Under consideration for publication in Math. Struct. in Comp. Science

Spatiality for formal topologies

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Received June 28th, 2006

We define what it means for a formal topology to be spatial, and investigate properties related to spatiality both in general and in examples.

1. Introduction

The development of the theory of locales provides substantial evidence that much of classical topology can be developed assuming open sets rather than points as the primary object of study (Johnstone, 1982). Apart from its conceptual elegance, the point-free approach has led to important applications in the context of topos theory (Joyal and Tierney, 1984). This is because large parts of the theory of locales can be developed without the assumption of the axiom of choice, and hence can be internalised in any elementary topos (Johnstone, 1983; Johnstone, 1991). The point-free and the point-wise approaches are related by the adjunction between the category of locales and the category of topological spaces (Fourman and Scott, 1979; Johnstone, 1982). This adjunction restricts to an adjoint equivalence between the category of spatial locales and the category of sober topological spaces, so that one might say that the point-free and point-wise approach are essentially equivalent on these subcategories. For the precise history of this adjunction, see (Johnstone, 1982, p. 76).

In many examples of interest, the statement expressing the spatiality of a locale cannot be proved within the internal logic of an elementary topos (Fourman and Grayson, 1982; Fourman and Hyland, 1979). An extreme example of this situation is given by the spectrum of prime ideals of a commutative ring endowed with the Zariski topology. The existence of a prime ideal in an arbitrary non-trivial commutative ring is equivalent to the Boolean ultrafilter theorem (Banaschewski, 1983; Rav, 1977). Accordingly, the prime spectrum might not have any point at all within an elementary topos (Johnstone, 1977). Under these circumstances, it is generally the locale, rather than the topological space, that exhibits the desired topological properties. For example, the space of Dedekind reals cannot be shown to be locally compact constructively, while its point-free counterpart is a locally compact locale (Fourman and Grayson, 1982; Johnstone, 1982).

Our aim here is to begin a systematic study of spatiality for formal topologies. The notion of a formal topology, originally introduced in (Sambin, 1987), may be understood as a reformulation of the notion of a locale that makes it convenient to develop point-free topology within foundational settings that are even more restrictive than that of an elementary topos. The study of formal topologies can be carried out within the dependent type theories of Martin-Löf (Martin-Löf, 1984; Nordström et al., 2000) and their variants (Gambino and Aczel, 2006; Maietti and Sambin, 2005), or within the constructive set theories of Myhill (Myhill, 1975) and Aczel (Aczel and Rathjen, 2001). Despite the constraints necessary to work within these settings, parts of locale theory have already been reconstructed (Aczel, 2006; Gambino, 2006; Maietti and Valentini, 2004; Sambin, 1987) Furthermore, interesting connections to the theory of inductive and coinductive definitions start to emerge (Coquand et al., 2003; Fox, 2005; Sambin, 2003; Schuster, 2006).

In the following, we work within Constructive Zermelo-Fraenkel set theory (Aczel and Rathjen, 2001). This allows us to exploit the usual set-theoretic notation that is used in common mathematical practice, as well as to treat carefully some predicatively meaningful distinctions. Mathematics in Constructive Zermelo-Fraenkel set theory (CZF) differs substantially not only from classical mathematics in Zermelo-Fraenkel set theory, but also from constructive mathematics in Intuitionistic Zermelo-Fraenkel set theory or in an elementary topos. This is because the axiom system of CZF does not include the Power Set axiom, and assumes the Restricted Separation axiom instead of the Full Separation axiom. As a consequence of this, it is essential to work also with proper classes when developing constructive mathematics in CZF. For example, the class of points of a formal topology is sometimes too large to form a set. Nevertheless, since the language of CZF contains unrestricted quantifiers, we are allowed to use quantifiers ranging over the elements of a class (Aczel and Rathjen, 2001). In particular, we can quantify over the class of points of a formal topology.

The starting point for our study is the very definition of the notion of spatiality for a formal topology. Thanks to Peter Aczel's reformulation in CZF of the adjunction between topological spaces and locales (Aczel, 2006), which we review in Section 2, it is possible to provide a definition of spatiality for formal topologies that is is completely analogous to the one for locales (Fourman and Scott, 1979; Johnstone, 1982). We will then show how this conceptually clear definition relates to the definition of spatiality for a formal topology existing in the literature (Sambin, 1987). The rest of the paper is devoted to the study of various concepts related to spatiality, and of several examples. In particular, we will extend the results in (Schuster, 2006) concerning the spatiality of the formal Zariski topology. We will also rephrase in the context of formal topologies the characterisations of the spatiality for the point-free counterparts of the Baire space and the Cantor space that have been obtained for locales in (Fourman and Grayson, 1982). In the context of formal topologies, the link between the spatiality of the formal Baire space and the principle of monotone bar induction was already described in (Sambin, 1987).

Remark. The notion of category that we assume in the paper allows for objects and maps to have the size of classes. For background on CZF, the reader is invited to refer to (Aczel and Rathjen, 2001).

2. Point-free and point-wise topology in constructive set theory

Formal topologies

Let us recall some terminology and abbreviations concerning ordered structures that will be used throughout the paper. By a preordered class we mean a class A equipped with a class relation $\leq$ on A: that is, a subclass $\leq$ of $A \times A$ which is a preorder. If A is a set, we assume implicitly that the preorder relation is actually a subset of $A \times A$. By the Restricted Separation axiom of CZF, this is always the case when the preorder is defined by a restricted formula: that is, a formula in which all bound variables range over sets. If $(A, \leq)$ is a preordered class, for $a \in A$ let $\downarrow a =_{\mathrm{def}} \{x \in A \mid x \leq a\}$, and for $a, b \in A$, let $a \downarrow b =_{\mathrm{def}} \downarrow a \cap \downarrow b$. We can extend these definitions to subsets of A as follows: for $U \subseteq A$ let $\downarrow U =_{\mathrm{def}} \bigcup_{x \in U} \downarrow x$, and for for $U, V \subseteq A$ let $U \downarrow V =_{\mathrm{def}} \downarrow U \cap \downarrow V$. A subset $U \subseteq A$ is said to be downward closed if $\downarrow U \subseteq U$, and hence $U = \downarrow U$.

A *formal topology* consists of a preordered set $(A, \leq)$ equipped with a class relation $a \lhd U$ between elements $a \in A$ and subsets $U \subseteq A$, called the cover relation, such that for every subset $U \subseteq A$ the class $j(U) =_{\mathrm{def}} \{x \in A \mid x \lhd U\}$ is a set, and the following axioms hold:

$$\downarrow U \lhd U \,; \qquad \text{(Reflexivity)}$$

$$a \lhd U \,,\; U \lhd V \Longrightarrow a \lhd V \,; \qquad \text{(Transitivity)}$$

$$a \lhd U_1 \,,\; a \lhd U_2 \Longrightarrow a \lhd U_1 \downarrow U_2 \,; \qquad \text{(Stability)}$$

where we use the standard abbreviation $U \lhd V =_{\mathrm{def}} (\forall x \in U)\,(x \lhd V)$. A continuous map $p : A \to B$ between formal topologies is a relation, given by a subset $p \subseteq A \times B$, satisfying four axioms. To express these axioms let us define, for $U \subseteq A$ and $V \subseteq B$,

$$p_*(U) =_{\mathrm{def}} \{y \in B \mid (\forall x \in U) p(x, y)\} \,, \quad p^*(V) =_{\mathrm{def}} \{x \in A \mid (\exists y \in V) p(x, y)\} \,.$$

The relation $p \subseteq A \times B$ is a continuous map if the following hold:

$$p(a, b) \,,\; b \lhd V \Longrightarrow a \lhd p^*(V) \,;$$

$$a \lhd U \,,\; b \in p_*(U) \Longrightarrow p(a, b) \,;$$

$$A \lhd p^*(B) \,;$$

$$a \lhd p^*(V_1) \,,\; a \lhd p^*(V_2) \Longrightarrow a \lhd p^*(V_1 \downarrow V_2) \,.$$

We write **FTop** for the category of formal topologies and continuous maps. The identity map $i_A : A \to A$ is the relation $i_A(a, b) =_{\mathrm{def}} a \lhd \{b\}$. The composition of $p : A \to B$ and $q : B \to C$ is the relation $q \cdot p : A \to C$ defined by

$$(q \cdot p)(a, c) =_{\mathrm{def}} a \lhd \{x \in A \mid (\exists y \in B) p(a, y) \wedge q(y, c)\} \,.$$

As shown in (Aczel, 2006, Section 4) working in CZF, the category **FTop** is equivalent to the category **Loc** of set-generated locales and continuous maps. Given a formal topology A, following (Sambin, 1987), we say that a subset $U \subseteq A$ is saturated if $j(U) \subseteq U$, so that $U = j(U)$. The restriction of j to the set-generated frame of downward closed subsets of A is a nucleus (Johnstone, 1982; Simmons, 1978); whence the class $\mathrm{Sat}(A)$ of saturated subsets of A, preordered by inclusion, acquires the structure of a set-generated frame. For a continuous map $p : A \to B$ of formal topologies, there is a continuous map $\mathrm{Sat}(p) : \mathrm{Sat}(A) \to \mathrm{Sat}(B)$ of set-generated locales, given by the frame homomorphism mapping a saturated subset $V \subseteq B$ into $j\big(p^*(V)\big) \subseteq A$. These definitions determine a functor $\mathrm{Sat} : \mathbf{FTop} \to \mathbf{Loc}$, which is part of an equivalence of categories

$$\mathbf{FTop} \simeq \mathbf{Loc}\,. \tag{1}$$

For more information concerning the structure of the category **FTop**, see (Maietti and Valentini, 2004; Palmgren, 2005b).

Constructive topological spaces

We work here with the category **Top** of constructive topological spaces and continuous functions (Aczel, 2006, Definition 4). Following (Sambin and Gebellato, 1999; Vickers, 1989), a constructive topological space is given by a class X of points, a set A of neighbourhood indices, and a class relation between points $\xi \in X$ and neighbourhood indices $a \in A$, written as $\xi \Vdash a$, that expresses the idea that a point ξ lies in the neighbourhood indexed by a. This relation is subject to both topological and set-theoretical axioms: the former express structural properties; the latter require appropriate classes to be sets. If we define $X_a =_{\mathrm{def}} \{\xi \in X \mid \xi \Vdash a\}$, for $a \in A$, and $X_U =_{\mathrm{def}} \bigcup_{a \in U} X_a$, for $U \subseteq A$, there is a preorder on A defined by letting

$$a \leq b =_{\mathrm{def}} X_a \subseteq X_b\,. \tag{2}$$

This definition allows us to use the notation for preorders introduced earlier. The topological axioms for a constructive topological space can then be expressed as follows:

$$X = X_A\,, \qquad X_{a_1} \cap X_{a_2} = X_{a_1 \downarrow a_2}\,.$$

The set-theoretical axioms for a constructive topological space state that, for $\xi \in X$, the classes $A_\xi =_{\mathrm{def}} \{a \in A \mid \xi \Vdash a\}$ and $\{\psi \in X \mid A_\xi = A_\psi\}$ are sets.

It is then possible to define a functor $\mathrm{Pt} : \mathbf{FTop} \to \mathbf{Top}$, which maps a formal topology A into the constructive topological space of its formal points. Let us recall that a formal point of a formal topology A is defined to be a continuous map $p : 1 \to A$, where 1 is the formal topology on the singleton set $1 =_{\mathrm{def}} \{*\}$, preordered by the equality relation, with cover $a \lhd U =_{\mathrm{def}} a \in U$. It is immediate to see that a formal point can be identified with a subset $\xi \subseteq A$ such that, defining $\xi \Vdash a =_{\mathrm{def}} a \in \xi$ and $\xi \Vdash U =_{\mathrm{def}} (\exists x \in U)\xi \Vdash x$, the following three properties hold:

$$\xi \Vdash A\,; \qquad \xi \Vdash a\,,\ \xi \Vdash b \Longrightarrow \xi \Vdash a \downarrow b\,; \quad \xi \Vdash a\,,\ a \lhd U \Longrightarrow \xi \Vdash U\,.$$

These properties imply that there is a constructive topological space with the class $\mathrm{Pt}(A)$ of formal points of A as its class of points, and the set A as its set of neighbourhood indices. As we will see, it is possible to define formal topologies whose associated constructive topological spaces are homeomorphic to the Baire space, the Cantor space, and the Zariski spectrum.

The fundamental adjunction

The notion of a standard constructive topological space and of a standard continuous function are strengthenings of the notions of a constructive topological space and of a continuous function that are obtained by imposing further set-theoretical axioms. A constructive topological space is said to be standard if, for each subset $U \subseteq A$, the class $\{a \in A \mid X_a \subseteq X_U\}$ is a set (Aczel, 2006, Definition 5). In this way, one obtains the subcategory $\mathbf{Top}'$ of $\mathbf{Top}$ consisting of standard constructive topological spaces and standard continuous maps[†]. The main reason for the interest in these notions is that they allow the definition of a functor

$$\mathbf{Top}' \xrightarrow{\quad\Omega\quad} \mathbf{FTop}\,.$$

Indeed, for standard constructive topological space with class of points X and set of neighbourhood indices A, the set A, equipped with the preorder defined in (2) and the cover relation defined by $a \lhd U =_{\mathrm{def}} X_a \subseteq X_U$, is a formal topology.

Using these notions, Aczel was able to reformulate in CZF the familiar adjunction between topological spaces and locales. He considered the subcategory $\mathbf{FTop}'$ of $\mathbf{FTop}$ consisting of those formal topologies and continuous maps that are mapped by the functor $\mathrm{Pt} : \mathbf{FTop} \to \mathbf{Top}$ into standard constructive topological spaces and standard continuous functions, respectively. The functor $\Omega : \mathbf{Top}' \to \mathbf{FTop}$ factors through the inclusion of $\mathbf{FTop}'$ into $\mathbf{FTop}$, and so, as proved in (Aczel, 2006, Section 5), it is possible to obtain an adjunction of the form

$$\mathbf{Top}' \underset{\mathrm{Pt}'}{\overset{\Omega'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{FTop}'\,. \tag{3}$$

Furthermore, assuming the Power Set and Full Separation axioms of IZF, the categories $\mathbf{Top}'$ and $\mathbf{FTop}'$ are equivalent to the usual categories of topological spaces and of locales, respectively (Aczel, 2006, Theorem 9 and Proposition 13). It is straightforward to define a subcategory $\mathbf{Loc}'$ of $\mathbf{Loc}$ such that the equivalence in (1) restricts to an equivalence

$$\mathbf{FTop}' \simeq \mathbf{Loc}'\,. \tag{4}$$

We can define $\mathbf{Loc}'$ as the category of standard set-generated locales and standard continuous maps, where a set-generated locale and a continuous map are said to be standard if the formal topology and continuous map corresponding to them via the equivalence in (1) are standard.

[†] Note that $\mathbf{Top}'$ does not seem to coincide with the full subcategory of $\mathbf{Top}$ consisting of standard constructive topological spaces. This is because it does not seem possible to show that an arbitrary continuous function between standard constructive topological spaces is standard (Aczel, 2006).

3. Spatial formal topologies

Spatiality

Let us now focus on the comonad on $\mathbf{FTop}'$ induced by the adjunction in (3). We write $\mathrm{Sp} : \mathbf{FTop}' \to \mathbf{FTop}'$ for its functor part. Unfolding the relevant definitions, we have that for a standard formal topology A, the standard formal topology $\mathrm{Sp}(A)$, called the spatial coreflection of A, has underlying set A with preorder given by $a \leq_{\mathrm{Sp}(A)} b =_{\mathrm{def}} (\forall \xi \in \mathrm{Pt}(A))(\xi \Vdash a \Longrightarrow \xi \Vdash b)$. Its cover relation is defined by letting

$$a \lhd_{\mathrm{Pt}(A)} U =_{\mathrm{def}} (\forall \xi \in \mathrm{Pt}(A))(\xi \Vdash a \Longrightarrow \xi \Vdash U)\,. \tag{5}$$

The component at A of the counit of the adjunction is the standard continuous map of formal topologies $\varepsilon_A : \mathrm{Sp}(A) \to A$ defined by

$$\varepsilon_A(a, b) =_{\mathrm{def}} a \lhd_{\mathrm{Pt}(A)} \{b\}\,.$$

Note that, for $U \subseteq A$, the definition of the notion of a formal point implies that

$$(\forall a \in A)\big(a \lhd U \Longrightarrow a \lhd_{\mathrm{Pt}(A)} U\big)\,. \tag{6}$$

This fact can be used to prove that the relation ε_A is indeed a continuous map of formal topologies. We can now define the notion of spatiality for formal topologies analogously to how it is defined for locales (Fourman and Scott, 1979; Johnstone, 1982).

Definition 3.1. A standard formal topology A is *spatial* if $\varepsilon_A : \mathrm{Sp}(A) \to A$ is an isomorphism.

If a formal topology is spatial, then it is isomorphic to the formal topology associated to a constructive topological space, namely the constructive topological space of its formal points. In Section 4 we will study logical equivalents of the statements asserting that various examples of formal topologies are spatial. In order to do so, it is convenient to characterise spatial formal topologies purely in terms of their cover relations. The converse implication to (6), which does not necessarily hold, allows us to characterise spatial formal topologies as follows.

Proposition 3.2. A standard formal topology A is spatial if and only if it holds

$$(\forall a \in A)\big(a \lhd_{\mathrm{Pt}(A)} U \Longrightarrow a \lhd_A U\big)\,. \tag{7}$$

for all subsets $U \subseteq A$.

Proof. Assume that (7) holds for all $U \subseteq A$. Then define a relation δ_A between A and $\mathrm{Sp}(A)$ by letting $\delta_A(a, b) =_{\mathrm{def}} a \lhd_A \{b\}$. The assumption (7) implies that this defines a continuous map $\delta_A : A \to \mathrm{Sp}(A)$. The definition of composition and of identities in the category $\mathbf{FTop}'$ imply that δ_A and ε_A are mutually inverse.

For the converse implication, let us assume that $\varepsilon_A : \mathrm{Sp}(A) \to A$ is an isomorphism. Via the equivalence of categories between standard formal topologies and standard set-generated locales, we have that the frame homomorphism $\mathrm{Sat}(\varepsilon_A) : \mathrm{Sat}(A) \to \mathrm{Sat}(\mathrm{Sp}(A))$ is an isomorphism. Unfolding the definitions and using the axioms for a formal topology, it is immediate to see that this frame homomorphism maps a saturated subset $U \subseteq A$

into $\{x \in A \mid x \lhd_{\mathrm{Pt}(A)} U\}$. The statement in (7) follows from the fact that this map, being an isomorphism, is a monomorphism. □

Note that the formula in (7) makes sense also for formal topologies that are not standard. In the following, we will therefore assume—as in (Sambin, 1987)—that the statement in (7) is the definition of spatiality for formal topologies that are not standard.

Variants of spatiality

The property in (6) for the empty subset $\emptyset \subseteq A$ can easily be shown to be equivalent to

$$(\forall a \in A)\big(\mathrm{Neg}(a) \Longrightarrow \neg\mathrm{Sat}(a)\big) \tag{8}$$

where

$$\mathrm{Neg}(a) =_{\mathrm{def}} a \lhd \emptyset\,, \qquad \mathrm{Sat}(a) =_{\mathrm{def}} (\exists \xi)(\xi \Vdash a)\,.$$

We read $\mathrm{Neg}(a)$ by saying that a is negative, and $\mathrm{Sat}(a)$ by saying that a is satisfiable. The statement in (8) then expresses that if a is negative then it is not satisfiable. Its classical contrapositive, given in (9), holds as well.

$$(\forall a \in A)\big(\mathrm{Sat}(a) \Longrightarrow \neg\mathrm{Neg}(a)\big)\,. \tag{9}$$

Neither of the converses of the implications in (8) and (9) is generally derivable. We refer to these converses as to the Sufficiency (10) and Existentiality (11) properties, respectively.

$$(\forall a \in A)\big(\neg\mathrm{Sat}(a) \Longrightarrow \mathrm{Neg}(a)\big) \tag{10}$$

$$(\forall a \in A)\big(\neg\mathrm{Neg}(a) \Longrightarrow \mathrm{Sat}(a)\big) \tag{11}$$

Note that Sufficiency is exactly the property in (7) for $U = \emptyset$. The next proposition, whose proof is an exercise in intuitionistic logic, describes some aspects of the relationship between Sufficiency and Existentiality.

Proposition 3.3.

(a) Sufficiency is equivalent to its contrapositive together with the stability of Neg.
(b) The contrapositives of Sufficiency and Existentiality are equivalent.
(c) Sufficiency is equivalent to the contrapositive of Existentiality together with the stability of Neg.

The analogous results can be obtained when Sufficiency and Neg are exchanged with Existentiality and Sat, respectively: Existentiality is equivalent to its contrapositive together with the stability of Sat, and thus also to the contrapositive of Sufficiency together with the stability of Sat.

Subspaces

Although Existentiality is often related to assumptions like the axiom of choice, the failure of Sufficiency in a constructive framework can already be explained by Brouwerian counterexamples. To illustrate this, we need to obtain some results concerning closed subspaces. Given a formal topology A with preorder $\leq$ and cover relation $\lhd$, the closed subspace A_U of A associated to any given $U \subseteq A$ is defined to be the formal topology with A as the underlying set, the same preorder $\leq$, and the cover relation $\lhd_U$ defined by

$$a \lhd_U V =_{\text{def}} a \lhd U \cup V \, .$$

It is easy to see that this definition determines a cover relation. In particular, the closed subspace of A associated to $U = \emptyset$ coincides with A. The next lemma characterises the formal points of a closed subspaces, which can be found without proof as Proposition 3.3 of (Palmgren, 2005a), allows us to regard A_U as the formal counterpart of the closed complement of the open $X_U = \bigcup_{x \in U} X_x$ in the constructive topological space of points of A.

Lemma 3.4 (Palmgren). Let A be a formal topology and $U \subseteq A$. A subset $\xi \subseteq A$ is a formal point of A_U if and only if it is a formal point of A such that $\xi \cap U = \emptyset$.

Proof. Let first $\xi \subseteq A$ be a formal point of A_U. If $a \in \xi \cap U$, then $\xi \Vdash a$ and $a \lhd_U \emptyset$, so that $\xi \Vdash \emptyset$, which is absurd; whence $\xi \cap U = \emptyset$. If $\xi \Vdash a$ and $a \lhd V$, then also $a \lhd_U V$ and thus $\xi \Vdash V$. Next, let $\xi \subseteq A$ be a formal point of A. If $\xi \Vdash a$ and $a \lhd_U V$, which means $a \lhd U \cup V$, then $\xi \Vdash U \cup V$ and thus $\xi \Vdash V$ if, in addition, we have $\xi \cap U = \emptyset$. For the remaining parts of the definition of a formal point, there is nothing to prove. □

Let $\mathfrak{A}$ be a class of formal topologies. We say that $\mathfrak{A}$ is closed under forming closed subspaces if $A \in \mathfrak{A}$ implies that $A_U \in \mathfrak{A}$ for all $U \subseteq A$. We say that Sufficiency (resp. Existentiality) holds for $\mathfrak{A}$ whenever Sufficiency (resp. Existentiality) holds for all $A \in \mathfrak{A}$ and $a \in A$.

Proposition 3.5. If $\mathfrak{A}$ is closed under forming closed subspaces, then every formal topology in $\mathfrak{A}$ is spatial if and only if Sufficiency holds for $\mathfrak{A}$.

Proof. Sufficiency is the special case $U = \emptyset$ of (7). By Proposition 3.2, $\mathfrak{A}$ consists of spatial formal topologies if and only if (7) holds for all its elements. The statement in (7) for $U \subseteq A$ follows from Sufficiency for A_U as follows. Suppose that $\xi \Vdash a$ implies $\xi \Vdash U$ for all formal points ξ of A. By contraposition, $\neg(\xi \Vdash U)$ implies $\neg(\xi \Vdash a)$, for all formal points ξ of A. But $\neg(\xi \Vdash U)$ is equivalent to $\xi \cap U = \emptyset$; whence by Lemma 3.4 we have $\neg(\xi \Vdash a)$ for all formal points ξ of A_U. By Sufficiency for A_U and $a \in A$, we arrive at $a \lhd_U \emptyset$, which is to say that $a \lhd U$. □

We next extend a result from (Schuster, 2006), which goes back to (Negri, 2002). A formal topology A is finitary if $a \lhd V$ implies the existence of $U \subseteq_\omega V$ such that $a \lhd U$. Here and in the following, $U \subseteq_\omega V$ means that U is a finite subset of V. For the sake of brevity, we use finite set to mean what is generally called a finitely enumerable set in the literature, namely a set U for which there exists a natural number n and a surjective mapping from $\{1, 2, \ldots, n\}$ to U. Thus a finite set U is either empty or inhabited depending on whether $n = 0$ or $n \geq 1$.

Proposition 3.6. Let A be a formal topology. For $a \in A$, define

$$U_a = \{x \in A \mid a = x\,,\ \neg\mathrm{Neg}(a)\}\,.$$

For the statements

(i) $\mathrm{Neg}(a)$ is decidable,

(ii) $a \lhd U_a$,

(iii) if $a \lhd_{\mathrm{Pt}(A)} U_a$, then $a \lhd_A U_a$,

the following implications hold:

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)}\,.$$

Furthermore, if A is finitary, then the reverse implications hold.

Proof. If $\mathrm{Neg}(a)$ then $a \lhd U_a$ by Transitivity, whereas if $\neg\mathrm{Neg}(a)$ then $a \in U_a$ and thus $a \lhd U_a$ by Reflexivity. This shows that (i) implies (ii); the implication from (ii) to (iii) is trivial. To conclude the proof, assume that A is finitary, and that (iii) be valid. For every formal point ξ, if $\xi \Vdash a$, then $\neg\mathrm{Neg}(a)$ by (9), so that $a \in U_a$ and thus $\xi \Vdash U_a$, with witness $a \in \xi \cap U_a$. By (iii), we have $a \lhd U_a$. Since A is finitary, there is $U_0 \subseteq U_a$ with U_0 finite and $a \lhd U_0$. As a finite set, U_0 is either empty or inhabited. In the former case, we have $\mathrm{Neg}(a)$, whereas in the latter case we also have that U_a is inhabited, which is to say $\neg\mathrm{Neg}(a)$. □

When reading the hypothesis of the following, observe that although constructively it is not generally possible to show that every subset of a finite set is finite, the finitary formal topologies are closed under forming closed subspaces. In fact, if $W \subseteq_\omega U \cup V$, then $W = U' \cup V'$ for suitably defined $U' \subseteq_\omega U$ and $V' \subseteq_\omega V$; whence if A is a finitary formal topology, then so is A_U for every $U \subseteq A$. Another class of finitary formal topologies that is closed under forming closed subspaces will be considered later: the class of formal Zariski topologies (Proposition 4.4).

Corollary 3.7. Let $\mathfrak{A}$ be a class of finitary formal topologies that is closed under forming closed subspaces. If Sufficiency holds for $\mathfrak{A}$, then $\mathrm{Neg}(a)$ is decidable for all $A \in \mathfrak{A}$ and $a \in A$.

4. Principles related to spatiality

Inductively generated formal topologies

As for locales (Johnstone, 1982; MacLane and Moerdijk, 1994; Simmons, 2004), definitions by 'generators and relations' provide a convenient method to obtain examples of formal topologies (Aczel, 2006; Battillotti and Sambin, 2006; Coquand et al., 2003). For formal topologies, this method is closely related to the theory of inductive definitions. In particular, to carry over these definitions in Constructive Set Theory, it is convenient to work in the extension of CZF that includes the Regular Extension Axiom (REA), originally introduced in (Aczel, 1986). This axiom ensures that, for a wide class of inductive definitions, inductively-defined classes are sets (Aczel and Rathjen, 2001). From now on, we work assuming REA.

Given a preordered set $(A, \leq)$, a covering system consists of a function mapping each element $a \in A$ into a set $C(a)$ of subsets $U \subseteq A$ such that the following hold:

(1) for all $U \in C(a)$, $U \subseteq {\downarrow} a$,
(2) if $a \leq b$, then for every $V \in C(b)$ there exists $U \in C(a)$ such that $U \subseteq {\downarrow} V$.

Given such a covering system, we can define inductively a cover relation on A. Observe that the cover relation is completely determined once we define for each $U \subseteq A$ the set $j(U) =_{\text{def}} \{x \in A \mid x \lhd U\}$. The set $j(U)$ is then defined inductively as the smallest downward closed set X containing U such that $a \in X$ whenever there exists $U \in C(a)$ such that $U \subseteq X$. This definition implies that Reflexivity is automatically verified. Furthermore, Transitivity and Stability can be shown to hold (Aczel, 2006; Coquand et al., 2003).

The formal points of an inductively generated formal topology can be characterised purely in terms of the covering system that generates the formal topology. For a preordered set $(A, \leq)$ with a covering system C on it, a subset $\xi \subseteq A$ is a formal point of the induced formal topology induced if and only if, defining $\xi \Vdash a =_{\text{def}} a \in \xi$ for $a \in A$ and $\xi \Vdash U =_{\text{def}} (\exists a \in U)\xi \Vdash a$ for $U \subseteq A$, the following properties hold for all $a, b \in A$ and $U \in C(a)$:

$$\xi \Vdash A\,; \qquad \xi \Vdash a\,,\ \xi \Vdash b \Longrightarrow \xi \Vdash a \downarrow b\,;$$
$$\xi \Vdash a\,,\ a \leq b \Longrightarrow \xi \Vdash b\,; \qquad \xi \Vdash a\,,\ U \in C(a) \Longrightarrow \xi \Vdash U\,.$$

Baire space

To define the formal Baire space, we define a partial order on the set $\mathbb{N}^*$ of finite sequences of natural numbers by letting

$$a \leq b =_{\text{def}} b \text{ is an initial segment of } a\,.$$

For $a \in \mathbb{N}^*$, we can then let $C(a)$ be the set whose unique element is the set of sequences $[a, n]$ obtained by appending a natural number n at the end of the sequence a. We may observe that this is a covering system: property (1) follows immediately by the definition of the partial order on $\mathbb{N}^*$; property (2) follows by observing that, if $b \in \mathbb{N}^*$ is an initial segment of $a \in \mathbb{N}^*$, then for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $[b, m]$ is an initial segment of $[a, n]$. We refer to the resulting formal topology $\mathcal{B}$ as the formal Baire space. It is possible to show that the constructive topological space $\mathrm{Pt}(\mathcal{B})$ of the formal points of $\mathcal{B}$ is homeomorphic to the Baire space (Gambino, 2002). In particular, we identify a formal point ξ with a sequence $(\xi_i)_{i \in \mathbb{N}}$ of natural numbers, and an element $a \in \mathbb{N}^*$ with a neighbourhood index for the Baire space. We then define a relation between points and neighbourhood indices by letting $\xi \Vdash a$ mean that a is an initial segment of ξ.

We shall now investigate the spatiality of the formal Baire space. To do so, let us recall from (Dummett, 2000) that a subset $U \subseteq \mathbb{N}^*$ is called monotonic if

$$(\forall a \in \mathbb{N}^*)\big(a \in U \Longrightarrow (\forall n \in \mathbb{N})\, [a, n] \in U\big)\,,$$

and it is called inductive if

$$(\forall a \in \mathbb{N}^*)\big((\forall n \in \mathbb{N})[a, n] \in U\big) \Longrightarrow a \in U\,.$$

Observe that the definition of the covering system for the formal Baire space $\mathcal{B}$ implies that a set is monotonic and inductive if and only if is a saturated subset of the formal topology $\mathcal{B}$. The principle of monotone bar induction asserts that for all monotonic and inductive subsets $U \subseteq \mathbb{N}^*$ we have

$$(\forall a \in \mathbb{N}^*)\big(a \lhd_{\mathrm{Pt}(\mathcal{B})} U \Longrightarrow a \in U\big)\,,$$

where $a \lhd_{\mathrm{Pt}(\mathcal{B})} U$ is defined as in (5) and it is usually read in this example as saying that U is a bar above a.

Proposition 4.1. Spatiality of the formal Baire space is equivalent to the principle of monotone bar induction.

Proof. Let us assume that $\mathcal{B}$ is spatial, and derive the principle of monotone bar induction. Let $U \subseteq \mathbb{N}^*$ be monotonic and inductive. By the spatiality of $\mathcal{B}$ we have

$$(\forall a \in \mathbb{N}^*)\big(a \lhd_{\mathrm{Pt}(\mathcal{B})} U \Longrightarrow a \lhd_{\mathcal{B}} U\big)\,. \tag{12}$$

Since U is monotonic and inductive, it is saturated. Therefore $a \lhd_{\mathcal{B}} U$ implies $a \in U$, as required. For the converse, it is sufficient to apply the principle of monotone bar induction to the set $\{x \in \mathcal{B} \mid x \lhd_{\mathcal{B}} U\}$, which is saturated and hence monotone and inductive. By Reflexivity, it contains U. □

Cantor space

The formal Cantor space can be defined in an analogous way to the formal Baire space. We now deal with sequences (finite or infinite) of elements of $\{0\,,1\}$ rather than of natural numbers. To analyse the spatiality of the formal Cantor space, we identify the points of the formal Cantor space with elements of $\{0\,,\ 1\}^{\mathbb{N}}$. The principle related to its spatiality is the Fan Theorem, which we now recall. For $U \subseteq \{0,\ 1\}^*$ and $n \in \mathbb{N}$, define

$$U_n =_{\mathrm{def}} \{x \in U \mid \mathrm{length}(x) \leq n\}\,.$$

The following lemma is the key to prove that the spatiality of the formal Cantor space is equivalent to the Fan Theorem.

Lemma 4.2. Let $a \in \{0,\ 1\}^*$, and $U \subseteq \{0,\ 1\}^*$. For every $n \in \mathbb{N}$ we have

$$a \lhd_{\mathrm{Pt}(\mathcal{C})} U_n \Longrightarrow a \lhd_{\mathcal{C}} U_n\,.$$

Proof. We distinguish two cases. If $n < \mathrm{length}(a)$ then let ξ be a point such that $\xi \Vdash a$. Assuming $a \lhd_{\mathrm{Pt}(\mathcal{C})} U_n$, we derive that there exists $x \in U$ such that $\mathrm{length}(x) \leq n$ and $\xi \Vdash x$. Therefore we obtain that

$$\mathrm{length}(x) \leq n < \mathrm{length}(a)\,.$$

Furthermore, we have $\xi \Vdash a$ and $\xi \Vdash x$ and therefore x has to be an initial segment of a. Since $x \in U_n$, we have $a \lhd U_n$, as required. We now consider the second case which arises when $\mathrm{length}(a) \leq n$. We wish to show

$$\{x \mid x \leq a\,,\ \mathrm{length}(x) \leq n\} \lhd U_n\,. \tag{13}$$

To see this, let $x \leq a$. We first consider the case that x has length n. If $\xi \Vdash x$, then $\xi \Vdash a$ and therefore there is $y \in U_n$ such that $\xi \Vdash y$. By a reasoning analogous to the one employed in the first case, we get that y is an initial segment of x, and hence $x \lhd U_n$. If the length of x is $n-1$, the reasoning above can be adapted to show that both $x \cdot 0 \lhd U_n$ and $x \cdot 1 \lhd U_n$ hold. This implies that $x \lhd U_n$ by the inductive definition of the cover. We can proceed in this fashion to prove (13). Since length$(a) \leq n$, we get $a \lhd_{\mathcal{C}} U_n$, as required. □

Let $\mathcal{C}$ be the formal Cantor space. The following implication, which should be read as saying that every bar above a has a bound relative to a, is generally referred to as the Fan Theorem.

$$(\forall a \in \{0,1\}^*)\big(a \lhd_{\mathrm{Pt}(\mathcal{C})} U \Longrightarrow (\exists n \in \mathbb{N})(a \lhd_{\mathrm{Pt}(\mathcal{C})} U_n)\big)\,.$$

Proposition 4.3. Spatiality of the formal Cantor space is equivalent to the Fan Theorem.

Proof. Assume the Fan Theorem. If $a \lhd_{\mathrm{Pt}(\mathcal{C})} U$, an application of the Fan Theorem implies that there exists $n \in \mathbb{N}$ such that $a \lhd_{\mathrm{Pt}(\mathcal{C})} U_n$. With an application of Lemma 4.2, it follows that $a \lhd_{\mathcal{C}} U_n$. Since $U_n \subseteq U$, we get $a \lhd_{\mathcal{C}} U$, as required.

For the converse implication, assume that the formal Cantor space is spatial. We derive the desired conclusion by adapting the argument showing that the principle of monotone bar induction implies the Fan Theorem (Dummett, 2000, Section 3.2). Assume that $a \lhd_{\mathrm{Pt}(\mathcal{C})} U$, and use the spatiality of the formal Cantor space to deduce $a \lhd_{\mathcal{C}} U$. Then define

$$V =_{\mathrm{def}} \{x \in \{0,1\}^* \mid (\exists n \in \mathbb{N})\,(x \lhd_{\mathrm{Pt}(\mathcal{C})} U_n)\}\,.$$

Direct calculations show that $U \subseteq V$, and therefore, using $a \lhd_{\mathcal{C}} U$, we get $a \lhd_{\mathcal{C}} V$ by Reflexivity and Transitivity. Now observe that V is a saturated set for the formal Cantor space (because it is monotonic and inductive). It therefore follows $a \in V$, and hence the conclusion of the Fan Theorem, as required. □

Zariski spectrum

As recalled in detail in (Schuster, 2006), the Zariski topology has a counterpart in formal topology. Following (Sigstam, 1995), given a commutative ring A, define a preorder on it by letting

$$a \leq b =_{\mathrm{def}} (\exists n \in \mathbb{N})(\exists x \in A)(a^n = bx)\,.$$

Then define a covering system on $(A, \leq)$ by letting

$$C(a) =_{\mathrm{def}} \big\{\{b_1, \ldots, b_k\} \subseteq {\downarrow} a \mid k \geq 0\,, b_1 + \ldots + b_k = a\}\big\}\,.$$

With this covering system, the inductive definition of the cover relation of the formal topology gives us

$$a \lhd U \Longleftrightarrow (\exists n \geq 1)\ (a^n \in I(U))\,,$$

where $I(U)$ is the ideal of A generated by U. Note that $I(\emptyset) = 0$, so that

$$a \lhd \emptyset \iff (\exists n \geq 1)\ (a^n = 0)\,.$$

Let us note that for a commutative ring A the formal points of the formal Zariski topology associated to A are exactly the prime filters of A. In (Schuster, 2006, Proposition 27), this characterisation was proved with a slightly different notion of formal topology, where the underlying set has the structure of a monoid rather than of a preorder. To prove the equivalence between the two definitions, it suffices to show

$$\xi \Vdash ab \iff \xi \Vdash a \downarrow b\,.$$

Assume $\xi \Vdash a \downarrow b$, so let $x \in A$ such that $\xi \Vdash x$, and $x \leq a$, $x \leq b$. Hence $x^n = ay$ and $x^m = bz$ for some $n, m \in \mathbb{N}$ and $y, z \in A$. We then get $x^{n+m} = (ab)(xy)$. Since $\xi \Vdash x$ and $x \leq x^{n+m}$, we have $\xi \Vdash (ab)(xy)$; and therefore we get $\xi \Vdash ab$ by $(ab)(xy) \leq ab$. The converse implication is obvious.

The formal Zariski topology is an example of a finitary formal topology. Following (Sambin, 2003), one can say that a finitary formal topology is a predicative presentation of a coherent frame (or equivalently of a spectral locale). In this vein one may think of the basic opens as of the compact elements of the locale. The concept of a spectral locale, which is the point-free counterpart of the notion of a spectral space (Hochster, 1969), is crucial for non-Hausdorff point-free topology in general, and for its applications in domain theory in particular (Vickers, 1989, Chapter 9). The notion of a spectral space comprises the characteristic properties of the Zariski spectrum of a commutative ring: that is, the whole of its prime ideals endowed with the Zariski topology.

Proposition 4.4. The formal Zariski topologies are closed under forming closed subspaces.

Proof. Let A be the formal Zariski topology of a commutative ring. Given $U \subseteq A$, consider the quotient ring $A/I(U)$ of A modulo the ideal $I(U)$ generated by U. The closed subspace A_U is isomorphic to the formal Zariski topology of $A/I(U)$. To see this, note first that to give an element of $A/I(U)$ is the same as to give an element of A. As for the cover relations, let $a \in A$ and $V \subseteq A$. By definition, $a \lhd_U V$ is $a \lhd U \cup V$, which is to say that $a^n \in I(U \cup V)$ for some $n \geq 1$. Since $I(U \cup V) = I(U) + I(V)$, this amounts to $a^n \in I(V)$ modulo $I(U)$ for some $n \geq 1$, which is nothing but $a \lhd V$ in the formal Zariski topology of $A/I(U)$. □

Corollary 4.5. All formal Zariski topologies are spatial if and only if Sufficiency holds for all formal Zariski topologies.

Proof. This is a consequence of Proposition 3.5 and Proposition 4.4. □

Let us note that for the formal Zariski topology on a commutative ring A, Sufficiency is equivalent to

$$(\forall \xi)(a \notin \xi) \Longrightarrow (\exists n \in \mathbb{N})(a^n = 0)\,.$$

This is the classical contrapositive of

$$(\forall n \in \mathbb{N})(a^n \neq 0) \Longrightarrow (\exists \xi)(a \in \xi)\,.$$

which is classically equivalent to

$$(\forall n \in \mathbb{N})(a^n \neq 0) \Longrightarrow (\exists \pi \subseteq A)(\pi \text{ prime ideal}\,,\ a \notin \pi)\,.$$

The validity of the latter implication for all $a \in A$ and every commutative ring A is equivalent to the assertion that every nontrivial commutative ring has a prime ideal.

In (Schuster, 2006) it was shown that spatiality of all formal Zariski topologies, expressed as in (7), implies the Limited Principle of Omniscience (LPO), which asserts that if $(\lambda_n)_{n\in\mathbb{N}}$ is an increasing binary sequence, then either $\lambda_n = 1$ for some $n \in \mathbb{N}$ or else $\lambda_n = 0$ for all $n \in \mathbb{N}$. In view of Corollary 4.5, this result can be put as follows; we give a proof for the sake of completeness.

Corollary 4.6. Sufficiency for all formal Zariski topologies implies the limited principle of omniscience.

Proof. Given an increasing binary sequence (λ_n), consider the commutative ring

$$A = \mathbb{Z}[T]/I\left(\{\lambda_n T^n : n \in \mathbb{N}\}\right)\,.$$

Writing a for the equivalence class of T, we have

$$a^n = 0 \iff \lambda_n = 1 \quad \text{and} \quad a^n \neq 0 \iff \lambda_n = 0\,.$$

Hence $\mathrm{Neg}(a)$ amounts to the existence of an index n with $\lambda_n = 1$. To conclude, use Corollary 3.7 and Proposition 4.4. □

5. Future work

Positivity

A natural direction for future research is the extension of the results presented here to the context of formal topologies equipped with a positivity predicate, which are related to open locales. This direction is currently being pursued by the second author in collaboration with Giovanni Sambin.

Independence results

The principles of spatiality for the formal topologies in the examples have been shown to be independent from the internal logic of a topos using sheaf models (Fourman and Hyland, 1979; Moerdijk, 1984). It seems therefore natural to investigate whether those proofs can be recast within the theory of sheaf models for Constructive Set Theory (Gambino, 2002; Gambino, 2006).

Points

Another potentially interesting direction for further research is represented by relationship between spatiality and the study of conditions on a formal topology for it to have a a 'small' set of points (Aczel, 2006; Curi, 2003; Curi, 2006), a condition to which we shall refer as smallness. There seems to be an essential difference between spatiality and smallness: spatiality is a structural property, regarding the relationship between a formal topology and its space of its points, while smallness does not depend on the topology on the space of its points. Smallness, however, has been shown to be related to topological properties of the formal topology (Aczel, 2006; Curi, 2006), and these topological properties, at least in the context of locales, are related to spatiality (Johnstone, 1982). Let us note however, that even for the formal topologies that enjoy smallness, such as those in the examples presented here, the spaces of their points do not generally possess the desirable topological properties of their point-free counterparts.

Acknowledgements Nicola Gambino acknowledges gratefully the support from the Manchester Institute for Mathematical Sciences (MIMS), where this paper was completed, and wishes to thank Peter Aczel and Harold Simmons for helpful discussions. Both authors wish to thank Laura Crosilla for giving the initial impetus to this paper while the first author was invited to Munich by the DFG-hosted Graduiertenkolleg Logik in der Informatik.

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