p-Automata: New Foundations for Discrete-Time Probabilistic Verification[☆]

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Abstract

We introduce p-Automata, which are automata that accept languages of Markov chains, by adapting notions and techniques from alternating tree automata to the realm of Markov chains. The set of languages of p-automata is closed under Boolean operations, and for every PCTL formula it contains the language of the set of models of the formula. Furthermore, the language of every p-automaton is closed under probabilistic bisimulation. Similar to tree automata, whose acceptance is defined via two-player games, we define acceptance of Markov chains by p-automata through two-player stochastic games. We show that acceptance is solvable in EXPTIME; but for automata that arise from PCTL formulas acceptance matches that of PCTL model checking, namely, linear in the formula and polynomial in the Markov chain. We also derive a notion of simulation between p-automata that approximates language containment in EXPTIME and is complete for Markov chains. These foundations therefore enable abstraction-based probabilistic model checking for probabilistic specifications that subsume Markov chains, and LTL and CTL* like logics.

Keywords: Markov chains, probabilistic computation tree logic, game theory, fairness conditions, probabilistic evidence
2000 MSC: 60J10 (Markov chains with discrete parameter), 03B44 (temporal

[☆]A preliminary version appered in [1].

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logic), 91A05 (2-person games), 91A15 (stochastic games)

1. Introduction

Markov chains are a very important modeling formalism in many areas of science. In computing, Markov chains form the basis of central techniques such as performance modeling, and the design and analysis of randomized algorithms used in security and communication protocols. Recognizing this prominent role of Markov chains, the formal-methods community has devoted significant attention to these models, e.g., in developing model checking for *qualitative* [2, 3, 4] and *quantitative* [5] properties, logics for reasoning about Markov chains [6, 7], and probabilistic simulation and bisimulation [8, 7]. Model-checking tools such as PRISM [9] and LiQuor [10] support such reasoning about Markov chains and have users in many fields of computer science and beyond.

In the non-probabilistic setting, the automata-theoretic approach to verification unifies such reasoning support for systems modeled as Kripke structures. Automata furnish the foundations for reasoning about these models: they can show decidability of satisfiability for a corresponding logic (decidable non-emptiness checks [11, 12, 13]), support algorithms that decide whether a model satisfies a formula (model checking [13, 14]), enable algorithms that generate a model satisfying a satisfiable formula (design synthesis [15]), and offer techniques of abstracting a model so that formulas holding for the abstract model also hold for the model they abstract (abstraction-based model checking [16, 17]).

Alternating tree automata [18] were introduced to prove the decidability of satisfiability for monadic, second-order logic and they provide a unifying framework for branching-time temporal logics such as μ-calculus, CTL, and CTL*. Of particular interest to us is that alternating tree automata afford a complete framework for abstraction with respect to branching-time logic [19, 20]. Thus, in this context, alternating automata form the right basis for abstraction, *the* technique that makes model checking scale to realistic designs in the hardware and software industry. For Markov chains, their aforementioned techniques lack such a unifying framework and the quest for robust notions of abstraction is an active line of research. Here, we define p-automata and show that they render such a framework.

p-automata are devices that read an entire Markov chain as input and either accept it or reject it. The definition of p-automata is motivated by PCTL [21], the de-facto standard logic for model checking Markov chains, and alternating tree automata: it combines the rich combinatorial structure of alternating automata

with PCTL's ability to quantify the probabilities of regular sets of paths. The acceptance of Kripke structures by an alternating tree automaton is decided by solving games (cf. [18]). In that spirit, acceptance of a Markov chain by a p-automaton is decided by solving *stochastic* games.

We now highlight the main results on p-automata developed in this paper.

- The language of Markov chains accepted by a p-automaton is semantically robust in that it is closed under probabilistic bisimulation.
- One can embed a Markov chain as a p-automaton accepting the language of Markov chains that are bisimilar to it.
- The *set* of languages of p-automata is closed under Boolean operations.
- Acceptance of finite Markov chains can be determined in exponential time.
- PCTL formulas can be expressed as p-automata whose complexity of acceptance of Markov chains matches the complexity of PCTL model checking.
- We define a simulation for p-automata that approximates language containment in EXPTIME and is exact for p-automata arising from Markov chains.
- p-automata are the first complete abstraction framework for PCTL model checking on Markov chains.

The latter means that if an infinite-state Markov chain satisfies a PCTL formula, then there is a finite p-automaton that abstracts (i.e. simulates) this Markov chain and whose language is contained in that of the formula.

The problem of emptiness of the language of a p-automaton generalizes the long-standing open problem of decidability for PCTL satisfiability, and is here left open.

The embedding of Markov chains as p-automata uses a new probabilistic separation operator, denoted by $*$, that decomposes the witness path set for a probability threshold into disjoint subsets. Use of this operator, however, has a certain price in the complexity of the resulting acceptance games.

Finally, probabilistic versions of LTL, CTL*, or desired ω-regular probabilistic extensions of these logics can also be expressed as p-automata but such a formal development is beyond the scope of this paper.

1.1. Outline of paper

In Section 2 notation is fixed and needed concepts are recalled. p-automata are introduced in Section 3, their acceptance games defined in Section 4, and expressiveness results featured in Section 5. Simulation and its salient properties are presented in Section 6 and used to prove that p-automata are a complete abstraction framework. In Section 7 related and future work are discussed. Section 8 contains our conclusions.

2. Background

A *countable labeled Markov chain* M over a set of atomic propositions $\mathbb{AP}$ is a tuple (S, P, L, s^{in}), where S is a countable set of *locations*, $P\colon S \times S \to [0,1]$ a stochastic matrix, $s^{\text{in}} \in S$ the *initial* location, and $L\colon S \to 2^{\mathbb{AP}}$ a *labeling function* with $L(s)$ the set of propositions true in location s. Let $\mathsf{succ}(s)$ be the set $\{s' \in S \mid P(s,s') > 0\}$ of *successors* of s. All Markov chains are assumed to be *finitely branching*, i.e. $\mathsf{succ}(s)$ is finite for all $s \in S$. We write $\mathsf{MC}_{\mathbb{AP}}$ for the set of all (finitely branching) Markov chains over $\mathbb{AP}$. A *path* π from location s in M is an infinite sequence of locations $s_0 s_1 \ldots$ with $s_0 = s$ and $P(s_i, s_{i+1}) > 0$ for all $i \geq 0$. For $Y \subseteq S$, let $P(s,Y)$ abbreviate $\sum_{s' \in Y} P(s,s')$. Given a Markov chain M with set of states S, an *open set* in S^ω is a set $\{s_0 \cdot w\} \cdot S^\omega$ for some $w \in S^*$. A set is *Borel* if it is in the σ-algebra defined by these open sets. The measure of every Borel set α is defined as usual in this σ-algebra [22, 23]. We denote the measure of a set α as $\mathsf{Prob}_M(\alpha)$.

For Markov chain $M = (S, P, L, s^{\text{in}})$, a (probabilistic) *bisimulation* [8] is an equivalence relation $H \subseteq S \times S$ where $(s,s') \in H$ implies (i) $L(s) = L(s')$ and (ii) $P(s,C) = P(s',C)$ for all equivalence classes $C \in S/H$. The union of all bisimulations for M is the greatest bisimulation $\sim$; locations s and s' are *bisimilar* iff $s \sim s'$. This definition extends to Markov chains M_1 and M_2 by considering bisimilarity of their initial locations in the disjoint union of M_1 and M_2.

Without loss of generality [24], one may define the probabilistic temporal logic PCTL [21] in "Greater Than Negation Normal Form": only propositions can be negated and probabilistic bounds are either $\geq$ or $>$. PCTL formulas are defined as follows, where $\mathsf{a} \in \mathbb{AP}$, $p \in [0,1]$, and $\bowtie \in \{>, \geq\}$:

ϕ, ψ	::=	*PCTL formulas*	α	::=	*Path formulas*
$\mathsf{a}, \neg\mathsf{a}$		Literals	$\mathsf{X}\,\phi$		Next
$\phi \wedge \psi$		Conjunction	$\phi\,\mathsf{U}\,\psi$		Until
$\phi \vee \psi$		Disjunction	$\phi\,\mathsf{W}\,\psi$		Weak Until
$[\alpha]_{\bowtie p}$		Path Probability			

Our semantics of PCTL is as in [21]: path formulas α are interpreted as predicates over paths in M, and wrap PCTL formulas into "LTL" operators for Next, (strong) Until, and Weak Until. The semantics $\|\phi\| \subseteq S$ of PCTL formula ϕ lifts path formulas to state formulas: $s \in \|[\alpha]_{\bowtie p}\|$ iff $\mathsf{Prob}_M(s, \alpha)$, the probability of the measurable set [25] $\mathsf{Path}(s, \alpha)$ of paths $ss_1s_2\ldots$ in M with $ss_1s_2\ldots \models \alpha$, satisfies $\bowtie p$. Markov chain M satisfies ϕ, denoted $M \models \phi$, if $s^{\text{in}} \in \|\phi\|$.

Weak Games. A tuple $G = ((V, E), (V_0, V_1, V_p), \kappa, \alpha)$ is a *stochastic weak game* if (V, E) is a directed graph, (V_0, V_1, V_p) a partition of V, and function κ associates with every $v \in V_p$ a distribution $\kappa(v)$ of mass 1 over $E(v) = \{v' \mid (v, v') \in E\}$ such that $(v, v') \in E$ iff $\kappa(v)(v') \neq 0$; we write $\kappa(v, v')$ instead of $\kappa(v)(v')$. Set $\alpha \subseteq V$ is the winning condition. Set V_p contains the probabilistic configurations of G. For $i = 0, 1$ set V_i contains the Player i configurations. We work with *weak* games: all maximal, strongly connected components (MSCCs) V' in (V, E) satisfy $V' \subseteq \alpha$ or $V' \cap \alpha = \{\}$. If $V_p = \{\}$, we call G a *weak game*. Markov chains can be thought of as stochastic weak games where $V_0 = V_1 = \{\}$ and $\alpha = V$.

A play in G is a maximal sequence $v_0v_1\ldots$ of configurations with $(v_i, v_{i+1}) \in E$ for all $i \in \mathbb{N}$. A play is winning for Player 0 if it is finite and ends in a Player 1 configuration, or if it is infinite and ends in a suffix of states in α. Otherwise, that play is winning for Player 1. A strategy for Player 0 is a function $\sigma\colon V_0 \to V$ with $(v, \sigma(v)) \in E$ for all $v \in V_0$ for which $\sigma(v)$ is defined. Play $v_0v_1\ldots$ is consistent with strategy σ if $v_{i+1} = \sigma(v_i)$ whenever $v_i \in V_0$. Strategies for Player 1 are defined analogously. Let $\mathbf{\Sigma}$ (resp. $\mathbf{\Pi}$) be the set of all strategies for Player 0 (resp. Player 1).

Each $(\sigma, \pi) \in \mathbf{\Sigma} \times \mathbf{\Pi}$ from game G determines a Markov chain $M^{\sigma,\pi}$ (with sinks for dead-ends in G) whose paths are plays in G consistent with σ and π. The set of plays from $v \in V$ that Player 0 wins is measurable in $M^{\sigma,\pi}$. Let $\mathsf{val}_0^{\sigma,\pi}(v)$ be that measure, and $\mathsf{val}_1^{\sigma,\pi}(v) = 1 - \mathsf{val}_0^{\sigma,\pi}(v)$. Then $\mathsf{val}_0(v) = \sup_{\sigma\in\mathbf{\Sigma}} \inf_{\pi\in\mathbf{\Pi}} \mathsf{val}_0^{\sigma,\pi}(v) \in [0, 1]$ and $\mathsf{val}_1(v) = \sup_{\pi\in\mathbf{\Pi}} \inf_{\sigma\in\mathbf{\Sigma}} \mathsf{val}_1^{\sigma,\pi}(v) \in [0, 1]$ are the game values. Strategies that achieve these values are *optimal*.

Theorem 1 [26, 27, 28] *Let $G = ((V, \cdot), \ldots)$ be a stochastic weak game and $v \in V$. Then $\mathsf{val}_0(v) + \mathsf{val}_1(v) = 1$. If G is finite, $\mathsf{val}_0(v)$ is computable in EXPTIME, and optimal strategies exist for both players. If G is a weak game, $\mathsf{val}_0(v)$ is in $\{0, 1\}$ and linear-time computable.*

These results generalize to the setting where configurations may have pre-seeded game values. That is, when we set in advance the value for some of the

configurations of the game and ignore how one may continue to play from them. These values are in $[0,1]$ for stochastic weak games and in $\{0,1\}$ for weak games.

3. Uniform Weak p-Automata

We introduce p-automata and their uniform weak variant. Each uniform weak p-automaton is then shown to accept a language of Markov chains. We assume familiarity with basic notions of trees and (alternating) tree automata (cf. [18]). For set T, let $B^+(T)$ be the set of positive Boolean *formulas* generated from elements $t \in T$, constants tt and ff, and disjunctions and conjunctions:

$$\varphi, \psi ::= t \mid \mathsf{tt} \mid \mathsf{ff} \mid \varphi \vee \psi \mid \varphi \wedge \psi \tag{1}$$

Formulas in $B^+(T)$ are finite even if T is not.

For set Q, the set of states of a p-automaton, we define *term* sets as follows:

$$\begin{aligned}
[\![Q]\!]_{>} &= \{[\![q]\!]_{\bowtie p} \mid q \in Q, \bowtie \in \{\geq, >\}, p \in [0,1]\} \\
[\![Q]\!]^{*} &= \{*(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i\colon t_i \in [\![Q]\!]_{>}\} \\
[\![Q]\!]^{\Diamond\!\!\!\!-} &= \{\Diamond\!\!\!\!-(t_1, \ldots, t_n) \mid n \in \mathbb{N}, \forall i\colon t_i \in [\![Q]\!]_{>}\} \\
[\![Q]\!] &= [\![Q]\!]^{*} \cup [\![Q]\!]^{\Diamond\!\!\!\!-}
\end{aligned}$$

This uses n-ary operators $*_n$ and $\Diamond\!\!\!\!-_n$ for every $n \in \mathbb{N}$, which we write as $*$ and $\Diamond\!\!\!\!-$ throughout as n will be clear from context. Also, we freely write $*(t_i \mid i \in X)$ for $*(t_1, t_2, \ldots, t_n)$ and so implicitly refer to $1, 2, \ldots, n$ as some enumeration of X.

Intuitively, a state $q \in Q$ of a p-automaton and its transition structure model a probabilistic path set. So $[\![q]\!]_{\bowtie p}$ holds in location s if the measure of paths that begin in s and satisfy q is $\bowtie p$. Now, $*([\![q_1]\!]_{>p_1}, [\![q_2]\!]_{\geq p_2})$, e.g., means q_1 and q_2 hold with probability $> p_1$ and $\geq p_2$, respectively; and that the sets supplying these probabilities are disjoint. Dually, $\Diamond\!\!\!\!-([\![q_1]\!]_{\geq p_1}, [\![q_2]\!]_{\geq p_2})$ means not only that either the probabilty of q_1 is $\geq p_1$ or the probability of q_2 is $\geq p_2$ but that this holds regardless of how we try to partition the sets supplying the full probability between them. So $*$ and $\Diamond\!\!\!\!-$ model a "disjoint and" and "intersecting or" operator, respectively. We may write $[\![q]\!]_{\bowtie p}$ for $*([\![q]\!]_{\bowtie p})$, and similarly for $\Diamond\!\!\!\!-$.

An element of $Q \cup [\![Q]\!]$ is a state of the p-automaton, a $*$ composition of terms $[\![q_i]\!]_{\bowtie p_i}$, or a $\Diamond\!\!\!\!-$ composition of such terms. For $\varphi \in B^+(Q \cup [\![Q]\!])$, its closure $\mathsf{cl}(\varphi)$ is the set of all subformulas of φ according to (1). In particular, $*(t_1, t_2) \in \mathsf{cl}(\varphi)$ does not imply $t_1, t_2 \in \mathsf{cl}(\varphi)$. For a set Φ of formulas, let $\mathsf{cl}(\Phi) = \bigcup_{\varphi \in \Phi} \mathsf{cl}(\varphi)$.

Definition 1 *A p-automaton A is a tuple $\langle \Sigma, Q, \delta, \varphi^{\text{in}}, \alpha \rangle$, where Σ is a finite input alphabet, Q a set of states, $\delta \colon Q \times \Sigma \to B^+(Q \cup [\![Q]\!])$ the transition function, $\varphi^{\text{in}} \in B^+([\![Q]\!])$ the initial condition, and $\alpha \subseteq Q$ an acceptance condition.*

As a convention, p-automata have *states*, Markov chains have *locations*, and weak stochastic games have *configurations*.

Example 1 *p-automaton $A = \langle 2^{\{\mathsf{a},\mathsf{b}\}}, \{q_1, q_2\}, \delta, [\![q_1]\!]_{\geq \frac{1}{2}}, \{q_2\} \rangle$ has δ given by*

$$\begin{array}{l} \delta(q_1, \{\mathsf{a}, \mathsf{b}\}) = \delta(q_1, \{\mathsf{a}\}) = q_1 \vee [\![q_2]\!]_{\geq \frac{1}{2}} \\ \delta(q_2, \{\mathsf{b}\}) = \delta(q_2, \{\mathsf{a}, \mathsf{b}\}) = [\![q_2]\!]_{\geq \frac{1}{2}} \\ \delta(q_1, \{\}) = \delta(q_1, \{\mathsf{b}\}) = \delta(q_2, \{\}) = \delta(q_2, \{\mathsf{a}\}) = \mathsf{ff} \end{array}$$

The winning condition $\{q_2\}$ (along with the loops in the transition of this automaton) means that only sequences of states in which q_2 is eventually reached and repeats forever are fair. It follows that term $[\![q_2]\!]_{\geq \frac{1}{2}}$ represents the recursive property ϕ, that atomic proposition b *holds at the location presently read by q_2, and that ϕ will hold with probability at least $\frac{1}{2}$ in the next locations. State q_1 asserts it is possible to get to a location that satisfies $[\![q_2]\!]_{\geq \frac{1}{2}}$ along a path that satisfies atomic proposition* a*. The initial condition $[\![q_1]\!]_{\geq \frac{1}{2}}$ means the set of paths satisfying* $\mathsf{a} \,\mathsf{U}\, \phi$ *has probability at least $\frac{1}{2}$.*

In order to be able to decide acceptance of input for p-automata through the solution of weak stochastic games, we restrict the cycles in the transition graph of p-automata. In doing so, we differentiate states q' appearing within a term in $[\![Q]\!]$ (bounded transition) from q' appearing "free" in the transition of a state q (unbounded transition). In this way, a p-automaton $A = \langle \Sigma, Q, \delta, \ldots \rangle$ determines a labeled, directed graph $\mathsf{G}_A = \langle Q', E, E_b, E_u \rangle$:

$$\begin{array}{rcl} Q' & = & Q \cup \mathsf{cl}(\delta(Q, \Sigma)) \\ E & = & \{(\varphi_1 \wedge \varphi_2, \varphi_i) \mid \varphi_i \in Q' \setminus Q, 1 \leq i \leq 2\} \\ & & \cup \quad \{(\varphi_1 \vee \varphi_2, \varphi_i) \mid \varphi_i \in Q' \setminus Q, 1 \leq i \leq 2\} \\ & & \cup \quad \{(q, \delta(q, \sigma)) \mid q \in Q, \sigma \in \Sigma\} \\ E_u & = & \{(\varphi \wedge q, q), (q \wedge \varphi, q), (\varphi \vee q, q), (q \vee \varphi, q) \mid \varphi \in Q', q \in Q\} \\ E_b & = & \{(\varphi, q) \mid \varphi \in [\![Q]\!] \text{ and } q \in \mathsf{gs}(\varphi)\} \end{array}$$

where $\delta(Q, \Sigma) = \{\delta(q, \sigma) \mid q \in Q \text{ and } \sigma \in \Sigma\} \cup \{\varphi^{\text{in}}\}$ and $\mathsf{gs}(\varphi)$ is the set of *guarded* states of φ: all $q \in Q$ occurring in some term in φ. Elements $(\varphi, q) \in E_u$ are *unbounded* transitions; elements $(\varphi, q) \in E_b$ are *bounded* transitions; and elements of E are *simple* transitions. We mark $(\varphi, q) \in E_b$ with $*$ (and respectively,

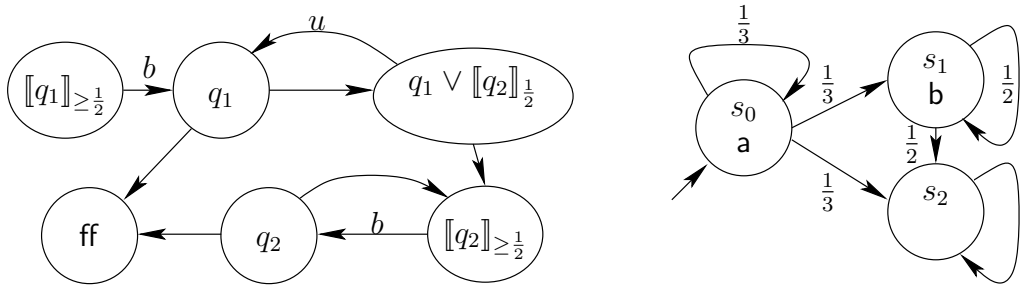

Figure 1: (a) Graph G_A of automaton A from Example 1 and (b) a Markov chain M

with $\dot{\forall}$) if $\varphi \in [\![Q]\!]^*$ (respectively, $\varphi \in [\![Q]\!]^{\dot{\forall}}$). Note that E, E_u, and E_b are pairwise disjoint. Let $\varphi \preceq_A \tilde{\varphi}$ iff there is a finite path from φ to $\tilde{\varphi}$ in $E \cup E_b \cup E_u$. Let $\equiv_A$ be $\preceq_A \cap \preceq_A^{-1}$ and $(\!(\varphi)\!)$ the equivalence class of φ with respect to $\equiv_A$. Each $(\!(\varphi)\!)$ is an MSCC in graph G_A.

Definition 2 *A p-automaton A is called* uniform *if:*

- *For each cycle in G_A, its set of transitions is either in $E \cup E_b$ or in $E \cup E_u$.*
- *For each cycle in $\langle Q, E \cup E_b \rangle$, its set of markings is either $\{\}$, $\{*\}$ or $\{\dot{\forall}\}$.*
- *The set of equivalence classes $\{(\!(\varphi)\!) \mid \varphi \in Q \cup \mathsf{cl}(\delta(Q, \Sigma))\}$ is finite.*

A (not necessarily uniform) p-automaton A is called weak *if for all $q \in Q$, either $(\!(q)\!) \cap Q \subseteq \alpha$ or $(\!(q)\!) \cap \alpha = \{\}$.*

Then, A is uniform, if the full subgraph of every equivalence class in $\equiv_A$ contains only one type of non-simple transitions and at most one kind of marking $*$ or $\dot{\forall}$. In particular, all states $q' \in Q$ or formulas φ occurring in $\delta(q, \sigma)$ for some $q \in Q$ and $\sigma \in \Sigma$ can be classified as unbounded, bounded with $*$, bounded with $\dot{\forall}$, or simple – according to MSCC $(\!(q)\!)$.

Example 2 *Figure 1*(a) *depicts G_A for A of Example 1. Automaton A is uniform: $(\!(q_1)\!) = \{q_1, q_1 \vee [\![q_2]\!]_{\geq\frac{1}{2}}\}$ and $(\!(q_2)\!) = \{q_2, [\![q_2]\!]_{\geq\frac{1}{2}}\}$; in $(\!(q_1)\!)$ there are no bounded edges, in $(\!(q_2)\!)$ there are no unbounded edges; and G_A has only $*$ markings (we treat $[\![q_1]\!]_{\geq\frac{1}{2}}$ as $*([\![q_1]\!]_{\geq\frac{1}{2}})$). The MSCC $(\!([\![q_1]\!]_{\geq\frac{1}{2}})\!) = \{[\![q_1]\!]_{\geq\frac{1}{2}}\}$ is trivial. In addition, A is weak as $\alpha = \{q_2\}$.*

Intuitively, the cycles in the structure of a uniform p-automaton A take either no bounded edges or no unbounded edges, and cycles that take bounded edges do not have both markings $*$ and $\dot{\forall}$. Below, all p-automata are uniform weak and so we often refer to them simply as "p-automata". Uniformity allows to define acceptance of input for p-automata through the solution of stochastic games.

But, a more relaxed notion of uniformity is what really drives the proof of well-definedness: every ascending chain in the partial order on MSCCs on the graph of a p-automaton has only finitely many alternations between bounded and unbounded MSCCs.

The requirement of weakness is made merely to simplify the presentation. Using a parity condition instead, e.g., would still allow us to decide acceptance of input for uniform p-automata, by solving stochastic parity games.

4. Acceptance Games

For some set of atomic propositions $\mathbb{AP}$, p-automaton $A = \langle 2^{\mathbb{AP}}, Q, \delta, \varphi^{\text{in}}, \alpha \rangle$ has $\mathsf{MC}_{\mathbb{AP}}$ as set of inputs. For $M = (S, P, L, s^{\text{in}})$ in $\mathsf{MC}_{\mathbb{AP}}$, we exploit the uniform weak structure of A to reduce the decision of whether A accepts M to solving a sequence of weak games and stochastic weak games. Intuitively, unbounded cycles in G_A correspond to weak stochastic games and bounded cycles to weak games. Then the language of A is $\mathcal{L}(A) = \{M \in \mathsf{MC}_{\mathbb{AP}} \mid A \text{ accepts } M\}$.

Just as in acceptance games of alternating tree automata, all states of A and all subformulas appearing in its transitions form part of acceptance games. For A as above, let $T = Q \cup \mathsf{cl}(\delta(Q, 2^{\mathbb{AP}}))$. We now (gradually) compute the values $\mathsf{val}(s, t)$, where $s \in S$ is a state of the Markov chain and $t \in T$ is a subformula appearing in the transition of A, which determine whether A accepts M. Initially, we set $\mathsf{val}(s, t) = \bot$ for all $s \in S$ and $t \in T$. Gradually, as the computation progresses, $\mathsf{val}(s, t)$ is reset for more and more pairs of states and subformulas. Partial order $(T/\equiv_A, \leq_A)$ has set $\{(\!(t)\!) \mid t \in T\}$ ordered by $(\!(\tilde{t})\!) \leq_A (\!(t)\!)$ iff $\tilde{t} \preceq_A t$. As A is uniform $\leq_A$ induces a finite partial order. For M as above, each $(\!(t)\!)$ determines a game $G_{M,(\!(t)\!)} = ((V, E), (V_0, V_1, V_p), \kappa, \tilde{\alpha})$. Most of its configurations are in $S \times T$. The construction is such that $(s^{\text{in}}, \varphi^{\text{in}})$ occurs in exactly one of these games $G_{M,(\!(t)\!)}$, and $\mathsf{val}(s^{\text{in}}, \varphi^{\text{in}}) \in [0, 1]$. Then A accepts M iff $\mathsf{val}(s^{\text{in}}, \varphi^{\text{in}}) = 1$.

We define these games as follows. Since A is uniform weak, each $(\!(t)\!)$ is of one of three types and each type determines a weak game or weak stochastic game as detailed in the three cases below. All game values already computed for games $G_{M,(\!(\tilde{t})\!)}$ of MSCCs $(\!(\tilde{t})\!)$ higher up with respect to $\leq_A$ (i.e. by induction) are used as pre-seeded values in $G_{M,(\!(t)\!)}$. As mentioned, we initially set $\mathsf{val}(s, \varphi) = \bot$ and as we progress, $\mathsf{val}(s, \varphi)$ is computed and reset. Then, if the pair (s, φ) appears again in some later game, the precomputed value $\mathsf{val}(s, \varphi)$ is used as a pre-seeded value in the later game. For every $s \in S$ we set $\mathsf{val}(s, \mathsf{ff}) = 0$ and $\mathsf{val}(s, \mathsf{tt}) = 1$.

Case 1. Let $(\!(t)\!)$ be a nontrivial MSCC such that all the transitions in the subgraph of G_A induced by $(\!(t)\!)$ are not in E_u and none have $\Vdots$ markings. For each $\varphi \in (\!(t)\!) \cap [\![Q]\!]^*$ of form $*([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ we define, for each $s \in S$, sets $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$. Then

$$V_0 = \bigcup_{s,\varphi} V_0^{s,\varphi} \qquad V_1 = \bigcup_{s,\varphi} V_1^{s,\varphi} \qquad V_p = \{\}$$
$$E = \bigcup_{s,\varphi} E^{s,\varphi} \qquad\qquad \tilde{\alpha} = \{\} \text{ or } V$$

defines the weak game $G_{M,(\!(t)\!)}$ – where $\tilde{\alpha}$ is V if some $q \in (\!(t)\!)$ is in α, and is empty otherwise. It remains to define $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$, for which we use pre-seeded values $\mathsf{val}(s, \tilde{t})$ for all $s \in S$ and all $\tilde{t} \notin (\!(t)\!)$ with $(\!(t)\!) \leq_A (\!(\tilde{t})\!)$.

As $\mathsf{succ}(s)$ and $\delta(q_i, L(s))$ are finite, so are

$$R_{s,\varphi} = \bigcup_{i=1}^{n} \{(s', \varphi') \mid s' \in \mathsf{succ}(s), \varphi' \in \mathsf{cl}(\delta(q_i, L(s)))\}$$
$$\mathsf{Val}_{s,\varphi} = \{0, 1\} \cup \{\mathsf{val}(s', \varphi') \mid (s', \varphi') \in R_{s,\varphi}, \mathsf{val}(s', \varphi') \neq \bot\}$$

Intuitively, $R_{s,\varphi}$ is the set of configurations reachable from (s, φ) using *one* transition of a state in φ. Thus, s' are the successors of s and φ' are subformulas of $\delta(q_i, L(s))$. Set $\mathsf{Val}_{s,\varphi}$ includes 0, 1, and values of configurations in $R_{s,\varphi}$. In game $G_{M,(\!(t)\!)}$, a play proceeding from (s, φ) reaches either a configuration whose value is pre-seeded (and therefore in $\mathsf{Val}_{s,\varphi}$) or a configuration (s, ψ) for $\psi \in (\!(t)\!)$.

For $n \in \mathbb{N}$, let $[n] = \{1, \ldots, n\}$. Sets $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$ are defined as

follows (whereby the definition of $\mathcal{F}^*_{s,\varphi}$ is deferred for now):

$$
\begin{aligned}
V_0^{s,\varphi} = \;& \{(s,\varphi)\} \\
& \cup \{(s',\varphi',v) \in R_{s,\varphi} \times \mathsf{Val}_{s,\varphi} \mid \bot \neq \mathsf{val}(s',\varphi') < v\} \\
& \cup \{(s',\varphi_1 \vee \varphi_2,v) \in R_{s,\varphi} \times \mathsf{Val}_{s,\varphi} \mid \mathsf{val}(s',\varphi_1 \vee \varphi_2) = \bot\} \\
V_1^{s,\varphi} = \;& \{(s,\varphi,f) \mid f \in \mathcal{F}^*_{s,\varphi}\} \\
& \cup \{(s',\varphi',v) \in R_{s,\varphi} \times \mathsf{Val}_{s,\varphi} \mid \bot \neq \mathsf{val}(s',\varphi') \geq v\} \\
& \cup \{(s',\varphi_1 \wedge \varphi_2,v) \in R_{s,\varphi} \times \mathsf{Val}_{s,\varphi} \mid \mathsf{val}(s',\varphi_1 \wedge \varphi_2) = \bot\} \\
E^{s,\varphi} = \;& \{((s,\varphi),(s,\varphi,f)) \mid f \in \mathcal{F}^*_{s,\varphi}\} \\
& \cup \{((s',\varphi',v),(s',\varphi')) \mid s' \in \mathsf{succ}(s),\ \varphi' \in [\![Q]\!], v \in \mathsf{Val}_{s,\varphi}, \\
& \qquad\qquad \mathsf{val}(s',\varphi') = \bot\} \\
& \cup \{((s,\varphi,f),(s',\delta(q_i,L(s)),f(i,s'))) \mid s' \in \mathsf{succ}(s), i \in [n], \\
& \qquad\qquad f(i,s') > 0\} \\
& \cup \{((s',\varphi_1 \circ \varphi_2,v),(s',\varphi_i,v)) \mid \circ \in \{\wedge,\vee\}, 1 \leq i \leq 2, \\
& \qquad (s',\varphi_1 \circ \varphi_2,v) \in R_{s,\varphi} \times \mathsf{Val}_{s,\varphi}, \mathsf{val}(s',\varphi_1 \circ \varphi_2) = \bot\}
\end{aligned}
\tag{2}
$$

The intuition behind this weak game is as follows: Configuration (s,φ) means that the transition of each q_i holds with probability $\bowtie_i p_i$ where the sets X_i measured by these probabilities are pairwise disjoint. In order to check that, given configuration (s,φ), Player 0 chooses a function $f \in \mathcal{F}^*_{s,\varphi}$ that associates with location $s' \in \mathsf{succ}(s)$ and state q_i the value Player 0 promises to achieve playing from $(s',\delta(q_i,L(s)))$. The play continues with Player 1 choosing a successor s' of s and a state q_i, and the play then reaches configuration $(s',\delta(q_i,L(s)),f(i,s'))$. From such value-annotated configurations, Player 0 and Player 1 choose successors according to the usual resolution of $\vee$ and $\wedge$:

- In a configuration whose value v was already determined, either $f(i,s') \bowtie_i v$, i.e. Player 0 achieved the promised value and wins immediately; or Player 0 failed to achieve the promised value and loses immediately.
- Otherwise, the play ends up in a configuration of form (s',φ') for $\varphi' \in [\![Q]\!]^*$ and the play continues and ignores value $f(i,s')$ (as obviously $f(i,s') \leq 1$).

If the play continues ad infinitum, the winner is determined according to acceptance condition $\tilde{\alpha}$.

We now define the function space $\mathcal{F}^*_{s,\varphi}$ that captures terms built from the separation operator $*$. Throughout, let $X \to Y$ be the set of total functions from set X to set Y. Let $\mathcal{F}_{s,\varphi}$ be $[n] \times \mathsf{succ}(s) \to \mathsf{Val}_{s,\varphi}$, the set of functions from pairs consisting of 'sub-stars' of φ and successors of s to values in $\mathsf{Val}_{s,\varphi}$. Also, an $f \in \mathcal{F}_{s,\varphi}$ is *disjoint* if there are $\{a_{i,s'} \in [0,1] \mid i \in [n] \text{ and } s' \in \mathsf{succ}(s)\}$ that

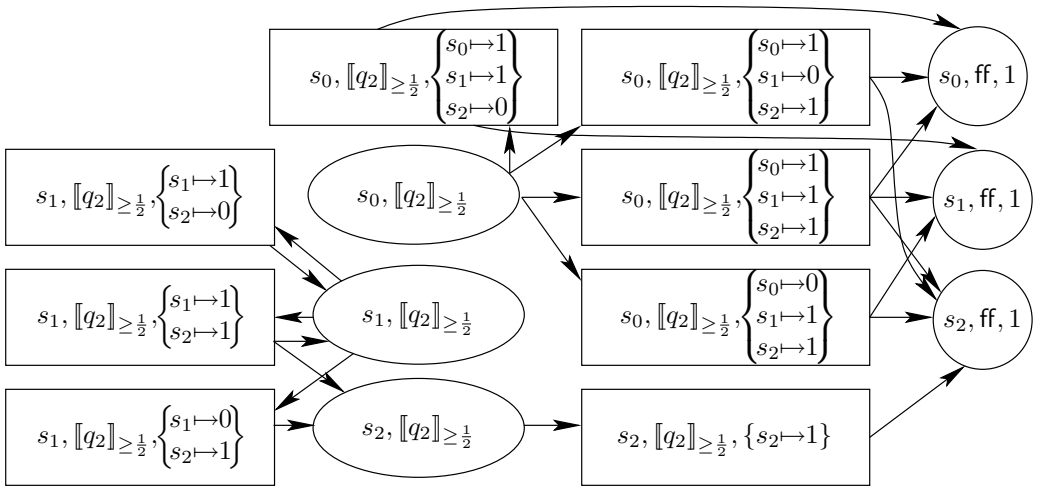

Figure 2: Case 1 of acceptance game

satisfy the following conditions:

(i) $\sum_{s' \in \mathsf{succ}(s)} a_{i,s'} f(i,s') P(s,s') \bowtie_i p_i$ for all $i \in [n]$ and

(ii) $\sum_{i \in [n]} a_{i,s'} = 1$ for all $s' \in \mathsf{succ}(s)$.

Let $\mathcal{F}^*_{s,\varphi}$ be the set of disjoint functions.

Intuitively, a function $f \in \mathcal{F}_{s,\varphi}$ associates with $q_1, \ldots, q_n$ and s' the value that Player 0 can achieve from configuration $(s', \delta(q_i, L(s)))$. Values in $\mathsf{Val}_{s,\varphi}$ suffice, as no others are directly reachable. We call f "disjoint", as all the requirements from the different q_i's can be achieved using a partition (realized by the existence of the above $a_{i,s'}$) of the probability of all successors.

By Theorem 1, V partitions into winning regions W_0 and W_1 of configurations for Player 0 and Player 1, respectively. We set $\mathsf{val}(c) = 1$ for $c \in W_0$ and $\mathsf{val}(c) = 0$ for $c \in W_1$.

Example 3 *We start verifing $M \in \mathcal{L}(A)$ for A from Example 1 and M from Fig. 1(b), where locations are labeled by propositions – e.g., $L(s_0) = \{\mathsf{a}\}$. The weak game of MSCC $(\!(q_2)\!)$, shown in Fig. 2, has only accepting configurations or dead ends. So Player 0 wins only $(s_1, [\![q_2]\!]_{\geq \frac{1}{2}})$ and $(s_1, [\![q_2]\!]_{\geq \frac{1}{2}}, \{s_1 \mapsto 1, s_2 \mapsto 0\})$.*

Case 2. Let $(\!(t)\!)$ be a nontrivial MSCC such that all transitions in the subgraph of G_A induced by $(\!(t)\!)$ are not in E_u and none has $*$ markings. For $\varphi \in (\!(t)\!) \cap [\![Q]\!]^{\Diamond}$ of form $\Diamond(\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ we reuse the definitions of $R_{s,\varphi}$, $\mathsf{Val}_{s,\varphi}$, and $\mathcal{F}_{s,\varphi}$. Weak game $G_{M,(\!(t)\!)}$ is defined as in Case 1. Sets $V_0^{s,\varphi}$, $V_1^{s,\varphi}$, and $E^{s,\varphi}$ are defined

as in (2), except that functions f don't range over $\mathcal{F}^*_{s,\varphi}$ but now range over $\mathcal{F}^{\Diamond}_{s,\varphi}$, the set of intersecting functions and the dual of $\mathcal{F}^*_{s,\varphi}$ of Case 1:

Function $f \in \mathcal{F}_{s,\varphi}$ is *intersecting* if for all sets $\{a_{i,s'} \in [0,1] \mid i \in [n] \text{ and } s' \in \mathsf{succ}(s)\}$ either

(i) there is $i \in [n]$ with $\sum_{s' \in \mathsf{succ}(s)} a_{i,s'} f(i,s') P(s,s') \bowtie_i p_i$ or

(ii) there is $s' \in \mathsf{succ}(s)$ with $\sum_{i \in [n]} a_{i,s'} \neq 1$.

As in Case 1, wins for Player 0 have value 1, wins for Player 1 have value 0. The intuition for this weak game is verbatim that of the weak game in Case 1, except that Player 0 chooses a function f that is in $\mathcal{F}^{\Diamond}_{s,\varphi}$ instead of in $\mathcal{F}^*_{s,\varphi}$.

We point out that when n above is 1, i.e. in handling $\varphi = [\![q_1]\!]_{\bowtie_1 p_1}$, the definitions of $*$ and $\Diamond$ coincide. Indeed, there is then exactly one option for choosing set $\{a_{1,s'} \mid s' \in \mathsf{succ}(s)\}$ that does not satisfy the second condition above: the value $a_{1,s'}$ has to be 1 for all $s' \in \mathsf{succ}(s)$. This justifies dropping the $*$ or $\Diamond$ when applied to one operand.

Case 3. For a nontrivial MSCC $(\!(t)\!)$ such that all transitions in the subgraph of G_A induced by $(\!(t)\!)$ are not in E_b, game $G_{M,(\!(t)\!)}$ is a stochastic weak game with

$$
\begin{array}{ll}
V = \{(s,\tilde{t}) \mid s \in S \text{ and } t \preceq_A \tilde{t}\} & V_0 = \{(s,\varphi_1 \vee \varphi_2) \in V\} \\
V_1 = \{(s,\varphi_1 \wedge \varphi_2) \in V\} & V_p = (S \times Q) \cap V \\
\kappa((s,q),(s',\delta(q,L(s)))) = P(s,s') & \tilde{\alpha} = \{\} \text{ or } V
\end{array}
$$

$$
\begin{array}{rl}
 & \{((s,\varphi_1 \wedge \varphi_2),(s,\varphi_i)) \in V \times V \mid 1 \leq i \leq 2\} \cup \\
E = & \{((s,\varphi_1 \vee \varphi_2),(s,\varphi_i)) \in V \times V \mid 1 \leq i \leq 2\} \cup \\
 & \{((s,q),(s',\delta(q,L(s)))) \in V \times V \mid P(s,s') > 0\}
\end{array}
$$

where $\tilde{\alpha}$ equals V if some state q in $(\!(t)\!)$ is in α, and equals $\{\}$ otherwise. By Theorem 1, $\mathsf{val}_0(c)$ is in $[0,1]$ for all configurations $c \in V$. We set $\mathsf{val}(c) = \mathsf{val}_0(c)$.

Example 4 *Continuing with the verification $M \in \mathcal{L}(A)$ for A from Example 1 and M from Fig. 1(b). The stochastic weak game $G_{M,(\!(q_1)\!)}$ for the MSCC $(\!(q_1)\!)$, shown in Fig. 3, depicts stochastic configurations with a diamond and configurations from other MSCCs are put into hexagons (with the hexagon labeled $(s_1, [\![q_2]\!]_{\geq \frac{1}{2}})$ having value 1 and all others having value 0). As none of the configurations are accepting, Player 0 can only win by reaching optimal hexagons. Hexagon $(s_1, [\![q_2]\!]_{\geq \frac{1}{2}})$ has value 1 and is the optimal choice for Player 0 from configuration $(s_1, q_1 \vee [\![q_2]\!]_{\geq \frac{1}{2}})$. Player 0 configuration $(s_2, q_1 \vee [\![q_2]\!]_{\geq \frac{1}{2}})$ has value 0. So the value for Player 0 of diamond configuration (s_0, q_1) is $\frac{1}{2}$.*

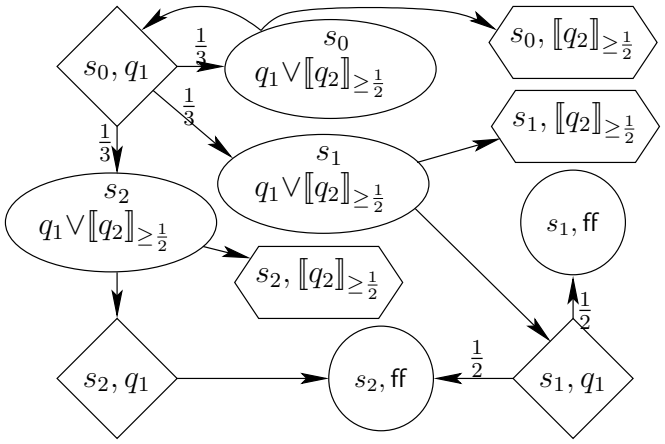

Figure 3: Case 3 of acceptance game

Trivial MSCCs $(\!(t)\!)$, are handled as one of the cases above. In case more than one case matches, the ambiguity is unproblematic as game values in $G_{M,(\!(t)\!)}$ are then determined via propagation of pre-seeded game values. In particular, the case of a configuration (s, φ), where $\varphi = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ is handled as in Case 1. The definitions of $\mathsf{Val}_{s,\varphi}$ and $\mathcal{F}_{s,\varphi}$ are as before and the configuration (s, φ, f) is connected to configurations of the form (s', φ', v), which form dead-ends in $G_{M,(\!(t)\!)}$.[1] The case of $\Diamond$ is handled as in Case 2.

Example 5 *Finally, we establish that $M \in \mathcal{L}(A)$ for A from Example 1 and M from Fig. 1(b). The initial configuration $(s_0, [\![q_1]\!]_{\geq \frac{1}{2}})$ makes up a trivial bounded MSCC. Consider the function $f = \{s_0 \mapsto \frac{1}{2},\ s_1 \mapsto 1,\ s_2 \mapsto 0\}$. It is disjoint as witnessed by $\{a_{1,s} = 1\}_{s \in S}$, which satisfies $1 \cdot \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot 1 \cdot \frac{1}{3} + 1 \cdot 0 \cdot \frac{1}{3} = \frac{1}{2}$ as required. Then, configuration $(s_0, q_1 \vee [\![q_2]\!]_{\geq \frac{1}{2}}, \frac{1}{2})$ is a dead end for Player 1 as $\mathsf{val}(s_0, q_1 \vee [\![q_2]\!]_{\geq \frac{1}{2}}) = \frac{1}{2}$ and similarly configuration $(s_1, q_1 \vee [\![q_2]\!]_{\geq \frac{1}{2}})$. Therefore, $M \in \mathcal{L}(A)$.*

We now state the well-definedness of languages for p-automata and the complexity of checking acceptance in case of finite automaton and Markov chain.

[1] Alternatively, for every state s and every $i \in [n]$ the value $\mathsf{val}(s, q_i)$ is pre-computed. Then, it is enough to find a set $\{a_i \mid i \in [n]\}$ such that $\sum_{i \in [n]} a_i = 1$ and for every $i \in [n]$ we have $a_i \cdot \mathsf{val}(s, q_i) \bowtie_i p_i$. A formal proof of this alternative definition is omitted.

Theorem 2 *Given a p-automaton $A = \langle 2^{\mathbb{AP}}, \ldots \rangle$, its language $\mathcal{L}(A)$ is well defined. If A and $M \in \mathsf{MC}_{\mathbb{AP}}$ are finite, $M \in \mathcal{L}(A)$ can be decided in EXPTIME.*

Proof: Well definedness of acceptance follows directly from Theorem 1. For finite Markov chain M and finite p-automata A we make two observations: First, the stochastic weak game arising from the combination of M and an unbounded MSCC of G_A can be solved in EXPTIME. Second, the weak game arising from the combination of M and a bounded MSCC of G_A may be exponential due to the large number of possible value assignment functions. Such a weak game can be solved in linear time leading to an EXPTIME upper bound. Since there are only linearly many such games in the sequence of weak games and stochastic weak games, acceptance can be solved in EXPTIME. □

For finite Markov chain M and p-automaton A with non-trivial, bounded MSCCs, checking acceptance $M \in \mathcal{L}(A)$ is exponential in the branching degree of M and in the branching degree of $*$ and $\overset{\forall}{\forall}$ operators of A, *but not in the number of states or locations*. If A has only trivial bounded-MSCCs, checking $M \in \mathcal{L}(A)$ reduces to solving a linear number of linear sized stochastic weak games.

5. Expressiveness of p-Automata

We now consider different aspects of the expressiveness of p-automata. We show that languages of p-automata are closed under Boolean operations. It follows that emptiness and containment of p-automata are equi-solvable. We then show that the language of every p-automaton is closed under bisimulation. For every Markov chain, we show how to construct a p-automaton accepting its bisimulation equivalence class. Finally, we show that each PCTL formula has a p-automaton whose language consists of all Markov chains satisfying that formula.

5.1. Closure of Languages

It is routine to see that the set of languages of p-automata is closed under union and intersection. But that set of languages is also closed under complementation: Given a p-automaton $A = \langle \Sigma, Q, \delta, \varphi^{\text{in}}, \alpha \rangle$, its dual $\mathsf{dual}(A)$ is $\langle \Sigma, \overline{Q}, \overline{\delta}, \mathsf{dual}(\varphi^{\text{in}}), \overline{Q \setminus \alpha} \rangle$ with bijection $\overline{Q} = \{\overline{q} \mid q \in Q\}$ and $\overline{\delta}(\overline{q}, \sigma) = \mathsf{dual}(\delta(q, \sigma))$, where $\mathsf{dual}(\varphi)$ is

defined as follows:

$$\begin{aligned}
\mathsf{dual}(\mathbin{\mathbb{V}}(t_1,\ldots,t_n)) &= *(\mathsf{dual}(t_1),\ldots,\mathsf{dual}(t_n))\\
\mathsf{dual}(*(t_1,\ldots,t_n)) &= \mathbin{\mathbb{V}}(\mathsf{dual}(t_1),\ldots,\mathsf{dual}(t_2))\\
\mathsf{dual}(\varphi_1 \wedge \varphi_2) &= \mathsf{dual}(\varphi_1) \vee \mathsf{dual}(\varphi_2)\\
\mathsf{dual}(\varphi_1 \vee \varphi_2) &= \mathsf{dual}(\varphi_1) \wedge \mathsf{dual}(\varphi_2)\\
\mathsf{dual}(q) &= \overline{q}\\
\mathsf{dual}(\overline{q}) &= q\\
\mathsf{dual}([\![q]\!]_{\bowtie p}) &= [\![\overline{q}]\!]_{\mathsf{dual}(\bowtie p)}\\
\mathsf{dual}(\geq p) &= > 1-p\\
\mathsf{dual}(> p) &= \geq 1-p
\end{aligned}$$

The structure of uniform weak p-automata ensures that $\mathsf{dual}(A)$ is also uniform weak. The languages of A and $\mathsf{dual}(A)$ are complements.

Theorem 3 *For every p-automaton A with $\Sigma = 2^{\mathbb{AP}}$, $\mathcal{L}(\mathsf{dual}(A))$ is $\mathsf{MC}_{\mathbb{AP}} \setminus \mathcal{L}(A)$.*

Proof: We prove a stronger claim, namely that $\mathsf{val}(s,\varphi) = 1 - \mathsf{val}(s,\mathsf{dual}(\varphi))$ for all $s \in S$ and $\varphi \in \mathsf{cl}(\delta(Q,\Sigma))$. The proof is by induction on the structure of the automaton. Consider an equivalence class $(\!(t)\!)$ in G_A. Assume by induction that the claim holds for all the MSCCs in G_A that are greater than $(\!(t)\!)$.

First, suppose that $(\!(t)\!)$ is a nontrivial MSCC and that no transition in $(\!(t)\!)$ is in the scope of $\mathbin{\mathbb{V}}$. It follows that $(\!(\mathsf{dual}(t))\!)$ is also a nontrivial MSCC and that no transition in $(\!(\mathsf{dual}(t))\!)$ is in the scope of $*$. Given a strategy for Player 0 in $G_{M,(\!(t)\!)}$, we show how to construct a strategy for Player 1 in $G_{M,(\!(\mathsf{dual}(t))\!)}$. The two strategies produce plays that are always in the same locations of the Markov chain M and same states of the automaton A (modulo dualization $t \mapsto \mathsf{dual}(t)$). For sake of brevity, we denote $G_{M,(\!(t)\!)}$ by G and $G_{M,(\!(\mathsf{dual}(t))\!)}$ by $\overline{G}$.

Consider two matching configurations (s,φ) and $(s,\mathsf{dual}(\varphi))$ in G and $\overline{G}$. Let φ be of form $*([\![q_1]\!]_{\bowtie_1 p_1},\ldots,[\![q_n]\!]_{\bowtie_n p_n})$, where $n > 1$. Consider the configuration $(s,\mathsf{dual}(\varphi))$. By playing for Player 1 in G we make Player 0 'reveal' her strategy in G and using her strategy we react to the moves of Player 0 in $\overline{G}$ by constructing a strategy for Player 1 in $\overline{G}$.

Consider two plays ending in (s,φ) and $(s,\mathsf{dual}(\varphi))$. Let $f\colon [n] \times \mathsf{succ}(s) \to \mathsf{Val}_{s,\varphi}$ be the function chosen by Player 0 in G and let $f'\colon [n] \times \mathsf{succ}(s) \to \mathsf{Val}_{s,\mathsf{dual}(\varphi)}$ be the function chosen by Player 0 in $\overline{G}$. By definition there are $\{a_{i,s'}\}$ that witness the disjointness of f and for every i we have

$$\sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot P(s,s') \cdot f(i,s') \bowtie_i p_i.$$

By using the same $\{a_{i,s'}\}$ stemming from the fact that f' is intersecting, we get that there is some i such that

$$\sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot P(s,s') \cdot f'(i,s') \mathsf{dual}(\bowtie_i p_i).$$

It follows that there is an $s' \in \mathsf{succ}(s)$ such that $f(i,s') + f'(i,s') > 1$.

It is now Player 1's turn to move in both G and $\overline{G}$. In G we make Player 1 choose $(s', \delta(q_i, L(s)), f(i,s'))$ and the strategy for Player 1 in $\overline{G}$ is extended by $(s', \mathsf{dual}(\delta(q_i, L(s))), f'(i,s'))$. We now proceed by utilizing the duality between $\vee$ and $\wedge$ to use Player 0's choices in $\overline{G}$ to suggest moves for Player 1 in G and use Player 0's strategy in G to suggest how to extend the strategy for Player 1 in $\overline{G}$.

If we reach configurations $(s', \varphi', f(i,s'))$ and $(s', \mathsf{dual}(\varphi'), f'(i,s'))$ such that $\mathsf{val}(s', \varphi') \neq \bot$ and $\mathsf{val}(s', \mathsf{dual}(\varphi')) \neq \bot$, then, by assumption $\mathsf{val}(s', \varphi') = 1 - \mathsf{val}(s', \mathsf{dual}(\varphi'))$. And if $\mathsf{val}(s', \varphi') \geq f(s')$, then $\mathsf{val}(s', \mathsf{dual}(\varphi')) < f'(s')$ must hold. Otherwise the game proceeds to a new configuration in $S \times [\![Q]\!]$. If the two plays are infinite, then by the duality of α and $\overline{Q \setminus \alpha}$ if Player 0 wins the play in G then Player 1 wins the play in $\overline{G}$.

That a win of Player 1 in G is translated to a win of Player 0 in $\overline{G}$ is shown similarly, and omitted.

Second, the case that $(\!(t)\!)$ is a nontrivial MSCC and that some transitions in $(\!(q)\!)$ are in scope of $\mathring{\forall}$ is similar, and omitted.

Third, suppose that $(\!(t)\!)$ is a nontrivial MSCC and that all transitions in $(\!(t)\!)$ are unbounded. Then, the claim follows from the dualization and the determinacy of stochastic weak games.

Finally, if $(\!(t)\!)$ is a trivial MSCC, the claim follows from the dualization and the duality of $\min$ and $\max$. □

Corollary 1 *Let $\Sigma = 2^{\mathbb{A}^{\mathbb{P}}}$. The set of languages accepted by p-automata with Σ is closed under Boolean operations. Language containment of p-automata with Σ reduces to language emptiness of such p-automata, and vice versa.*

Proof: By Theorem 3, this set of languages is closed under complement. Showing closure under intersection and union is routine, and omitted. That language containment and non-emptiness are equi-solvable is a standard argument, since p-automata have duals and since there are p-automata with empty language. □

5.2. Closure of Languages to Bisimulation

We now show that languages of p-automata are closed under bisimulation.

Lemma 1 *For p-automaton* $A{=}\langle 2^{\mathbb{AP}}, Q, \delta, \varphi^{\mathsf{in}}, \alpha\rangle$ *and* $M_1, M_2 \in \mathsf{MC}_{\mathbb{AP}}$ *with* $M_1 \sim M_2$*: Markov chain* M_1 *is in* $\mathcal{L}(A)$ *iff Markov chain* M_2 *is in* $\mathcal{L}(A)$*.*

Proof: Let $M_i = (S_i, P_i, L_i, s_i^{\mathsf{in}})$, for $i \in \{1, 2\}$, with the same set of atomic propositions $\mathbb{AP}$. Let $A = \langle \Sigma, Q, \delta, [\![q_0]\!]_{\bowtie p}, \alpha\rangle$, where $\Sigma = 2^{\mathbb{AP}}$. Let $\sim \subseteq S_1 \times S_2$ be the maximal bisimulation between M_1 and M_2.

We show that for every state $q \in Q$ and locations $s_1 \in S_1$, and $s_2 \in S_2$ such that $s_1 \sim s_2$, we have $\mathsf{val}(s_1, q) = \mathsf{val}(s_2, q)$. We prove this claim by induction on the partial order on the MSCCs in G_A. Suppose that the claim holds for all MSCCs greater than $(\!(q)\!)$ in the partial order. Consider the games $G_{M_1,(\!(q)\!)}$ and $G_{M_2,(\!(q)\!)}$. Consider a winning strategy σ for Player 0 in $G_{M_1,(\!(q)\!)}$. We show how this is also a winning strategy for Player 0 in $G_{M_2,(\!(q)\!)}$.

Consider a play in an unbounded MSCC $(\!(q)\!)$. We inductively construct a play in $G_{M_1,(\!(q)\!)}$ and a play in $G_{M_2,(\!(q)\!)}$ with the invariant that the plays end in configurations of the form (s_1, t) and (s_2, t) such that $s_1 \sim s_2$. Clearly, the initial configurations in both games satisfy this invariant. We show how to extend the play to maintain this invariant. If t is of the form $\varphi_1 \wedge \varphi_2$ and Player 1 chooses φ_i in $G_{M_2,(\!(q)\!)}$, then we emulate the same choice in $G_{M_1,(\!(q)\!)}$. If t is of the form $\varphi_1 \vee \varphi_2$, then σ instructs Player 0 to choose φ_i in $G_{M_1,(\!(q)\!)}$ and we emulate the same choice in $G_{M_2,(\!(q)\!)}$. If t is of the form q' for some state $q' \in Q$ then choices in (s_1, q') and (s_2, q') are resolved by the stochastic player.

As $s_1 \sim s_2$, the successors of s_1 and s_2 can be partitioned into equivalence classes such that, for each equivalence class C_1 in M_1 and C_2 in M_2, we have $P_1(s_1, C_1) = P_2(s_2, C_2)$. Consider now the measure of plays that are winning according to this composed strategy. The plays can be partitioned according to bisimulation equivalence classes and every choice has the same weight. So the measure of winning plays is identical in both games.

Consider a play in a bounded MSCC $(\!(q)\!)$ where no transition uses $\Diamond$. Disjunctions and conjunctions are handled as above. Consider a pair of configurations (s_1, t) and (s_2, t), where $s_1 \sim s_2$ and t is of the form $*([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$. Let f_1 be the function chosen by Player 0 in $G_{M_1,(\!(q)\!)}$. As $s_1 \sim s_2$, we can find a function f_2 such that for every s_2' we have $f_2(i, s_2') = f_1(i, s_1')$ for some $s_1' \sim s_2'$ that satisfies the requirement of the game. Next, Player 1 chooses a state $s' \in \mathsf{succ}(s_2)$ and a state q_i. The same choice can be mimicked in $G_{M_1,(\!(q)\!)}$. As

$s_1 \sim s_2$, it follows that $L(s_1) = L(s_2)$ and the automaton component in both configurations remains the same.

The treatment of a play in a bounded MSCC $(\!(q)\!)$ with $\Mightarrow$ markings is similar. □

5.3. Embedding of Markov Chains

A Markov chain $M = (S, P, L, s^{\text{in}}) \in \mathsf{MC}_{\mathbb{AP}}$ can be converted into a p-automaton $A_M = \langle 2^{\mathbb{AP}}, Q, \delta, \varphi^{\text{in}}, \alpha \rangle$ whose language $\mathcal{L}(A_M)$ is the set of Markov chains bisimilar to M:

$$\begin{aligned} Q &= \{(s,s') \in S \times S \mid P(s,s') > 0\} \\ \delta((s,s'),L(s)) &= *(\llbracket (s',s'') \rrbracket_{\geq P(s',s'')} \mid s'' \in \mathsf{succ}(s')) \\ \delta((s,s'),\sigma) &= \mathsf{ff} \qquad \text{if } \sigma \neq L(s) \\ \varphi^{\text{in}} &= *(\llbracket (s^{\text{in}},s') \rrbracket_{\geq P(s^{\text{in}},s')} \mid P(s^{\text{in}},s') > 0) \\ \alpha &= Q \end{aligned}$$

State (s, s') represents the transition from s to s'. Labels are compared for location s. Location s' is used to require that there are successors of probability at least $P(s', s'')$. This p-automaton A_M has only bounded transitions and uses only the $*$ operator. In particular, it is uniform weak.

Theorem 4 *For every Markov chain $M \in \mathsf{MC}_{\mathbb{AP}}$, the language $\mathcal{L}(A_M)$ is the bisimulation equivalence class of M.*

Proof: By Lemma 1, we know that $M' \sim M$ implies $M' \in \mathcal{L}(A_M)$ as soon as we have that $M \in \mathcal{L}(A_M)$. To simplify the presentation of the proof of $M \in \mathcal{L}(A_M)$, we assume that all locations of M are in one MSCC. Consider a location $s \in S$ and $(s, s') \in Q$. Let $\varphi_s = *(\llbracket (s,s') \rrbracket_{\geq P(s,s')} \mid s' \in \mathsf{succ}(s))$. We show that from a configuration of the form (s, φ_s), Player 0 has a strategy that keeps returning to configurations of this form. As $\alpha = Q$, Player 0 can continue playing forever and wins. We start from the configuration (s, φ_s). Then Player 0 chooses the function $f\colon [n] \times \mathsf{succ}(s) \to \{0, 1\}$ such that $f(i, s') = 1$ iff $s_i = s'$. The trivial assignment $a_{i,s'} = 1$ iff $s_i = s'$ shows that f is disjoint. Then, Player 1 chooses a successor $(s_i, \delta((s, s_i), L(s)), 1)$. As $\delta((s, s_i), L(s)) = \varphi_{s_i}$ the claim follows and Player 0 has a strategy to continue the play forever.

The initial configuration in the game is $*(\llbracket (s^{\text{in}},s') \rrbracket_{\geq P(s^{\text{in}},s')} \mid s' \in \mathsf{succ}(s^{\text{in}}))$. The same intuition shows that this is winning for Player 0 as well.

Conversely, if $M' \not\sim M$ we show that $M' \notin \mathcal{L}(A_M)$. Let $M = (S, P, L, s^{\text{in}})$ and $M' = (T, P, L, t^{\text{in}})$. To simplify notations we assume that $S \cap T = \{\}$ and use P and L for the probability distribution and labeling of both Markov chains. We use the partition refinement algorithm that computes the bisimulation equivalence sets for a Markov chain. Let $\Xi_0 = \{S' \subseteq S \cup T \mid \forall s, s' \in S' \colon L(s) = L(s')$ and S' is maximal with respect to that$\}$. Clearly, Ξ_0 is a partition of $S \cup T$. Let Ξ_{i+1} be the coarsest partition of $S \cup T$ that refines Ξ_i and in addition for every $G \in \Xi_{i+1}$, for all $s, s' \in G$, and for all $G' \in \Xi_i$ we have $P(s, G') = P(s', G')$. It is known that if $s \not\sim s'$ there is $i_{s,s'}$ where s and s' are in different sets in $\Xi_{i_{s,s'}}$.

By assumption, $s^{\text{in}} \not\sim t^{\text{in}}$. Let i_0 be minimal such that s^{in} and t^{in} are in different sets in Ξ_{i_0}. Denote $s_{i_0} = s^{\text{in}}$, $t_{i_0} = t^{\text{in}}$, $\varphi_{i_0} = \varphi^{\text{in}}$, and $c_{i_0} = (t_{i_0}, \varphi_{i_0})$. Consider the configuration $c_{i_j} = (t_{i_j}, \varphi_{i_j})$, where $\varphi_{i_j} = *(\llbracket (s_{i_j}, s') \rrbracket_{\geq P(s_{i_j}, s')} \mid s' \in \mathsf{succ}(s_{i_j}))$ and s_{i_j} and t_{i_j} are in different sets in Ξ_{i_j}. We show that from configuration c_{i_j} Player 1 either wins immediately or finds a similar configuration for $i_{j+1} < i_j$.

If $i_j = 0$, then $L(t_{i_j}) \neq L(s_{i_j})$. Regardless of the immediate choices of Player 0, we have $\delta((s_{i_j}, s'), L(t_{i_j})) = \mathsf{ff}$ and Player 1 wins.

Otherwise, $i_j > 0$. By assumption, there is some $i_{j+1} < i_j$ and $G \in \Xi_{i_{j+1}}$ such that $P(s_{i_j}, G) \neq P(t_{i_j}, G)$. Without loss of generality we assume that $P(s_{i_j}, G) > P(t_{i_j}, G)$. Indeed, if $P(s_{i_j}, G) < P(t_{i_j}, G)$, then as $P(s_{i_j}, S) = 1$ there must be a different set $G' \in \Xi_{i_{j+1}}$ such that $P(s_{i_j}, G') > P(t_{i_j}, G')$.

Let $S_{i_{j+1}} = G \cap S$. Let $(t_{i_j}, \varphi_{i_j}, f)$ be the configuration chosen by Player 0. By disjointness of f, and as $P(t_{i_j}, G) < P(s_{i_j}, G)$, there must be $s_{i_{j+1}} \in G$ and $t_{i_{j+1}} \notin G$ such that $f(t_{i_{j+1}}, s_{i_{j+1}}) > 0$. Player 1 chooses $c_{i_{j+1}} = (t_{i_{j+1}}, \varphi_{i_{j+1}}, v)$, where $\varphi_{i_{j+1}} = \delta((s_{i_j}, s_{i_{j+1}}), L(t_{i_j}))$. As $t_{i_{j+1}} \notin G$, Player 1 has forced the game to a similar configuration with $i_{j+1} < i_j$ and eventually wins by reaching Ξ_0. $\square$

The construction of A_M for infinite Markov chains was the only reason why we allow p-automata with infinite state sets. Finite state sets suffice for embedding finite Markov chains. The construction of A_M was also our initial reason for introducing the $*$ and $\dot{\forall}$ operators. But we believe that the separation of concerns expressed in these operators is useful in p-automata in general. In the construction of A_M, the conjunctive operator $*$ effectively hides an exponential blowup.

If a Markov chain is deterministic (all successors of every location disagree on their labelings), we can eliminate the use of $*$ in A_M and still secure Theorem 4. But this embedding does break Theorem 4 for non-deterministic Markov chains if we replace $*$ with the much simpler $\wedge$ in the definition of $\delta((s, s'), L(s))$ and φ^{in} for A_M. We refer to this modified p-automaton as A_M^w subsequently.

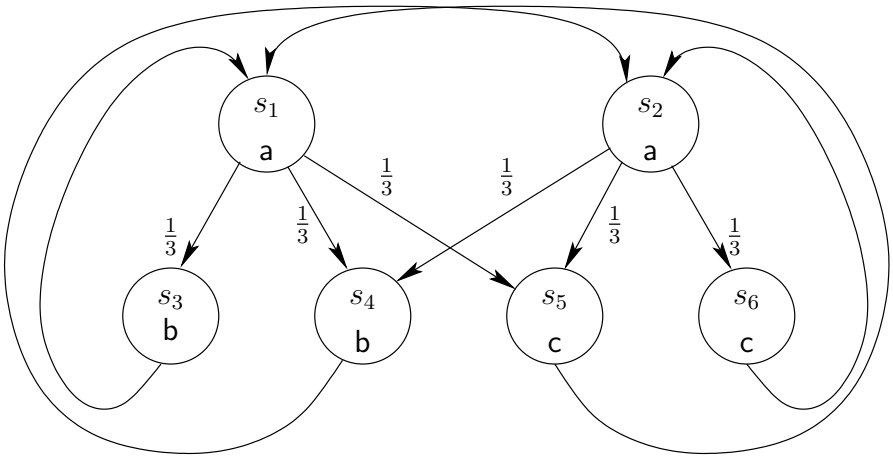

Figure 4: Markov chain whose uniform weak embedding accepts non-bisimilar Markov chains

Consider the Markov chain M in Figure 4 and let M_1 be M with s_1 as initial location, and let M_2 be M with s_2 as initial location. First, we note that M_1 and M_2 are not bisimilar since the transitions from s_1 to locations whose label is b have probability $\frac{2}{3}$ and the transitions from s_2 to locations whose label is b have probability $\frac{1}{3}$. In fact, no two locations in M are bisimilar.

Second, we observe that $A^w_{M_1}$ accepts M_2. To see the latter, the initial configuration is $(s_2, \varphi^{\text{in}})$. As φ^{in} is a conjunction, Player 1 can choose one of three successor configurations: $(s_2, [\![(s_1, s_3)]\!]_{\geq \frac{1}{3}})$, $(s_2, [\![(s_1, s_4)]\!]_{\geq \frac{1}{3}})$, and $(s_2, [\![(s_1, s_5)]\!]_{\geq \frac{1}{3}})$. One can see that Player 0 wins from the latter two. In the other case, Player 1 chooses the configuration $(s_2, [\![(s_1, s_3)]\!]_{\geq \frac{1}{3}})$. Then Player 0 chooses the configuration $(s_2, [\![(s_1, s_3)]\!]_{\geq \frac{1}{3}}, f)$ where f is the function that sets $f(1, s_4) = 1$ and $f(1, s_5) = f(1, s_6) = 0$. The next configuration is $(s_4, [\![(s_3, s_1)]\!]_{\geq 1}, 1)$. We complete a cycle by going back to configuration $(s_2, \varphi^{\text{in}})$. This completes a winning strategy for Player 0.

5.4. Embedding of PCTL Formulas

A PCTL formula ϕ over $\mathbb{AP}$ yields a p-automaton A_ϕ without $*$ markings, $\langle 2^{\mathbb{AP}}, \mathsf{cl_p}(\phi) \cup \mathbb{AP}, \rho_{\mathsf{x}}, \rho_\epsilon(\phi), F\rangle$, that accepts exactly the Markov chains satisfying ϕ. The construction resembles the translation from CTL to alternating tree automata:

- $\mathsf{cl_p}(\phi)$ denotes the set of path subformulas of ϕ
- F consists of $\mathbb{AP}$ and their negations, and all ψ of $\mathsf{cl_p}(\phi)$ *not* of form $\psi_1 \,\mathsf{U}\, \psi_2$

- functions ρ_{x} and ρ_ϵ are defined in (3), where we interpret the evaluation of $\mathsf{a} \in \sigma$ as a truth constant $\mathtt{tt}$ or $\mathtt{ff}$:

$$\begin{array}{rcll}
\rho_{\mathsf{x}}(\mathsf{a}, \sigma) & = & (\mathsf{a} \in \sigma) & \\
\rho_{\mathsf{x}}(\neg\mathsf{a}, \sigma) & = & \neg(\mathsf{a} \in \sigma) & \\
\rho_{\mathsf{x}}(\mathsf{X}\,\varphi_1, \sigma) & = & \rho_\epsilon(\varphi_1) & \\
\rho_{\mathsf{x}}(\varphi_1 \,\mathsf{U}\, \varphi_2, \sigma) & = & (\rho_\epsilon(\varphi_1) \wedge \varphi_1 \,\mathsf{U}\, \varphi_2) \vee \rho_\epsilon(\varphi_2) & \\
\rho_{\mathsf{x}}(\varphi_1 \,\mathsf{W}\, \varphi_2, \sigma) & = & (\rho_\epsilon(\varphi_1) \wedge \varphi_1 \,\mathsf{W}\, \varphi_2) \vee \rho_\epsilon(\varphi_2) & \\
\rho_\epsilon(\mathsf{a}) & = & \mathsf{a} & \\
\rho_\epsilon(\neg\mathsf{a}) & = & \neg\mathsf{a} & \\
\rho_\epsilon(\varphi_1 \circ \varphi_2) & = & \rho_\epsilon(\varphi_1) \circ \rho_\epsilon(\varphi_2) & \text{where } \circ \in \{\wedge, \vee\} \\
\rho_\epsilon([\,\mathsf{X}\,\varphi_1]_{\bowtie p}) & = & [\![\,\mathsf{X}\,\varphi_1]\!]_{\bowtie p} & \\
\rho_\epsilon([\varphi_1 \,\mathsf{U}\, \varphi_2]_{\bowtie p}) & = & (\rho_\epsilon(\varphi_1) \wedge [\![\varphi_1 \,\mathsf{U}\, \varphi_2]\!]_{\bowtie p}) \vee \rho_\epsilon(\varphi_2) & \\
\rho_\epsilon([\varphi_1 \,\mathsf{W}\, \varphi_2]_{\bowtie p}) & = & (\rho_\epsilon(\varphi_1) \wedge [\![\varphi_1 \,\mathsf{W}\, \varphi_2]\!]_{\bowtie p}) \vee \rho_\epsilon(\varphi_2) & (3)
\end{array}$$

Function ρ_{x} records whether or not literals of the formula are consistent with the input symbol, unfolds fix-points, and replaces the threshold context $[\cdot]_{\bowtie p}$ with $[\![\cdot]\!]_{\bowtie p}$ (through a recursive call to ρ_ϵ). That replacement is also done by function ρ_ϵ for the initial condition. The effect of these functions is similar to that achieved by using ϵ transitions to translate CTL formulas into two-way tree automata [29].

We now have that $\psi \in \mathsf{cl_p}(\phi)$ for subformulas $[\psi]_{\bowtie p}$ of ϕ. Also, $[\psi_1 \,\mathsf{U}\, \psi_2]_{\bowtie p}$ may appear inside an element in $\mathsf{cl_p}(\phi)$ whereas $[\![\psi_1 \,\mathsf{U}\, \psi_2]\!]_{\bowtie p}$ can only be an element of $[\![\mathsf{cl_p}(\phi)]\!]_>$, it wraps $\psi_1 \,\mathsf{U}\, \psi_2 \in \mathsf{cl_p}(\phi)$ in the probabilistic quantification $[\![\cdot]\!]_{\bowtie p}$ of A_ϕ.

Example 6 *Let* $\varphi = [\mathsf{a}\,\mathsf{U}\,[\,\mathsf{X}\,\mathsf{b}]_{>\frac{1}{2}}]_{\geq 0.3}$. *Automaton* A_φ *is* $\langle 2^{\{\mathsf{a},\mathsf{b}\}}, \mathsf{cl_p}(\varphi) \cup \{\mathsf{a}, \mathsf{b}\}, \rho_{\mathsf{x}}, \rho_\epsilon(\varphi), F\rangle$, *where* $\mathsf{cl_p}(\varphi) = \{\mathsf{a}\,\mathsf{U}\,[\,\mathsf{X}\,\mathsf{b}]_{>\frac{1}{2}},\ \mathsf{X}\,\mathsf{b}\}$, F *is* $\{\,\mathsf{X}\,\mathsf{b}, \mathsf{a}, \mathsf{b}\}$, $\rho_\epsilon(\varphi)$ *is* $(\mathsf{a} \wedge [\![\mathsf{a}\,\mathsf{U}\,[\,\mathsf{X}\,\mathsf{b}]_{>\frac{1}{2}}]\!]_{\geq 0.3}) \vee [\![\,\mathsf{X}\,\mathsf{b}]\!]_{>\frac{1}{2}}$, $\rho_{\mathsf{x}}(\,\mathsf{X}\,\mathsf{b})$ *is* b, *and* $\rho_{\mathsf{x}}(\mathsf{a}\,\mathsf{U}\,[\,\mathsf{X}\,\mathsf{b}]_{>\frac{1}{2}})$ *is* $(\mathsf{a} \wedge \mathsf{a}\,\mathsf{U}\,[\,\mathsf{X}\,\mathsf{b}]_{>\frac{1}{2}}) \vee [\![\,\mathsf{X}\,\mathsf{b}]\!]_{>\frac{1}{2}}$.

Our acceptance game captures PCTL model checking, with same complexity.

Theorem 5 *For* $M \in \mathsf{MC}_{\mathbb{AP}}$ *and PCTL formula* ϕ *over* $\mathbb{AP}$, $M \models \phi$ *iff* $M \in \mathcal{L}(A_\phi)$. *Deciding the latter is polynomial in the size of* M, *linear in the size of* ϕ.

Proof:

1. We show the first statement of the theorem by proving

> For all locations s of M and PCTL (state) subformulas φ' of φ: configuration $(s, \rho_\epsilon(\varphi'))$ has value 1 for Player 0 in acceptance game of A_φ on M iff $M, s \models \varphi'$. Furthermore, the value is 0 otherwise.

by induction on the structure of the formula.

1.1. For a proposition a, notice that the value of (s, a) depends on the values of $(s', \rho_{\mathsf{x}}(\mathsf{a}, L(s)))$ for successors s' of s. By definition, $\rho_{\mathsf{x}}(\mathsf{a}, L(s)) = \mathsf{tt}$ if $\mathsf{a} \in L(s)$ and ff otherwise. The claim holds similarly for the other Boolean operators.

1.2. Consider a subformula of the form $\varphi' = [\mathsf{X}\,\psi]_{\bowtie p}$. By induction $M, s' \models \psi$ iff the configuration $(s', \rho_\epsilon(\psi))$ is winning for Player 0. By definition $\rho_\epsilon([\mathsf{X}\,\psi]_{\bowtie p}) = [\![\mathsf{X}\,\psi]\!]_{\bowtie p}$. Consider a state s such that $s \models \varphi'$. Let $Y = \{s' \mid s' \models \psi\}$. It follows that $P(s, Y) \bowtie p$. Then, the function $f\colon [1] \times \mathsf{succ}(s) \to [0, 1]$ with $f(1, s') = 1$ iff $\mathsf{val}(s', \rho_\epsilon(\psi)) = 1$ and $f(1, s') = 0$ otherwise satisfies $\sum_{s' \in \mathsf{succ}(s)} f(1, s')P(s, s') \bowtie p$. It follows that for every s' such that $f(1, s') > 0$ we have $(s', \rho_\epsilon(\psi), 1)$ is a Player 1 configuration that is a dead-end, and hence winning for Player 0. It follows that $(s, \rho_\epsilon(\varphi'))$ has value 1 for Player 0. In the other direction, suppose that $(s, \rho_\epsilon([\mathsf{X}\,\psi]_{\bowtie p}))$ has value 1 for Player 0 in the acceptance game of A_φ on M. It follows that there is a function $f : [1] \times \mathsf{succ}(s) \to [0, 1]$ such that $\sum_{s' \in \mathsf{succ}(s)} f(1, s')P(s, s') \bowtie p$ and that Player 0 wins from $(s, \rho_\epsilon([\mathsf{X}\,\psi]_{\bowtie p}), f)$. Thus, for every s' such that $f(1, s') > 0$ we have $\mathsf{val}(s', \rho_\epsilon(\psi)) \geq f(1, s') > 0$. However, by induction assumption, $\mathsf{val}(s', \rho_\epsilon(\psi)) \in \{0, 1\}$ and $\mathsf{val}(s', \rho_\epsilon(\psi)) = 1$ iff $s' \models \psi$. Thus, as $\sum_{s' \in \mathsf{succ}(s)} f(1, s')P(s, s') \bowtie p$, we conclude that $P(s, Y) \bowtie p$, where $Y = \{s' \in \mathsf{succ}(s) \mid s' \models \psi\}$, proving that $s \models [\mathsf{X}\,\psi]_{\bowtie p}$.

1.3. Consider a formula of the form $\varphi' = [\psi_1\,\mathsf{U}\,\psi_2]_{\bowtie p}$. By induction $M, s \models \psi_i$ iff the configuration $(s, \rho_\epsilon(\psi_i))$ is winning for Player 0, for $i \in \{1, 2\}$. Consider the stochastic weak game induced by the MSCC $(\!(\psi_1\,\mathsf{U}\,\psi_2)\!)$ in G_{A_φ}. The optimal strategy for both players is memoryless and pure. Restricting our attention to these memoryless pure strategies we can think about the game as restricted to configurations of the form $(s', \rho_\epsilon(\psi_1))$, where all configurations are probabilistic. A play that is winning for Player 0 is exactly a play that remains in states s' such that $M, s' \models \psi_1$ until reaching states s'' such that $M, s'' \models \psi_2$ (as $\psi_1\,\mathsf{U}\,\psi_2$ is unfair). It follows that the value of $(s, \psi_1\,\mathsf{U}\,\psi_2)$ in the stochastic game is exactly $\mathsf{Prob}_M(s, \psi_1\,\mathsf{U}\,\psi_2)$.

We note that $\rho_\epsilon([\psi_1 \mathsf{U} \psi_2]_{\bowtie p}) = \rho_\epsilon(\psi_1) \wedge [\![\psi_1 \mathsf{U} \psi_2]\!]_{\bowtie p} \vee \rho_\epsilon(\psi_2)$.

Consider a location s such that $s \models [\psi_1 \mathsf{U} \psi_2]_{\bowtie p}$. We have to show that Player 0 wins from configuration $(s, \rho_\epsilon([\psi_1 \mathsf{U} \psi_2]_{\bowtie p}))$. In case that $s \models \psi_2$ then, by induction $\mathsf{val}(s, \rho_\epsilon(\psi_2)) = 1$. It follows that the value of $(s, \rho_\epsilon(\psi_1) \wedge [\![\psi_1 \mathsf{U} \psi_2]\!]_{\bowtie p} \vee \rho_\epsilon(\psi_2))$ is also 1. In case that $s \not\models \psi_2$ then it must be the case that $s \models \psi_1$ and that the set of paths that satisfy $\psi_1 \mathsf{U} \psi_2$ and start with s is $\bowtie p$. It follows that the value of $(s, \rho_\epsilon(\psi_1))$ is 1. By our claim regarding the game above for every successor s' of s we have $\mathsf{val}(s', \psi_1 \mathsf{U} \psi_2)$ is $\mathsf{Prob}_M(s', \psi_1 \mathsf{U} \psi_2)$. Also, as $s \models [\psi_1 \mathsf{U} \psi_2]_{\bowtie p}$ it follows that $\sum_{s' \in \mathsf{succ}(s)} P(s, s') \mathsf{Prob}_M(s', \psi_1 \mathsf{U} \psi_2) \bowtie p$. Hence, the function f that associates $\mathsf{Prob}_M(s', \psi_1 \mathsf{U} \psi_2)$ with s' is a disjoint function that leads to Player 0 winning from $(s, \rho_\epsilon([\psi_1 \mathsf{U} \psi_2]_{\bowtie p}))$.

In the other direction, consider a location s such that Player 0 wins from $(s, \rho_\epsilon([\psi_1 \mathsf{U} \psi_2]_{\bowtie p}))$. Then one of the following two cases holds. Either Player 0 wins from $(s, \rho_\epsilon(\psi_2))$ and by induction $s \models \psi_2$. Or Player 0 wins from $(s, \rho_\epsilon(\psi_1))$ and Player 0 wins from $(s, [\![\psi_1 \mathsf{U} \psi_2]\!]_{\bowtie p})$. By induction $s \models \psi_1$. Furthermore, there is a function $f : \mathsf{succ}(s) \to [0, 1]$ such that $\sum_{s' \in \mathsf{succ}(s)} P(s, s') f(s') \bowtie p$ and Player 0 wins from $(s', \delta(\psi_1 \mathsf{U} \psi_2, L(s)), f(s'))$. However, by the observation about the stochastic games above, it follows that $\mathsf{Prob}_M(s', \psi_1 \mathsf{U} \psi_2) \geq f(s')$. Thus, this shows that $s \models [\psi_1 \mathsf{U} \psi_2]_{\bowtie p}$.

Finally, as $\rho_\epsilon([\psi_1 \mathsf{U} \psi_2]_{\bowtie p}) = \rho_\epsilon(\psi_1) \wedge [\![\psi_1 \mathsf{U} \psi_2]\!]_{\bowtie p} \vee \rho_\epsilon(\psi_2)$ it follows that the value of $(s, \rho_\epsilon([\psi_1 \mathsf{U} \psi_2]_{\bowtie p}))$ is in $\{0, 1\}$.

1.4. The cases of a formula of form $\varphi' = [\psi_1 \mathsf{W} \psi_2]_{\bowtie p}$ is similar.

2. Finally, we provide the complexity analysis showing that the acceptance game for $M \in \mathcal{L}(A_\phi)$ is linear in the PCTL formula ϕ and polynomial in M.

The number of MSCCs in G_{A_ϕ} is linear in the size of ϕ. Most MSCCs are trivial and consist of a single subformula of ϕ. The corresponding games can be solved by propagating pre-seeded game values. The only nontrivial MSCCs in G_{A_ϕ} arise from states of the form $\phi_1 \mathsf{U} \phi_2$ and $\phi_1 \mathsf{W} \phi_2$. It follows that for each nontrivial MSCC, the choices for each of the players in the corresponding weak stochastic game are determined by pre-seeded game values. Therefore, these games have effectively only probabilistic configurations and can be solved in polynomial time in the size of the Markov chain.

Overall, the number of games is linear in the size of ϕ and the solution of each game is polynomial in the size of the Markov chain. □

We note that our definition of PCTL does not include the bounded versions of the Strong Until and Weak Until. However, the techniques above can be extended

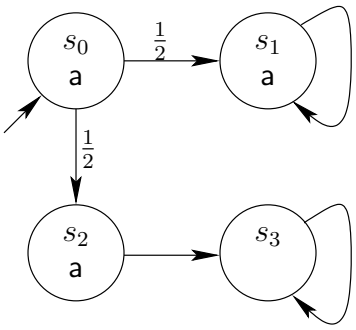

Figure 5: Markov chain M in $\mathcal{L}(A_R) \setminus \mathcal{L}(A_\eta)$

to handle these operators. Essentially, in order to handle $\psi_1 \,\mathsf{U}^{\leq k} \psi_2$ the automaton would have states corresponding to $\psi_1 \,\mathsf{U}^{\leq k'} \psi_2$ for all $0 \leq k' \leq k$. The case of bounded Weak Until would be similar. In particular, the proof of case **1.3** above gives all the necessary ingredients for handling these cases as well.

Corollary 1 and Theorem 5 imply that the satisfiability of PCTL [6, 30] reduces to both the language emptiness and containment of p-automata. The decidability status is open for all these problems at the time of writing.

In comparing automata and temporal logic, automata usually can count but temporal logics cannot. Thus, just as alternating tree automata are more expressive than CTL and CTL*, p-automata are more expressive than PCTL.

Also, p-automata can encode recursive, probabilistic properties that we believe are not expressible in PCTL: $A_R = \langle 2^{\{\mathsf{a}\}}, \{q_2\}, \delta, [\![q_2]\!]_{>0}, \{q_2\}\rangle$ with $\delta(q_2, \{\mathsf{a}\}) = [\![q_2]\!]_{\geq \frac{1}{2}}$ and $\delta(q_2, \{\}) = \mathsf{ff}$, asserts the recursive, probabilistic property that a location is labeled a, and that the probability of its successors with the same property is $\geq \frac{1}{2}$. A naive attempt of expressing this in PCTL is through formula $\eta = \mathsf{a} \wedge [(\neg\mathsf{a} \vee [\,\mathsf{X}\,\mathsf{a}]_{\geq \frac{1}{2}})\,\mathsf{W}\,\neg\mathsf{a}]_{\geq 1}$. Then $\mathcal{L}(A_\eta) \subset \mathcal{L}(A_R)$ but this inclusion is strict; e.g., the Markov chain in Fig. 5 is in $\mathcal{L}(A_R)$ but not in the language of A_η.

6. Simulation of p-Automata

We now define simulation of p-automata that under-approximates language containment: if p-automaton B simulates p-automaton A (denoted $A \leq B$), then $\mathcal{L}(A)$ is contained in $\mathcal{L}(B)$, under qualifications detailed in the formal theorem below. This simulation is defined as a combination of fair simulation [17], simulation for alternating word automata [31], probabilistic bisimulation [8], and the

games defined in Section 3. The simulation takes into account the structure of the automata, their acceptance condition, and local probabilistic constraints. We show that whether B simulates A can be decided in EXPTIME and that simulation under-approximates language containment.

We determine whether B simulates A through solving a series of games $G_{\leq}$ on the product of states and transitions of A and B: state u of B simulates state r of A iff Player 0 wins from configuration (r, u) in the corresponding game. More general configurations (α, β) are such that α is part of a transition of A and β is part of a transition of B. The classification of α and β as unbounded, bounded with $*$, bounded with $\Diamond$, or simple classifies (α, β) as one of 16 cases. As before, simple cases where either α or β belong to trivial MSCCs in their respective automata reduce to one of the previous cases. Thus we are left with "only" 9 interesting cases.

Here, we restrict our attention to A and B that do not use the $\Diamond$ operator. These restrictions, and concentrating on the non-simple cases, reduce the number of cases to consider to 4. This restriction is sufficient for handling simulation of automata that result from embedding PCTL formulas or Markov chains, as such automata do not use $\Diamond$ transitions. We also assume that $\mathtt{tt}$ and $\mathtt{ff}$ do not appear in transitions of A and B. Clearly, this does not restrict the expressive power of automata.

For simplicity of presentation, p-automata $A = \langle \Sigma, Q, \delta, \varphi_a^{\text{in}}, F \rangle$ and $B = \langle \Sigma, U, \delta, \psi_b^{\text{in}}, F \rangle$ satisfy $Q \cap U = \{\}$. We also use δ for the transition function of both automata and F for both acceptance conditions. The strict versions of the partial orders on equivalence classes of G_A and G_B are well founded. We consider their pointwise extension $\leq_{A,B}$ as an ordering on the MSCCs of the resulting game. Namely, $(((\varphi)), ((\psi))) \leq_{A,B} (((\tilde{\varphi})), ((\tilde{\psi})))$ if $((\varphi)) \leq_A ((\tilde{\varphi}))$ and $((\psi)) \leq_B ((\tilde{\psi}))$ and $(((\varphi)), ((\psi))) <_{A,B} (((\tilde{\varphi})), ((\tilde{\psi})))$ if $(((\varphi)), ((\psi))) \leq_{A,B} (((\tilde{\varphi})), ((\tilde{\psi})))$ and either $((\varphi)) <_A ((\tilde{\varphi}))$ or $((\psi)) <_B ((\tilde{\psi}))$. Clearly, order $\leq_{A,B}$ is well founded as well.

As before, we start by setting $\mathsf{val}(\varphi, \psi) = \bot$ for all pairs $(\varphi, \psi) \in \delta(Q, \Sigma) \times \delta(U, \Sigma)$ and gradually set concrete values to all of them. Consider a pair of equivalence classes $(((\varphi)), ((\psi)))$, where φ is in A and ψ is in B. Suppose that all pairs larger than $(((\varphi)), ((\psi)))$ with respect to $\leq_{A,B}$ have already been handled: for every φ' and ψ' with $(((\varphi)), ((\psi))) <_{A,B} (((\varphi')), ((\psi')))$ value $\mathsf{val}(\varphi', \psi') \neq \bot$ is pre-seeded.

Case 1. Let $((\varphi))$ and $((\psi))$ be MSCCs where $((\varphi))$ has no transitions in E_b, and $((\psi))$ no transitions in E_u and no $\Diamond$ markings. We set $\mathsf{val}(\varphi, \psi) = 0$; bounded-

with-$*$ states cannot simulate unbounded states.

Case 2. Let $(\!(\varphi)\!)$ and $(\!(\psi)\!)$ be MSCCs such that both $(\!(\varphi)\!)$ and $(\!(\psi)\!)$ have no transitions in E_b. Then $G_{\leq}((\!(\varphi)\!),(\!(\psi)\!))$ is a stochastic weak game with

$$
\begin{array}{rcl}
V & = & \{(\tilde{\varphi},\tilde{\psi}) \mid \varphi \preceq_A \tilde{\varphi}, \psi \preceq_B \tilde{\psi}\} \\
V_p & = & \{\} \\
V_0 & = & \{c \in V \mid \exists \varphi_i, \psi_i \colon c = (\varphi_1 \wedge \varphi_2, \psi_1 \vee \psi_2)\} \\
 & \cup & \{c \in V \mid \exists q' \exists \psi_i \colon c = (q', \psi_1 \vee \psi_2)\} \\
 & \cup & \{c \in V \mid \exists \varphi_i \exists u' \colon c = (\varphi_1 \wedge \varphi_2, u')\} \\
V_1 & = & \{c \in V \mid \exists q', u' \colon c = (q', u')\} \\
 & \cup & \{c \in V \mid \exists \varphi_i, \psi \colon c = (\varphi_1 \vee \varphi_2, \psi)\} \\
 & \cup & \{c \in V \mid \exists \varphi, \psi_i \colon c = (\varphi, \psi_1 \wedge \psi_2)\} \\
E & = & \{((\varphi_1 \wedge \varphi_2, \psi_2 \vee \psi_2), (\varphi_i, \psi_j)) \in V \times V \mid 1 \leq i,j \leq 2\} \\
 & \cup & \{((q', \psi_1 \vee \psi_2), (q', \psi_i)) \in V \times V \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\varphi_1 \wedge \varphi_2, u'), (\varphi_i, u')) \in V \times V \mid 1 \leq i \leq 2\} \\
 & \cup & \{((q', u'), (\delta(q', \sigma), \delta(u', \sigma))) \in V \times V \mid \sigma \in \Sigma\} \\
 & \cup & \{((\varphi_1 \vee \varphi_2, \psi), (\varphi_i, \psi)) \in V \times V \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\varphi, \psi_1 \wedge \psi_2), (\varphi, \psi_i)) \in V \times V \mid 1 \leq i \leq 2\}
\end{array}
\tag{4}
$$

The game $G_{\leq}((\!(\varphi)\!),(\!(\psi)\!))$ does not have probabilistic configurations. However, pre-seeded values $\mathsf{val}(\tilde{\varphi},\tilde{\psi})$ for configurations $(\tilde{\varphi},\tilde{\psi})$ with $((\!(\varphi)\!),(\!(\psi)\!)) <_{A,B} ((\!(\tilde{\varphi})\!),(\!(\tilde{\psi})\!))$ may be in the range $(0,1)$. Thus, we treat $G_{\leq}((\!(\varphi)\!),(\!(\psi)\!))$ as a stochastic weak game.

Intuitively, Player 1 resolves disjunctions on the left and conjunctions on the right and does this before Player 0 needs to move. Player 0 resolves conjunctions on the left and disjunctions on the right when Player 1 cannot move. From configurations of the form (q',u'), where q' is a state of A and u' is a state of B, Player 1 chooses a letter $\sigma \in \Sigma$ and applies the transitions of q' and u' reading σ.

Finally, an infinite play in $G_{\leq}((\!(q)\!),(\!(u)\!))$ is winning for Player 0 if $(\!(\varphi)\!) \cap Q \subseteq F$ implies $(\!(\psi)\!) \cap U \subseteq F$. By Theorem 1 every configuration c has a value for Player 0. We set $\mathsf{val}(c)$ to that value.

Case 3. Let $(\!(\varphi)\!)$ and $(\!(\psi)\!)$ be MSCCs such that both have neither transitions in E_u nor $\Vdash$ markings. Below, let $\tilde{\varphi}$ have form $*([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$, $\tilde{\psi}$ have form $*([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_m]\!]_{\bowtie'_m p'_m})$. Clearly, for every q_i, u_j and $\sigma \in \Sigma$, we have $\delta(q_i,\sigma)$

and $\delta(u_j, \sigma)$ are finite. Then, so are

$$R_{\tilde{\varphi},\tilde{\psi}} = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\bigcup_{\sigma\in\Sigma}\{(\alpha,\beta) \mid \alpha \in \mathsf{cl}(\delta(q_i,\sigma)), \beta \in \mathsf{cl}(\delta(u_j,\sigma))\}$$
$$\mathsf{Val}_{\tilde{\varphi},\tilde{\psi}} = \{0,1\} \cup \{\mathsf{val}(\alpha,\beta) \mid (\alpha,\beta) \in R_{\tilde{\varphi},\tilde{\psi}}, \mathsf{val}(\alpha,\beta) \neq \bot\}$$

As before, $R_{\tilde{\varphi},\tilde{\psi}}$ is the set of configurations reachable from $(\tilde{\varphi},\tilde{\psi})$ using *one* transition in δ. Set $\mathsf{Val}_{\tilde{\varphi},\tilde{\psi}}$ includes 0, 1, and values of configurations in $R_{s,\varphi}$. We define the set $\mathcal{F}_{\tilde{\varphi},\tilde{\psi}} = [n] \times [m] \to \mathsf{Val}_{\tilde{\varphi},\tilde{\psi}}$.

Also, $f \in \mathcal{F}_{\tilde{\varphi},\tilde{\psi}}$ is *disjoint* if there is $\{a_{i,j} \in [0,1] \mid i \in [n] \text{ and } j \in [m]\}$ such that the following holds:

(i) for all $i \in [n]$ we have $\sum_{j\in[m]} a_{i,j} = 1$ and

(ii) for all $j \in [m]$ we have either $\sum_{i\in[n]} a_{i,j} \cdot p_i \cdot f(i,j) > p'_j$ or $\sum_{i\in[n]} a_{i,j} \cdot p_i \cdot f(i,j) = p'_j$ and either $\bowtie'_j$ is $\geq$ or there is i' with $a_{i',j} > 0$ such that $\bowtie_{i'} = >$.

Let $\mathcal{F}^*_{\tilde{\varphi},\tilde{\psi}}$ be the set of disjoint functions in $\mathcal{F}_{\tilde{\varphi},\tilde{\psi}}$. Weak game $G_{\leq}((\!(\varphi)\!), (\!(\psi)\!))$ is defined as

$$\begin{array}{rcl}
V & = & \{(\tilde{\varphi},\tilde{\psi},\sigma), (\tilde{\varphi},\tilde{\psi},\sigma,f) \mid \tilde{\varphi} \in (\!(\varphi)\!), \tilde{\psi} \in (\!(\psi)\!), \sigma \in \Sigma, f \in \mathcal{F}^*_{\tilde{\varphi},\tilde{\psi}}\} \\
 & \cup & \{(\tilde{\varphi},\tilde{\psi}), (\tilde{\varphi},\tilde{\psi},v) \mid \varphi \preceq_A \tilde{\varphi}, \psi \preceq_B \tilde{\psi}, v \in \mathsf{Val}_{\tilde{\varphi},\tilde{\psi}}\} \\
V_0 & = & \{(\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2, v), (\alpha_1 \wedge \alpha_2, \epsilon, v), (\gamma, \beta_1 \vee \beta_2, v), (\gamma, \epsilon, \sigma)\} \\
 & \cup & \{(\alpha,\beta,v) \mid \bot \neq \mathsf{val}(\alpha,\beta) < v\} \\
V_1 & = & \{(\gamma,\epsilon,v), (\gamma,\epsilon), (\gamma,\epsilon,f), (\alpha_1 \vee \alpha_2, \beta, v), (\alpha, \beta_1 \wedge \beta_2, v)\} \\
 & \cup & \{(\alpha,\beta,v) \mid \bot \neq \mathsf{val}(\alpha,\beta) \geq v\} \\
E & = & \{((\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2, v), (\alpha_i, \beta_j, v)) \mid 1 \leq i,j \leq 2\} \\
 & \cup & \{((\alpha_1 \wedge \alpha_2, \epsilon, v), (\alpha_i, \epsilon, v)) \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\gamma, \beta_1 \vee \beta_2, v), (\gamma, \beta_i, v)) \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\gamma,\epsilon,\sigma), (\gamma,\epsilon,\sigma,f))\} \\
 & \cup & \{((\gamma,\epsilon,v), (\gamma,\epsilon))\} \\
 & \cup & \{((\gamma,\epsilon), (\gamma,\epsilon,\sigma)) \mid \sigma \in \Sigma\} \\
 & \cup & \{((\gamma,\epsilon,\sigma,f), (\delta(q_i,\sigma), \delta(u_j,\sigma), f(i,j))) \mid f(i,j) > 0\} \\
 & \cup & \{((\alpha_1 \vee \alpha_2, \beta, v), (\alpha_i, \beta, v)) \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\alpha, \beta_1 \wedge \beta_2, v), (\alpha, \beta_i, v)) \mid 1 \leq i \leq 2\}
\end{array} \tag{5}$$

where α and β range over formulas in transitions of A and B, respectively, and γ and ϵ range over formulas in $(\!(\varphi)\!) \cap [\![Q]\!]^*$ and $(\!(\psi)\!) \cap [\![U]\!]^*$, respectively.

For $(\gamma,\epsilon) \in [\![Q]\!]^* \times [\![U]\!]^*$ where γ is of form $*([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and ϵ is of form $*([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_m]\!]_{\bowtie'_m p'_m})$, for Player 0 to demonstrate that ϵ simulates γ, she is required to show that the probability of ϵ (and its partition) can be supported

by γ. Accordingly, from (γ, ϵ) Player 1 chooses a letter $\sigma \in \Sigma$ and moves to configuration $(\gamma, \epsilon, \sigma)$. Then Player 0 chooses $f\colon [n] \times [m] \to [0,1]$ and moves to configuration $(\gamma, \epsilon, \sigma, f)$. Such a configuration relates to the claim that q_i is related to u_j with proportion $f(i,j)$ and that f can be partitioned (using the $\{a_{i,j}\}$ to support the different u_j's). Then, Player 1 chooses i and j such that $f(i,j) > 0$ and proceeds to $(\delta(q_i,\sigma), \delta(u_j,\sigma), f(i,j))$.[2] Conjunctions and disjunctions are resolved in the usual way until either reaching another configuration in $[\![Q]\!]^* \times [\![U]\!]^* \times [0,1]$, in which case the value $f(i,j)$ is ignored (as $f(i,j) \leq 1$), or until the play reaches a configuration with a pre-seeded value v. Then, if $f(i) \leq v$ Player 0 has fulfilled her obligation and she wins. If $f(i) > v$, Player 0 failed and she loses. An infinite play in $G_{\leq}(((\varphi)), ((\psi)))$ is winning for Player 0 if $((\varphi)) \cap Q \subseteq F$ implies $((\psi)) \cap U \subseteq F$. By Theorem 1, every $c \in V$ has a value in $\{0,1\}$ for Player 0. We set $\mathsf{val}(c)$ to that value.

Case 4. Let $((\varphi))$ and $((\psi))$ be MSCCs where $((\varphi))$ has no E_u transitions or $\Diamond\!\!\!\!\!\!\diamond$ markings, and $((\psi))$ has no E_b transitions. Stochastic weak game $G_{\leq}(((\varphi)), ((\psi)))$ is

$$
\begin{array}{rcl}
V & = & \{(\tilde{\varphi}, \tilde{\psi}) \mid \varphi \preceq_A \tilde{\varphi}, \psi \preceq_B \tilde{\psi}\} \\
 & \cup & ([\![Q]\!] \times U \times \Sigma) \cap (((\varphi)) \times ((\psi)) \times \Sigma) \\
V_0 & = & \{(\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2), (\alpha_1 \wedge \alpha_2, u), (\gamma, \beta_1 \vee \beta_2)\} \\
V_1 & = & \{(\alpha_1 \vee \alpha_2, \beta), (\alpha, \beta_1 \wedge \beta_2), (\gamma, u)\} \\
V_p & = & ([\![Q]\!]^* \times U \times \Sigma) \cap V \\
E & = & \{((\alpha_1 \wedge \alpha_2, \beta_1 \vee \beta_2), (\alpha_i, \beta_j)) \mid 1 \leq i,j \leq 2\} \\
 & \cup & \{((\alpha_1 \wedge \alpha_2, u), (\alpha_i, u)) \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\gamma, \beta_1 \vee \beta_2), (\gamma, \beta_j)) \mid 1 \leq j \leq 2\} \\
 & \cup & \{((\alpha_1 \vee \alpha_2, \beta), (\alpha_i, \beta)) \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\alpha, \beta_1 \wedge \beta_2), (\alpha, \beta_i)) \mid 1 \leq i \leq 2\} \\
 & \cup & \{((\gamma, u), (\gamma, u, \sigma))\} \\
 & \cup & \{((\gamma, u, \sigma), (\delta(q_i, \sigma), \delta(u, \sigma)))\}
\end{array}
\tag{6}
$$

where $\kappa((\gamma, u, \sigma), (\delta(q_i,\sigma), \delta(u,\sigma))) = p_i$, in the implicit set comprehensions α, α_i and β, β_i range over formulas in transitions of A and B (resp.), whereas γ and u range over $[\![Q]\!]^*$ and U (resp.). For probabilities p_i that don't sum up to 1, we add a sink state (losing for Player 0) filling that gap.

An infinite play in $G_{\leq}(((\varphi)), ((\psi)))$ is winning for Player 0 if $((\varphi)) \cap Q \subseteq F$

[2] We note that forcing Player 1 to choose the letter before Player 0 chooses the disjoint function gives Player 0 more power. This is still strong enough to imply language containment and relaxes the notion of simulation.

implies $(\!(\psi)\!) \cap U \subseteq F$. By Theorem 1 every configuration c has a value for Player 0. We set $\mathsf{val}(c)$ to that value.

Intuitively, a state u measures the probability of some regular set of paths, and a state $[\![q]\!]_{\bowtie p}$ can restrict the immediate steps taken by a Markov chain as well as enforce some regular structure on paths. Thus, this stochastic weak game establishes the conditions under which a Markov chain accepted from $[\![q]\!]_{\bowtie p}$ can be also accepted from u.

The case when $(\!(\varphi)\!)$ or $(\!(\psi)\!)$ is a trivial MSCC is subsumed by at least one of the four preceding cases. As for the acceptance game, this ambiguity is unproblematic as the game values in $G_{\leq}((\!(\varphi)\!), (\!(\psi)\!))$ are then determined by the propagation of pre-seeded game values.

Definition 3 *We say that B simulates A, denoted $A \leq B$, if the value of configuration $(\varphi_a^{\mathsf{in}}, \psi_b^{\mathsf{in}})$, computed in the previous sequence of games, is* 1.

We can now state and prove our main theorem on simulation of p-automata.

Theorem 6 *For A and B p-automata over $2^{\mathbb{A}^{\mathbb{P}}}$ with no occurrence of $\forall$: If A and B are finite, $A \leq B$ can be decided in EXPTIME and implies $\mathcal{L}(A) \subseteq \mathcal{L}(B)$. If A is A_M for an $M \in \mathsf{MC}_{\mathbb{A}^{\mathbb{P}}}$, then $A \leq B$ iff $\mathcal{L}(A) \subseteq \mathcal{L}(B)$ for all B over $2^{\mathbb{A}^{\mathbb{P}}}$.*

Proof: Let A and B be finite p-automata. To see that $A \leq B$ can be decided in EXPTIME, we make two observations: First, that the stochastic weak game arising from the combination of a bounded MSCC with an unbounded MSCC or from the combination of two unbounded MSCCs is linear in each of the automata and can be solved in EXPTIME. Second, that the weak game arising from the combination of two bounded MSCCs may be exponential due to the large number of possible value assignment functions. Such a weak game can be solved in linear time leading to an EXPTIME upper bound. As there are only linearly many such games, the sequence of weak games and stochastic weak games can be solved in EXPTIME.

To show the last claim of the theorem, we note that when A equals A_M for some $M \in \mathsf{MC}_{\mathbb{A}^{\mathbb{P}}}$, the simulation game for $A_M \leq B$ and the acceptance game for $M \in \mathcal{L}(B)$ collapse to the same game. Thus, regardless of whether A_M or B is infinite-state we have $A_M \leq B$ iff $M \in \mathcal{L}(B)$. And the latter is equivalent to $\mathcal{L}(A_M) \subseteq \mathcal{L}(B)$ by Lemma 1 and Theorem 4.

It remains to show that $A \leq B$ implies $\mathcal{L}(A) \subseteq \mathcal{L}(B)$ for finite-state A and B. To that end, consider a Markov chain $M = (S, P, L, s^{\mathsf{in}})$ and two formulas φ and ψ such that φ appears in the transition of A and ψ appears in the transition

of B. Let G_A and G_B be the acceptance games to test whether $M \in \mathcal{L}(A)$ and whether $M \in \mathcal{L}(B)$, respectively. We try to derive a result regarding G_B from the known results in G_A and $G_{\leq}$. Thus, in G_A and $G_{\leq}$ there are optimal strategies for Player 0. We construct from them a strategy for Player 0 in G_B. As we have strategies in G_A and $G_{\leq}$ that work against all possible strategies of Player 1, we can "drive" Player 1 in these games in order to explore these optimal strategies. It follows that we devise a strategy that answers actions of Player 1 in G_B by emulating Player 1 in G_A and $G_{\leq}$ and using the strategies for Player 0 in G_A and $G_{\leq}$. This gives as a strategy for Player 0 in G_B. In general, for a given play in G_B we construct matching plays in G_A and $G_{\leq}$ that start from (s, ψ), (s, φ), and (φ, ψ), respectively. We then show that the values obtained by the strategy we construct satisfy $\mathsf{val}(s, \varphi) \cdot \mathsf{val}(\varphi, \psi) \leq \mathsf{val}(s, \psi)$. Thus, we prove that $M \in \mathcal{L}(A)$ implies $M \in \mathcal{L}(B)$.

Suppose that the claim holds by induction for configurations $((\!(\tilde{\varphi})\!), (\!(\tilde{\psi})\!))$, where $((\!(\varphi)\!), (\!(\psi)\!)) <_{A,B} ((\!(\tilde{\varphi})\!), (\!(\tilde{\psi})\!))$.

1. If $\varphi \in Q$ and $\psi \in [\![U]\!]^*$, then $\mathsf{val}(\varphi, \psi) = 0$ and the claim holds trivially.

2. Suppose that both φ and ψ are in unbounded MSCCs. Then $G_{M,(\!(\varphi)\!)}$, $G_{M,(\!(\psi)\!)}$, and $G_{(\!(\varphi)\!),(\!(\psi)\!)}$ are all stochastic weak games. To simplify notations we refer to $G_{M,(\!(\varphi)\!)}$ as G_A, $G_{M,(\!(\psi)\!)}$ as G_B, and $G_{(\!(\varphi)\!),(\!(\psi)\!)}$ as $G_{\leq}$. We know that the optimal strategy of Player 0 in G_A secures at least $\mathsf{val}(s, \varphi)$. We have to show a strategy in G_B that secures at least $\mathsf{val}(s, \varphi) \cdot \mathsf{val}(\varphi, \psi)$. We note, however, that there are no stochastic probabilistic condigurations in $G_{\leq}$. Thus, the combination of the optimal strategy for Player 0 in $G_{\leq}$ with some strategy for Player 1 produces a unique play. It follows that the value of this play must be at least $\mathsf{val}(\varphi, \psi)$. Hence, either this play is infinite and winning (and $\mathsf{val}(\varphi, \psi) = 1$) or this play is finite, leading outside of $G_{(\!(\varphi)\!),(\!(\psi)\!)}$ and ending in a configuration whose value is at least $\mathsf{val}(\varphi, \psi)$.

Consider the configurations (s, φ), (s, ψ), and (φ, ψ) in the games G_A, G_B, and $G_{\leq}$, respectively.

If φ is a disjunction, (s, φ) is a Player 0 configuration in G_A and (φ, ψ) is a Player 1 configuration in $G_{\leq}$. Then, Player 0's strategy in G_A instructs her to choose a disjunct φ_i of φ. As (φ, ψ) is a Player 1 configuration in $G_{\leq}$ we can emulate Player 1 by choosing the successor configuration (φ_i, ψ). If ψ is a conjunction, then (s, ψ) is a Player 1 configuration in G_B and (φ, ψ) is a Player 1 configuration in $G_{\leq}$. Then, Player 1 chooses a successor (s, ψ_i) of (s, ψ) in G_B. We emulate Player 1 in $G_{\leq}$ by choosing the successor (φ, ψ_i) in $G_{\leq}$. If φ is a

conjunction and ψ is not a conjunction, then (s, φ) is a Player 1 configuration in G_A and (φ, ψ) a Player 0 configuration in $G_{\leq}$. The strategy of Player 0 in $G_{\leq}$ instructs Player 0 to choose a conjunct φ_i of φ. We emulate Player 1 in G_A by choosing the same conjunct leading to configuration (s, φ_i) in G_A. If φ is not a disjunction and ψ is a disjunction, then (φ, ψ) is a Player 0 configuration in $G_{\leq}$ and (s, ψ) is a Player 0 configuration in G_B. The strategy of Player 0 in $G_{\leq}$ instructs her to choose a disjunct ψ_i of ψ. We use the same choice as the strategy for Player 0 in G_B leading to configuration (s, ψ_i). Finally, if φ is a state of A and ψ is a state of B then (s, φ) and (s, ψ) are stochastic configurations in G_A and G_B and (φ, ψ) is a Player 1 configuration in $G_{\leq}$. We advance $G_{\leq}$ by emulating the choice of Player 1 for the letter $L(s)$ advancing to configuration $(\delta(\varphi, L(s)), \delta(\psi, L(s)))$. In G_A and G_B the next configurations are of the form $(s', \delta(\varphi, L(s)))$ and $(s', \delta(\psi, L(s)))$ for $s' \in \mathsf{succ}(s)$.

It follows that the games G_A and G_B produce Markov chains that have the same probability distributions. Furthermore, there is a 1-1 and onto mapping between paths in the Markov chain induced by G_A to paths in $G_{\leq}$ and paths in the Markov chain induced by G_B. Consider three matching paths in the three respective games. If all three are infinite, then if the path is winning for Player 0 in G_A then by the winning condition of $G_{\leq}$ the path in G_B is winning for Player 0 as well. If one of the paths is finite, then the end configurations of the paths get out from $G_{\leq}$ and reach the triplet (s'', φ'), (φ', ψ') and (s'', ψ'). We deduce that $\mathsf{val}(\varphi', \psi')$ in $G_{\leq}$ is at least $\mathsf{val}(\varphi, \psi)$. Furthermore, by our assumptions regarding all configurations in $G_{\leq}$ such that $(\varphi', \psi') \notin (\!(\varphi)\!) \times (\!(\psi)\!)$ it follows that $\mathsf{val}(s'', \varphi') \cdot \mathsf{val}(\varphi', \psi') \leq \mathsf{val}(s'', \psi')$. In particular, as $\mathsf{val}(\varphi', \psi') \geq \mathsf{val}(\varphi, \psi)$ it follows that $\mathsf{val}(s'', \varphi') \cdot \mathsf{val}(\varphi, \psi) \leq \mathsf{val}(s'', \psi')$. Therefore, the inequality in the claim holds for every matching paths in the two Markov chains and thus it must hold for the value of the Markov chains. Therefore, $\mathsf{val}(s, \varphi) \cdot \mathsf{val}(\varphi, \psi) \leq \mathsf{val}(s, \psi)$ as required.

3. Suppose that both φ and ψ are in bounded MSCCs. Then $G_{M,(\!(\varphi)\!)}$, $G_{M,(\!(\psi)\!)}$, and $G_{(\!(\varphi)\!),(\!(\psi)\!)}$ are all weak games. We again use the notations G_A, G_B, and $G_{\leq}$. Consider the three configurations (s, φ), (φ, ψ) and (s, ψ). By definition $\mathsf{val}(s, \varphi) \in \{0, 1\}$ and similarly $\mathsf{val}(\varphi, \psi)$ and $\mathsf{val}(s, \psi)$. If either $\mathsf{val}(s, \varphi)$ or $\mathsf{val}(\varphi, \psi)$ is 0, then the claim holds trivially. Consider the case that $\mathsf{val}(s, \varphi) = 1$ and $\mathsf{val}(\varphi, \psi) = 1$. That is, both in G_A and $G_{\leq}$ there is a winning strategy for Player 0 such that regardless of how Player 1 resolves her choice Player 0 wins. We now give a strategy for Player 0 in G_B that establishes $\mathsf{val}(s, \psi) = 1$.

Let $\varphi = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and $\psi = *([\![u_1]\!]_{\bowtie'_1 p'_1}, \ldots, [\![u_m]\!]_{\bowtie'_m p'_m})$. It

follows that configuration (s, φ) is a Player 0 configuration in G_A, configuration (φ, ψ) is a Player 1 configuration in $G_{\leq}$ and configuration (s, ψ) is a Player 0 configuration in G_B. First, we make Player 1 in $G_{\leq}$ choose the successor $(\varphi, \psi, L(s))$. Then, Player 0's strategy in G_A instructs her to choose successor configuration (s, φ, f), where $f\colon [n] \times \mathsf{succ}(s) \to [0,1]$. Player 0's strategy in $G_{\leq}$ instructs her to choose successor configuration $(\varphi, \psi, L(s), f')$, where $f'\colon [n] \times [m] \to [0,1]$. Configuration (s, ψ) is a Player 0 configuration in G_B and we set her strategy to choose the successor (s, ψ, f''), where $f''\colon [m] \times \mathsf{succ}(s) \to [0,1]$ such that for every $j \in [m]$ and every $s' \in \mathsf{succ}(s)$ we have $f''(j, s')$ is the minimal value in $\mathsf{Val}_{s,\psi}$ that is at least $\max_{i \in [n]} f(i, s') \cdot f'(i, j)$.

We have to show that f'' is disjoint. To that end, let $a_{j,s'} = \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j}$. First, one can see that for every $s' \in \mathsf{succ}(s)$ we have

$$\sum_{j \in [m]} a_{j,s'} = \sum_{j \in [m]} \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} = \sum_{i \in [n]} a_{i,s'} \sum_{j \in [m]} a_{i,j} = \sum_{i \in [n]} a_{i,s'} = 1$$

Second, consider some $j \in [m]$. Then,

$$\begin{aligned}
&\sum_{s' \in \mathsf{succ}(s)} a_{j,s'} \cdot f''(j, s') \cdot P(s, s') \\
= &\sum_{s' \in \mathsf{succ}(s)} \left(\sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \right) \cdot f''(j, s') \cdot P(s, s') \\
\geq &\sum_{s' \in \mathsf{succ}(s)} \left(\sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \right) \cdot \max_{i \in [n]} (f(i, s') \cdot f'(i, j)) \cdot P(s, s') \\
\geq &\sum_{s' \in \mathsf{succ}(s)} \sum_{i \in [n]} a_{i,s'} \cdot a_{i,j} \cdot f(i, s') \cdot f'(i, j) \cdot P(s, s') \\
= &\sum_{i \in [n]} \sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot a_{i,j} \cdot f(i, s') \cdot f'(i, j) \cdot P(s, s') \\
= &\sum_{i \in [n]} a_{i,j} \cdot f'(i, j) \cdot \sum_{s' \in \mathsf{succ}(s)} a_{i,s'} \cdot f(i, s') \cdot P(s, s') \\
\bowtie &\sum_{i \in [n]} a_{i,j} \cdot f'(i, j) \cdot p_i \bowtie' p_j
\end{aligned}$$

and $\bowtie$ is $>$ if for some $i \in [n]$ we have $\bowtie_i$ equals $>$ and then $\bowtie'$ is $\geq$, otherwise either $\bowtie'$ is $>$ or $\bowtie'_j$ is $\geq$ and the proof is complete.

With f'' established as being disjoint, we get back to the games. In G_B, Player 1 chooses j and $s' \in \mathsf{succ}(s)$ and moves to state $(s', \delta(u_j, L(s)), f''(j, s'))$.

We emulate Player 1 in G_A and make her choose the state q_i such that $f(i, s') \cdot f'(i, j)$ is maximal and move to $(s', \delta(q_i, L(s)), f(i, s'))$. We emulate Player 1 in $G_\leq$ as well and make her choose the states q_i and u_j leading to configuration $(\delta(q_i, L(s)), \delta(u_j, L(s)), f'(i, j))$.

Consider three matching plays produced by following this strategy. Suppose that the plays stay inside the same MSCC in $G_\leq$ indefinitely. Then, as Player 0 is winning in G_A it follows that the play in G_A is winning. As Player 0 is winning in $G_\leq$ it follows that the play in $G_\leq$ is winning as well. However, this implies that the sequence of states of B on the right-hand-side of the configurations in $G_\leq$ is winning. Thus, the infinite play in G_B is winning as well.

Suppose that the plays exit the MSCC in $G_\leq$ and reach the triplet of configurations (s'', φ'', v_1), (s'', ψ'', v_2), and (φ'', ψ'', v). By induction, $\mathsf{val}(s'', \varphi'') \cdot \mathsf{val}(\varphi'', \psi'') \leq \mathsf{val}(s'', \psi'')$ holds. Furthermore, we have to show that $\mathsf{val}(s'', \psi'') \geq v_2$. Let (s', φ'), (s', ψ') and (φ', ψ') be the last configurations that are part of the MSCC before reaching the above triplet of configurations. It follows that $\mathsf{val}(s'', \psi'') \in \mathsf{Val}_{s', \psi'}$. By the choices of f, f' and f'' we know that v is the minimal value in $\mathsf{Val}_{s', \psi'}$ that is at least $\max_{i \in [n]} f(i, s'') \cdot f'(i, j)$. In addition, the last choice in G_A was exactly the state q_i such that i is maximal. As G_A and $G_\leq$ are won, we know that $\mathsf{val}(s'', \varphi'') \geq v_1$ and that $\mathsf{val}(\varphi'', \psi'') \geq v$. It follows that $\mathsf{val}(s'', \varphi'') \cdot \mathsf{val}(\varphi'', \psi'') \geq v_1 \cdot v$. But, v_2 is the minimal possible value in $\mathsf{Val}_{s, \psi}$ that is at least $v_1 \cdot v$. Thus, $\mathsf{val}(s'', \psi'') \geq v_2$.

4. Suppose that φ is in a bounded MSCC and ψ is in an unbounded MSCC. Then, $G_{M, (\!(\varphi)\!)}$ is a weak game and $G_{(\!(\varphi)\!), (\!(\psi)\!)}$ and $G_{M, (\!(\psi)\!)}$ are stochastic weak games. As before, we use the notations G_A, G_B, and $G_\leq$. As G_A is a weak game, the case that $\mathsf{val}(s, \varphi) = 0$ is not interesting. Thus, we assume that $\mathsf{val}(s, \varphi) = 1$. It follows that in G_A Player 0 has a winning strategy such that all possible plays in G_A are winning for Player 0.

Given a strategy of Player 1 in G_B, we show how to use the winning strategies of Player 0 in G_A and $G_\leq$ to produce a winning strategy for Player 0 in G_B. Consider a triplet of configurations (s, φ), (φ, u), (s, u), where $\varphi \in [\![Q]\!]^*$ and $u \in U$. The configurations (φ, u) and (s, u) are probabilistic configurations in their stochastic games. The successors of configurations of the form (φ, u) in $G_\leq$ are of the form $(\delta([\![q_i]\!]_{\bowtie_i p_i}, L(s)), \delta(u, L(s)))$. The successors of configurations of the form (s, u) in G_B are of the form $(s', \delta(u, L(s)))$, where $s' \in \mathsf{succ}(s)$. In order to continue using the association between the three games (and the strategies in G_A and $G_\leq$) to give a strategy for Player 0 in G_B we have to associate succeessors of (s, u) to successors of (φ, u). However, it is not clear which mapping is most

beneficial. Hence, we leave this mapping option open for a while. Instead of elaborating G_B and $G_\leq$ to Markov chains (by fixing strategies for Player 0 and Player 1) we elaborate them to Markov decision processes. Thus, based on the different mapping options, Player 0 is going to have multiple ways to proceed in G_B and Player 1 is going to have multiple ways to proceed in $G_\leq$. These MDPs capture all possible evolutions of plays in G_B and $G_\leq$ according to possible mapping choices between configurations in $G_\leq$ and G_B. We then use these MDPs to prove that the claim holds.

Consider three configurations (s, φ') in G_A, (φ', ψ') in $G_\leq$, and (s, ψ') in G_B. If ψ' is a conjunction, then (s, ψ') is a Player 1 configuration in G_B. It follows that Player 1 chooses a conjunct ψ_i of ψ' and proceeds to configuration (s, ψ_i). The configuration (φ', ψ') is a Player 1 configuration in $G_\leq$. We emulate Player 1 in $G_\leq$ by making her choose (φ', ψ_i). If ψ' is a disjunction, then (s, ψ') is a Player 0 configuration in G_B. There are now a few cases:

- If φ' is a conjunction, then (φ', ψ') is a Player 0 configuration in $G_\leq$. Then, Player 0's strategy in $G_\leq$ instructs her to choose a conjunct φ_i of φ' and proceed to configuration (φ_i, ψ'). We note that configuration (s, φ') is a Player 1 configuration in G_A and we emulate Player 1 by making her choose (s, φ_i).
- If φ' is a disjunction, then (s, φ') is a Player 0 configuration in G_A. Player 0's winning strategy in G_A instructs her to choose a disjunct φ_i and proceed to (s, φ_i). Configuration (φ', ψ') is a Player 1 configuration in $G_\leq$. We emulate Player 1 in $G_\leq$ and make her choose (φ_i, ψ').
- If φ' is in $[\![Q]\!]^*$ then (φ', ψ') is a Player 0 configuration in $G_\leq$. Then, the strategy of Player 0 in $G_\leq$ instructs her to choose a disjunct ψ_i of ψ' and proceed to configuration (φ', ψ_i). We set Player 0's strategy in G_B to make the same choice and proceed to configuration (s, ψ_i).

The only remaining case is where $(\varphi', \psi') \in [\![Q]\!]^* \times U \cap V$. We denote $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and $\psi' = u$. The configuration (s, φ') in G_A is a Player 0 configuration, the configuration (s, u) in G_B is probabilistic, and the configuration (φ', u) in $G_\leq$ is a Player 1 configuration. We emulate Player 1 in $G_\leq$ by making her choose $L(s)$ proceeding to the probabilistic configuration $(\varphi', u, L(s))$. The strategy of Player 0 in G_A instructs her to choose a disjoint function $f : [n] \times \mathsf{succ}(s) \to [0, 1]$ and proceed to (s, φ', f). Let $\{a_{i,s'}\}$ be the witnesses to the disjointness of f.

Consider a location s' that is chosen with probability $P(s, s')$ in G_B. Now, for every possible index i such that $a_{i,s'} > 0$ the successor $(s', \delta(q_i, L(s)), f(i, s'))$ is a possible successor of (s', φ', f) in G_A. Also, $(\delta(q_i, L(s)), \delta(u, L(s)))$ is a suc-

cessor of $(\varphi', u, L(s))$ in $G_\leq$ and the probability to get to it is p_i. Here, we make multiple possible choices of continuing in the games, giving rise to MDPs (with a matching between the choices in them). Consider all indices i such that $a_{i,s'} > 0$. It follows that for every such index there is a way to continue unraveling the plays by making Player 1 in G_A choose the successor $(s', \delta(q_i, L(s)), f(i, s'))$ and continuing to configurations $(\delta(q_i, L(s)), \delta(u, L(s)))$ in $G_\leq$ and $(s', \delta(u, L(s)))$ in G_B. Notice that the choice in G_B is implicitly based on the choice of f in G_A. As the future elaboration of the strategy in G_B depends on the association between configurations in the three games the choice of an index i such that $a_{i,s'} > 0$ is effectively also a choice in G_B that determines the way the strategy is extended.

By using these strategies and these associations between the games, this effectively creates from G_B and $G_\leq$ MDPs where the choices are angelic in G_B and demonic in $G_\leq$. That is, the actual value of G_B is the best possible value in the MDP arising from G_B and the value in $G_\leq$ is the worst possible value in $G_\leq$. Hence, it is enough to show one choice such that the value in the MDP arising from G_B satisfies the requirement of the claim. Indeed, the actual value in G_B could only be higher while the actual value in $G_\leq$ could only be lower.

Consider now three configurations (s, φ'), (φ', ψ'), and (s, ψ'), and the resulting MDPs from (φ', ψ') and (s, ψ'). By the construction of the strategy, every play starting in (s, ψ') is associated with plays that start in (s, φ') and (φ', ψ') such that at every stage the three configurations use the same state of the Markov chain and formulas in the transitions of A and B. We consider four cases:

4.1. A triplet of configurations (s, φ'), (φ', ψ'), (s, ψ') where (φ', ψ') is not in the equivalence class of $((\!(\varphi)\!), (\!(\psi)\!))$. By induction $\mathsf{val}(s, \psi') \geq \mathsf{val}(s, \varphi') \cdot \mathsf{val}(\varphi', \psi')$.

4.2. A triplet of configurations (s, φ'), (φ', ψ'), (s, ψ') with some choice in the MDP that arises from G_B where all plays starting in (φ', ψ') remain in $((\!(\varphi)\!), (\!(\psi)\!))$ and are winning for Player 0 in $G_\leq$. The matching choice of plays starting from (s, ψ') are winning for Player 0 in G_B. Indeed, if this were not the case, there would be a play in G_B that is losing. It follows that the corresponding play in $G_\leq$ does not satisfy the acceptance of A and that the play in G_A is losing. However G_A is a weak game and so this is impossible.

4.3. A triplet of configurations (s, φ'), (φ', ψ'), (s, ψ') where, for all choices in the MDP that arises from $G_\leq$, plays starting in (φ', ψ') remain in $((\!(\varphi)\!), (\!(\psi)\!))$ and are losing for Player 0 in $G_\leq$. One can see that then $\mathsf{val}(s, \psi') \geq 0$.

4.4. A triplet (s, φ'), (φ', ψ'), (s, ψ') with $(\varphi', \psi') \in ((\!(\varphi)\!), (\!(\psi)\!))$ where (i) for no choice in the MDP arising from $G_{\leq}$ are all paths winning for Player 0 and (ii) for all such choices the probability for Player 0 to win is positive. As automaton and Markov chain are finite, so are the resulting MDPs. It follows that the probability of winning in $G_{\leq}$ equals the probability of getting to one of the previous three types of configurations. That is, we have the MDP resulting from G_B and we are searching for a strategy for Player 0 with reachability objective to reach one of the previous end components. We show that the probability to reach one of the three previous types of configurations in n steps satisfies the requirements of the Theorem, for every n. The requirement of the claim will follow. For that, we compute the probability P_n to reach from a configuration (φ, ψ) in $G_{\leq}$ and from a configuration (s, ψ) in G_B one of these three types of configurations in n steps. We compute P_n by induction starting from P_0.

For each triplet (s', φ'), (φ', ψ'), (s', ψ') we let $P_0(\varphi', \psi')$ be $\mathsf{val}(\varphi', \psi')$ and $P_0(s', \psi')$ be $\mathsf{val}(s', \psi')$ if (φ', ψ') is one of the three types of configurations mentioned above. Let $P_0(\varphi', \psi')$ and $P_0(s', \psi')$ be 0, otherwise.

Let $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1}, \ldots, [\![q_n]\!]_{\bowtie_n p_n})$ and consider triplet (s, φ'), (φ', ψ'), (s, ψ') where $\psi' = u$, $P_0(\varphi', \psi') = P_0(s, \psi') = 0$ but there are $s' \in \mathsf{succ}(s)$ and i such that $P_0(\delta(q_i, L(s')), \delta(u, L(s))) > 0$ and $P_0(s', \delta(u, L(s'))) > 0$. Then, P_1 satisfies the requirement reasoned as follows. First,

$$P_1(s, u) = \sum_{s' \in \mathsf{succ}(s) | \exists i \in [n] \colon a_{i,s'} > 0} P(s, s') P_0(s', \delta(u, L(s)))$$

For every s' let $i_{s'}$ be such that $P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s)))$ is maximal among all $i \in [n]$ such that $a_{i,s'} > 0$. We know that $P_0(s', \delta(u, L(s))) \geq P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s)))$. Indeed, if $P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) > 0$ this holds by our proof for P_0. If $P_0(\delta(q_{i_{s;}}, L(s)), \delta(u, L(s))) = 0$ then this holds trivially. We obtain

$$P_1(s, u) \geq$$

$$\sum_{s' \in \mathsf{succ}(s) | \exists i \in [n] \colon a_{i,s'} > 0} P(s, s') \cdot P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s)))$$

By f being disjoint, we have $\sum_{i\in[n]} a_{i,s} = 1$. Therefore:

$$P_1(s,u) \geq$$

$$\sum_{s'\in \mathsf{succ}(s) | \exists i \in [n] \colon a_{i,s'}>0} P(s,s') \cdot (\sum_{i\in[n]} a_{i,s'}) \cdot P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s)))$$

But $i_{s'}$ maximizes $P_0(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s)))$, so that

$$P_1(s,u) \geq$$

$$\sum_{s'\in \mathsf{succ}(s) | \exists i \in I \colon a_{i,s'}>0} P(s,s') \cdot \sum_{i\in[n]} a_{i,s'} \cdot P_0(\delta(q_i, L(s)), \delta(u, L(s))) \cdot \mathsf{val}(s', \delta(q_i, L(s)))$$

From the inequalities $\mathsf{val}(s', \delta(q_i, L(s))) \geq f(i,s')$, which hold from the choice of f and win in G_A this yields

$$P_1(s,u) \geq$$

$$\sum_{s'\in \mathsf{succ}(s) | \exists i \in [n] \colon a_{i,s'}>0} P(s,s') \cdot \sum_{i\in[n]} a_{i,s'} \cdot f(i,s') \cdot P_0(\delta(q_i, L(s)), \delta(u, L(s)))$$

Moving $P(s,s')$ into the second sum and changing the order of summation, we obtain

$$P_1(s,u) \geq$$

$$\sum_{i\in[n]} P_0(\delta(q_i, L(s)), \delta(u, L(s))) \cdot \sum_{s'\in \mathsf{succ}(s) | \exists i \in [n] \colon a_{i,s'}>0} f(i,s') \cdot a_{i,s'} \cdot P(s,s')$$

Crucially, since f is disjoint, we get the inequalities $\sum_{s'\in \mathsf{succ}(s) | \exists i \in [n] \colon a_{i,s'}>0} f(i,s') \cdot a_{i,s'} \cdot P(s,s') \geq p_i$. Notice that the successors s' that are not included in this sum have $a_{i,s'} = 0$ for all i. Hence, adding other successors s' to the sum does not change anything and indeed the successors we considered in the equation are sufficient. We thus obtain

$$P_1(s,u) \geq \sum_{i\in I} P_0(\delta(q_i, L(s)), \delta(u, L(s))) \cdot p_i$$

where the righthand side is $P_1(\varphi', u)$ as desired.

Assume now that the claim holds for all configurations and for P_n. That is, the probability to reach end components, where values are defined as before, satisfies the requirement of the claim for reachability in n steps. We now show that it holds for reachability in $n+1$ steps, i.e., for P_{n+1}. For a triplet (s,φ'), (φ',ψ'), (s,ψ'), the strategy defined makes most such configurations deterministic in their respective MDPs. The only interesting case is when $\varphi' \in [\![Q]\!]$ and $\psi' \in U$. In this case (φ',ψ') and (s,ψ') are probabilistic configurations and the strategy above includes some choice in the matching between successors of (φ',ψ') and (s,ψ'). Let $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1},\ldots,[\![q_n]\!]_{\bowtie_n p_n})$ and $\psi' = u$. Then,

$$P_{n+1}(s,u) = \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot P_n(s', \delta(u, L(s)))$$

Recall that the way to extend the game from configuration (φ',u) – matching a move to $\delta(q_i, L(s))$ with the move to $(s', \delta(u, L(s)))$ – depends on which $a_{i,s'}$ are positive in a disjoint function f. We thus have:

$$P_{n+1}(\varphi',u) = \sum_{i\in[n]} \max_{i:a_{i,s'}>0} \mathsf{val}(s', \delta(q_i, L(s))) \cdot P_n(\delta(q_i, L(s)), \delta(u, L(s)))$$

By induction, for possible matching triplet (s,φ'), (φ',ψ'), and (s,ψ') we have:

$$P_n(s,\psi') \geq \mathsf{val}(s,\varphi') \cdot P_n(\varphi',\psi')$$

We then prove the same inequality for P_{n+1}. We concentrate on the only interesting case, where $\varphi' = *([\![q_1]\!]_{\bowtie_1 p_1},\ldots,[\![q_n]\!]_{\bowtie_n p_n})$ and $\psi' = u$. The desired inequality for $P_{n+1}(s,u)$ is derived as follows. First,

$$P_{n+1}(s,u) = \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot P_n(s', \delta(u, L(s)))$$

For every s' let $i_{s'}$ be such that $\mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$ is maximal among all $i \in [n]$. By induction

$$P_n(s', \delta(u, L(s))) \geq \mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

and we infer

$$P_{n+1}(s,u) \geq \sum_{s' \in \mathsf{succ}(s)} P(s,s') \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

By f being disjoint, we have $\sum_{i\in[n]} a_{i,s'} = 1$, so

$$P_{n+1}(s,u) \geq$$
$$\sum_{s'\in \mathsf{succ}(s)} P(s,s') \cdot (\sum_{i\in[n]} a_{i,s'}) \cdot \mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$$

But $i_{s'}$ maximizes $\mathsf{val}(s', \delta(q_{i_{s'}}, L(s))) \cdot P_n(\delta(q_{i_{s'}}, L(s)), \delta(u, L(s)))$, so that

$$P_{n+1}(s,u) \geq$$
$$\sum_{s'\in \mathsf{succ}(s)} P(s,s') \cdot \sum_{i\in[n]} a_{i,s'} \cdot \mathsf{val}(s', \delta(q_i, L(s))) \cdot P_n(\delta(q_i, L(s)), \delta(u, L(s)))$$

From the inequality $\mathsf{val}(s', \delta(q_i, L(s))) \geq f(i,s')$, which holds from the choice of f and win in G_A, this yields

$$P_{n+1}(s,u) \geq \sum_{s'\in \mathsf{succ}(s)} P(s,s') \cdot \sum_{i\in[n]} a_{i,s'} \cdot f(i,s') \cdot P_n(\delta(q_i, L(s)), \delta(u, L(s)))$$

After a change of summation order, we get

$$P_{n+1}(s,u) \geq \sum_{i\in[n]} P_n(\delta(q_i, L(s)), \delta(u, L(s))) \cdot \sum_{s'\in \mathsf{succ}(s)} a_{i,s'} \cdot f(i,s') \cdot P(s,s')$$

Again, using that f is disjoint, we obtain the inequalities $\sum_{s'\in \mathsf{succ}(s)} a_{i,s'} \cdot f(i,s') \cdot P(s,s') \geq p_i$ from which we infer

$$P_{n+1}(s,u) \geq \sum_{i\in[n]} P_n(\delta(q_i, L(s)), \delta(u, L(s))) \cdot p_i$$

where the righthand side is the desired $P_{n+1}(\varphi', u)$. □

We now get sound and complete verification of model checks through simulations, in the sense of Dams and Namjoshi [19].

Corollary 2 *For all infinite Markov chains M in $\mathsf{MC}_{\mathbb{AP}}$ and PCTL formulas ϕ over $\mathbb{AP}$: $M \models \phi$ iff there is a finite p-automaton A with $A_M \leq A$ and $A \leq A_\phi$.*

To see this, every such A implies $\mathcal{L}(A_M) \subseteq \mathcal{L}(A)$ and $\mathcal{L}(A) \subseteq \mathcal{L}(A_\phi)$ by both parts of Theorem 6. Thus, $M \models \phi$ holds by Theorems 4 and 5. By construction, for every Markov chain M the automaton A_M contains only $*$-transitions. Similarly, for every PCTL formula ϕ we have A_ϕ does not contain occurrences of $\forall$. It follows that simulation (without $\forall$) is sufficient.

Conversely, if there is no such A, then A_ϕ can also not be such an A. As $A_\phi \leq A_\phi$ this implies $A_M \not\leq A_\phi$ and so $\mathcal{L}(A_M) \not\subseteq \mathcal{L}(A_\phi)$. So there is some $M' \sim M$ with $M' \not\models \phi$. Since $M' \sim M$, we get $M \not\models \phi$ as well by Lemma 1.

This method for deciding $M \models \phi$ via simulations is thus complete in the sense of [19] – to our knowledge the first such result for PCTL and Markov chains.

7. Related and Future Work

Automata for coalgebras [32], for the functor whose coalgebras are Markov chains, have a corresponding logic that enjoys the finite model property. Since PCTL does not have that property, these automata cannot express PCTL – notably its path modalities. Probabilistic processes [7] use automata-theoretic techniques for refinement checking only. Probabilistic automata [33] give only rise to probabilistic languages of non-probabilistic models, for example the language of all Kripke structures that are accepted with probability at least $\frac{1}{2}$. And probabilistic verification of specifications written in linear-time temporal logic (LTL) [4] uses automata-theoretic machinery but cannot reason about combinations of LTL operators and probability thresholds as found in PCTL. The stochastic games of [34] abstract Markov decision processes as a 2-person game where two sources of nondeterminism, stemming from the MDP and the state space partition respectively, are controlled by different players. This separation allows for more precision of abstractions but is not complete in the sense of [19], as shown in [35]. In [24], a Hintikka game was defined for satisfaction, i.e. whether a Markov chain satisfies a PCTL formula. That game resembles our acceptance game for p-automata that are embeddings of PCTL formulas.

Concurrently with our own work, Caillaud et al. introduced *constraint Markov chains* as a specification framework for Markov chains [36]. They generalize interval Markov chains, where every transition is labeled by a possible interval of probabilities [37]. A concrete Markov chain implements an interval Markov chain if a bisimulation game between the two can be carried out in a way that probabilities fall in intervals. Constraint Markov chains have formulas constraining the probabilities of successors, and the implementation relation is defined through a game that is similar to those showing implementations of interval Markov chains.

They show that the resulting theory supports the notions of specification, implementation, refinement, conjunction, and parallel composition. The constraints themselves are akin to our $*$-operator, also relate to disjoint probabilities, but do not have the ability to use parts of the probability of the same successor for different purposes. Furthermore, constraint Markov chains cannot reason about paths in the Markov chain and, as a consequence, have no relation to model checking. It would be interesting to explore a combination of $*$-operators with constraints, leading, perhaps, to a normal form of possible transitions.

7.1. Future work

p-automata suggest a new approach to understanding the open problem of decidability of PCTL satisfiability. Algorithms for checking emptiness of alternating tree automata and solving satisfiability of monadic second-order logic, μ-calculus, CTL*, and dynamic logic convert automata into non-deterministic versions, for which non-emptiness is then decidable with standard techniques. We mean to investigate whether a notion of non-deterministic p-automata exists such that (i) all p-automata can be converted into non-deterministic versions, and (ii) all non-deterministic p-automata have decidable non-emptiness checks. We aim to develop a generalization of stochastic games such that acceptance of input for *non-uniform* p-automata can be decided by solving a *single* such game, as opposed to a sequence of such games. We also want to research the effectiveness of p-automata in supporting counter-example guided abstraction refinement.

8. Conclusions

We presented a novel kind of automata, p-automata, that read in an entire Markov chain and either accept or reject that input. We showed how this acceptance can be decided by a series of stochastic weak games and weak games, at worst case exponential in the size of the automaton and of the Markov chain.

We proved p-automata to be closed under Boolean operations, that language containment and emptiness are equi-solvable, and that a p-automaton's language is closed under bisimulation. Bisimulation equivalence classes of every Markov chain as well as the set of models of every PCTL formula were shown to be expressible as such languages. In particular, the complexity of the acceptance game matches that of probabilistic model checking for such formulas. Therefore the emptiness, universality, and containment of p-automata seem all to be tightly related to the open problem of decidability of PCTL satisfiability.

We developed a (fair) simulation between p-automata that stem from Markov chains or PCTL formulas. We proved simulation to be decidable in EXPTIME and to under-approximate language containment. In particular, p-automata are a complete abstraction framework for PCTL: if an infinite Markov chain satisfies a PCTL formula, there is a finite p-automaton that abstracts this Markov chain and whose language is contained in that of the p-automaton for that PCTL formula.

Acknowledgments. We acknowledge the kind support of the UK EPSRC project *Complete and Efficient Checks for Branching-Time Abstractions* (EP/E028985/1). We thank the anonymous referees for many useful comments, which improved readability of the manuscript.

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