x \wedge y \in F$,
- F is completely prime, i.e. $(\forall u \in \mathit{Pow}\, A) \bigvee u \in F \to (\exists x \in u) x \in F$.

Proposition 2.13. *Let $\mathcal{A}$ be a set-generated frame. There is a bijective correspondence between set-generated completely prime filters of $\mathcal{A}$ and frame morphisms from $\mathcal{A}$ to Ω.*

The notion of a formal point [30, 31], that we recall in the next definition, can be related to the one of a set-generated completely prime filter. The proof of the Proposition 2.15 consists of simple calculations, and is therefore left to the reader.

Definition 2.14. Let $\mathcal{S} = (S, \leq, \lhd)$ be a formal topology. A subset $\alpha \subseteq S$ is said to be a *formal point* if

- α is inhabited,
- α is an upper subset of S,
- α is stable, i.e. $(\forall x, y \in S) x \in \alpha \,\&\, y \in \alpha \to (\exists z \in \alpha) z \leq x \,\&\, z \leq y$,
- α is prime, i.e. $(\forall x \in S)(\forall u \in \mathit{Pow}\, S) x \in \alpha \,\&\, x \lhd u \to (\exists y \in u) y \in \alpha$.

Proposition 2.15. *Let $\mathcal{S}$ be a formal topology, and let $\mathcal{A}$ be the set-generated frame determined by it. There is a bijective correspondence between the formal points of $\mathcal{S}$ and the set-generated completely prime filters of $\mathcal{A}$.*

2.4 Posites and inductive definitions

The technique of defining frames via 'generators and relations' is folklore in locale theory [23, Section 2.11], and was adapted to formal topology, working in the setting of Martin-Löf type theory [12]. We review how this method works in a constructive set theory, exploiting the theory of inductive definitions [6, Chapter 5]. In the next definition, we call a *posite* what is referred to as a covering system in [26, pages 524 – 525]. This is essentially just a variation over the notion of a site [23, Section 2.11].

Definition 2.16. A *posite* $(S, \leq, \mathit{Cov})$ is given by a poset $(S, \leq)$, and function $\mathit{Cov} : S \to \mathit{Pow}(\mathit{Pow}\, S)$, called the *coverage*, that satisfies the following properties:

- $(\forall u \in \mathit{Cov}\, a) u \subseteq \downarrow\{a\}$
- $(\forall x, y \in S) y \leq x \to (\forall u \in \mathit{Cov}\, x)(\exists v \in \mathit{Cov}\, y) v \leqq u$,

where $v \leqq u =_{\mathrm{def}} (\forall y \in v)(\exists x \in u) y \leq x$, for subsets $u, v \subseteq S$.

Let $(S, \leq, \mathit{Cov})$ be a posite and let $\mathcal{A}$ be a set-generated frame. We say that a function $f : S \to A$ is a *coverage map* if it holds that

- f respects top element, i.e. $\top \leq \bigvee\{f(x) \mid x \in S\}$,
- f is monotone,
- f respects meets, i.e. $(\forall x, y \in S) fx \wedge fy \leq \bigvee\{fz \mid z \leq x\,,\ z \leq y\}$,
- f sends covers to joins, i.e. $(\forall x \in S)(\forall u \in \mathit{Cov}\, x) fx = \bigvee\{fy \mid y \in u\}$.

From now on we consider an arbitrary but fixed posite $(S, \leq \mathit{Cov})$. A lower subclass $X \subseteq S$ is an *ideal* if it holds that

$$(\exists u \in \mathit{Cov}\, a)(u \subseteq X) \to a \in X\,.$$

A *set-ideal* is an ideal that is a set, and we write $Idl(S)$ for the class of set-ideals. For an inductive definition Φ on S, that is a subset of $Pow(S) \times S$ [6, Chapter 5], a subclass $X \subseteq S$ is said to be Φ-*closed* if it holds that

$$p \subseteq X \to a \in X$$

for all $(p, a) \in \Phi$. For a subset $p \subseteq S$, we define $I(\Phi, p)$ to be the smallest class containing p that is Φ-closed. This class exists by Theorem 5.1 of [6]. Assuming the Regular Extension axiom (REA), the class $I(\Phi, p)$ is a set, for any subset $p \subseteq S$, by Theorem 5.7 of [6]. For the remainder of this section, we assume REA and exploit this fact.

The inductive definition Φ on S defined by

$$\Phi =_{\text{def}} \{(\{y\}, x) \mid x, y \in S, x \leq y\} \cup \{(u, x) \mid x \in S, u \in Cov\, x\}$$

is such that the Φ-closed subclasses of S are exactly the ideals of the posite. For $a \in S$ and a subset $p \subseteq S$, we then define

$$a \lhd p =_{\text{def}} a \in I(\Phi, p)\,, \tag{16}$$

and let $jp =_{\text{def}} I(\Phi, p)$. The next lemma is an immediate consequence of the definition of j.

Lemma 2.17 (Induction principle). *For a subset $p \subseteq S$, and a subclass $X \subseteq S$, if X is an ideal and $p \subseteq X$, then $jp \subseteq X$.*

The induction principle leads to the following result, whose proof can be carried over in CZF$^+$. An analogous result in the setting of Martin-Löf type theory is given in [7].

Theorem 2.18 (Johnstone's coverage theorem). *Let $(S, \leq, Cov)$ be a posite. Then $Idl(S)$ is a set-generated frame, and there is a coverage map $\gamma : S \to Idl(S)$ defined by letting, for $a \in S$, $\gamma(a) =_{\text{def}} j\{a\}$. If $\mathcal{A}$ is a set-generated frame, then for every coverage map $f : S \to \mathcal{A}$ there exists a unique frame morphism $\Phi_f : Idl(S) \to \mathcal{A}$ such that the following diagram*

$$\begin{tikzcd} S \arrow[r, "\gamma"] \arrow[dr, "f"'] & Idl(S) \arrow[d, "\Phi_f"] \\ & \mathcal{A} \end{tikzcd}$$

commutes.

Proof. The proof can be obtained following the pattern of the proof of Proposition 2.11 of [23], using repeatedly Lemma 2.17. For the first claim, one should observe that the relation defined in (16) is a formal topology and therefore we get a nucleus j on $Low(S)$. We have that $Low(S)_j = Idl(S)$, since set-ideals are Φ-closed sets. Therefore $\mathit{Idl}(S)$ is a set-generated frame, because so is $\mathit{Low}(S)_j$. For the second part, given a coverage map $f : S \to A$ define a function $\Phi_f : \mathit{Idl}(S) \to A$ by letting, for $p \in \mathit{Idl}(S)$,

$$\Phi_f(p) =_{\text{def}} \bigvee \{fx \mid x \in p\} .$$

The required properties can be verified exploiting the fact that $\mathit{Idl}(S)$ has a generating set. $\square$

Example 2.19. In [19, Section 4.6] it is shown how posites determine pointfree versions not only of the Baire and Cantor spaces, but also of the Dedekind reals as defined in CST [6, Section 3.6]. We refer to these posites as the *formal Baire, Cantor and Dedekind space* and write B, C and D for the set-generated frames associated to them. The well-known definitions of these posites [16, 23, 30] or [26, pages 524 – 525] can be carried over working in CST. Theorem 4.3 will show that the double-negation formal topology of Example 2.6 cannot be obtained using posites.

We illustrate some further consequences of the assumption of REA. For a formal topology $(S, \leq, \lhd)$, let $a \in S$ and consider the class of 'covers' of a, i.e. the subsets $p \subseteq S$ such that $a \lhd p$. In general this class is not a set, but for formal topologies defined inductively, it is possible to replace it with a set, in the sense specified by the next definition.

Definition 2.20. A formal topology $(S, \leq \lhd)$ is said to be *set-presentable* if there exists a *set-presentation* for it, i.e. a function $R : S \to \mathit{Pow}(\mathit{Pow}\, S)$ that is a set and such that

$$a \lhd p \leftrightarrow \big(\exists u \in R(a)\big) u \subseteq p$$

holds, for all $a \in S$ and all subsets $p \subseteq S$.

An application of the Set Compactness Theorem [6, Theorem 5.11], which can be proved using REA, leads to the following result.

Proposition 2.21 (Aczel). *The formal topologies determined by a posite are set-presentable.*

We now provide a characterisation of the points of $\mathit{Idl}(S)$.

Definition 2.22. We say that a subset $\chi \subseteq S$ is a *coverage filter* if

- χ is inhabited,
- χ is an upper subset of S,
- χ is stable, i.e. $(\forall x, y \in S) x \in \chi \,\&\, y \in \chi \to (\exists z \in \chi) z \leq x \,\&\, z \leq y$,
- χ is closed, i.e. $(\forall x \in S)(\forall u \in \mathit{Cov}\, x) x \in \chi \leftrightarrow (\exists y \in u) y \in \chi$.

Proposition 2.23. *Let $(S, \leq, Cov)$ be a posite. There is a bijective correspondence between coverage filters of S and set-generated completely prime filters of $Idl(S)$.*

Proof. The coverage filters are in bijective correspondence with coverage maps into the set-generated frame Ω. The claim then follows by Proposition 2.13 and Theorem 2.18. $\square$

3 Heyting-valued interpretations

3.1 Definition of the interpretation

From now on we work informally in CZF^-, and consider an arbitrary but fixed formal topology $\mathcal{S} = (S, \leq, \lhd)$. Let j be the nucleus j on $Low(S)$ that associated to the formal topology. For a function f, $dom(f)$ and $ran(f)$ denote its domain and range, respectively. The class $V^{(\mathcal{S})}$ of 'Heyting-valued sets' that is used to interpret sets, is defined via an inductive definition: we let $V^{(\mathcal{S})}$ be the smallest class X such that if f is a function with $dom(f) \subseteq X$ and $ran(f) \subseteq Low(S)_j$, then $f \in X$. This inductive definition determines a class within CZF^- by Theorem 5.1 of [6]. It is worth highlighting the content of this inductive definition as a lemma, whose proof is a direct consequence of the inductive definition of $V^{(\mathcal{S})}$.

Lemma 3.1. *Let a be a function. If $dom(a) \subseteq V^{(\mathcal{S})}$ and, for all $x \in dom(a)$, $ax \in Low(S)_j$, then $a \in V^{(\mathcal{S})}$.*

Our metatheory, i.e. the theory in which the interpretation is defined, is the constructive set theory CZF^-. We keep the notational conventions used until now and reserve the letters x, y, z, u, v, w (possibly with indexes or subscripts) for variables. The object theories, i.e. the theories that are interpreted, are CZF^- and extensions of its. In order to define the Heyting-valued interpretation, it is convenient to assume that the object theories are formulated an extension $\mathcal{L}^{(\mathcal{S})}$ of the language $\mathcal{L}$ with constants $a, b, c, \ldots$ for elements of $V^{(\mathcal{S})}$. Observe that the symbol a plays two roles: it is a constant of the object language $\mathcal{L}^{(\mathcal{S})}$, and it denotes a set in $V^{(\mathcal{S})}$ in the metatheory. With a slight abuse of language, if ϕ is a formula of $\mathcal{L}^{(\mathcal{S})}$ with $\mathrm{FV}\phi = \{x\}$ then we understand x both as a variable in the object language and as a variable in the metalanguage.

Let $a \in V^{(\mathcal{S})}$ and $(P_x)_{x \in dom(a)}$ be a family of subclasses of S such that for all $x \in dom(a)$, $J(P_x) = P_x$. We define

$$\bigvee_{x:a} P_x =_{\mathrm{def}} \bigvee_{x \in dom(a)} ax \wedge P_x \tag{17}$$

$$\bigwedge_{x:a} P_x =_{\mathrm{def}} \bigwedge_{x \in dom(a)} ax \rightarrow P_x \tag{18}$$

Observe that the supremum and infimum on the right-hand side of these defining equations exist because they are of the form in (14) and (15). Given $a, b \in V^{(\mathcal{S})}$, double set recursion allows us to define an element $a =_A b$ of $Low(S)_j$ such that the equation

$$a =_A b = \Big(\bigwedge_{x:a} \bigvee_{y:b} x =_A y\Big) \;\wedge\; \Big(\bigwedge_{y:b} \bigvee_{x:a} x =_A y\Big) \tag{19}$$

holds [22, Section 2.2]. The definition of the Heyting-valued interpretation is given by structural induction on formulas of the language $\mathcal{L}^{(\mathcal{S})}$. We let

$$\begin{aligned} [\![\bot]\!] &=_{\text{def}} \bot \\ [\![a = b]\!] &=_{\text{def}} a =_A b \end{aligned}$$

To interpret the binary logical connectives, we use the operations defined in (11), (12) and (13), and let

$$[\![\phi * \psi]\!] =_{\text{def}} [\![\phi]\!] * [\![\psi]\!]$$

where $* \in \{\wedge, \vee, \rightarrow\}$. The interpretation of restricted quantifiers uses the notation introduced in (17) and (18), and the suprema and infima required to interpret the unrestricted quantifiers are of the form in (14) and (15):

$$\begin{aligned} [\![(\exists x \in a)\phi]\!] &=_{\text{def}} \bigvee_{x:a} [\![\phi]\!] \\ [\![(\forall x \in a)\phi]\!] &=_{\text{def}} \bigwedge_{x:a} [\![\phi]\!] \\ [\![(\exists x)\phi]\!] &=_{\text{def}} \bigvee_{x \in V^{(\mathcal{S})}} [\![\phi]\!] \\ [\![(\forall x)\phi]\!] &=_{\text{def}} \bigwedge_{x \in V^{(\mathcal{S})}} [\![\phi]\!] \end{aligned}$$

A sentence ϕ of $\mathcal{L}^{(\mathcal{S})}$ said to be *valid* in $V^{(\mathcal{S})}$ if $[\![\phi]\!] = \top$. We say that an axiom scheme is *valid* if all of its instances with parameters that are elements of $V^{(\mathcal{S})}$ are valid.

Proposition 3.2. *Let θ, ϕ be sentences of $\mathcal{L}^{(\mathcal{S})}$, and assume that θ is restricted.*

(i) *$[\![\phi]\!]$ is a subclass of S such that $J[\![\phi]\!] = [\![\phi]\!]$.*

(ii) *$[\![\theta]\!]$ is a subset of S such that $j[\![\theta]\!] = [\![\theta]\!]$, and so $[\![\theta]\!] \in Low(S)_j$.*

Proof. Lemma 2.10 and Lemma 2.11 imply (i). For (ii), show by structural induction that the operations of the set-generated frame $(Low\ S)_j$ suffice to define the interpretation of a restricted formula. □

3.2 Validity of the basic axioms

We continue to work informally in CZF^-. Lemma 3.3, Lemma 2.10 and Lemma 2.11 imply that the axioms for intuitionistic logic and for restricted quantifiers are valid.

Lemma 3.3. *Let $a, b \in V^{(\mathcal{S})}$ and ϕ a formula with $\mathrm{FV}\phi = \{x\}$. Then it holds that*

$$[\![\phi[a/x]]\!] \wedge [\![a = b]\!] \leq [\![\phi[b/x]]\!] \, .$$

Proof. An argument by structural induction proves the claim. □

Proposition 3.4. *Extensionality and Set Induction are valid in $V^{(\mathcal{S})}$.*

Proof. Validity of Extensionality follows by the equivalence in (19). Validity of Set Induction is direct consequence of the inductive definition of $V^{(\mathcal{S})}$. □

We define an embedding from the class of all sets into $V^{(\mathcal{S})}$. For a set a, define by set recursion a function $\widehat{a}$ with domain $\{\widehat{x} \mid x \in a\}$ by letting, for $x \in a$

$$\widehat{a}(\widehat{x}) =_{\mathrm{def}} \top \, ,$$

and observe that $\widehat{a} \in V^{(\mathcal{S})}$ by Lemma 3.1. The next definition uses this embedding to define a notion that will be used in the applications of Heyting-valued interpretations in Section 4.

Definition 3.5. A formula ϕ with $\mathrm{FV}\phi = \{x_1, \ldots, x_n\}$ is said to be *absolute* if for all $a_1, \ldots, a_n$ the equivalence

$$\phi[a_1, \ldots, a_n/x_1, \ldots, x_n] \leftrightarrow [\![\phi[\widehat{a}_1, \ldots, \widehat{a}_n/x_1, \ldots, x_n]]\!] = \top$$

holds.

Lemma 3.6. *Let a, b be sets. The following equivalences*

(i) $[\![\widehat{a} = \widehat{b}]\!] = \top$ *if and only if* $a = b$,

(ii) $[\![\widehat{a} \in \widehat{b}]\!] = \top$ *if and only if* $a \in b$,

hold.

Proof. Direct calculations using Set Induction. □

Proposition 3.7. *All restricted formulas are absolute.*

Proof. The proof of Theorem 1.23 in [8] for Boolean-valued interpretations of Classical Set Theory carries over. In particular, the set-generated frame Ω plays in our context the same role that the complete Boolean algebra 2 plays in the classical context. □

Proposition 3.8. *Pairing, Union, Infinity and Restricted Separation are valid in $V^{(\mathcal{S})}$.*

Proof. The Heyting-valued interpretation of Pairing and Union can be shown to be valid following the proof used in the context of ZF or IZF [8, 10]. Validity of Infinity follows by embedding an infinite set in $V^{(\mathcal{S})}$. By means of illustration we present the proof of the validity of Restricted Separation in some detail. Let $a \in V^{(\mathcal{S})}$, and let θ a restricted formula with $\mathrm{FV}\phi = \{x\}$. Define a function b with the same domain of a by letting, for $x \in dom(a)$,

$$bx =_{\mathrm{def}} ax \wedge [\![\theta]\!] \, .$$

By part (ii) of Lemma 2.11 and Restricted Separation, bx is a set and, for all $x \in dom(a)$, we have $j(bx) = bx$. Hence we have that $b \in V^{(\mathcal{S})}$. For $x \in dom(a)$, we have $x \in dom(b)$ and $ax \wedge [\![\theta]\!] \leq bx$, and therefore

$$ax \wedge [\![\theta]\!] \leq [\![x \in b]\!] \, .$$

This implies the validity of $(\forall x \in a)(\theta \rightarrow x \in b)$. For $x \in dom(b)$ it holds that $x \in dom(a)$ and $bx \leq ax \wedge [\![\theta]\!]$ hold, by the definition of b. Hence we obtain $bx \leq [\![x \in a]\!] \wedge [\![\theta]\!]$. Validity of $(\forall x \in b)(x \in a \wedge \theta)$ follows by direct calculations and the definition of the Heyting-valued interpretation. □

3.3 Validity of the collection axioms

It does not seem possible to replace the use of Full Separation in the proof of the validity of the Collection axiom of IZF in Heyting-valued models [10] with an application of Restricted Separation. We can still prove the validity of Strong Collection without assuming Full Separation, but rather exploiting Strong Collection.

Lemma 3.9. *Let a in $V^{(\mathcal{S})}$ and let ϕ be a formula of $\mathcal{L}^{(\mathcal{S})}$ with $\mathrm{FV}\phi = \{x\}$.*

$$(\forall u \in Low(S)_j)\big(u \leq [\![(\forall x \in a)\phi]\!] \leftrightarrow (\forall x \in dom\, a) u \wedge ax \leq [\![\phi]\!]\big) \, .$$

Proof. Direct calculations suffice to prove the claim. □

Lemma 3.10. *Let $a \in V^{(\mathcal{S})}$ and let ϕ a formula of $\mathcal{L}^{(\mathcal{S})}$ with $\mathrm{FV}\phi = \{x, y\}$. Let $p \in Low(S)_j$ and define*

$$P =_{\mathrm{def}} \{(x, y, z) \mid x \in dom\, a \, , \; y \in V^{(\mathcal{S})} \, , \; z \in p \wedge ax \wedge [\![\phi]\!]\} \, .$$

Assume that $p \subseteq [\![(\forall x \in a)(\exists y)\phi]\!]$. Then there exists a subset $r \subseteq P$ such that

$$(\forall x \in dom\, a) p \wedge a(x) \subseteq j\{z \mid (\exists y)(x, y, z) \in r\} \, .$$

Proof. Let us introduce some notation, and define

$$Q =_{\mathrm{def}} \{(x,z) \mid (\exists y \in V^{(\mathcal{S})})(x,y,z) \in P\}\,,$$

and then, for x in $dom(a)$, define $Q_x =_{\mathrm{def}} \{z \mid (x,z) \in Q\}$. For x in $dom(a)$, v in $Pow(S)$ define $\psi =_{\mathrm{def}} p \wedge ax \subseteq jv$. Lemma 2.8 implies that

$$(\forall x \in dom\, a)\big((\exists v)(v \subseteq Q_x \wedge \psi) \wedge (\forall v)(\forall w)(v \subseteq w \wedge \psi \to \psi[w/v])\big)\,.$$

Proposition 1.2 implies that there is a function f with domain $dom(a)$ such that

$$(\forall x \in dom\, a)\big(fx \subseteq Q_x \wedge \psi[fx/v]\big)\,. \tag{20}$$

In view of an application of Proposition 1.1, let us define

$$q =_{\mathrm{def}} \{(x,z) \mid x \in dom(a)\,,\ z \in fx\}$$

and, for x in $dom(a)$, y in $V^{(\mathcal{S})}$ and z in S define $\xi =_{\mathrm{def}} (x,y,z) \in P$. By the definitions just introduced and (20) we obtain $(\forall (x,z) \in q)(\exists y)\xi$. We can then apply Proposition 1.1 and get a function g with domain q such that

$$(\forall (x,z) \in q)\big((\exists y)(y \in g(x,z)) \wedge (\forall y \in g(x,z))\xi\big)\,.$$

Once we define $r =_{\mathrm{def}} \{(x,y,z) \mid (x,z) \in q\,,\ y \in g(x,z)\}$, the desired conclusion is reached with direct calculations. □

Proposition 3.11. *Strong Collection is valid.*

Proof. We use the same notation and definitions used in Lemma 3.10. Let $a \in V^{(\mathcal{S})}$ and let ϕ be a formula with $\mathrm{FV}\phi = \{x,y\}$. Let $p \in Low(S)_j$ and assume that

$$p \subseteq [\![(\forall x \in a)(\exists y)\phi]\!]\,.$$

By Lemma 3.10 there is a subset $r \subseteq P$ such that

$$(\forall x \in dom\, a) p \wedge ax \subseteq j\{z \mid (\exists y)(x,y,z) \in r\}\,. \tag{21}$$

To define $b \in V^{(\mathcal{S})}$ such that $p \subseteq [\![coll(x \in a, y \in b, \phi)]\!]$ consider the function with domain

$$dom(b) =_{\mathrm{def}} \{y \mid (\exists x)(\exists z)(x,y,z) \in r\}\,.$$

and defined by letting, for y in $dom(b)$,

$$by =_{\mathrm{def}} j\{z \mid (\exists x)(x,y,z) \in r\}\,.$$

The conclusion now follows from (21). □

As we will see, it is not possible to prove without further assumptions on the formal topology $\mathcal{S}$ that Subset Collection is valid, even assuming Subset Collection in the metatheory. We therefore assume that the formal topology is set-presentable, in the sense of Definition 2.20, and let R be a set-presentation for it. Define r as the image of the function R by letting

$$r =_{\mathrm{def}} \{u \mid (\exists x \in S)\, u \in R(x)\} \,.$$

By the definition of set-presentable formal topology, for $a \in S$ and a subset $p \subseteq S$ we have that $a \in jp$ holds if and only if $(\exists u \in r)(a \in ju \wedge u \subseteq p)$ does.

Lemma 3.12. *For $a \in S$ and a subclass $P \subseteq S$, we have*

$$a \in JP \leftrightarrow (\exists u \in r)(a \in ju \wedge u \subseteq P) \,.$$

Proof. Direct calculations suffice to prove the claim. □

Define $g =_{\mathrm{def}} \{j\{x\} \mid x \in S\}$ and recall that g is a generating set for the set-generated frame $Low(S)_j$. The next lemma is proved assuming Subset Collection and exploiting Proposition 1.4.

Lemma 3.13. *Let $a, b \in V^{(\mathcal{S})}$ and let ϕ be a formula with* $\mathrm{FV}\phi = \{x, y, z\}$. *There exists a subset $d \subseteq V^{(\mathcal{S})}$ such that for all $z \in V^{(\mathcal{S})}$ and for all $p \in g$ if*

$$p \subseteq [\![(\forall x \in a)(\exists y \in b)\phi]\!] \,,$$

then there exists $e \in d$ such that $p \subseteq [\![coll(x \in a, y \in e, \phi)]\!]$.

Proof. Let $p \in g$ and assume

$$p \subseteq [\![(\forall x \in a)(\exists y \in b)\phi]\!] \,. \tag{22}$$

We will apply Proposition 1.4 twice, and so it is convenient to define sets a', b' as follows:

$$\begin{aligned} a' &=_{\mathrm{def}} \{(x, w') \in dom(a) \times S \mid w' \in p \cap ax\} \,, \\ b' &=_{\mathrm{def}} dom(b) \times S \,. \end{aligned}$$

The set a' will be used in the second application of Proposition 1.4, while the set b' will be used in the first. For $x \in dom(a)$, $y \in dom(b)$ and $z \in V^{(\mathcal{S})}$ define the class

$$P_{x,y} =_{\mathrm{def}} ax \wedge by \wedge [\![\phi]\!] \,.$$

Let $x' \in a'$. By the definition of a' we get $x \in dom(a)$ and $w' \in p \cap ax$ such that $x' = (x, w')$. We now define the formula that will be used in our first application of Proposition 1.4. For $q \in r$, $w \in q$ and $y' \in b'$ define

$$\psi =_{\mathrm{def}} (\exists y \in dom\, b)\big(y' = (y, w) \wedge w \in q \cap P_{x,y}\big) \,.$$

From (22) and Lemma 3.12 we derive that there is $q \in r$ such that $w' \in jq$ and

$$(\forall w \in q)(\exists y' \in b')\psi \,.$$

By Proposition 3.12, we obtain a set c', independent of p, x', q and z, such that there is $u \in c'$ for which

$$\mathit{coll}(w \in q, y' \in u, \psi) \tag{23}$$

holds. We now define the formula used in the second application of Proposition 3.12. For x', x, w', q and u define

$$\xi =_{\text{def}} (\exists q \in r)(\exists x \in \mathit{dom}\, a)(\exists w \in p \cap a(x) \cap jq)\chi$$

where $\chi =_{\text{def}} x' = (x, w') \wedge \mathit{coll}(w \in q, y' \in u, \psi)$. Discharging the assumption of $x' \in a'$, we obtain

$$(\forall x' \in a')(\exists u \in c')\xi \,.$$

A second application of Proposition 3.12 implies that there is a set c, independent of p and z, such that there exists $v \in c$ for which

$$\mathit{coll}(x' \in a', u \in v, \xi) \tag{24}$$

holds. For $v \in c$ define a function f_v with domain $\mathit{dom}(b)$ by letting, for $y \in \mathit{dom}\, b$

$$f_v(y) =_{\text{def}} j\{w \mid (y, w) \in \bigcup v\} \,.$$

Define $d =_{\text{def}} \{f_v \mid v \in c\}$ and observe that d is a subset of $V^{(\mathcal{S})}$. To conclude the proof, let $v \in c$ and assume that it satisfies (24). Define $e =_{\text{def}} f_v$ so that we have $e \in d$. We show that

$$p \subseteq [\![\mathit{coll}(x \in a, y \in e, \phi)]\!]$$

holds in two steps. For the first step, let $x \in \mathit{dom}(a)$ and $w' \in p \cap ax$. Using (24) and (23) we obtain that there is $q \in r$ such that

$$\big(w' \in jq\big) \wedge \big(q \subseteq \bigcup_{y \in \mathit{dom}\, e} e(y) \cap [\![\phi]\!]\big) \,.$$

We then get $p \subseteq [\![(\forall x \in a)(\exists y \in e)\phi]\!]$ and this concludes the first step. For the second step, let $y \in \mathit{dom}(e)$ and define

$$t =_{\text{def}} p \cap \{w \in s \mid (y, w) \in \bigcup v\} \,.$$

We have

$$\big(p \cap ey \subseteq jt\big) \wedge \big(t \subseteq [\![(\exists x \in a)\phi]\!]\big) \,,$$

using (24) and (23). Therefore we get $p \subseteq [\![(\forall y \in e)(\exists x \in a)\phi]\!]$ and this concludes the second step. Putting together the conclusions reached at the end of the two steps, we get the desired result. $\square$

The next proposition is proved assuming Subset Collection.

Proposition 3.14. *Subset Collection is valid in $V^{(\mathcal{S})}$.*

Proof. Let $a, b \in V^{(\mathcal{S})}$ and let ϕ be a formula with $\mathrm{FV}\phi = \{x, y, z\}$. We can assume to have a set d as in the conclusion of Lemma 3.13. Then define a function c with domain d by letting, for $v \in d$, $cv =_{\mathrm{def}} \top$. Direct calculations lead to the validity of Subset Collection. □

The next theorem summarises the results of this section.

Theorem 3.15. *Let $\mathcal{S} = (S, \leq, \lhd)$ be a formal topology.*

(i) *The Heyting-valued interpretation of* CZF$^-$ *in $V^{(\mathcal{S})}$ is valid.*

(ii) *Assuming Subset Collection, if $\mathcal{S}$ is set-presentable, then the Heyting-valued interpretation of* CZF *in $V^{(\mathcal{S})}$ is valid.*

4 Applications

4.1 Proof-theoretic applications

Let $\mathcal{S}$ be the double-negation formal topology of Example 2.6, and note that the nucleus associated to it is defined by letting

$$jp =_{\mathrm{def}} \{x \in 1 \mid \neg\neg x \in p\} ,$$

for $p \in \Omega$. The nucleus j can be extended to an operator J on subclasses of 1 following the definition in (10). For a subclass $P \subseteq 1$ define

$$JP =_{\mathrm{def}} \bigcup \{jv \mid v \subseteq P\} .$$

It holds that $\{x \in 1 \mid \neg\neg x \in P\} \subseteq JP$, but it does not seem possible to prove the reverse inclusion without further assumptions on P. We now consider the Heyting-valued interpretation in $V^{(\mathcal{S})}$. The law of *restricted excluded middle*, REM, is the scheme

$$\theta \vee \neg\theta ,$$

where θ is a restricted formula. Observe that REM is equivalent to the sentence

$$(\forall v \in \Omega)(v = 1 \vee \neg v = 1) .$$

In [11] the set theory CZF$^-$ + REM was given an interpretation into a semi-classical system W that can in turn be interpreted in a Martin-Löf type theory with well-ordering types. Here we use Heyting-valued interpretations to obtain a direct interpretation of CZF$^-$ + REM into a theory with intuitionistic logic.

Lemma 4.1. *Let $\mathcal{S}$ be the double-negation formal topology. The Heyting-valued interpretation of* $\mathrm{CZF}^- + \mathrm{REM}$ *in* $V^{(\mathcal{S})}$ *is valid.*

Proof. Let θ be a restricted sentence and observe that $\neg\neg(\neg\neg\,\theta \vee \neg\theta)$ is derivable in intuitionistic logic. For p in Ω_j define

$$\neg p =_{\mathrm{def}} p \to \bot\,,$$

and observe that

$$\top = \neg\neg(\neg\neg[\![\theta]\!] \cup [\![\neg\theta]\!])\,,$$

by the validity of Heyting-valued interpretations and direct calculations. We have that $[\![\theta]\!]$ is in Ω_j by Lemma 3.2, and thus $[\![\theta]\!] = \neg\neg[\![\theta]\!]$. We therefore obtain

$$\top = [\![\theta \vee \neg\theta]\!]\,,$$

which shows the validity of REM. The validity of the axioms of CZF^- is part (i) of Theorem 3.15. □

By standard coding, for a set theory T there is a sentence $\mathit{Con}(\mathrm{T})$ in the language of first-order arithmetic asserting the consistency of T. A set theory T_1 is *reducible* to another set theory T_2 if $\mathit{Con}(\mathrm{T}_2) \to \mathit{Con}(\mathrm{T}_1)$ is provable in first-order arithmetic. Theorem 1.19 of [8] shows that Boolean-valued interpretations give relative consistency proofs for extensions of ZF. The theorem carries over also to Heyting-valued interpretations and therefore we obtain the next result, that is a direct consequence of Lemma 4.1.

Theorem 4.2. $\mathrm{CZF}^- + \mathrm{REM}$ *is reducible to* CZF^-.

The independence result we prove next was suggested to us by Thierry Coquand, and seems to have been first expected in [21]. Let us now consider the theory $\mathrm{CZF} + \mathrm{REM}$. Recall from [6, Chapter 9] that this set theory has at least the proof-theoretic strength of second-order arithmetic and therefore

$$\mathrm{CZF} + \mathrm{REM} \vdash \mathit{Con}(\mathrm{CZF})\,.$$

Theorem 4.3. *The sentence asserting that the double-negation formal topology is set-presentable cannot be proved in* CZF.

Proof. Let ϕ be the sentence asserting that the double-negation formal topology is set-presentable and assume

$$\mathrm{CZF} \vdash \phi\,. \tag{25}$$

Theorem 3.15 shows that the Heyting-valued interpretation of CZF in $V^{(\mathcal{S})}$ is valid. Furthermore we have seen that REM is valid. Combining these two facts we obtain that $\mathit{Con}(\mathrm{CZF})$ is valid in $V^{(\mathcal{S})}$. Since $\mathit{Con}(\mathrm{CZF})$ is an absolute formula, we have $\mathrm{CZF} \vdash \mathit{Con}(\mathrm{CZF})$ by Proposition 3.7. But this is a contradiction to Gödel's second incompleteness theorem. We have therefore proved that the assumption (25) leads to a contradiction, hence the conclusion. □

A consequence of Theorem 4.3 and Proposition 2.21 is that the double-negation formal topology cannot be described using posites and inductive definitions. If this was the case, then the formal topology would indeed be set-presentable. Another example of formal topology that cannot be defined using posites is given in [12].

4.2 Sheaf-theoretic applications

In the theory of sheaf toposes there is a strong correspondence between internal notions, i.e. notions defined in the internal logic of a topos, and external notions, i.e. notions defined in the formal system in which sheaf toposes are considered. For example, the internal Dedekind reals in the topos of sheaves over a topological space $(X, \mathcal{O}(X))$ correspond to external continuous functions from $(X, \mathcal{O}(X))$ to the Dedekind reals $(R, \mathcal{O}(R))$ [26, Section VI.8], and similar theorems can be proved for localic toposes [17].

We transfer these results in the context of constructive set theories, replacing concrete spaces with their pointfree counterparts. This allows us to obtain representation of internal points as external frame morphisms without assuming additional principles. From now on, we will work with a fixed formal topology $\mathcal{T}$.

Definition 4.4. Let ϕ be a formula of $\mathcal{L}^{(\mathcal{T})}$ with free variables $x_1, \dots, x_n$. We say that elements $a_1, \dots, a_n$ of $V^{(\mathcal{T})}$ *satisfy* ϕ *in* $V^{(\mathcal{T})}$ if

$$[\![\phi[a_1, \dots, a_n / x_1, \dots, x_n]]\!] = \top \,.$$

We say that the elements of a definable collection of classes *represent* the elements of $V^{(\mathcal{T})}$ that satisfy ϕ if there is a definable operation assigning to each class P in the collection an element b_P of $V^{(\mathcal{T})}$ such that for all a in $V^{(\mathcal{T})}$ that satisfy ϕ in $V^{(\mathcal{T})}$ there is a unique class P in the collection such that $[\![a = b_P]\!] = \top$.

Recall from Subsection 3.2 that there is an embedding assigning an element $\widehat{a}$ of $V^{(\mathcal{T})}$ to any set a. Let θ be the formula of $\mathcal{L}^{(\mathcal{T})}$ with $\mathrm{FV}\theta = \{x\}$ asserting that x is a posite. By the definition of posite, θ is a restricted formula and therefore if x is a posite then $\widehat{x}$ satisfies θ in $V^{(\mathcal{T})}$ by Proposition 3.7. Let x be a posite, and let ϕ be the formula with a free variable y expressing that y is a coverage filter of $\widehat{x}$. We refer to the elements of $V^{(\mathcal{T})}$ that satisfy ϕ as the *internal points* of the posite x in $V^{(\mathcal{T})}$.

Theorem 4.5. *Let $\mathcal{T}$ be a formal topology, and let $\mathcal{B} =_{\mathrm{def}} Low(T)_j$ be the set-generated frame associated to it. For any posite $(S, \leq, Cov)$, frame morphisms from $Idl(S)$ to $\mathcal{B}$ represent internal points of $(S, \leq, Cov)$ in $V^{(\mathcal{T})}$.*

Proof. By Theorem 2.18 it is sufficient to show that coverage maps from S to B represent internal points of (S, Cov) in $V^{(\mathcal{T})}$. Given a coverage map f

from S to B, define an element χ_f of $V^{(\mathcal{T})}$ as follows: χ_f is a function with domain $\{\widehat{x} \mid x \in S\}$ defined by letting, for x in S

$$\chi_f(\widehat{x}) =_{\mathrm{def}} f(x)$$

and observe that χ_f is in $V^{(\mathcal{T})}$ because its domain is a subset of $V^{(\mathcal{T})}$ and its range is a subset of B. The proof that χ_f is a coverage filter of $\widehat{S}$ is a consequence of the assumption that f is a coverage map. Now, let χ be a coverage filter of $\widehat{S}$ in $V^{(\mathcal{T})}$. We need to find a coverage map f from S to B such that

$$[\![\chi = \chi_f]\!] = \top\,. \tag{26}$$

Define f_χ as the function with domain S defined by letting, for x in S

$$f_\chi(x) =_{\mathrm{def}} [\![\widehat{x} \in \chi]\!]\,,$$

and observe that f_χ is a coverage map because χ is a coverage filter of $\widehat{S}$. The calculations to show this involve applications of Proposition 3.7, but are straightforward. To show $[\![\chi = \chi_{f_\chi}]\!] = \top$ we use the validity of Extensionality in $V^{(\mathcal{T})}$, as follows. Let x in S and observe that

$$[\![\widehat{x} \in \chi]\!] = f_\chi(x) = [\![\widehat{x} \in \chi_{f_\chi}]\!]\,.$$

Finally, to show that f_χ is unique among the maps f for which (26) holds, observe that for all coverage maps f we have $f_{\chi_f} = f$. □

A direct consequence of Theorem 4.5 is the following representation of the internal points of the spaces discussed in Example 2.19.

Corollary 4.6. *Let $\mathcal{T}$ be a formal topology, and let $\mathcal{B}$ be the set-generated frame associated to it. Frame morphisms from* B, C *and* D *to $\mathcal{B}$ represent the internal points in $V^{(\mathcal{T})}$ of the formal Baire, Cantor and Dedekind space, respectively.*

4.3 Future work

Theorem 4.5 and Corollary 4.6 represent the first steps to obtain for constructive set theories the relative consistency and independence results obtained for intuitionistic set theories in [17, 32, 33]. For example, the Heyting-valued interpretations developed in this paper could be applied to prove the independence from CZF of various choice principles, like dependent and countable choice, and of principles of intuitionistic analysis, like the monotone bar induction and fan theorem principles [17].

We expect Heyting-valued interpretations to allow also further applications. Investigations into notions of real numbers in intuitionistic mathematics provide examples of interesting open problems. In [14] it is shown that, alongside the well-known notions of Cauchy and Dedekind reals, there is also another class of real numbers that is of interest: the Cauchy completion of the rationals [6, Section 3.6]. It is known that, assuming the principle of countable choice, the three notions are equivalent [14]. Heyting-valued interpretations for intuitionistic set theories have been applied to show that the Dedekind and the Cauchy reals are distinct by defining interpretations in which the countable choice principle fails [17]. Heyting-valued interpretations for constructive set theories seem a natural method to investigate the open problem of whether the Cauchy reals and the Cauchy completion of the rationals are distinct.

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