p-Automata: New Foundations for Discrete-Time Probabilistic Verification

Michael Huth\textsuperscript{a}, Nir Piterman\textsuperscript{b,\*}, Daniel Wagner\textsuperscript{a}

\textsuperscript{a}Department of Computing, Imperial College London, South Kensington campus, London, SW7 2AZ, United Kingdom
\textsuperscript{b}Department of Computer Science, University of Leicester, University Road, Leicester, LE1 7RH, United Kingdom

Abstract

We introduce p-Automata, which are automata that accept languages of Markov chains, by adapting notions and techniques from alternating tree automata to the realm of Markov chains. The set of languages of p-automata is closed under Boolean operations, and for every PCTL formula it contains the language of the set of models of the formula. Furthermore, the language of every p-automaton is closed under probabilistic bisimulation. Similar to tree automata, whose acceptance is defined via two-player games, we define acceptance of Markov chains by p-automata through two-player stochastic games. We show that acceptance is solvable in EXPTIME; but for automata that arise from PCTL formulas acceptance matches that of PCTL model checking, namely, linear in the formula and polynomial in the Markov chain. We also derive a notion of simulation between p-automata that approximates language containment in EXPTIME and is complete for Markov chains. These foundations therefore enable abstraction-based probabilistic model checking for probabilistic specifications that subsume Markov chains, and LTL and CTL* like logics.

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\*Corresponding author

Email addresses: M.Huth@imperial.ac.uk (Michael Huth),
Nir.Piterman@leicester.ac.uk (Nir Piterman), D.Wagner06@imperial.ac.uk (Daniel Wagner)
1. Introduction

Markov chains are a very important modeling formalism in many areas of science. In computing, Markov chains form the basis of central techniques such as performance modeling, and the design and analysis of randomized algorithms used in security and communication protocols. Recognizing this prominent role of Markov chains, the formal-methods community has devoted significant attention to these models, e.g., in developing model checking for qualitative [2, 3, 4] and quantitative [5] properties, logics for reasoning about Markov chains [6, 7], and probabilistic simulation and bisimulation [8, 7]. Model-checking tools such as PRISM [9] and LiQuor [10] support such reasoning about Markov chains and have users in many fields of computer science and beyond.

In the non-probabilistic setting, the automata-theoretic approach to verification unifies such reasoning support for systems modeled as Kripke structures. Automata furnish the foundations for reasoning about these models: they can show decidability of satisfiability for a corresponding logic (decidable non-emptiness checks [11, 12, 13]), support algorithms that decide whether a model satisfies a formula (model checking [13, 14]), enable algorithms that generate a model satisfying a satisfiable formula (design synthesis [15]), and offer techniques of abstracting a model so that formulas holding for the abstract model also hold for the model they abstract (abstraction-based model checking [16, 17]).

Alternating tree automata [18] were introduced to prove the decidability of satisfiability for monadic, second-order logic and they provide a unifying framework for branching-time temporal logics such as $\mu$-calculus, CTL, and CTL*. Of particular interest to us is that alternating tree automata afford a complete framework for abstraction with respect to branching-time logic [19, 20]. Thus, in this context, alternating automata form the right basis for abstraction, the technique that makes model checking scale to realistic designs in the hardware and software industry. For Markov chains, their aforementioned techniques lack such a unifying framework and the quest for robust notions of abstraction is an active line of research. Here, we define p-automata and show that they render such a framework.

p-automata are devices that read an entire Markov chain as input and either accept it or reject it. The definition of p-automata is motivated by PCTL [21], the de-facto standard logic for model checking Markov chains, and alternating tree automata: it combines the rich combinatorial structure of alternating automata
with PCTL’s ability to quantify the probabilities of regular sets of paths. The acceptance of Kripke structures by an alternating tree automaton is decided by solving games (cf. [18]). In that spirit, acceptance of a Markov chain by a p-automaton is decided by solving stochastic games.

We now highlight the main results on p-automata developed in this paper.

- The language of Markov chains accepted by a p-automaton is semantically robust in that it is closed under probabilistic bisimulation.
- One can embed a Markov chain as a p-automaton accepting the language of Markov chains that are bisimilar to it.
- The set of languages of p-automata is closed under Boolean operations.
- Acceptance of finite Markov chains can be determined in exponential time.
- PCTL formulas can be expressed as p-automata whose complexity of acceptance of Markov chains matches the complexity of PCTL model checking.
- We define a simulation for p-automata that approximates language containment in EXPTIME and is exact for p-automata arising from Markov chains.
- p-automata are the first complete abstraction framework for PCTL model checking on Markov chains.

The latter means that if an infinite-state Markov chain satisfies a PCTL formula, then there is a finite p-automaton that abstracts (i.e. simulates) this Markov chain and whose language is contained in that of the formula.

The problem of emptiness of the language of a p-automaton generalizes the long-standing open problem of decidability for PCTL satisfiability, and is here left open.

The embedding of Markov chains as p-automata uses a new probabilistic separation operator, denoted by $\ast$, that decomposes the witness path set for a probability threshold into disjoint subsets. Use of this operator, however, has a certain price in the complexity of the resulting acceptance games.

Finally, probabilistic versions of LTL, CTL*, or desired $\omega$-regular probabilistic extensions of these logics can also be expressed as p-automata but such a formal development is beyond the scope of this paper.
1.1. Outline of paper

In Section 2 notation is fixed and needed concepts are recalled. p-automata are introduced in Section 3, their acceptance games defined in Section 4, and expressiveness results featured in Section 5. Simulation and its salient properties are presented in Section 6 and used to prove that p-automata are a complete abstraction framework. In Section 7 related and future work are discussed. Section 8 contains our conclusions.

2. Background

A countable labeled Markov chain $M$ over a set of atomic propositions $\mathbb{AP}$ is a tuple $(S, P, L, s_{\text{in}})$, where $S$ is a countable set of locations, $P : S \times S \to [0, 1]$ a stochastic matrix, $s_{\text{in}} \in S$ the initial location, and $L : S \to 2^{\mathbb{AP}}$ a labeling function with $L(s)$ the set of propositions true in location $s$. Let $\text{succ}(s)$ be the set $\{s' \in S \mid P(s, s') > 0\}$ of successors of $s$. All Markov chains are assumed to be finitely branching, i.e. $\text{succ}(s)$ is finite for all $s \in S$. We write $\text{MC}_{\mathbb{AP}}$ for the set of all (finitely branching) Markov chains over $\mathbb{AP}$. A path $\pi$ from location $s$ in $M$ is an infinite sequence of locations $s_0 s_1 \ldots$ with $s_0 = s$ and $P(s_i, s_{i+1}) > 0$ for all $i \geq 0$. For $Y \subseteq S$, let $P(s, Y)$ abbreviate $\sum_{s' \in Y} P(s, s')$. Given a Markov chain $M$ with set of states $S$, an open set in $S^\omega$ is a set $\{s_0 \cdots w \} \cdot S^\omega$ for some $w \in S^*$. A set is Borel if it is in the $\sigma$-algebra defined by these open sets. The measure of every Borel set $\alpha$ is defined as usual in this $\sigma$-algebra [22, 23]. We denote the measure of a set $\alpha$ as $\text{Prob}_M(\alpha)$.

For Markov chain $M = (S, P, L, s_{\text{in}})$, a (probabilistic) bisimulation [8] is an equivalence relation $H \subseteq S \times S$ where $(s, s') \in H$ implies (i) $L(s) = L(s')$ and (ii) $P(s, C) = P(s', C)$ for all equivalence classes $C \in S/H$. The union of all bisimulations for $M$ is the greatest bisimulation $\sim$; locations $s$ and $s'$ are bisimilar iff $s \sim s'$. This definition extends to Markov chains $M_1$ and $M_2$ by considering bisimilarity of their initial locations in the disjoint union of $M_1$ and $M_2$.

Without loss of generality [24], one may define the probabilistic temporal logic PCTL [21] in “Greater Than Negation Normal Form”: only propositions can be negated and probabilistic bounds are either $\geq$ or $>$. PCTL formulas are defined as follows, where $a \in \mathbb{AP}$, $p \in [0, 1]$, and $\bowtie \in \{>, \geq\}$:

$$
\begin{align*}
\phi, \psi & := \text{PCTL formulas} & \alpha & := \text{Path formulas} \\
\text{a, } & \neg a \quad \text{Literals} & X\phi & \text{Next} \\
\phi & \land \psi \quad \text{Conjunction} & \phi U \psi & \text{Until} \\
\phi & \lor \psi \quad \text{Disjunction} & \phi W \psi & \text{Weak Until} \\
[\alpha]_{\bowtie p} & \quad \text{Path Probability}
\end{align*}
$$
Our semantics of PCTL is as in [21]: path formulas $\alpha$ are interpreted as predicates over paths in $M$, and wrap PCTL formulas into “LTL” operators for Next, (strong) Until, and Weak Until. The semantics $\|\phi\| \subseteq S$ of PCTL formula $\phi$ lifts path formulas to state formulas: $s \in \|\|\phi\|\|$ wff $\mathrm{Prob}_M(s, \alpha)$, the probability of the measurable set [25] $\text{Path}(s, \alpha)$ of paths $ss_1s_2\ldots$ in $M$ with $ss_1s_2\ldots \parallel \alpha$, satisfies $\bowtie p$. Markov chain $M$ satisfies $\phi$, denoted $M \models \phi$, if $s^\top \in \|\phi\|$.

**Weak Games.** A tuple $G = ((V, E), (V_0, V_1, V_p), \kappa, \alpha)$ is a stochastic weak game if $(V, E)$ is a directed graph, $(V_0, V_1, V_p)$ a partition of $V$, and function $\kappa$ associates with every $v \in V_p$ a distribution $\kappa(v)$ of mass 1 over $E(v) = \{v' \mid (v, v') \in E\}$ such that $(v, v') \in E$ iff $\kappa(v)(v') \neq 0$; we write $\kappa(v, v')$ instead of $\kappa(v)(v')$. Set $\alpha \subseteq V$ is the winning condition. Set $V_p$ contains the probabilistic configurations of $G$. For $i = 0, 1$ set $V_i$ contains the Player $i$ configurations. We work with weak games: all maximal, strongly connected components (MSCCs) $V'$ in $(V, E)$ satisfy $V' \subseteq \alpha$ or $V' \cap \alpha = \emptyset$. If $V_p = \emptyset$, we call $G$ a weak game. Markov chains can be thought of as stochastic weak games where $V_0 = V_1 = \emptyset$ and $\alpha = V$.

A play in $G$ is a maximal sequence $v_0v_1\ldots$ of configurations with $(v_i, v_{i+1}) \in E$ for all $i \in \mathbb{N}$. A play is winning for Player 0 if it is finite and ends in a Player 1 configuration, or if it is infinite and ends in a suffix of states in $\alpha$. Otherwise, that play is winning for Player 1. A strategy for Player 0 is a function $\sigma: V_0 \rightarrow V$ with $(v, \sigma(v)) \in E$ for all $v \in V_0$ for which $\sigma(v)$ is defined. Play $v_0v_1\ldots$ is consistent with strategy $\sigma$ if $v_{i+1} = \sigma(v_i)$ whenever $v_i \in V_0$. Strategies for Player 1 are defined analogously. Let $\Sigma$ (resp. $\Pi$) be the set of all strategies for Player 0 (resp. Player 1).

Each $(\sigma, \pi) \in \Sigma \times \Pi$ from game $G$ determines a Markov chain $M^{\sigma, \pi}$ (with sinks for dead-ends in $G$) whose paths are plays in $G$ consistent with $\sigma$ and $\pi$. The set of plays from $v \in V$ that Player 0 wins is measurable in $M^{\sigma, \pi}$. Let $\mathrm{val}_0^{\sigma, \pi}(v)$ be that measure, and $\mathrm{val}_1^{\sigma, \pi}(v) = 1 - \mathrm{val}_0^{\sigma, \pi}(v)$. Then $\mathrm{val}_0(v) = \sup_{\sigma \in \Sigma} \inf_{\pi \in \Pi} \mathrm{val}_0^{\sigma, \pi}(v) \in [0, 1]$ and $\mathrm{val}_1(v) = \sup_{\pi \in \Pi} \inf_{\sigma \in \Sigma} \mathrm{val}_1^{\sigma, \pi}(v) \in [0, 1]$ are the game values. Strategies that achieve these values are optimal.

**Theorem 1** [26, 27, 28] Let $G = ((V, \cdot), \ldots)$ be a stochastic weak game and $v \in V$. Then $\mathrm{val}_0(v) + \mathrm{val}_1(v) = 1$. If $G$ is finite, $\mathrm{val}_0(v)$ is computable in EXPTIME, and optimal strategies exist for both players. If $G$ is a weak game, $\mathrm{val}_0(v)$ is in $\{0, 1\}$ and linear-time computable.

These results generalize to the setting where configurations may have preseeded game values. That is, when we set in advance the value for some of the
configurations of the game and ignore how one may continue to play from them. These values are in $[0, 1]$ for stochastic weak games and in $\{0, 1\}$ for weak games.

3. Uniform Weak $p$-Automata

We introduce $p$-automata and their uniform weak variant. Each uniform weak $p$-automaton is then shown to accept a language of Markov chains. We assume familiarity with basic notions of trees and (alternating) tree automata (cf. [18]).

For set $T$, let $B^+(T)$ be the set of positive Boolean formulas generated from elements $t \in T$, constants $tt$ and $ff$, and disjunctions and conjunctions:

$$
\phi, \psi ::= t \mid tt \mid ff \mid \phi \lor \psi \mid \phi \land \psi
$$

Formulas in $B^+(T)$ are finite even if $T$ is not.

For set $Q$, the set of states of a $p$-automaton, we define term sets as follows:

$$
\begin{align*}
[Q]_> &= \{ [q]_{\geq p} \mid q \in Q, \exists n \in \{\geq, >\}, p \in [0, 1]\} \\
[Q]^* &= \{ *(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i : t_i \in [Q]_> \} \\
[Q]^\Psi &= \{ \Psi(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i : t_i \in [Q]_> \} \\
[Q] &= [Q]^* \cup [Q]^\Psi
\end{align*}
$$

This uses $n$-ary operators $*^n$ and $\Psi_n$ for every $n \in \mathbb{N}$, which we write as $*$ and $\Psi$ throughout as $n$ will be clear from context. Also, we freely write $*(t_i \mid i \in X)$ for $*(t_1, t_2, \ldots, t_n)$ and so implicitly refer to $1, 2, \ldots, n$ as some enumeration of $X$.

Intuitively, a state $q \in Q$ of a $p$-automaton and its transition structure model a probabilistic path set. So $[q]_{\geq p}$ holds in location $s$ if the measure of paths that begin in $s$ and satisfy $q$ is $\geq p$. Now, $*(\{[q_1]_{\geq p_1}, [q_2]_{\geq p_2}\})$, e.g., means $q_1$ and $q_2$ hold with probability $\geq p_1$ and $\geq p_2$, respectively; and that the sets supplying these probabilities are disjoint. Dually, $\Psi(\{[q_1]_{\geq p_1}, [q_2]_{\geq p_2}\})$ means not only that either the probability of $q_1$ is $\geq p_1$ or the probability of $q_2$ is $\geq p_2$ but that this holds regardless of how we try to partition the sets supplying the full probability between them. So $*$ and $\Psi$ model a “disjoint and” and “intersecting or” operator, respectively. We may write $[q]_{\geq p}$ for $*(\{q\}_{\geq p})$, and similarly for $\Psi$.

An element of $Q \cup [Q]$ is a state of the $p$-automaton, a $*$ composition of terms $[q_i]_{\geq p_i}$, or a $\Psi$ composition of such terms. For $\phi \in B^+(Q \cup [Q])$, its closure $\text{cl}(\phi)$ is the set of all subformulas of $\phi$ according to (1). In particular, $*(t_1, t_2) \in \text{cl}(\phi)$ does not imply $t_1, t_2 \in \text{cl}(\phi)$. For a set $\Phi$ of formulas, let $\text{cl}(\Phi) = \bigcup_{\phi \in \Phi} \text{cl}(\phi)$. 

6
Definition 1 A p-automaton $A$ is a tuple $<\Sigma, Q, \delta, \phi^\text{in}, \alpha>$, where $\Sigma$ is a finite input alphabet, $Q$ a set of states, $\delta: Q \times \Sigma \rightarrow B^+(Q \cup \{\})$ the transition function, $\phi^\text{in} \in B^+(\{\})$ the initial condition, and $\alpha \subseteq Q$ an acceptance condition.

As a convention, p-automata have states, Markov chains have locations, and weak stochastic games have configurations.

Example 1 p-automaton $A = <2\{a,b\}, \{q_1, q_2\}, \delta, [q_1]\geq \frac{1}{2}, \{q_2\}>$ has $\delta$ given by

\[
\begin{align*}
\delta(q_1, \{a, b\}) &= \delta(q_1, \{a\}) = q_1 \lor [q_2]\geq \frac{1}{2} \\
\delta(q_2, \{b\}) &= \delta(q_2, \{a, b\}) = [q_2]\geq \frac{1}{2} \\
\delta(q_1, \{\}\}) &= \delta(q_1, \{b\}) = \delta(q_2, \{\}) = \delta(q_2, \{a\}) = \text{ff}
\end{align*}
\]

The winning condition $\{q_2\}$ (along with the loops in the transition of this automaton) means that only sequences of states in which $q_2$ is eventually reached and repeats forever are fair. It follows that term $[q_2]\geq \frac{1}{2}$ represents the recursive property $\phi$, that atomic proposition $b$ holds at the location presently read by $q_2$, and that $\phi$ will hold with probability at least $\frac{1}{2}$ in the next locations. State $q_1$ asserts it is possible to get to a location that satisfies $[q_2]\geq \frac{1}{2}$ along a path that satisfies atomic proposition $a$. The initial condition $[q_1]\geq \frac{1}{2}$ means the set of paths satisfying $a \lor \phi$ has probability at least $\frac{1}{2}$.

In order to be able to decide acceptance of input for p-automata through the solution of weak stochastic games, we restrict the cycles in the transition graph of p-automata. In doing so, we differentiate states $q'$ appearing within a term in $[Q]$ (bounded transition) from $q'$ appearing “free” in the transition of a state $q$ (unbounded transition). In this way, a p-automaton $A = <\Sigma, Q, \delta, \ldots>$ determines a labeled, directed graph $G_A = \langle Q', E, E_b, E_u \rangle$:

\[
\begin{align*}
Q' &= Q \cup \text{cl}(\delta(Q, \Sigma)) \\
E &= \{(\varphi_1 \land \varphi_2, \varphi_i) \mid \varphi_i \in Q' \setminus Q, 1 \leq i \leq 2\} \\
&\quad \cup \{(\varphi_1 \lor \varphi_2, \varphi_i) \mid \varphi_i \in Q' \setminus Q, 1 \leq i \leq 2\} \\
&\quad \cup \{(q, \delta(q, \sigma)) \mid q \in Q, \sigma \in \Sigma\} \\
E_u &= \{(\varphi \land q, q), (q \land \varphi, q), (\varphi \lor q, q), (q \lor \varphi, q) \mid \varphi \in Q', q \in Q\} \\
E_b &= \{(\varphi, q) \mid \varphi \in \text{gs}(\varphi) \text{ and } q \in \varphi^\text{in}\}
\end{align*}
\]

where $\delta(Q, \Sigma) = \{\delta(q, \sigma) \mid q \in Q \text{ and } \sigma \in \Sigma\} \cup \{\varphi^\text{in}\}$ and $\text{gs}(\varphi)$ is the set of guarded states of $\varphi$: all $q \in Q$ occurring in some term in $\varphi$. Elements $(\varphi, q) \in E_u$ are unbounded transitions; elements $(\varphi, q) \in E_b$ are bounded transitions; and elements of $E$ are simple transitions. We mark $(\varphi, q) \in E_b$ with $*$ (and respectively,
with \( \hat{\varphi} \) if \( \varphi \in [Q]^* \) (respectively, \( \varphi \in [Q]^{\hat{\varphi}} \)). Note that \( E, E_u, \) and \( E_b \) are pairwise disjoint. Let \( \varphi \preceq_A \tilde{\varphi} \) iff there is a finite path from \( \varphi \) to \( \tilde{\varphi} \) in \( E \cup E_b \cup E_u \). Let \( \equiv_A \) be \( \preceq_A \cap \preceq_A^{-1} \) and \( [\varphi] \) the equivalence class of \( \varphi \) with respect to \( \equiv_A \). Each \( [\varphi] \) is an MSCC in graph \( G_A \).

**Definition 2** A \( p \)-automaton \( A \) is called uniform if:

- For each cycle in \( G_A \), its set of transitions is either in \( E \cup E_b \) or in \( E \cup E_u \).
- For each cycle in \( q, E \cup E_b \), its set of markings is either \( \{\} \), \( \{\ast\} \) or \( \{\hat{\varphi}\} \).
- The set of equivalence classes \( \{[\varphi] \mid \varphi \in Q \cup \text{cl}(\delta(Q, \Sigma))\} \) is finite.

A (not necessarily uniform) \( p \)-automaton \( A \) is called weak if for all \( q \in Q \), either \( [q] \cap Q \subseteq \alpha \) or \( [q] \cap \alpha = \{\} \).

Then, \( A \) is uniform, if the full subgraph of every equivalence class in \( \equiv_A \) contains only one type of non-simple transitions and at most one kind of marking \( \ast \) or \( \hat{\varphi} \). In particular, all states \( q' \in Q \) or formulas \( \varphi \) occurring in \( \delta(q, \sigma) \) for some \( q \in Q \) and \( \sigma \in \Sigma \) can be classified as unbounded, bounded with \( \ast \), bounded with \( \hat{\varphi} \), or simple – according to MSCC \( [q] \).

**Example 2** Figure 1(a) depicts \( G_A \) for \( A \) of Example 1. Automaton \( A \) is uniform: \( [q_1] = \{q_1, q_1 \lor [q_2] \geq \frac{1}{2}\} \) and \( [q_2] = \{q_2, [q_2] \geq \frac{1}{2}\} \); in \( [q_1] \) there are no bounded edges, in \( [q_2] \) there are no unbounded edges; and \( G_A \) has only \( \ast \) markings (we treat \( [q_1] \geq \frac{1}{2} \) as \( \ast([q_1] \geq \frac{1}{2}) \)). The MSCC \( [q_1] = \{[q_1] \geq \frac{1}{2}\} \) is trivial. In addition, \( A \) is weak as \( \alpha = \{q_2\} \).

Intuitively, the cycles in the structure of a uniform \( p \)-automaton \( A \) take either no bounded edges or no unbounded edges, and cycles that take bounded edges do not have both markings \( \ast \) and \( \hat{\varphi} \). Below, all \( p \)-automata are uniform weak and so we often refer to them simply as “\( p \)-automata”. Uniformity allows to define acceptance of input for \( p \)-automata through the solution of stochastic games.
But, a more relaxed notion of uniformity is what really drives the proof of well-definedness: every ascending chain in the partial order on MSCCs on the graph of a p-automaton has only finitely many alternations between bounded and unbounded MSCCs.

The requirement of weakness is made merely to simplify the presentation. Using a parity condition instead, e.g., would still allow us to decide acceptance of input for uniform p-automata, by solving stochastic parity games.

4. Acceptance Games

For some set of atomic propositions $AP$, p-automaton $A = (2^{AP}, Q, \delta, \varphi^\text{in}, \alpha)$ has $MC_{AP}$ as set of inputs. For $M = (S, P, L, s^\text{in})$ in $MC_{AP}$, we exploit the uniform weak structure of $A$ to reduce the decision of whether $A$ accepts $M$ to solving a sequence of weak games and stochastic weak games. Intuitively, unbounded cycles in $G_A$ correspond to weak stochastic games and bounded cycles to weak games. Then the language of $A$ is $L(A) = \{ M \in MC_{AP} \mid A \text{ accepts } M \}$.

Just as in acceptance games of alternating tree automata, all states of $A$ and all subformulas appearing in its transitions form part of acceptance games. For $A$ as above, let $T = Q \cup cl(\delta(Q, 2^{AP}))$. We now (gradually) compute the values $\text{val}(s, t)$, where $s \in S$ is a state of the Markov chain and $t \in T$ is a subformula appearing in the transition of $A$, which determine whether $A$ accepts $M$. Initially, we set $\text{val}(s, t) = \perp$ for all $s \in S$ and $t \in T$. Gradually, as the computation progresses, $\text{val}(s, t)$ is reset for more and more pairs of states and subformulas. Partial order $(T/\equiv_A, \preceq_A)$ has set $\{(t) \mid t \in T\}$ ordered by $(\tilde{t}) \preceq_A (t)$ iff $\tilde{t} \preceq_A t$. As $A$ is uniform $\preceq_A$ induces a finite partial order. For $M$ as above, each $(t)$ determines a game $G_{M,(t)} = ((V, E), (V_0, V_1, V_p), \kappa, \bar{\alpha})$. Most of its configurations are in $S \times T$. The construction is such that $(s^\text{in}, \varphi^\text{in})$ occurs in exactly one of these games $G_{M,(t)}$, and $\text{val}(s^\text{in}, \varphi^\text{in}) \in [0, 1]$. Then $A$ accepts $M$ iff $\text{val}(s^\text{in}, \varphi^\text{in}) = 1$.

We define these games as follows. Since $A$ is uniform weak, each $(t)$ is of one of three types and each type determines a weak game or weak stochastic game as detailed in the three cases below. All game values already computed for games $G_{M,(\tilde{t})}$ of MSCCs $(\tilde{t})$ higher up with respect to $\preceq_A$ (i.e. by induction) are used as pre-seeded values in $G_{M,(t)}$. As mentioned, we initially set $\text{val}(s, \varphi) = \perp$ and as we progress, $\text{val}(s, \varphi)$ is computed and reset. Then, if the pair $(s, \varphi)$ appears again in some later game, the precomputed value $\text{val}(s, \varphi)$ is used as a pre-seeded value in the later game. For every $s \in S$ we set $\text{val}(s, ff) = 0$ and $\text{val}(s, tt) = 1$. 
**Case 1.** Let \( ((t)) \) be a nontrivial MSCC such that all the transitions in the subgraph of \( G_A \) induced by \( ((t)) \) are not in \( E_u \) and none have \( \Psi \) markings. For each \( \varphi \in ((t)) \cap [Q]^* \) of form \( *([q_1]_{p_1}, \ldots, [q_n]_{p_n}) \) we define, for each \( s \in S \), sets \( V_{0s}^{s,\varphi} \), \( V_{1s}^{s,\varphi} \), and \( E_{0s}^{s,\varphi} \). Then

\[
V_0 = \bigcup_{s,\varphi} V_{0s}^{s,\varphi} \quad V_1 = \bigcup_{s,\varphi} V_{1s}^{s,\varphi} \quad V_p = \{\}
\]

\[
E = \bigcup_{s,\varphi} E_{0s}^{s,\varphi} \quad \tilde{\alpha} = \{\} \text{ or } V
\]

defines the weak game \( G_{M,((t))} \) – where \( \tilde{\alpha} \) is \( V \) if some \( q \in ((t)) \) is in \( \alpha \), and is empty otherwise. It remains to define \( V_{0s}^{s,\varphi} \), \( V_{1s}^{s,\varphi} \), and \( E_{0s}^{s,\varphi} \), for which we use pre-seeded values \( \text{val}(s, \tilde{t}) \) for all \( s \in S \) and all \( \tilde{t} \notin ((t)) \) with \( ((t)) \leq_A ((\tilde{t})) \).

As \( \text{succ}(s) \) and \( \delta(q_i, L(s)) \) are finite, so are

\[
R_{s,\varphi} = \bigcup_{i=1}^{n} \{(s', \varphi') \mid s' \in \text{succ}(s), \varphi' \in \text{cl}(\delta(q_i, L(s)))\}
\]

\[
\text{Val}_{s,\varphi} = \{0,1\} \cup \left\{ \text{val}(s', \varphi') \mid (s', \varphi') \in R_{s,\varphi}, \text{val}(s', \varphi') \neq \bot \right\}
\]

Intuitively, \( R_{s,\varphi} \) is the set of configurations reachable from \( (s, \varphi) \) using one transition of a state in \( \varphi \). Thus, \( s' \) are the successors of \( s \) and \( \varphi' \) are subformulas of \( \delta(q_i, L(s)) \). Set \( \text{Val}_{s,\varphi} \) includes 0, 1, and values of configurations in \( R_{s,\varphi} \). In game \( G_{M,((t))} \), a play proceeding from \( (s, \varphi) \) reaches either a configuration whose value is pre-seeded (and therefore in \( \text{Val}_{s,\varphi} \)) or a configuration \( (s, \psi) \) for \( \psi \in ((t)) \).

For \( n \in \mathbb{N} \), let \([n] = \{1, \ldots, n\} \). Sets \( V_{0s}^{s,\psi} \), \( V_{1s}^{s,\psi} \), and \( E_{0s}^{s,\psi} \) are defined as
follows (whereby the definition of $\mathcal{F}_{s,\varphi}$ is deferred for now):

$$V_0^{s,\varphi} = \{ (s, \varphi) \}$$
$$\cup \{(s', \varphi', v) \in R_{s,\varphi} \times \text{Val}_{s,\varphi} \mid \perp \not\in \text{val}(s', \varphi') < v \}$$
$$\cup \{(s', \varphi_1 \lor \varphi_2, v) \in R_{s,\varphi} \times \text{Val}_{s,\varphi} \mid \text{val}(s', \varphi_1 \lor \varphi_2) = \perp \}$$

$$V_1^{s,\varphi} = \{ (s, \varphi, f) \mid f \in \mathcal{F}_{s,\varphi}^* \}$$
$$\cup \{(s', \varphi', v) \in R_{s,\varphi} \times \text{Val}_{s,\varphi} \mid \perp \not\in \text{val}(s', \varphi') \geq v \}$$
$$\cup \{(s', \varphi_1 \land \varphi_2, v) \in R_{s,\varphi} \times \text{Val}_{s,\varphi} \mid \text{val}(s', \varphi_1 \land \varphi_2) = \perp \}$$

$$E^{s,\varphi} = \{ ((s, \varphi), (s, \varphi, f)) \mid f \in \mathcal{F}_{s,\varphi}^* \}$$
$$\cup \{((s', \varphi', v), (s', \varphi')) \mid s' \in \text{succ}(s), \varphi' \in [Q], v \in \text{Val}_{s,\varphi}, \text{val}(s', \varphi') = \perp \}$$
$$\cup \{((s, \varphi, f), (s', \delta(q_i, L(s)), f(i, s'))) \mid s' \in \text{succ}(s), i \in [n], f(i, s') > 0 \}$$
$$\cup \{((s', \varphi_1 \land \varphi_2, v), (s', \varphi_1, v)) \mid \in \{ \land, \lor \}, 1 \leq i \leq 2, (s', \varphi_1 \land \varphi_2, v) \in R_{s,\varphi} \times \text{Val}_{s,\varphi}, \text{val}(s', \varphi_1 \land \varphi_2) = \perp \}$$

(2)

The intuition behind this weak game is as follows: Configuration $(s, \varphi)$ means that the transition of each $q_i$ holds with probability $\vDash_i p_i$ where the sets $X_i$ measured by these probabilities are pairwise disjoint. In order to check that, given configuration $(s, \varphi)$, Player 0 chooses a function $f \in \mathcal{F}_{s,\varphi}^*$ that associates with location $s' \in \text{succ}(s)$ and state $q_i$ the value Player 0 promises to achieve playing from $(s', \delta(q_i, L(s)))$. The play continues with Player 1 choosing a successor $s'$ of $s$ and a state $q_i$, and the play then reaches configuration $(s', \delta(q_i, L(s)), f(i, s'))$. From such value-annotated configurations, Player 0 and Player 1 choose successors according to the usual resolution of $\lor$ and $\land$:

- In a configuration whose value $v$ was already determined, either $f(i, s') \vDash_i v$, i.e. Player 0 achieved the promised value and wins immediately; or Player 0 failed to achieve the promised value and loses immediately.
- Otherwise, the play ends up in a configuration of form $(s', \varphi')$ for $\varphi' \in [[Q]]^*$ and the play continues and ignores value $f(i, s')$ (as obviously $f(i, s') \leq 1$).

If the play continues ad infinitum, the winner is determined according to acceptance condition $\bar{\alpha}$.

We now define the function space $\mathcal{F}_{s,\varphi}^*$ that captures terms built from the separation operator $\ast$. Throughout, let $X \rightarrow Y$ be the set of total functions from set $X$ to set $Y$. Let $\mathcal{F}_{s,\varphi}$ be $[n] \times \text{succ}(s) \rightarrow \text{Val}_{s,\varphi}$, the set of functions from pairs consisting of ‘sub-stars’ of $\varphi$ and successors of $s$ to values in $\text{Val}_{s,\varphi}$. Also, an $f \in \mathcal{F}_{s,\varphi}$ is disjoint if there are $\{a_i, s' \in [0, 1] \mid i \in [n] \text{ and } s' \in \text{succ}(s)\}$ that
Case 2. Let \((|t|)\) be a nontrivial MSCC such that all transitions in the subgraph of \(G_A\) induced by \((|t|)\) are not in \(E_a\) and none has \(*\) markings. For \(\varphi \in (|t|) \cap \{Q\}^\emptyset\) of form \(\Psi([q_1]_{\ge 1}, \ldots, [q_n]_{\ge 1})\) we reuse the definitions of \(R_{s,\varphi}, Val_{s,\varphi},\) and \(F_{s,\varphi}^s\). Weak game \(G_{M,(|t|)}\) is defined as in Case 1. Sets \(V_0^{s,\varphi}, V_1^{s,\varphi},\) and \(E^{s,\varphi}\) are defined.
as in (2), except that functions \( f \) don’t range over \( \mathcal{F}_{\ast,\varphi}^s \) but now range over \( \mathcal{F}_{s,\varphi}^\ast \), the set of intersecting functions and the dual of \( \mathcal{F}_{s,\ast}^\varphi \) of Case 1:

Function \( f \in \mathcal{F}_{s,\varphi}^\ast \) is intersecting if for all sets \( \{a_{i,s'} \in [0,1] \mid i \in [n] \text{ and } s' \in \text{succ}(s)\} \) either

(i) there is \( i \in [n] \) with \( \sum_{s' \in \text{succ}(s)} a_{i,s'} f(i,s') P(s,s') \succ_a p_i \) or

(ii) there is \( s' \in \text{succ}(s) \) with \( \sum_{i \in [n]} a_{i,s'} \neq 1 \).

As in Case 1, wins for Player 0 have value 1, wins for Player 1 have value 0. The intuition for this weak game is verbatim that of the weak game in Case 1, except that Player 0 chooses a function \( f \) that is in \( \mathcal{F}_{s,\varphi}^\ast \) instead of in \( \mathcal{F}_{s,\ast}^\varphi \).

We point out that when \( n \) above is 1, i.e. in handling \( \varphi = [q_1]_{\sim_0,p_1} \), the definitions of \( \ast \) and \( \psi \) coincide. Indeed, there is then exactly one option for choosing set \( \{a_{1,s'} \mid s' \in \text{succ}(s)\} \) that does not satisfy the second condition above: the value \( a_{1,s'} \) has to be 1 for all \( s' \in \text{succ}(s) \). This justifies dropping the \( \ast \) or \( \psi \) when applied to one operand.

**Case 3.** For a nontrivial MSCC \( \langle t \rangle \) such that all transitions in the subgraph of \( G_A \) induced by \( \langle t \rangle \) are not in \( E_b \), game \( G_{M,\langle t \rangle} \) is a stochastic weak game with

\[
\begin{align*}
V &= \{(s,\tilde{t}) \mid s \in S \text{ and } t \preceq_A \tilde{t}\} \quad V_0 = \{(s,\varphi_1 \lor \varphi_2) \in V\} \\
V_1 &= \{(s,\varphi_1 \land \varphi_2) \in V\} \\
V_p &= (S \times Q) \cap V \\
\kappa((s,q),(s',\delta(q,L(s)))) &= P(s,s') \\
\tilde{\alpha} &= \{\} \text{ or } V \\
E &= \{(s,\varphi_1 \land \varphi_2) \in V \times V \mid 1 \leq i \leq 2\} \cup \\
&\{(s,\varphi_1 \lor \varphi_2) \in V \times V \mid 1 \leq i \leq 2\} \cup \\
&\{(s,q),(s',\delta(q,L(s))) \in V \times V \mid P(s,s') > 0\}
\end{align*}
\]

where \( \tilde{\alpha} \) equals \( V \) if some state \( q \) in \( \langle t \rangle \) is in \( \alpha \), and equals \( \{\} \) otherwise. By Theorem 1, \( \text{val}_0(c) \) is in \([0,1]\) for all configurations \( c \in V \). We set \( \text{val}(c) = \text{val}_0(c) \).

**Example 4** Continuing with the verification \( M \in \mathcal{L}(A) \) for \( A \) from Example 1 and \( M \) from Fig. 1(b). The stochastic weak game \( G_{M,\langle q_1 \rangle} \) for the MSCC \( \langle q_1 \rangle \), shown in Fig. 3, depicts stochastic configurations with a diamond and configurations from other MSCCs are put into hexagons (with the hexagon labeled \( (s_1,\sqsupseteq_{q_2} \frac{1}{2}) \) having value 1 and all others having value 0). As none of the configurations are accepting, Player 0 can only win by reaching optimal hexagons. Hexagon \( (s_1,\sqsupseteq_{q_2} \frac{1}{2}) \) has value 1 and is the optimal choice for Player 0 from configuration \( (s_1,q_1 \lor \sqsupseteq_{q_2} \frac{1}{2}) \). Player 0 configuration \( (s_2,q_1 \lor \sqsupseteq_{q_2} \frac{1}{2}) \) has value 0. So the value for Player 0 of diamond configuration \( (s_0,q_1) \) is \( \frac{1}{2} \).
Trivial MSCCs \((t)\), are handled as one of the cases above. In case more than one case matches, the ambiguity is unproblematic as game values in \(G_{M,(t)}\) are then determined via propagation of pre-seeded game values. In particular, the case of a configuration \((s, \varphi)\), where \(\varphi = *([q_1]_{\geq p_1}, \ldots, [q_n]_{\geq p_n})\) is handled as in Case 1. The definitions of \(\text{Val}_{s,\varphi}\) and \(\mathcal{F}_{s,\varphi}\) are as before and the configuration \((s, \varphi, f)\) is connected to configurations of the form \((s', \varphi', v)\), which form dead-ends in \(G_{M,(t)}\). The case of \(\Psi\) is handled as in Case 2.

**Example 5** Finally, we establish that \(M \in \mathcal{L}(A)\) for \(A\) from Example 1 and \(M\) from Fig. 1(b). The initial configuration \((s_0, [q_1]_{\geq \frac{1}{2}})\) makes up a trivial bounded MSCC. Consider the function \(f = \{s_0 \mapsto \frac{1}{2}, s_1 \mapsto 1, s_2 \mapsto 0\}\). It is disjoint as witnessed by \(\{a_{1,s} = 1\}_{s \in S}\), which satisfies \(1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + 1 \cdot 0 \cdot \frac{1}{3} = \frac{1}{2}\) as required. Then, configuration \((s_0, q_1 \lor [q_2]_{\geq \frac{1}{2}}, \frac{1}{2})\) is a dead end for Player 1 as \(\text{val}(s_0, q_1 \lor [q_2]_{\geq \frac{1}{2}}) = \frac{1}{2}\) and similarly configuration \((s_1, q_1 \lor [q_2]_{\geq \frac{1}{2}})\). Therefore, \(M \in \mathcal{L}(A)\).

We now state the well-definedness of languages for p-automata and the complexity of checking acceptance in case of finite automaton and Markov chain.

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1 Alternatively, for every state \(s\) and every \(i \in [n]\) the value \(\text{val}(s, q_i)\) is pre-computed. Then, it is enough to find a set \(\{a_i \mid i \in [n]\}\) such that \(\sum_{i \in [n]} a_i = 1\) and for every \(i \in [n]\) we have \(a_i \cdot \text{val}(s, q_i) \cong_i p_i\). A formal proof of this alternative definition is omitted.
Theorem 2  Given a p-automaton \( A = \langle 2^\mathbb{AP}, \ldots \rangle \), its language \( \mathcal{L}(A) \) is well defined. If \( A \) and \( M \in MC_{AP} \) are finite, \( M \in \mathcal{L}(A) \) can be decided in EXPTIME.

Proof: Well definedness of acceptance follows directly from Theorem 1. For finite Markov chain \( M \) and finite p-automata \( A \) we make two observations: First, the stochastic weak game arising from the combination of \( M \) and an unbounded MSCC of \( G_A \) can be solved in EXPTIME. Second, the weak game arising from the combination of \( M \) and a bounded MSCC of \( G_A \) may be exponential due to the large number of possible value assignment functions. Such a weak game can be solved in linear time leading to an EXPTIME upper bound. Since there are only linearly many such games in the sequence of weak games and stochastic weak games, acceptance can be solved in EXPTIME.

For finite Markov chain \( M \) and p-automaton \( A \) with non-trivial, bounded MSCCs, checking acceptance \( M \in \mathcal{L}(A) \) is exponential in the branching degree of \( M \) and in the branching degree of \( \star \) and \( \forall \) operators of \( A \), but not in the number of states or locations. If \( A \) has only trivial bounded-MSCCs, checking \( M \in \mathcal{L}(A) \) reduces to solving a linear number of linear sized stochastic weak games.

5. Expressiveness of p-Automata

We now consider different aspects of the expressiveness of p-automata. We show that languages of p-automata are closed under Boolean operations. It follows that emptiness and containment of p-automata are equi-solvable. We then show that the language of every p-automaton is closed under bisimulation. For every Markov chain, we show how to construct a p-automaton accepting its bisimulation equivalence class. Finally, we show that each PCTL formula has a p-automaton whose language consists of all Markov chains satisfying that formula.

5.1. Closure of Languages

It is routine to see that the set of languages of p-automata is closed under union and intersection. But that set of languages is also closed under complementation: Given a p-automaton \( A = \langle \Sigma, Q, \delta, \varphi^{in}, \alpha \rangle \), its dual \( \text{dual}(A) = \langle \Sigma, \overline{Q}, \overline{\delta}, \text{dual}(\varphi^{in}), \overline{Q} \setminus \alpha \rangle \) with bijection \( \overline{Q} = \{ \overline{q} \mid q \in Q \} \) and \( \overline{\delta}(\overline{q}, \sigma) = \text{dual}(\delta(q, \sigma)) \), where \( \text{dual}(\varphi) \) is
defined as follows:
\[
\begin{align*}
\text{dual}(\mathcal{V}(t_1, \ldots, t_n)) &= *\text{dual}(t_1, \ldots, \text{dual}(t_n)) \\
\text{dual}(\mathcal{E}(t_1, \ldots, t_n)) &= \mathcal{V}(\text{dual}(t_1), \ldots, \text{dual}(t_2)) \\
\text{dual}(\varphi_1 \land \varphi_2) &= \text{dual}(\varphi_1) \lor \text{dual}(\varphi_2) \\
\text{dual}(\varphi_1 \lor \varphi_2) &= \text{dual}(\varphi_1) \land \text{dual}(\varphi_2) \\
\text{dual}(q) &= \overline{q} \\
\text{dual}(\overline{q}) &= q \\
\text{dual}([q]_{\geq p}) &= \overline{\text{dual}([q]_{< p})} \\
\text{dual}(\geq p) &= 1 - p \\
\text{dual}(< p) &= \geq 1 - p
\end{align*}
\]

The structure of uniform weak p-automata ensures that dual(A) is also uniform weak. The languages of A and dual(A) are complements.

**Theorem 3** For every p-automaton A with \(\Sigma = 2^{AP}\), \(L(\text{dual}(A))\) is \(MC_{AP} \setminus L(A)\).

**Proof:** We prove a stronger claim, namely that \(\text{val}(s, \varphi) = 1 - \text{val}(s, \text{dual}(\varphi))\) for all \(s \in S\) and \(\varphi \in \text{cl}(\delta(Q, \Sigma))\). The proof is by induction on the structure of the automaton. Consider an equivalence class \(\langle t \rangle\) in \(G_A\). Assume by induction that the claim holds for all the MSCCs in \(G_A\) that are greater than \(\langle t \rangle\).

First, suppose that \(\langle t \rangle\) is a nontrivial MSCC and that no transition in \(\langle t \rangle\) is in the scope of \(\mathcal{V}\). It follows that \(\langle \text{dual}(\langle t \rangle) \rangle\) is also a nontrivial MSCC and that no transition in \(\langle \text{dual}(\langle t \rangle) \rangle\) is in the scope of \(\mathcal{E}\). Given a strategy for Player 0 in \(G_{M,\langle t \rangle}\), we show how to construct a strategy for Player 1 in \(G_{M,\langle \text{dual}(\langle t \rangle) \rangle}\). The two strategies produce plays that are always in the same locations of the Markov chain \(M\) and same states of the automaton \(A\) (modulo dualization \(t \mapsto \text{dual}(t)\)). For sake of brevity, we denote \(G_{M,\langle \text{dual}(\langle t \rangle) \rangle}\) by \(G\) and \(G_{M,\langle t \rangle}\) by \(G\).

Consider two matching configurations \((s, \varphi)\) and \((s, \text{dual}(\varphi))\) in \(G\) and \(\overline{G}\). Let \(\varphi\) be of form \(*([q_{1}]_{\geq p_1}, \ldots, [q_{n}]_{< p_n})\), where \(n > 1\). Consider the configuration \((s, \text{dual}(\varphi))\). By playing for Player 1 in \(G\) we make Player 0 'reveal' her strategy in \(G\) and using her strategy we react to the moves of Player 0 in \(\overline{G}\) by constructing a strategy for Player 1 in \(\overline{G}\).

Consider two plays ending in \((s, \varphi)\) and \((s, \text{dual}(\varphi))\). Let \(f : [n] \times \text{succ}(s) \rightarrow \text{Val}_{s,\varphi}\) be the function chosen by Player 0 in \(G\) and let \(f' : [n] \times \text{succ}(s) \rightarrow \text{Val}_{s,\text{dual}(\varphi)}\) be the function chosen by Player 0 in \(\overline{G}\). By definition there are \(\{a_{i,s'}\}\) that witness the disjointness of \(f\) and for every \(i\) we have
\[
\sum_{s' \in \text{succ}(s)} a_{i,s'} \cdot P(s, s') \cdot f(i, s') \lor \mathbb{q}_i p_i.
\]
By using the same \( \{a_{i,s'}\} \) stemming from the fact that \( f' \) is intersecting, we get that there is some \( i \) such that
\[
\sum_{s' \in \text{succ}(s)} a_{i,s'} \cdot P(s,s') \cdot f'(i,s') \text{dual}(\triangledown a_i p_i).
\]

It follows that there is an \( s' \in \text{succ}(s) \) such that \( f(i,s') + f'(i,s') > 1 \).

It is now Player 1’s turn to move in both \( G \) and \( \overline{G} \). In \( G \) we make Player 1 choose \((s',\delta(q_i, L(s)), f(i,s'))\) and the strategy for Player 1 in \( \overline{G} \) is extended by \((s', \text{dual}(\delta(q_i, L(s))), f'(i,s'))\). We now proceed by utilizing the duality between \( \lor \) and \( \land \) to use Player 0’s choices in \( \overline{G} \) to suggest moves for Player 1 in \( G \) and use Player 0’s strategy in \( G \) to suggest how to extend the strategy for Player 1 in \( \overline{G} \).

If we reach configurations \((s', \varphi', f(i,s'))\) and \((s', \text{dual}(\varphi'), f'(i,s'))\) such that \( \text{val}(s', \varphi') \neq \bot \) and \( \text{val}(s', \text{dual}(\varphi')) \neq \bot \), then, by assumption \( \text{val}(s', \varphi') = 1 - \text{val}(s', \text{dual}(\varphi')) \). And if \( \text{val}(s', \varphi') \geq f(s') \), then \( \text{val}(s', \text{dual}(\varphi')) < f'(s') \) must hold. Otherwise the game proceeds to a new configuration in \( S \times [\left[ Q \right]] \). If the two plays are infinite, then by the duality of \( \alpha \) and \( \overline{Q} \setminus \alpha \) if Player 0 wins the play in \( G \) then Player 1 wins the play in \( \overline{G} \).

That a win of Player 1 in \( G \) is translated to a win of Player 0 in \( \overline{G} \) is shown similarly, and omitted.

Second, the case that \((t)\) is a nontrivial MSCC and that some transitions in \((q)\) are in scope of \( \Psi \) is similar, and omitted.

Third, suppose that \((t)\) is a nontrivial MSCC and that all transitions in \((t)\) are unbounded. Then, the claim follows from the dualization and the determinacy of stochastic weak games.

Finally, if \((t)\) is a trivial MSCC, the claim follows from the dualization and the duality of min and max. \( \square \)

**Corollary 1** Let \( \Sigma = 2^\text{AP} \). The set of languages accepted by p-automata with \( \Sigma \) is closed under Boolean operations. Language containment of p-automata with \( \Sigma \) reduces to language emptiness of such p-automata, and vice versa.

**Proof:** By Theorem 3, this set of languages is closed under complement. Showing closure under intersection and union is routine, and omitted. That language containment and non-emptiness are equi-solvable is a standard argument, since p-automata have duals and since there are p-automata with empty language. \( \square \)
5.2. Closure of Languages to Bisimulation

We now show that languages of p-automata are closed under bisimulation.

Lemma 1 For p-automaton $A=(2^{AP}, Q, δ, \varphi^{in}, \alpha)$ and $M_1, M_2 \in \text{MC}_{AP}$, with $M_1 \sim M_2$: Markov chain $M_1$ is in $L(A)$ iff Markov chain $M_2$ is in $L(A)$.

Proof: Let $M_i = (S_i, P_i, L_i, s_i^{in})$, for $i \in \{1, 2\}$, with the same set of atomic propositions $\Pi$. Let $A = (\Sigma, Q, δ, [[q_0]]_{\triangleright p}, \alpha)$, where $\Sigma = 2^{AP}$. Let $\sim \subseteq S_1 \times S_2$ be the maximal bisimulation between $M_1$ and $M_2$.

We show that for every state $q \in Q$ and locations $s_1 \in S_1$, and $s_2 \in S_2$ such that $s_1 \sim s_2$, we have $\text{val}(s_1, q) = \text{val}(s_2, q)$. We prove this claim by induction on the partial order on the MSCCs in $G_A$. Suppose that the claim holds for all MSCCs greater than $\langle \langle q \rangle \rangle$ in the partial order. Consider the games $G_{M_1, \langle q \rangle}$ and $G_{M_2, \langle q \rangle}$. Consider a winning strategy $σ$ for Player 0 in $G_{M_1, \langle q \rangle}$. We show how this is also a winning strategy for Player 0 in $G_{M_2, \langle q \rangle}$.

Consider a play in an unbounded MSCC $\langle \langle q \rangle \rangle$. We inductively construct a play in $G_{M_1, \langle q \rangle}$ and a play in $G_{M_2, \langle q \rangle}$ with the invariant that the plays end in configurations of the form $⟨s_1, t⟩$ and $⟨s_2, t⟩$ such that $s_1 \sim s_2$. Clearly, the initial configurations in both games satisfy this invariant. We show how to extend the play to maintain this invariant. If $t$ is of the form $ϕ_1 \land ϕ_2$ and Player 1 chooses $ϕ_i$ in $G_{M_2, \langle q \rangle}$, then we emulate the same choice in $G_{M_1, \langle q \rangle}$. If $t$ is of the form $ϕ_1 \lor ϕ_2$, then $σ$ instructs Player 0 to choose $ϕ_i$ in $G_{M_1, \langle q \rangle}$ and we emulate the same choice in $G_{M_2, \langle q \rangle}$. If $t$ is of the form $q'$ for some state $q' \in Q$ then choices in $⟨s_1, q'⟩$ and $⟨s_2, q'⟩$ are resolved by the stochastic player.

As $s_1 \sim s_2$, the successors of $s_1$ and $s_2$ can be partitioned into equivalence classes such that, for each equivalence class $C_1$ in $M_1$ and $C_2$ in $M_2$, we have $P_1(s_1, C_1) = P_2(s_2, C_2)$. Consider now the measure of plays that are winning according to this composed strategy. The plays can be partitioned according to bisimulation equivalence classes and every choice has the same weight. So the measure of winning plays is identical in both games.

Consider a play in a bounded MSCC $\langle \langle q \rangle \rangle$ where no transition uses $\triangleright$. Disjunctions and conjunctions are handled as above. Consider a pair of configurations $(s_1, t)$ and $(s_2, t)$, where $s_1 \sim s_2$ and $t$ is of the form $*([q_1]_{p_1}, \ldots, [q_n]_{\triangleright p_n})$. Let $f_1$ be the function chosen by Player 0 in $G_{M_1, \langle q \rangle}$. As $s_1 \sim s_2$, we can find a function $f_2$ such that for every $s'_1$ we have $f_2(i, s'_2) = f_1(i, s'_1)$ for some $s'_1 \sim s'_2$ that satisfies the requirement of the game. Next, Player 1 chooses a state $s' \in \text{succ}(s_2)$ and a state $q_1$. The same choice can be mimicked in $G_{M_1, \langle q \rangle}$. As
It follows that $L(s_1) = L(s_2)$ and the automaton component in both configurations remains the same.

The treatment of a play in a bounded MSCC $(q)$ with $\forall$ markings is similar.

5.3. Embedding of Markov Chains

A Markov chain $M = (S, P, L, s^\text{in}) \in \mathsf{MC}_{\mathsf{AP}}$ can be converted into a p-automaton $A_M = (2\mathsf{AP}, Q, \delta, \varphi^\text{in}, \alpha)$ whose language $L(A_M)$ is the set of Markov chains bisimilar to $M$:

- $Q = \{(s, s') \in S \times S \mid P(s, s') > 0\}$
- $\delta((s, s'), L(s)) = \ast([[(s', s'')]_{\geq P(s', s'')} \mid s'' \in \text{succ}(s'))$
- $\delta((s, s'), \sigma) = \text{ff}$ if $\sigma \neq L(s)$
- $\varphi^\text{in} = \ast([[(s^\text{in}, s')]_{\geq P(s^\text{in}, s')} \mid P(s^\text{in}, s') > 0)$
- $\alpha = Q$

State $(s, s')$ represents the transition from $s$ to $s'$. Labels are compared for location $s$. Location $s'$ is used to require that there are successors of probability at least $P(s', s'')$. This p-automaton $A_M$ has only bounded transitions and uses only the $\ast$ operator. In particular, it is uniform weak.

**Theorem 4** For every Markov chain $M \in \mathsf{MC}_{\mathsf{AP}}$, the language $L(A_M)$ is the bisimulation equivalence class of $M$.

**Proof:** By Lemma 1, we know that $M' \sim M$ implies $M' \in L(A_M)$ as soon as we have that $M \in L(A_M)$. To simplify the presentation of the proof of $M \in L(A_M)$, we assume that all locations of $M$ are in one MSCC. Consider a location $s \in S$ and $(s, s') \in Q$. Let $\varphi_s = \ast([[(s, s')]_{\geq P(s, s')} \mid s' \in \text{succ}(s))$. We show that from a configuration of the form $(s, \varphi_s)$, Player 0 has a strategy that keeps returning to configurations of this form. As $\alpha = Q$, Player 0 can continue playing forever and wins. We start from the configuration $(s, \varphi_s)$. Then Player 0 chooses the function $f: [n] \times \text{succ}(s) \rightarrow \{0, 1\}$ such that $f(i, s') = 1$ iff $s_i = s'$. The trivial assignment $a_i = 1$ iff $s_i = s'$ shows that $f$ is disjoint. Then, Player 1 chooses a successor $(s_i, \delta((s, s_i), L(s)), 1)$. As $\delta((s, s_i), L(s)) = \varphi_{s_i}$ the claim follows and Player 0 has a strategy to continue the play forever.

The initial configuration in the game is $\ast([[(s^\text{in}, s')]_{\geq P(s^\text{in}, s')} \mid s' \in \text{succ}(s^\text{in})]$. The same intuition shows that this is winning for Player 0 as well.
Conversely, if $M' \not\sim M$ we show that $M' \not\in \mathcal{L}(A_M)$. Let $M = (S, P, L, s^m)$ and $M' = (T, P, L, t^m)$. To simplify notations we assume that $S \cap T = \emptyset$ and use $P$ and $L$ for the probability distribution and labeling of both Markov chains. We use the partition refinement algorithm that computes the bisimulation equivalence sets for a Markov chain. Let $\Xi_0 = \{S' \subseteq S \cup T \mid \forall s, s' \in S': L(s) = L(s') \text{ and } S' \text{ is maximal with respect to that}\}$. Clearly, $\Xi_0$ is a partition of $S \cup T$. Let $\Xi_{i+1}$ be the coarsest partition of $S \cup T$ that refines $\Xi_i$ and in addition for every $G \in \Xi_{i+1}$, for all $s, s' \in G$, and for all $G' \in \Xi_i$ we have $P(s, G') = P(s', G')$. It is known that if $s \not\sim s'$ there is $i, s, s'$ where $s$ and $s'$ are in different sets in $\Xi_{i+1}$.

By assumption, $s^m \not\sim t^m$. Let $i_0$ be minimal such that $s^m$ and $t^m$ are in different sets in $\Xi_{i_0}$. Denote $s_{i_0} = s^m$, $t_{i_0} = t^m$, $\varphi_{i_0} = \varphi^m$, and $c_{i_0} = (t_{i_0}, \varphi_{i_0})$. Consider the configuration $c_{i_0} = (t_{i_0}, \varphi_{i_0})$, where $\varphi_{i_0} = \varphi(s^m) = \varphi(t^m, s^m) = \varphi^m$. By assumption, there is some $i_0$ such that $P(s_{i_0}, G) > P(t_{i_0}, G)$. Indeed, if $P(s_{i_0}, G) < P(t_{i_0}, G)$, then as $P(s_{i_0}, S) = 1$ there must be a different set $G' \in \Xi_{i_0+1}$ such that $P(s_{i_0}, G') > P(t_{i_0}, G')$.

Let $S_{i_0+1} = G \cap S$. Let $(t_{i_0}, \varphi_{i_0}, f)$ be the configuration chosen by Player 0. By disjointness of $f$, and as $P(t_{i_0}, G) < P(s_{i_0}, G)$, there must be $s_{i_0+1} \in G$ and $t_{i_0+1} \not\in G$ such that $f(t_{i_0+1}, s_{i_0+1} > 0$. Player 1 chooses $c_{i_0+1} = (t_{i_0+1}, \varphi_{i_0+1}, v)$, where $\varphi_{i_0+1} = \delta((s_{i_0}, s_{i_0+1}), L(t_{i_0}))$. As $t_{i_0+1} \not\in G$, Player 1 has forced the game to a similar configuration with $i_0+1 < i_0$ and eventually wins by reaching $\Xi_0$. □

The construction of $A_M$ for infinite Markov chains was the only reason why we allow p-automata with infinite state sets. Finite state sets suffice for embedding finite Markov chains. The construction of $A_M$ was also our initial reason for introducing the $\ast$ and $\psi$ operators. But we believe that the separation of concerns expressed in these operators is useful in p-automata in general. In the construction of $A_M$, the conjunctive operator $\ast$ effectively hides an exponential blowup.

If a Markov chain is deterministic (all successors of every location disagree on their labelings), we can eliminate the use of $\ast$ in $A_M$ and still secure Theorem 4. But this embedding does break Theorem 4 for non-deterministic Markov chains if we replace $\ast$ with the much simpler $\land$ in the definition of $\delta((s, s'), L(s))$ and $\varphi^m$ for $A_M$. We refer to this modified p-automaton as $A_M^w$ subsequently.
Consider the Markov chain $M$ in Figure 4 and let $M_1$ be $M$ with $s_1$ as initial location, and let $M_2$ be $M$ with $s_2$ as initial location. First, we note that $M_1$ and $M_2$ are not bisimilar since the transitions from $s_1$ to locations whose label is $b$ have probability $\frac{2}{3}$ and the transitions from $s_2$ to locations whose label is $b$ have probability $\frac{1}{3}$. In fact, no two locations in $M$ are bisimilar.

Second, we observe that $A_{M_1}^{w}$ accepts $M_2$. To see the latter, the initial configuration is $(s_2, \varphi_{in})$. As $\varphi_{in}$ is a conjunction, Player 1 can choose one of three successor configurations: $(s_2, \left[\left[ (s_1, s_3) \right] \geq \frac{1}{3} \right])$, $(s_2, \left[\left[ (s_1, s_4) \right] \geq \frac{1}{3} \right])$, and $(s_2, \left[\left[ (s_1, s_5) \right] \geq \frac{1}{3} \right])$. One can see that Player 0 wins from the latter two. In the other case, Player 1 chooses the configuration $(s_2, \left[\left[ (s_1, s_3) \right] \geq \frac{1}{3} \right])$. Then Player 0 chooses the configuration $(s_2, \left[\left[ (s_1, s_3) \right] \geq \frac{1}{3} \right], f)$ where $f$ is the function that sets $f(1, s_4) = 1$ and $f(1, s_5) = f(1, s_6) = 0$. The next configuration is $(s_4, \left[\left[ (s_3, s_1) \right] \geq 1, 1 \right])$. We complete a cycle by going back to configuration $(s_2, \varphi_{in})$. This completes a winning strategy for Player 0.

### 5.4. Embedding of PCTL Formulas

A PCTL formula $\phi$ over $\mathbb{AP}$ yields a p-automaton $A_\phi$ without $\ast$ markings, $(2^{\mathbb{AP}}, cl_p(\phi) \cup \mathbb{AP}, \rho_\epsilon, \rho_\epsilon(\phi), F)$, that accepts exactly the Markov chains satisfying $\phi$. The construction resembles the translation from CTL to alternating tree automata:

- $\text{cl}_p(\phi)$ denotes the set of path subformulas of $\phi$
- $F$ consists of $\mathbb{AP}$ and their negations, and all $\psi$ of $\text{cl}_p(\phi)$ not of form $\psi_1 \cup \psi_2$
• functions $\rho_a$ and $\rho_x$ are defined in (3), where we interpret the evaluation of $a \in \sigma$ as a truth constant $\text{tt}$ or $\text{ff}$:

$$
\rho_a(a, \sigma) = \begin{cases} 
\text{tt} & (a \in \sigma) \\
\text{ff} & (a \notin \sigma)
\end{cases}
$$

$$
\rho_x(\neg a, \sigma) = \neg(a \in \sigma)
$$

$$
\rho_x(\varphi_1, \sigma) = \rho_e(\varphi_1)
$$

$$
\rho_x(\varphi_1 \cup \varphi_2, \sigma) = (\rho_e(\varphi_1) \land \varphi_1 \cup \varphi_2) \lor \rho_e(\varphi_2)
$$

$$
\rho_x(\varphi_1 W \varphi_2, \sigma) = (\rho_e(\varphi_1) \land \varphi_1 W \varphi_2) \lor \rho_e(\varphi_2)
$$

$$
\rho_e(a) = a
$$

$$
\rho_e(\neg a) = \neg a
$$

$$
\rho_e(\varphi_1 \circ \varphi_2) = \rho_e(\varphi_1) \circ \rho_e(\varphi_2) \quad \text{where } \circ \in \{\land, \lor\}
$$

$$
\rho_e(\varphi_1 [\lceil (a U \varphi_2) \rceil_{\text{op}}]_{\text{op}}) = (\rho_e(\varphi_1) \land \varphi_1 [\lceil (a U \varphi_2) \rceil_{\text{op}}]_{\text{op}}) \lor \rho_e(\varphi_2)
$$

$$
\rho_e(\varphi_1 [\lceil W \varphi_2 \rceil_{\text{op}}]_{\text{op}}) = (\rho_e(\varphi_1) \land \varphi_1 [\lceil W \varphi_2 \rceil_{\text{op}}]_{\text{op}}) \lor \rho_e(\varphi_2)
$$

(3)

Function $\rho_a$ records whether or not literals of the formula are consistent with the input symbol, unfolds fix-points, and replaces the threshold context $\lceil . \rceil_{\text{op}}$ with $[.]_{\text{op}}$ (through a recursive call to $\rho_e$). That replacement is also done by function $\rho_e$ for the initial condition. The effect of these functions is similar to that achieved by using $\epsilon$ transitions to translate CTL formulas into two-way tree automata [29].

We now have that $\psi \in \text{cl}_p(\phi)$ for subformulas $[\psi]_{\text{op}}$ of $\phi$. Also, $[\psi_1 U \psi_2]_{\text{op}}$ may appear inside an element in $\text{cl}_p(\phi)$ whereas $[\psi_1 \psi_2]_{\text{op}}$ can only be an element of $[\text{cl}_p(\phi)]_{\text{op}}$, it wraps $\psi_1 U \psi_2 \in \text{cl}_p(\phi)$ in the probabilistic quantification $[.]_{\text{op}}$ of $A_\phi$.

**Example 6** Let $\varphi = [a U [X b] \geq \frac{1}{2}]_{\geq 0.3}$. Automaton $A_\varphi$ is $(2^{\{a,b\}}, \text{cl}_p(\varphi) \cup \{a, b\},$

$\rho_x, \rho_e(\varphi), F)$, where $\text{cl}_p(\varphi) = \{a U [X b] \geq \frac{1}{2}, X b\}$, $F$ is $\{X b, a, b\}$, $\rho_e(\varphi)$ is $(a \land [a U [X b] \geq \frac{1}{2}]_{\geq 0.3}) \lor [X b] \geq \frac{1}{2}$, $\rho_x(X b)$ is $b$, and $\rho_x(a U [X b] \geq \frac{1}{2})$ is $(a \land a U [X b] \geq \frac{1}{2}) \lor [X b] \geq \frac{1}{2}$.

Our acceptance game captures PCTL model checking, with same complexity.

**Theorem 5** For $M \in \text{MC}_{\text{AP}}$ and PCTL formula $\phi$ over AP, $M \models \phi$ iff $M \in \mathcal{L}(A_\phi)$. Deciding the latter is polynomial in the size of $M$, linear in the size of $\phi$.

**Proof:**
1. We show the first statement of the theorem by proving

For all locations \( s \) of \( M \) and PCTL (state) subformulas \( \varphi' \) of \( \varphi \): configuration \( (s, \rho_s(\varphi')) \) has value 1 for Player 0 in acceptance game of \( A_\varphi \) on \( M \) iff \( M, s \models \varphi' \). Furthermore, the value is 0 otherwise.

by induction on the structure of the formula.

1.1. For a proposition \( a \), notice that the value of \( (s, a) \) depends on the values of \( (s', \rho_s(\psi)) \) for successors \( s' \) of \( s \). By definition, \( \rho_s(a, L(s)) = \top \) if \( a \in L(s) \) and \( \bot \) otherwise. The claim holds similarly for the other Boolean operators.

1.2. Consider a subformula of the form \( \varphi' = [X \psi]_\omega \). By induction \( M, s' \models \psi \) iff the configuration \( (s', \rho_s(\psi)) \) is winning for Player 0. By definition \( \rho_s([X \psi]_\omega) = [X \psi]_\omega \). Consider a state \( s \) such that \( s \models \varphi' \). Let \( Y = \{s' \mid s' \models \psi\} \). It follows that \( P(s, Y) \propto p \). Then, the function \( f : [1] \times \text{succ}(s) \rightarrow [0, 1] \) with \( f(1, s') = 1 \) iff \( \text{val}(s', \rho_s(\psi)) = 1 \) and \( f(1, s') = 0 \) otherwise satisfies \( \sum_{s' \in \text{succ}(s)} f(1, s') P(s, s') \propto p \). It follows that for every \( s' \) such that \( f(1, s') > 0 \) we have \( (s', \rho_s(\psi), 1) \) is a Player 1 configuration that is a dead-end, and hence winning for Player 0. It follows that \( (s, \rho_s(\varphi')) \) has value 1 for Player 0. In the other direction, suppose that \( (s, \rho_s([X \psi]_\omega)) \) has value 1 for Player 0 in the acceptance game of \( A_\varphi \) on \( M \). It follows that there is a function \( f : [1] \times \text{succ}(s) \rightarrow [0, 1] \) such that \( \sum_{s' \in \text{succ}(s)} f(1, s') P(s, s') \propto p \) and that Player 0 wins from \( (s, \rho_s([X \psi]_\omega), f) \). Thus, for every \( s' \) such that \( f(1, s') > 0 \) we have \( \text{val}(s', \rho_s(\psi)) \geq f(1, s') > 0 \). However, by induction assumption, \( \text{val}(s', \rho_s(\psi)) \in \{0, 1\} \) and \( \text{val}(s', \rho_s(\psi)) = 1 \) iff \( s' \models \psi \). Thus, as \( \sum_{s' \in \text{succ}(s)} f(1, s') P(s, s') \propto p \), we conclude that \( P(s, Y) \propto p \), where \( Y = \{s' \in \text{succ}(s) \mid s' \models \psi\} \), proving that \( s \models [X \psi]_\omega \).

1.3. Consider a formula of the form \( \varphi' = [\psi_1 \cup \psi_2]_\omega \). By induction \( M, s \models \psi_i \) iff the configuration \( (s, \rho_s(\psi_i)) \) is winning for Player 0, for \( i \in \{1, 2\} \). Consider the stochastic weak game induced by the MSCC \( (\psi_1 \cup \psi_2) \) in \( G_{A_\varphi} \). The optimal strategy for both players is memoryless and pure. Restricting our attention to these memoryless pure strategies we can think about the game as restricted to configurations of the form \( (s', \rho_s(\psi_1)) \), where all configurations are probabilistic. A play that is winning for Player 0 is exactly a play that remains in states \( s' \) such that \( M, s' \models \psi_1 \) until reaching states \( s'' \) such that \( M, s'' \models \psi_2 \) (as \( \psi_1 \cup \psi_2 \) is unfair). It follows that the value of \( (s, \psi_1 \cup \psi_2) \) in the stochastic game is exactly \( \text{Prob}_M(s, \psi_1 \cup \psi_2) \).
We note that $\rho_c([\psi_1 U \psi_2]_{s, \rho}) = \rho_c(\psi_1) \land [\psi_1 U \psi_2]_{s, \rho} \lor \rho_c(\psi_2)$.

Consider a location $s$ such that $s \models [\psi_1 U \psi_2]_{s, \rho}$. We have to show that Player 0 wins from configuration $(s, \rho_c([\psi_1 U \psi_2]_{s, \rho}))$. In case that $s \models \psi_2$ then, by induction $\text{val}(s, \rho_c(\psi_2)) = 1$. It follows that the value of $(s, \rho_c(\psi_1) \land [\psi_1 U \psi_2]_{s, \rho} \lor \rho_c(\psi_2))$ is also 1. In case that $s \not\models \psi_2$ then it must be the case that $s \models \psi_1$ and that the set of paths that satisfy $\psi_1$ and start with $s$ is $\propto p$. It follows that the value of $(s, \rho_c(\psi_1))$ is 1. By our claim regarding the game above for every successor $s'$ of $s$ we have $\text{val}(s', \psi_1 U \psi_2)$ is $\text{Prob}_M(s', \psi_1 U \psi_2)$. Also, as $s \models [\psi_1 U \psi_2]_{s, \rho}$ it follows that $\sum_{s' \in \text{succ}(s)} P(s, s') \text{Prob}_M(s', \psi_1 U \psi_2) \propto p$. Hence, the function $f$ that associates $\text{Prob}_M(s', \psi_1 U \psi_2)$ with $s'$ is a disjoint function that leads to Player 0 winning from $(s, \rho_c([\psi_1 U \psi_2]_{s, \rho}))$.

In the other direction, consider a location $s$ such that Player 0 wins from $(s, \rho_c([\psi_1 U \psi_2]_{s, \rho}))$. Then one of the following two cases holds. Either Player 0 wins from $(s, \rho_c(\psi_2))$ and by induction $s \models \psi_2$. Or Player 0 wins from $(s, \rho_c(\psi_1))$ and Player 0 wins from $(s, [\psi_1 U \psi_2]_{s, \rho})$. By induction $s \models \psi_1$. Furthermore, there is a function $f : \text{succ}(s) \to [0, 1]$ such that $\sum_{s' \in \text{succ}(s)} P(s, s') f(s') \propto p$ and Player 0 wins from $(s', \delta(\psi_1 U \psi_2, L(s)), f(s'))$. However, by the observation about the stochastic games above, it follows that $\text{Prob}_M(s', \psi_1 U \psi_2) \geq f(s')$. Thus, this shows that $s \models [\psi_1 U \psi_2]_{s, \rho}$.

Finally, as $\rho_c([\psi_1 U \psi_2]_{s, \rho}) = \rho_c(\psi_1) \land [\psi_1 U \psi_2]_{s, \rho} \lor \rho_c(\psi_2)$ it follows that the value of $(s, \rho_c([\psi_1 U \psi_2]_{s, \rho}))$ is in $\{0, 1\}$.

1.4. The cases of a formula of form $\varphi' = [\psi_1 W \psi_2]_{s, \rho}$ is similar.

2. Finally, we provide the complexity analysis showing that the acceptance game for $M \in L(A_\phi)$ is linear in the PCTL formula $\phi$ and polynomial in $M$.

The number of MSCCs in $G_{A_\phi}$ is linear in the size of $\phi$. Most MSCCs are trivial and consist of a single subformula of $\phi$. The corresponding games can be solved by propagating pre-seeded game values. The only nontrivial MSCCs in $G_{A_\phi}$ arise from states of the form $\phi_1 U \phi_2$ and $\phi_1 W \phi_2$. It follows that for each nontrivial MSCC, the choices for each of the players in the corresponding weak stochastic game are determined by pre-seeded game values. Therefore, these games have effectively only probabilistic configurations and can be solved in polynomial time in the size of the Markov chain.

Overall, the number of games is linear in the size of $\phi$ and the solution of each game is polynomial in the size of the Markov chain. \qed

We note that our definition of PCTL does not include the bounded versions of the Strong Until and Weak Until. However, the techniques above can be extended
to handle these operators. Essentially, in order to handle $\psi_1 U \leq_k \psi_2$ the automaton would have states corresponding to $\psi_1 U \leq_{k'} \psi_2$ for all $0 \leq k' \leq k$. The case of bounded Weak Until would be similar. In particular, the proof of case 1.3 above gives all the necessary ingredients for handling these cases as well.

Corollary 1 and Theorem 5 imply that the satisfiability of PCTL [6, 30] reduces to both the language emptiness and containment of p-automata. The decidability status is open for all these problems at the time of writing.

In comparing automata and temporal logic, automata usually can count but temporal logics cannot. Thus, just as alternating tree automata are more expressive than CTL and CTL*, p-automata are more expressive than PCTL.

Also, p-automata can encode recursive, probabilistic properties that we believe are not expressible in PCTL: $A_R = \langle 2\{a\}, \{q_2\}, \delta, [q_2]_{>0}, \{q_2\} \rangle$ with $\delta(q_2, \{a\}) = [q_2]_{\geq \frac{1}{2}}$ and $\delta(q_2, \{\}) = \text{ff}$, asserts the recursive, probabilistic property that a location is labeled $a$, and that the probability of its successors with the same property is $\geq \frac{1}{2}$. A naive attempt of expressing this in PCTL is through formula $\eta = a \land [(\neg a \lor [X a]_{\geq \frac{1}{2}}) W \neg a]_{>1}$. Then $L(A_\eta) \subset L(A_R)$ but this inclusion is strict; e.g., the Markov chain in Fig. 5 is in $L(A_R)$ but not in the language of $A_\eta$.

6. Simulation of p-Automata

We now define simulation of p-automata that under-approximates language containment: if p-automaton $B$ simulates p-automaton $A$ (denoted $A \preceq B$), then $L(A)$ is contained in $L(B)$, under qualifications detailed in the formal theorem below. This simulation is defined as a combination of fair simulation [17], simulation for alternating word automata [31], probabilistic bisimulation [8], and the
games defined in Section 3. The simulation takes into account the structure of the automata, their acceptance condition, and local probabilistic constraints. We show that whether $B$ simulates $A$ can be decided in EXPTIME and that simulation under-approximates language containment.

We determine whether $B$ simulates $A$ through solving a series of games $G_{\leq}$ on the product of states and transitions of $A$ and $B$: state $u$ of $B$ simulates state $r$ of $A$ iff Player 0 wins from configuration $(r, u)$ in the corresponding game. More general configurations $(\alpha, \beta)$ are such that $\alpha$ is part of a transition of $A$ and $\beta$ is part of a transition of $B$. The classification of $\alpha$ and $\beta$ as unbounded, bounded with $*$, bounded with $\Psi$, or simple classifies $(\alpha, \beta)$ as one of 16 cases. As before, simple cases where either $\alpha$ or $\beta$ belong to trivial MSCCs in their respective automata reduce to one of the previous cases. Thus we are left with “only” 9 interesting cases.

Here, we restrict our attention to $A$ and $B$ that do not use the $\Psi$ operator. These restrictions, and concentrating on the non-simple cases, reduce the number of cases to consider to 4. This restriction is sufficient for handling simulation of automata that result from embedding PCTL formulas or Markov chains, as such automata do not use $\Psi$ transitions. We also assume that $tt$ and $ff$ do not appear in transitions of $A$ and $B$. Clearly, this does not restrict the expressive power of automata.

For simplicity of presentation, p-automata $A = \langle \Sigma, Q, \delta, \varphi^a, F \rangle$ and $B = \langle \Sigma, U, \delta, \psi^a, F \rangle$ satisfy $Q \cap U = \{\}$. We also use $\delta$ for the transition function of both automata and $F$ for both acceptance conditions. The strict versions of the partial orders on equivalence classes of $G_A$ and $G_B$ are well founded. We consider their pointwise extension $\leq_{A,B}$ as an ordering on the MSCCs of the resulting game. Namely, $\langle (\varphi), (\psi) \rangle \leq_{A,B} \langle (\tilde{\varphi}), (\tilde{\psi}) \rangle$ if $\langle (\varphi) \rangle \leq_A \langle (\tilde{\varphi}) \rangle$ and $\langle (\psi) \rangle \leq_B \langle (\tilde{\psi}) \rangle$.

As before, we start by setting $\text{val}(\varphi, \psi) = \bot$ for all pairs $(\varphi, \psi) \in \delta(Q, \Sigma) \times \delta(U, \Sigma)$ and gradually set concrete values to all of them. Consider a pair of equivalence classes $\langle (\varphi), (\psi) \rangle$, where $\varphi$ is in $A$ and $\psi$ is in $B$. Suppose that all pairs larger than $\langle (\varphi), (\psi) \rangle$ with respect to $\leq_{A,B}$ have already been handled: for every $\varphi'$ and $\psi'$ with $\langle (\varphi'), (\psi') \rangle <_{A,B} \langle (\varphi), (\psi) \rangle$ value $\text{val}(\varphi', \psi') \neq \bot$ is pre-seeded.

Case 1. Let $\langle (\varphi) \rangle$ and $\langle (\psi) \rangle$ be MSCCs where $\langle (\varphi) \rangle$ has no transitions in $E_b$, and $\langle (\psi) \rangle$ no transitions in $E_u$ and no $\Psi$ markings. We set $\text{val}(\varphi, \psi) = 0$; bounded-
with-\* states cannot simulate unbounded states.

**Case 2.** Let $((\varphi))$ and $((\psi))$ be MSCCs such that both $((\varphi))$ and $((\psi))$ have no transitions in $E_u$. Then $G_{\leq}(((\varphi)),((\psi)))$ is a stochastic weak game with

$$
V = \{((\tilde{\varphi}, \tilde{\psi}) | \varphi \leq_A \tilde{\varphi}, \psi \leq_B \tilde{\psi}\}
$$

$$
V_p = \{\}
$$

$$
V_0 = \{c \in V | \exists\varphi_i, \psi_i: c = (\varphi_1 \land \varphi_2, \psi_1 \lor \psi_2) \}
$$

$$
\bigcup \{c \in V | \exists q' \exists \psi_i: c = (q', \psi_1 \lor \psi_2) \}
$$

$$
\bigcup \{c \in V | \exists \varphi_i \exists u': c = (\varphi_1 \land \varphi_2, u') \}
$$

$$
V_1 = \{c \in V | \exists q', u': c = (q', u') \}
$$

$$
\bigcup \{c \in V | \exists \varphi_i, \psi: c = (\varphi_1 \lor \varphi_2, \psi) \}
$$

$$
\bigcup \{c \in V | \exists \varphi, \psi_i: c = (\varphi, \psi_1 \land \psi_2) \}
$$

$$
E = \{(((\varphi_1 \land \varphi_2, \psi_2 \lor \psi_2), (\varphi_i, \psi_j)) \in V \times V | 1 \leq i, j \leq 2 \}
$$

$$
\bigcup \{((q', \psi_1 \lor \psi_2), (q', \psi_i)) \in V \times V | 1 \leq i \leq 2 \}
$$

$$
\bigcup \{((\varphi_1 \land \varphi_2, u'), (\varphi_1, u')) \in V \times V | 1 \leq i \leq 2 \}
$$

$$
\bigcup \{((q', u'), (\delta(q', \sigma), \delta(u', \sigma))) \in V \times V | \sigma \in \Sigma \}
$$

$$
\bigcup \{((\varphi_1 \lor \varphi_2, \psi), (\varphi_i, \psi)) \in V \times V | 1 \leq i \leq 2 \}
$$

$$
\bigcup \{((\varphi, \psi_1 \land \psi_2), (\varphi, \psi_i)) \in V \times V | 1 \leq i \leq 2 \}
$$

The game $G_{\leq}(((\varphi)),((\psi)))$ does not have probabilistic configurations. However, pre-seeded values $\text{val}(\tilde{\varphi}, \tilde{\psi})$ for configurations $(\tilde{\varphi}, \tilde{\psi})$ with $((\varphi)),((\psi)) <_{A,B} (((\tilde{\varphi})),((\tilde{\psi})))$ may be in the range $(0, 1)$. Thus, we treat $G_{\leq}(((\varphi)),((\psi)))$ as a stochastic weak game.

Intuitively, Player 1 resolves disjunctions on the left and conjunctions on the right and does this before Player 0 needs to move. Player 0 resolves conjunctions on the left and disjunctions on the right when Player 1 cannot move. From configurations of the form $(q', u')$, where $q'$ is a state of $A$ and $u'$ is a state of $B$, Player 1 chooses a letter $\sigma \in \Sigma$ and applies the transitions of $q'$ and $u'$ reading $\sigma$.

Finally, an infinite play in $G_{\leq}(((q)),((u)))$ is winning for Player 0 if $((\varphi)) \cap Q \subseteq F$ implies $((\psi)) \cap U \subseteq F$. By Theorem 1 every configuration $c$ has a value for Player 0. We set $\text{val}(c)$ to that value.

**Case 3.** Let $((\varphi))$ and $((\psi))$ be MSCCs such that both have neither transitions in $E_u$ nor $\Psi$ markings. Below, let $\tilde{\varphi}$ have form $*((q_1)_{> p_1}, \ldots, (q_n)_{> p_n}), \tilde{\psi}$ have form $*((u_1)_{> p'_1}, \ldots, (u_m)_{> p'_m})$. Clearly, for every $q_i, u_j$ and $\sigma \in \Sigma$, we have $\delta(q_i, \sigma)$
and \( \delta(u_j, \sigma) \) are finite. Then, so are

\[
R_{\tilde{\varphi}, \tilde{\psi}} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \{ (\alpha, \beta) \mid \alpha \in \text{cl}(\delta(q_i, \sigma)), \beta \in \text{cl}(\delta(u_j, \sigma)) \}
\]

\[
\text{Val}_{\tilde{\varphi}, \tilde{\psi}} = \{0, 1\} \cup \{ \text{val}(\alpha, \beta) \mid (\alpha, \beta) \in R_{\tilde{\varphi}, \tilde{\psi}}, \text{val}(\alpha, \beta) \neq \bot \}
\]

As before, \( R_{\tilde{\varphi}, \tilde{\psi}} \) is the set of configurations reachable from \((\tilde{\varphi}, \tilde{\psi})\) using one transition in \( \delta \). Set \( \text{Val}_{\tilde{\varphi}, \tilde{\psi}} \) includes 0, 1, and values of configurations in \( R_{\delta, \varphi} \). We define the set \( \mathcal{F}_{\tilde{\varphi}, \tilde{\psi}} = [n] \times [m] \rightarrow \text{Val}_{\tilde{\varphi}, \tilde{\psi}} \).

Also, \( f \in \mathcal{F}_{\tilde{\varphi}, \tilde{\psi}} \) is disjoint if there is \( \{a_{i,j} \in [0, 1] \mid i \in [n] \text{ and } j \in [m]\} \) such that the following holds:

(i) for all \( i \in [n] \) we have \( \sum_{j \in [m]} a_{i,j} = 1 \) and

(ii) for all \( j \in [m] \) we have either \( \sum_{i \in [n]} a_{i,j} \cdot f(i, j) > p'_j \) or \( \sum_{i \in [n]} a_{i,j} \cdot p'_i \cdot f(i, j) = p'_j \) and either \( \preceq_j \) is \( \geq \) or there is \( i' \) with \( a_{i',j} > 0 \) such that \( \preceq_{i'} = \succ \).

Let \( \mathcal{F}_{\tilde{\varphi}, \tilde{\psi}}^* \) be the set of disjoint functions in \( \mathcal{F}_{\tilde{\varphi}, \tilde{\psi}} \). Weak game \( G_{\leq}((\varphi), (\psi)) \) is defined as

\[
V = \{ (\tilde{\varphi}, \tilde{\psi}, \sigma) \mid (\tilde{\varphi}, \tilde{\psi}, \sigma, f) \in \mathcal{F}_{\tilde{\varphi}, \tilde{\psi}}^* \}
\]

\[
V_0 = \{ (\alpha_1 \land \alpha_2, \beta_1 \lor \beta_2, v) \mid (\alpha_1 \land \alpha_2, \epsilon, v), (\gamma, \beta_1 \lor \beta_2, v), (\gamma, \epsilon, \sigma) \}
\]

\[
V_1 = \{ (\gamma, \epsilon, v) \mid (\gamma, \epsilon, v), (\gamma, \epsilon, f), (\alpha_1 \lor \alpha_2, \beta_1 \land \beta_2, v) \}
\]

\[
E = \{ ((\alpha_1 \land \alpha_2, \beta_1 \lor \beta_2, v), (\alpha_i, \beta_j, v)) \mid 1 \leq i, j \leq 2 \}
\]

\[
\bigcup \{ ((\alpha_1 \land \alpha_2, \epsilon, v), (\alpha_i, \epsilon, v)) \mid 1 \leq i \leq 2 \}
\]

\[
\bigcup \{ ((\gamma, \beta_1 \lor \beta_2, v), (\gamma, \beta_j, v)) \mid 1 \leq i \leq 2 \}
\]

\[
\bigcup \{ ((\gamma, \epsilon, \sigma), (\gamma, \epsilon, \sigma, f)) \}
\]

\[
\bigcup \{ ((\gamma, \epsilon, v), (\gamma, \epsilon)) \}
\]

\[
\bigcup \{ ((\gamma, \epsilon, (\gamma, \epsilon, \sigma)) \mid \sigma \in \Sigma \}
\]

\[
\bigcup \{ ((\gamma, \epsilon, f), (\delta(q_i, \sigma), \delta(u_j, \sigma), f(i, j))) \mid f(i, j) > 0 \}
\]

\[
\bigcup \{ ((\alpha_1 \lor \alpha_2, \beta_1 \land \beta_2, v), (\alpha_i, \beta_j, v)) \mid 1 \leq i \leq 2 \}
\]

\[
\bigcup \{ ((\alpha, \beta_1 \lor \beta_2, v), (\alpha, \beta_i, v)) \mid 1 \leq i \leq 2 \}
\]

where \( \alpha \) and \( \beta \) range over formulas in transitions of \( A \) and \( B \), respectively, and \( \gamma \) and \( \epsilon \) range over formulas in \( ((\varphi)) \cap [[Q]]^* \) and \( ((\psi)) \cap [[U]]^* \), respectively.

For \( (\gamma, \epsilon) \in [[Q]]^* \times [[U]]^* \) where \( \gamma \) is of form \( *([q_1]_{p_1}, \ldots, [q_n]_{p_n}) \) and \( \epsilon \) is of form \( *([u_1]_{p_1}, \ldots, [u_m]_{p_m}) \), for Player 0 to demonstrate that \( \epsilon \) simulates \( \gamma \), she is required to show that the probability of \( \epsilon \) (and its partition) can be supported
by $\gamma$. Accordingly, from $(\gamma, \epsilon)$ Player 1 chooses a letter $\sigma \in \Sigma$ and moves to configuration $(\gamma, \epsilon, \sigma)$. Then Player 0 chooses $f : [n] \times [m] \rightarrow [0, 1]$ and moves to configuration $(\gamma, \epsilon, \sigma, f)$. Such a configuration relates to the claim that $q_i$ is related to $u_j$ with proportion $f(i, j)$ and that $f$ can be partitioned (using the $\{a_{i,j}\}$ to support the different $u_j$’s). Then, Player 1 chooses $i$ and $j$ such that $f(i, j) > 0$ and proceeds to $(\delta(q_i, \sigma), \delta(u_j, \sigma), f(i, j))$.\footnote{We note that forcing Player 1 to choose the letter before Player 0 chooses the disjoint function gives Player 0 more power. This is still strong enough to imply language containment and relaxes the notion of simulation.} Conjunctions and disjunctions are resolved in the usual way until either reaching another configuration in $[Q]^* \times [U]^* \times [0, 1]$, in which case the value $f(i, j)$ is ignored (as $f(i, j) \leq 1$), or until the play reaches a configuration with a pre-seeded value $v$. Then, if $f(i) \leq v$ Player 0 has fulfilled her obligation and she wins. If $f(i) > v$, Player 0 failed and she loses. An infinite play in $G_{\preceq}(\varphi), (\psi)$ is winning for Player 0 if $(\varphi) \cap Q \subseteq F$ implies $(\psi) \cap U \subseteq F$. By Theorem 1, every $c \in V$ has a value in $\{0, 1\}$ for Player 0. We set $\text{val}(c)$ to that value.

**Case 4.** Let $(\varphi)$ and $(\psi)$ be MSCCs where $(\varphi)$ has no $E_i$ transitions or $\Psi$ markings, and $(\psi)$ has no $E_0$ transitions. Stochastic weak game $G_{\preceq}(\varphi), (\psi))$ is

$$V = \{ (\varphi, \psi) \mid \varphi \preceq_A \varphi, \psi \preceq_B \psi \}$$

$$V_0 = \{ (\alpha_1 \land \alpha_2, \beta_1 \lor \beta_2), (\alpha_1 \land \alpha_2, u), (\gamma, \beta_1 \lor \beta_2) \}$$

$$V_1 = \{ (\alpha_1 \lor \alpha_2, \beta), (\alpha_1 \lor \beta_1 \land \beta_2), (\gamma, u) \}$$

$$V_p = ([Q]^* \times U \times \Sigma) \cap V$$

$$E = \{ ((\alpha_1 \land \alpha_2, \beta_1 \lor \beta_2), (\alpha_i, \beta_j)) \mid 1 \leq i, j \leq 2 \}$$

$$\cup \{ ((\alpha_1 \land \alpha_2, u), (\alpha_i, u)) \mid 1 \leq i \leq 2 \}$$

$$\cup \{ ((\gamma, \beta_1 \lor \beta_2), (\gamma, \beta_j)) \mid 1 \leq j \leq 2 \}$$

$$\cup \{ ((\alpha_1 \lor \alpha_2, \beta), (\alpha_i, \beta)) \mid 1 \leq i \leq 2 \}$$

$$\cup \{ ((\alpha_1 \lor \beta_1 \land \beta_2), (\alpha_i, \beta_i)) \mid 1 \leq i \leq 2 \}$$

$$\cup \{ ((\gamma, u), (\gamma, u, \sigma)) \}$$

$$\cup \{ ((\gamma, u, \sigma), (\delta(q_i, \sigma), \delta(u, \sigma))) \}$$

where $\kappa((\gamma, u, \sigma), (\delta(q_i, \sigma), \delta(u, \sigma))) = p_i$, in the implicit set comprehensions $\alpha, \alpha_i$ and $\beta, \beta_i$ range over formulas in transitions of $A$ and $B$ (resp.), whereas $\gamma$ and $u$ range over $[Q]^*$ and $U$ (resp.). For probabilities $p_i$ that don’t sum up to 1, we add a sink state (losing for Player 0) filling that gap.

An infinite play in $G_{\preceq}(\varphi), (\psi)$ is winning for Player 0 if $(\varphi) \cap Q \subseteq F$
implies \( (\psi) \cap U \subseteq F \). By Theorem 1 every configuration \( c \) has a value for Player 0. We set \( \text{val}(c) \) to that value.

Intuitively, a state \( u \) measures the probability of some regular set of paths, and a state \( [q]_{\varphi} \) can restrict the immediate steps taken by a Markov chain as well as enforce some regular structure on paths. Thus, this stochastic weak game establishes the conditions under which a Markov chain accepted from \( [q]_{\varphi} \) can be also accepted from \( u \).

The case when \( (\varphi) \) or \( (\psi) \) is a trivial MSCC is subsumed by at least one of the four preceding cases. As for the acceptance game, this ambiguity is unproblematic as the game values in \( G_{\leq}((\varphi), (\psi)) \) are then determined by the propagation of pre-seeded game values.

**Definition 3** We say that \( B \) simulates \( A \), denoted \( A \leq B \), if the value of configuration \( (\varphi, \psi) \), computed in the previous sequence of games, is 1.

We can now state and prove our main theorem on simulation of p-automata.

**Theorem 6** For \( A \) and \( B \) p-automata over \( 2^{AP} \) with no occurrence of \( \forall \): If \( A \) and \( B \) are finite, \( A \leq B \) can be decided in EXPTIME and implies \( L(A) \subseteq L(B) \). If \( A \) is \( A_M \) for an \( M \in MC_{AP} \), then \( A \leq B \) iff \( L(A) \subseteq L(B) \) for all \( B \) over \( 2^{AP} \).

**Proof:** Let \( A \) and \( B \) be finite p-automata. To see that \( A \leq B \) can be decided in EXPTIME, we make two observations: First, that the stochastic weak game arising from the combination of a bounded MSCC with an unbounded MSCC or from the combination of two unbounded MSCCs is linear in each of the automata and can be solved in EXPTIME. Second, that the weak game arising from the combination of two bounded MSCCs may be exponential due to the large number of possible value assignment functions. Such a weak game can be solved in linear time leading to an EXPTIME upper bound. As there are only linearly many such games, the sequence of weak games and stochastic weak games can be solved in EXPTIME.

To show the last claim of the theorem, we note that when \( A \) equals \( A_M \) for some \( M \in MC_{AP} \), the simulation game for \( A_M \leq B \) and the acceptance game for \( M \in L(B) \) collapse to the same game. Thus, regardless of whether \( A_M \) or \( B \) is infinite-state we have \( A_M \leq B \) if \( M \in L(B) \). And the latter is equivalent to \( L(A_M) \subseteq L(B) \) by Lemma 1 and Theorem 4.

It remains to show that \( A \leq B \) implies \( L(A) \subseteq L(B) \) for finite-state \( A \) and \( B \). To that end, consider a Markov chain \( M = (S, P, L, s^0) \) and two formulas \( \varphi \) and \( \psi \) such that \( \varphi \) appears in the transition of \( A \) and \( \psi \) appears in the transition
of $B$. Let $G_A$ and $G_B$ be the acceptance games to test whether $M \in \mathcal{L}(A)$ and whether $M \in \mathcal{L}(B)$, respectively. We try to derive a result regarding $G_B$ from the known results in $G_A$ and $G_{\leq}$. Thus, in $G_A$ and $G_{\leq}$ there are optimal strategies for Player 0. We construct from them a strategy for Player 0 in $G_B$. As we have strategies in $G_A$ and $G_{\leq}$ that work against all possible strategies of Player 1, we can “drive” Player 1 in these games in order to explore these optimal strategies. It follows that we devise a strategy that answers actions of Player 1 in $G_B$ and can “drive” Player 1 in these games in order to explore these optimal strategies.

We emulate Player 1 in $G_B$ strategies in $G_B$ for Player 0. We construct from them a strategy for Player 0 in $G_B$. It follows that we devise a strategy that answers actions of Player 1 in $G_B$ and using the strategies for Player 0 in $G_A$ and $G_{\leq}$. This gives as a strategy for Player 0 in $G_B$. In general, for a given play in $G_B$ we construct matching plays in $G_A$ and $G_{\leq}$ that start from $(s, \psi), (s, \varphi)$, and $(\varphi, \psi)$, respectively. We then show that the values obtained by the strategy we construct satisfy $\text{val}(s, \varphi) \cdot \text{val}(\varphi, \psi) \leq \text{val}(s, \psi)$. Thus, we prove that $M \in \mathcal{L}(A)$ implies $M \in \mathcal{L}(B)$.

Suppose that the claim holds by induction for configurations $((\bar{\varphi}), (\bar{\psi}))$, where $((\varphi), (\psi)) \prec_{A,B} ((\bar{\varphi}), (\bar{\psi}))$.

1. If $\varphi \in Q$ and $\psi \in [U]^*$, then $\text{val}(\varphi, \psi) = 0$ and the claim holds trivially.

2. Suppose that both $\varphi$ and $\psi$ are in unbounded MSCCs. Then $G_{M,((\varphi))}, G_{M,(\psi)}$, and $G_{((\varphi)),(\psi)}$ are all stochastic weak games. To simplify notations we refer to $G_{M,((\varphi))}$ as $G_A$, $G_{M,(\psi)}$ as $G_B$, and $G_{((\varphi)),(\psi)}$ as $G_{\leq}$. We know that the optimal strategy of Player 0 in $G_A$ secures at least $\text{val}(s, \varphi)$. We have to show a strategy in $G_B$ that secures at least $\text{val}(s, \varphi) \cdot \text{val}(\varphi, \psi)$. We note, however, that there are no stochastic probabilistic configurations in $G_{\leq}$. Thus, the combination of the optimal strategy for Player 0 in $G_{\leq}$ with some strategy for Player 1 produces a unique play. It follows that the value of this play must be at least $\text{val}(\varphi, \psi)$. Hence, either this play is infinite and winning (and $\text{val}(\varphi, \psi) = 1$) or this play is finite, leading outside of $G_{((\varphi)),(\psi)}$ and ending in a configuration whose value is at least $\text{val}(\varphi, \psi)$.

Consider the configurations $(s, \varphi), (s, \psi)$, and $(\varphi, \psi)$ in the games $G_A$, $G_B$, and $G_{\leq}$, respectively.

If $\varphi$ is a disjunction, $(s, \varphi)$ is a Player 0 configuration in $G_A$ and $(\varphi, \psi)$ is a Player 1 configuration in $G_{\leq}$. Then, Player 0’s strategy in $G_A$ instructs her to choose a disjunct $\varphi_i$ of $\varphi$. As $(\varphi, \psi)$ is a Player 1 configuration in $G_{\leq}$ we can emulate Player 1 by choosing the successor configuration $(\varphi_i, \psi)$. If $\psi$ is a conjunction, then $(s, \psi)$ is a Player 1 configuration in $G_B$ and $(\varphi, \psi)$ is a Player 1 configuration in $G_{\leq}$. Then, Player 1 chooses a successor $(s, \psi_i)$ of $(s, \psi)$ in $G_B$. We emulate Player 1 in $G_{\leq}$ by choosing the successor $(\varphi, \psi_i)$ in $G_{\leq}$. If $\varphi$ is a
conjunction and \( \psi \) is not a conjunction, then \((s, \varphi)\) is a Player 1 configuration in \( G_A \) and \((\varphi, \psi)\) a Player 0 configuration in \( G_\leq \). The strategy of Player 0 in \( G_\leq \) instructs Player 0 to choose a conjunct \( \varphi_i \) of \( \varphi \). We emulate Player 1 in \( G_A \) by choosing the same conjunct leading to configuration \((s, \varphi_i)\) in \( G_A \). If \( \varphi \) is not a disjunction and \( \psi \) is a disjunction, then \((\varphi, \psi)\) is a Player 0 configuration in \( G_\leq \) and \((s, \psi)\) is a Player 0 configuration in \( G_B \). The strategy of Player 0 in \( G_\leq \) instructs her to choose a disjunct \( \psi_i \) of \( \psi \). We use the same choice as the strategy for Player 0 in \( G_B \) leading to configuration \((s, \psi_i)\). Finally, if \( \varphi \) is a state of \( A \) and \( \psi \) is a state of \( B \) then \((s, \varphi)\) and \((s, \psi)\) are stochastic configurations in \( G_A \) and \( G_B \) and \((\varphi, \psi)\) is a Player 1 configuration in \( G_\leq \). We advance \( G_\leq \) by emulating the choice of Player 1 for the letter \( L(s) \) advancing to configuration \((\delta(\varphi, L(s)), \delta(\psi, L(s)))\). In \( G_A \) and \( G_B \) the next configurations are of the form \((s', \delta(\varphi, L(s)))\) and \((s', \delta(\psi, L(s)))\) for \( s' \in \text{succ}(s) \).

It follows that the games \( G_A \) and \( G_B \) produce Markov chains that have the same probability distributions. Furthermore, there is a 1-1 and onto mapping between paths in the Markov chain induced by \( G_A \) to paths in \( G_\leq \) and paths in the Markov chain induced by \( G_B \). Consider three matching paths in the three respective games. If all three are infinite, then if the path is winning for Player 0 in \( G_A \) then by the winning condition of \( G_\leq \) the path in \( G_B \) is winning for Player 0 as well. If one of the paths is finite, then the end configurations of the paths get out from \( G_\leq \) and reach the triplet \((s'', \varphi'), (\varphi', \psi') \) and \((s'', \psi')\). We deduce that \( \text{val}(\varphi', \psi') \in G_\leq \) is at least \( \text{val}(\varphi, \psi) \). Furthermore, by our assumptions regarding all configurations in \( G_\leq \) such that \((\varphi', \psi') \notin (\langle \varphi \rangle) \times (\langle \psi \rangle)\) it follows that \( \text{val}(s'', \varphi') \cdot \text{val}(\varphi', \psi') \leq \text{val}(s'', \psi') \). In particular, as \( \text{val}(\varphi', \psi') \geq \text{val}(\varphi, \psi) \) it follows that \( \text{val}(s'', \varphi') \cdot \text{val}(\varphi, \psi) \leq \text{val}(s'', \psi') \). Therefore, the inequality in the claim holds for every matching path in the two Markov chains and thus it must hold for the value of the Markov chains. Therefore, \( \text{val}(s, \varphi) \cdot \text{val}(\varphi, \psi) \leq \text{val}(s, \psi) \) as required.

3. Suppose that both \( \varphi \) and \( \psi \) are in bounded MSCCs. Then \( G_{M, \langle \varphi \rangle} \), \( G_{M, \langle \psi \rangle} \), and \( G_{\langle \varphi \rangle, \langle \psi \rangle} \) are all weak games. We again use the notations \( G_A \), \( G_B \), and \( G_\leq \). Consider the three configurations \((s, \varphi)\), \((\varphi, \psi)\) and \((s, \psi)\). By definition \( \text{val}(s, \varphi) \in \{0, 1\} \) and similarly \( \text{val}(\varphi, \psi) \) and \( \text{val}(s, \psi) \). If either \( \text{val}(s, \varphi) \) or \( \text{val}(\varphi, \psi) \) is 0, then the claim holds trivially. Consider the case that \( \text{val}(s, \varphi) = 1 \) and \( \text{val}(\varphi, \psi) = 1 \). That is, both in \( G_A \) and \( G_\leq \) there is a winning strategy for Player 0 such that regardless of how Player 1 resolves her choice Player 0 wins.

We now give a strategy for Player 0 in \( G_B \) that establishes \( \text{val}(s, \psi) = 1 \).

Let \( \varphi = *([q_1]_{p_1 q_1}, \ldots, [q_n]_{p_n q_n}) \) and \( \psi = *([u_1]_{p_1' q_1'}, \ldots, [u_m]_{p_m' q_m'}) \). It
follows that configuration \((s, \varphi)\) is a Player 0 configuration in \(G_A\), configuration \((\varphi, \psi)\) is a Player 1 configuration in \(G_{\leq}\) and configuration \((s, \psi)\) is a Player 0 configuration in \(G_B\). First, we make Player 1 choose the successor \((\varphi, \psi, L(s))\).

Then, Player 0’s strategy in \(G_A\) instructs her to choose successor configuration \((s, \varphi, f)\), where \(f : [n] \times \text{succ}(s) \to [0, 1]\). Player 0’s strategy in \(G_{\leq}\) instructs her to choose successor configuration \((\varphi, \psi, L(s), f')\), where \(f' : [n] \times [m] \to [0, 1]\). Configuration \((s, \psi)\) is a Player 0 configuration in \(G_B\) and we set her strategy to choose the successor \((s, \psi, f'')\), where \(f'' : [m] \times \text{succ}(s) \to [0, 1]\) such that for every \(j \in [m]\) and every \(s' \in \text{succ}(s)\) we have \(f''(j, s')\) is the minimal value in \(\text{Val}_{s, \psi}\) that is at least \(\max_{i \in [n]} f(i, s') \cdot f'(i, j)\).

We have to show that \(f''\) is disjoint. To that end, let \(a_{j,s'} = \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j}\). First, one can see that for every \(s' \in \text{succ}(s)\) we have

\[
\sum_{j \in [m]} a_{j,s'} = \sum_{j \in [m]} \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} = \sum_{i \in [n]} a_{i,s'} \cdot \sum_{j \in [m]} a_{i,j} = \sum_{i \in [n]} a_{i,s'} = 1
\]

Second, consider some \(j \in [m]\). Then,

\[
\sum_{s' \in \text{succ}(s)} a_{j,s'} \cdot f''(j, s') \cdot P(s, s') = \sum_{s' \in \text{succ}(s)} \left( \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \right) \cdot f''(j, s') \cdot P(s, s') \\
\geq \sum_{s' \in \text{succ}(s)} \left( \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \right) \cdot \max_{i \in [n]} (f(i, s') \cdot f'(i, j)) \cdot P(s, s') \\
\geq \sum_{s' \in \text{succ}(s)} \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \cdot f(i, s') \cdot f'(i, j) \cdot P(s, s') \\
= \sum_{i \in [n]} \sum_{s' \in \text{succ}(s)} a_{i,s'} \cdot a_{i,j} \cdot f'(i, j) \cdot \sum_{s' \in \text{succ}(s)} a_{i,s'} \cdot f(i, s') \cdot P(s, s') \\
\gg \sum_{i \in [n]} a_{i,j} \cdot f'(i, j) \cdot p_i \gg p_j
\]

and \(\gg\) is \(>\) if for some \(i \in [n]\) we have \(\gg_i\) equals \(>\) and then \(\gg'\) is \(\geq\), otherwise either \(\gg'\) is \(>\) or \(\gg'_j\) is \(\geq\) and the proof is complete.

With \(f''\) established as being disjoint, we get back to the games. In \(G_B\), Player 1 chooses \(j\) and \(s' \in \text{succ}(s)\) and moves to state \((s', \delta(u_j, L(s)), f''(j, s'))\).
We emulate Player 1 in \( G_A \) and make her choose the state \( q_i \) such that \( f(i,s') \cdot f'(i,j) \) is maximal and move to \((s',\delta(q_i,L(s)),f(i,s'))\). We emulate Player 1 in \( G_{\leq} \) as well and make her choose the states \( q_i \) and \( u_j \) leading to configuration \((\delta(q_i,L(s)),\delta(u_j,L(s)),f'(i,j))\).

Consider three matching plays produced by following this strategy. Suppose that the plays stay inside the same MSCC in \( G_{\leq} \) indefinitely. Then, as Player 0 is winning in \( G_A \) it follows that the play in \( G_A \) is winning. As Player 0 is winning in \( G_{\leq} \) it follows that the play in \( G_{\leq} \) is winning as well. However, this implies that the sequence of states of \( B \) on the right-hand-side of the configurations in \( G_{\leq} \) is winning. Thus, the infinite play in \( G_B \) is winning as well.

Suppose that the plays exit the MSCC in \( G_{\leq} \) and reach the triplet of configurations \((s'',\varphi'',v_1),(s'',\psi'',v_2)\) and \((\varphi'',\psi'',v)\). By induction, \( \text{val}(s'',\varphi'') \cdot \text{val}(\varphi'',\psi'') \leq \text{val}(s'',\psi'') \) holds. Furthermore, we have to show that \( \text{val}(s'',\psi'') \geq v_2 \). Let \((s',\varphi'),(s',\psi')\) and \((\varphi',\psi')\) be the last configurations that are part of the MSCC before reaching the above triplet of configurations. It follows that \( \text{val}(s'',\psi'') \in \text{Val}_{s',\psi'} \). By the choices of \( f, f' \) and \( f'' \) we know that \( v \) is the minimal value in \( \text{Val}_{s',\psi'} \) that is at least \( \max_{i\in[n]} f(i,s') \cdot f'(i,j) \). In addition, the last choice in \( G_A \) was exactly the state \( q_i \) such that \( i \) is maximal. As \( G_A \) and \( G_{\leq} \) are won, we know that \( \text{val}(s'',\varphi'') \geq v_1 \) and that \( \text{val}(\varphi'',\psi'') \geq v \). It follows that \( \text{val}(s'',\varphi'') \cdot \text{val}(\varphi'',\psi'') \geq v_1 \cdot v \). But, \( v_2 \) is the minimal possible value in \( \text{Val}_{s',\psi'} \) that is at least \( v_1 \cdot v \). Thus, \( \text{val}(s'',\psi'') \geq v_2 \).

4. Suppose that \( \varphi \) is in a bounded MSCC and \( \psi \) is in an unbounded MSCC. Then, \( G_{M,\langle \varphi \rangle} \) is a weak game and \( G_{\langle \varphi \rangle,\langle \psi \rangle} \) and \( G_{M,\langle \psi \rangle} \) are stochastic weak games. As before, we use the notations \( G_{A}, G_{B} \), and \( G_{\leq} \). As \( G_A \) is a weak game, the case that \( \text{val}(s,\varphi) = 0 \) is not interesting. Thus, we assume that \( \text{val}(s,\varphi) = 1 \). It follows that in \( G_A \) Player 0 has a winning strategy such that all possible plays in \( G_A \) are winning for Player 0.

Given a strategy of Player 1 in \( G_B \), we show how to use the winning strategies of Player 0 in \( G_A \) and \( G_{\leq} \) to produce a winning strategy for Player 0 in \( G_B \). Consider a triplet of configurations \((s,\varphi),(\varphi,u),(s,u)\), where \( \varphi \in [\mathbb{Q}]^* \) and \( u \in U \). The configurations \((\varphi,u)\) and \((s,u)\) are probabilistic configurations in their stochastic games. The successors of configurations of the form \((\varphi,u)\) in \( G_{\leq} \) are of the form \((\delta([q_i]_{p_0,p_1},L(s)),\delta(u,L(s)))\). The successors of configurations of the form \((s,u)\) in \( G_B \) are of the form \((s',\delta(u,L(s)))\), where \( s' \in \text{succ}(s) \). In order to continue using the association between the three games (and the strategies in \( G_A \) and \( G_{\leq} \)) to give a strategy for Player 0 in \( G_B \) we have to associate successors of \((s,u)\) to successors of \((\varphi,u)\). However, it is not clear which mapping is most
beneficial. Hence, we leave this mapping option open for a while. Instead of elaborating $G_B$ and $G_\leq$ to Markov chains (by fixing strategies for Player 0 and Player 1) we elaborate them to Markov decision processes. Thus, based on the different mapping options, Player 0 is going to have multiple ways to proceed in $G_B$ and Player 1 is going to have multiple ways to proceed in $G_\leq$. These MDPs capture all possible evolutions of plays in $G_B$ and $G_\leq$ according to possible mapping choices between configurations in $G_\leq$ and $G_B$. We then use these MDPs to prove that the claim holds.

Consider three configurations $(s, \varphi')$ in $G_A$, $(\varphi', \psi')$ in $G_\leq$, and $(s, \psi')$ in $G_B$. If $\psi'$ is a conjunction, then $(s, \psi')$ is a Player 1 configuration in $G_B$. It follows that Player 1 chooses a conjunct $\psi_i$ of $\psi'$ and proceeds to configuration $(s, \psi_i)$. The configuration $(\varphi', \psi')$ is a Player 1 configuration in $G_\leq$. We emulate Player 1 in $G_\leq$ by making her choose $(\varphi', \psi_i)$. If $\psi'$ is a disjunction, then $(s, \psi')$ is a Player 0 configuration in $G_B$. There are now a few cases:

- If $\varphi'$ is a conjunction, then $(\varphi', \psi')$ is a Player 0 configuration in $G_\leq$. Then, Player 0’s strategy in $G_\leq$ instructs her to choose a conjunct $\varphi_i$ of $\varphi'$ and proceed to configuration $(\varphi_i, \psi')$. We note that configuration $(s, \varphi')$ is a Player 1 configuration in $G_A$ and we emulate Player 1 by making her choose $(s, \varphi_i)$.
- If $\varphi'$ is a disjunction, then $(s, \varphi')$ is a Player 0 configuration in $G_A$. Player 0’s winning strategy in $G_A$ instructs her to choose a disjunct $\varphi_i$ and proceed to $(s, \varphi_i)$. Configuration $(\varphi', \psi')$ is a Player 1 configuration in $G_\leq$. We emulate Player 1 in $G_\leq$ and make her choose $(\varphi_i, \psi')$.
- If $\varphi'$ is in $[Q]^*$ then $(\varphi', \psi')$ is a Player 0 configuration in $G_\leq$. Then, the strategy of Player 0 in $G_\leq$ instructs her to choose a disjunct $\psi_i$ of $\psi'$ and proceed to configuration $(\varphi', \psi_i)$. We set Player 0’s strategy in $G_B$ to make the same choice and proceed to configuration $(s, \psi_i)$.

The only remaining case is where $(\varphi', \psi') \in [Q]^* \times U \cup V$. We denote $\varphi' = *([q_1]_{i=1:p_1}, \ldots, [q_n]_{i=1:p_n})$ and $\psi' = u$. The configuration $(s, \varphi')$ in $G_A$ is a Player 0 configuration, the configuration $(s, u)$ in $G_B$ is probabilistic, and the configuration $(\varphi', u)$ in $G_\leq$ is a Player 1 configuration. We emulate Player 1 in $G_\leq$ by making her choose $L(s)$ proceeding to the probabilistic configuration $(\varphi', u, L(s))$. The strategy of Player 0 in $G_A$ instructs her to choose a disjoint function $f : [n] \times \text{succ}(s) \rightarrow [0, 1]$ and proceed to $(s, \varphi', f)$. Let $\{a_{i,s'}\}$ be the witnesses to the disjointness of $f$.

Consider a location $s'$ that is chosen with probability $P(s, s')$ in $G_B$. Now, for every possible index $i$ such that $a_{i,s'} > 0$ the successor $(s', \delta(q_i, L(s)), f(i, s'))$ is a possible successor of $(s', \varphi', f)$ in $G_A$. Also, $(\delta(q_i, L(s)), \delta(u, L(s)))$ is a suc-
cessor of \((\varphi', u, L(s))\) in \(G_\leq\) and the probability to get to it is \(p_i\). Here, we make multiple possible choices of continuing in the games, giving rise to MDPs (with a matching between the choices in them). Consider all indices \(i\) such that \(a_{i,s''} > 0\). It follows that for every such index there is a way to continue unraveling the plays by making Player 1 in \(G_A\) choose the successor \((s', \delta(q_i, L(s)), f(i, s'))\) and continuing to configurations \((\delta(q_i, L(s)), \delta(u, L(s)))\) in \(G_\leq\) and \((s', \delta(u, L(s)))\) in \(G_B\). Notice that the choice in \(G_B\) is implicitly based on the choice of \(f\) in \(G_A\). As the future elaboration of the strategy in \(G_B\) depends on the association between configurations in the three games the choice of an index \(i\) such that \(a_{i,s''} > 0\) is effectively also a choice in \(G_B\) that determines the way the strategy is extended.

By using these strategies and these associations between the games, this effectively creates from \(G_B\) and \(G_\leq\) MDPs where the choices are angelic in \(G_B\) and demonic in \(G_\leq\). That is, the actual value of \(G_B\) is the best possible value in the MDP arising from \(G_B\) and the value in \(G_\leq\) is the worst possible value in \(G_\leq\). Hence, it is enough to show one choice such that the value in the MDP arising from \(G_B\) satisfies the requirement of the claim. Indeed, the actual value in \(G_B\) could only be higher while the actual value in \(G_\leq\) could only be lower.

Consider now three configurations \((s, \varphi'), (\varphi', \psi'), (s, \psi')\) and the resulting MDPs from \((\varphi', \psi')\) and \((s, \psi')\). By the construction of the strategy, every play starting in \((s, \psi')\) is associated with plays that start in \((s, \varphi')\) and \((\varphi', \psi')\) such that at every stage the three configurations use the same state of the Markov chain and formulas in the transitions of \(A\) and \(B\). We consider four cases:

4.1. A triplet of configurations \((s, \varphi'), (\varphi', \psi'), (s, \psi')\) where \((\varphi', \psi')\) is not in the equivalence class of \(((\varphi)), ((\psi))\). By induction \(\text{val}(s, \psi') \geq \text{val}(s, \varphi') \cdot \text{val}(\varphi', \psi')\).

4.2. A triplet of configurations \((s, \varphi'), (\varphi', \psi'), (s, \psi')\) with some choice in the MDP that arises from \(G_B\) where all plays starting in \((\varphi', \psi')\) remain in \(((\varphi)), ((\psi))\) and are winning for Player 0 in \(G_\leq\). The matching choice of plays starting from \((s, \psi')\) are winning for Player 0 in \(G_B\). Indeed, if this were not the case, there would be a play in \(G_B\) that is losing. It follows that the corresponding play in \(G_\leq\) does not satisfy the acceptance of \(A\) and that the play in \(G_A\) is losing. However \(G_A\) is a weak game and so this is impossible.

4.3. A triplet of configurations \((s, \varphi'), (\varphi', \psi'), (s, \psi')\) where, for all choices in the MDP that arises from \(G_\leq\), plays starting in \((\varphi', \psi')\) remain in \(((\varphi)), ((\psi))\) and are losing for Player 0 in \(G_\leq\). One can see that then \(\text{val}(s, \psi') \geq 0\).
4.4. A triplet \((s, \varphi'), (\varphi', \psi'), (s, \psi')\) with \((\varphi', \psi') \in (((\varphi)), ((\psi)))\) where (i) for no choice in the MDP arising from \(G_\leq\) are all paths winning for Player 0 and (ii) for all such choices the probability for Player 0 to win is positive. As automaton and Markov chain are finite, so are the resulting MDPs. It follows that the probability of winning in \(G_\leq\) equals the probability of getting to one of the previous three types of configurations. That is, we have the MDP resulting from \(G_B\) and we are searching for a strategy for Player 0 with reachability objective to reach one of the previous end components. We show that the probability to reach one of the three previous types of configurations in \(n\) steps satisfies the requirements of the theorem, for every \(n\). The requirement of the claim will follow. For that, we compute the probability \(P_n\) to reach from a configuration \((\varphi, \psi)\) in \(G_\leq\) and from a configuration \((s, \psi)\) in \(G_B\) one of these three types of configurations in \(n\) steps. We compute \(P_n\) by induction starting from \(P_0\).

For each triplet \((s', \varphi'), (\varphi', \psi'), (s', \psi')\) we let \(P_0(\varphi', \psi')\) be \(\text{val}(\varphi', \psi')\) and \(P_0(s', \psi')\) be \(\text{val}(\varphi', \psi')\) if \((\varphi', \psi')\) is one of the three types of configurations mentioned above. Let \(P_0(\varphi', \psi')\) and \(P_0(s', \psi')\) be 0, otherwise.

Let \(\varphi' = \ast([q_1]_{a_1 \in p_1}, \ldots, [q_n]_{a_n \in p_n})\) and consider triplet \((s, \varphi'), (\varphi', \psi'), (s, \psi')\) where \(\psi' = u\), \(P_0(\varphi', \psi') = P_0(s, \psi') = 0\) but there are \(s' \in \text{succ}(s)\) and \(i\) such that \(P_0(\delta(q_i, L(s'))), \delta(u, L(s))) > 0\) and \(P_0(s', \delta(u, L(s'))) > 0\). Then, \(P_1\) satisfies the requirement reasoned as follows. First,

\[
P_1(s, u) = \sum_{s' \in \text{succ}(s)\mid \exists i \in [n]: a_{i,s'} > 0} P(s, s')P_0(s', \delta(u, L(s)))
\]

For every \(s'\) let \(i_{s'}\) be such that \(P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) = \text{val}(s', \delta(q_{i_{s'}}, L(s)))\) is maximal among all \(i \in [n]\) such that \(a_{i,s'} > 0\). We know that \(P_0(s', \delta(u, L(s))) \geq P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))\) if \(P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) > 0\) this holds by our proof for \(P_0\). If \(P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) = 0\) then this holds trivially. We obtain

\[
P_1(s, u) \geq \sum_{s' \in \text{succ}(s)\mid \exists i \in [n]: a_{i,s'} > 0} P(s, s') \cdot P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) \cdot \text{val}(s', \delta(q_{i_{s'}}, L(s)))
\]
By \( f \) being disjoint, we have \( \sum_{i \in [n]} a_{i,s} = 1 \). Therefore:

\[
P_1(s, u) \geq \sum_{s' \in \text{succ}(s) \exists i \in [n]: a_{i,s'} > 0} P(s, s') \cdot (\sum_{i \in [n]} a_{i,s'}) \cdot P_0(\delta(q_{i,s'}, L(s)), \delta(u, L(s))) \cdot \text{val}(s', \delta(q_{i,s'}, L(s)))
\]

But \( i_{s'} \) maximizes \( P_0(\delta(q_{i,s'}, L(s)), \delta(u, L(s))) \cdot \text{val}(s', \delta(q_{i,s'}, L(s))) \), so that

\[
P_1(s, u) \geq \sum_{s' \in \text{succ}(s) \exists i \in [n]: a_{i,s'} > 0} P(s, s') \cdot \sum_{i \in [n]} a_{i,s'} \cdot P_0(\delta(q_{i,s'}, L(s)), \delta(u, L(s))) \cdot \text{val}(s', \delta(q_{i,s'}, L(s)))
\]

From the inequalities \( \text{val}(s', \delta(q_{i,s'}, L(s))) \geq f(i, s') \), which hold from the choice of \( f \) and win in \( G_A \) this yields

\[
P_1(s, u) \geq \sum_{s' \in \text{succ}(s) \exists i \in [n]: a_{i,s'} > 0} P(s, s') \cdot \sum_{i \in [n]} a_{i,s'} \cdot f(i, s') \cdot P_0(\delta(q_{i,s'}, L(s)), \delta(u, L(s)))
\]

Moving \( P(s, s') \) into the second sum and changing the order of summation, we obtain

\[
P_1(s, u) \geq \sum_{i \in [n]} P_0(\delta(q_{i,s}, L(s)), \delta(u, L(s))) \cdot \sum_{s' \in \text{succ}(s) \exists i \in [n]: a_{i,s'} > 0} f(i, s') \cdot a_{i,s'} \cdot P(s, s')
\]

Crucially, since \( f \) is disjoint, we get the inequalities \( \sum_{s' \in \text{succ}(s) \exists i \in [n]: a_{i,s'} > 0} f(i, s') \cdot a_{i,s'} \cdot P(s, s') \geq p_i \). Notice that the successors \( s' \) that are not included in this sum have \( a_{i,s'} = 0 \) for all \( i \). Hence, adding other successors \( s' \) to the sum does not change anything and indeed the successors we considered in the equation are sufficient. We thus obtain

\[
P_1(s, u) \geq \sum_{i \in I} P_0(\delta(q_{i,s}, L(s)), \delta(u, L(s))) \cdot p_i
\]

where the righthand side is \( P_1(\varphi', u) \) as desired.
Assume now that the claim holds for all configurations and for $P_n$. That is, the probability to reach end components, where values are defined as before, satisfies the requirement of the claim for reachability in $n$ steps. We now show that it holds for reachability in $n + 1$ steps, i.e., for $P_{n+1}$. For a triplet $(s, \varphi', (\varphi', \psi')]$, the strategy defined makes most such configurations deterministic in their respective MDPs. The only interesting case is when $\varphi' \in [Q]$ and $\psi' \in U$. In this case $(\varphi', \psi')$ and $(s, \psi')$ are probabilistic configurations and the strategy above includes some choice in the matching between successors of $(\varphi', \psi')$ and $(s, \psi')$. Let $\varphi' = *([q_1]_{\geq p_1}, \ldots, [q_n]_{\geq p_n})$ and $\psi' = u$. Then,

$$P_{n+1}(s, u) = \sum_{s' \in \text{succ}(s)} P(s, s') \cdot P_n(s', \delta(u, L(s)))$$

Recall that the way to extend the game from configuration $(\varphi', u)$ – matching a move to $\delta(q_i, L(s))$ with the move to $(s', \delta(u, L(s)))$ – depends on which $a_{i, s'}$ are positive in a disjoint function $f$. We thus have:

$$P_{n+1}(\varphi', u) = \sum_{i \in [n]} \max_{i, a_{i, s'} > 0} \text{val}(s', \delta(q_i, L(s))) \cdot P_n(\delta(q_i, L(s)), \delta(u, L(s)))$$

By induction, for possible matching triplet $(s, \varphi')$, $(\varphi', \psi')$, and $(s, \psi')$ we have:

$$P_n(s, \psi') \geq \text{val}(s, \varphi') \cdot P_n(\varphi', \psi')$$

We then prove the same inequality for $P_{n+1}$. We concentrate on the only interesting case, where $\varphi' = *([q_1]_{\geq p_1}, \ldots, [q_n]_{\geq p_n})$ and $\psi' = u$. The desired inequality for $P_{n+1}(s, u)$ is derived as follows. First,

$$P_{n+1}(s, u) = \sum_{s' \in \text{succ}(s)} P(s, s') \cdot P_n(s', \delta(u, L(s)))$$

For every $s'$ let $i_{s'}$ be such that $\text{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$ is maximal among all $i \in [n]$. By induction

$$P_n(s', \delta(u, L(s))) \geq \text{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

and we infer

$$P_{n+1}(s, u) \geq \sum_{s' \in \text{succ}(s)} P(s, s') \cdot \text{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

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By $f$ being disjoint, we have $\sum_{i \in [n]} a_{i,s'} = 1$, so

$$P_{n+1}(s,u) \geq \sum_{s' \in \text{succ}(s)} P(s,s') \cdot \left( \sum_{i \in [n]} a_{i,s'} \cdot \text{val}(s', \delta(q_{i,s'}, L(s))) \cdot P_n(\delta(q_{i,s'}, L(s)), \delta(u, L(s))) \right)$$

But $i_{s'}$ maximizes $\text{val}(s', \delta(q_{i,s'}, L(s))) \cdot P_n(\delta(q_{i,s'}, L(s)), \delta(u, L(s)))$, so that

$$P_{n+1}(s,u) \geq \sum_{s' \in \text{succ}(s)} P(s,s') \cdot \sum_{i \in [n]} a_{i,s'} \cdot \text{val}(s', \delta(q_{i,s'}, L(s))) \cdot P_n(\delta(q_{i,s'}, L(s)), \delta(u, L(s)))$$

From the inequality $\text{val}(s', \delta(q_{i,s'}, L(s))) \geq f(i, s')$, which holds from the choice of $f$ and win in $G_A$, this yields

$$P_{n+1}(s,u) \geq \sum_{s' \in \text{succ}(s)} P(s,s') \cdot \sum_{i \in [n]} a_{i,s'} \cdot f(i, s') \cdot P_n(\delta(q_{i,s'}, L(s)), \delta(u, L(s)))$$

After a change of summation order, we get

$$P_{n+1}(s,u) \geq \sum_{i \in [n]} P_n(\delta(q_{i,s'}, L(s)), \delta(u, L(s))) \cdot \sum_{s' \in \text{succ}(s)} a_{i,s'} \cdot f(i, s') \cdot P(s,s')$$

Again, using that $f$ is disjoint, we obtain the inequalities $\sum_{s' \in \text{succ}(s)} a_{i,s'} \cdot f(i, s') \cdot P(s,s') \geq p_i$ from which we infer

$$P_{n+1}(s,u) \geq \sum_{i \in [n]} P_n(\delta(q_{i,s'}, L(s)), \delta(u, L(s))) \cdot p_i$$

where the righthand side is the desired $P_{n+1}(\varphi', u)$. \hfill \Box

We now get sound and complete verification of model checks through simulations, in the sense of Dams and Namjoshi [19].

**Corollary 2** For all infinite Markov chains $M$ in $\text{MC}_{\mathbb{AP}}$ and PCTL formulas $\phi$ over $\mathbb{AP}$: $M \models \phi$ iff there is a finite $p$-automaton $A$ with $A_M \leq A$ and $A \leq A_\phi$. 

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To see this, every such $A$ implies $\mathcal{L}(A_M) \subseteq \mathcal{L}(A)$ and $\mathcal{L}(A) \subseteq \mathcal{L}(A_\phi)$ by both parts of Theorem 6. Thus, $M \models \phi$ holds by Theorems 4 and 5. By construction, for every Markov chain $M$ the automaton $A_M$ contains only $*$-transitions. Similarly, for every PCTL formula $\phi$ we have $A_\phi$ does not contain occurrences of $\not\exists\not\forall$. It follows that simulation (without $\not\exists\not\forall$) is sufficient.

Conversely, if there is no such $A$, then $A_\phi$ can also not be such an $A$. As $A_\phi \leq A_\phi$ this implies $A_M \not\leq A_\phi$ and so $\mathcal{L}(A_M) \not\subseteq \mathcal{L}(A_\phi)$. So there is some $M' \sim M$ with $M' \not\models \phi$. Since $M' \sim M$, we get $M \not\models \phi$ as well by Lemma 1.

This method for deciding $M \models \phi$ via simulations is thus complete in the sense of [19] – to our knowledge the first such result for PCTL and Markov chains.

7. Related and Future Work

Automata for coalgebras [32], for the functor whose coalgebras are Markov chains, have a corresponding logic that enjoys the finite model property. Since PCTL does not have that property, these automata cannot express PCTL – notably its path modalities. Probabilistic processes [7] use automata-theoretic techniques for refinement checking only. Probabilistic automata [33] give only rise to probabilistic languages of non-probabilistic models, for example the language of all Kripke structures that are accepted with probability at least $\frac{1}{2}$. And probabilistic verification of specifications written in linear-time temporal logic (LTL) [4] uses automata-theoretic machinery but cannot reason about combinations of LTL operators and probability thresholds as found in PCTL. The stochastic games of [34] abstract Markov decision processes as a 2-person game where two sources of non-determinism, stemming from the MDP and the state space partition respectively, are controlled by different players. This separation allows for more precision of abstractions but is not complete in the sense of [19], as shown in [35]. In [24], a Hintikka game was defined for satisfaction, i.e. whether a Markov chain satisfies a PCTL formula. That game resembles our acceptance game for p-automata that are embeddings of PCTL formulas.

Concurrently with our own work, Caillaud et al. introduced constraint Markov chains as a specification framework for Markov chains [36]. They generalize interval Markov chains, where every transition is labeled by a possible interval of probabilities [37]. A concrete Markov chain implements an interval Markov chain if a bisimulation game between the two can be carried out in a way that probabilities fall in intervals. Constraint Markov chains have formulas constraining the probabilities of successors, and the implementation relation is defined through a game that is similar to those showing implementations of interval Markov chains.
They show that the resulting theory supports the notions of specification, implementation, refinement, conjunction, and parallel composition. The constraints themselves are akin to our $\ast$-operator, also relate to disjoint probabilities, but do not have the ability to use parts of the probability of the same successor for different purposes. Furthermore, constraint Markov chains cannot reason about paths in the Markov chain and, as a consequence, have no relation to model checking. It would be interesting to explore a combination of $\ast$-operators with constraints, leading, perhaps, to a normal form of possible transitions.

7.1. Future work

$p$-automata suggest a new approach to understanding the open problem of decidability of PCTL satisfiability. Algorithms for checking emptiness of alternating tree automata and solving satisfiability of monadic second-order logic, $\mu$-calculus, CTL*, and dynamic logic convert automata into non-deterministic versions, for which non-emptiness is then decidable with standard techniques. We mean to investigate whether a notion of non-deterministic $p$-automata exists such that (i) all $p$-automata can be converted into non-deterministic versions, and (ii) all non-deterministic $p$-automata have decidable non-emptiness checks. We aim to develop a generalization of stochastic games such that acceptance of input for non-uniform $p$-automata can be decided by solving a single such game, as opposed to a sequence of such games. We also want to research the effectiveness of $p$-automata in supporting counter-example guided abstraction refinement.

8. Conclusions

We presented a novel kind of automata, $p$-automata, that read in an entire Markov chain and either accept or reject that input. We showed how this acceptance can be decided by a series of stochastic weak games and weak games, at worst case exponential in the size of the automaton and of the Markov chain.

We proved $p$-automata to be closed under Boolean operations, that language containment and emptiness are equi-solvable, and that a $p$-automaton’s language is closed under bisimulation. Bisimulation equivalence classes of every Markov chain as well as the set of models of every PCTL formula were shown to be expressible as such languages. In particular, the complexity of the acceptance game matches that of probabilistic model checking for such formulas. Therefore the emptiness, universality, and containment of $p$-automata seem all to be tightly related to the open problem of decidability of PCTL satisfiability.
We developed a (fair) simulation between p-automata that stem from Markov chains or PCTL formulas. We proved simulation to be decidable in EXPTIME and to under-approximate language containment. In particular, p-automata are a complete abstraction framework for PCTL: if an infinite Markov chain satisfies a PCTL formula, there is a finite p-automaton that abstracts this Markov chain and whose language is contained in that of the p-automaton for that PCTL formula.

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**References**


