A completeness result for gs-monoidal categories

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Discussions: Andrea Corradini
a roadmap

- some original motivations from graph rewriting
- an alternative presentation for cartesian categories
- a functorial characterization for partial and multi-algebras
- some facts on monoidal monads
- a completeness result for gs-monoidal cats [over semi-modules]
- a characterization for multiset-algebras

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Why graph rewriting (late Sixties, early Seventies)

- generalizes Chomsky grammars (adding data sharing)
- used in constraint solving and data structuring (70’s)
- applied as a (visual) specification technique (80’s-90’s)

but...

- no (obvious) algebraic structure (no induction)
- neither (temporal) logic nor calculus

Many data structures (HLR, adhesive...) for the same meta-approach
addressing the syntax...

a signature

$$\Sigma = \langle \{X, Y, Z\}, \{c \in \Sigma_{X,Y}, f \in \Sigma_{Y,Y,Z}\} \rangle$$

a rooted tree

a standard term

$$f(c(x), c(x))$$

how to obtain a term-like presentation for term graphs?

which are the associated models (algebras), if any?
let \( y \) be \( c(x) \) in \( f(y, y) \)

let \( \langle y_1, y_2 \rangle \) be \( \langle c(x), c(x) \rangle \) in \( f(y_1, y_2) \)
The algebraic theory $Th(\Sigma)$ is concretely defined as

- lists of vars as objects, (tuples of) typed terms as arrows
- term substitution as composition

(the theory is also the free cartesian category over $\Sigma$)

Algebras over $\Sigma$ and axioms in $E$ as functors

$$M \in [Th(\Sigma), \textbf{Set}]^\times_E$$

product and axioms preserving (homs as natural transfs.)
a completeness result

\[
\forall s, t \in Th(\Sigma). \{ s \equiv t \iff \forall M \in [Th(\Sigma), \text{Set}]_E. M(s) = M(t) \}
\]

from objects (arrows) to sets (functions)

\[
M(X) \xrightarrow{M(c) \times M(c)} M(Y) \times M(Y) \equiv M(Y \times Y) \xrightarrow{M(f)} M(Z)
\]
an alternative take

\[ Th(\Sigma) \text{ is the free symmetric (strict) monoidal category equipped with symmetric monoidal natural transformations} \]

\[ \nabla_a : a \rightarrow a \otimes a \]

\[ !_a : a \rightarrow e \]

(intuitively representing pairing tuple \( <x, x> \) and empty tuple)
explicit definition of a theory
two alternative takes

\[ \text{GSTh}(\Sigma) \]

\[ \text{GTh}(\Sigma) \]
(more of) running example...

\[ f \circ \nabla_Y \circ c \]

let \( y \) be \( c(x) \) in \( f(y, y) \)

\[ f \circ (c \otimes c) \circ \nabla_X \]

let \( \langle y_1, y_2 \rangle \) be \( \langle c(x), c(x) \rangle \) in \( f(y_1, y_2) \)
the (linear) CoSpan (bi-)category

$\text{TGrp}(\Sigma) = \text{TGrp} \downarrow \Sigma$

an arrow

composition

[holding for any cocomplete cat]
some characterization results

- arrows in $GSTh(\Sigma)$ are (isomorphic classes of linear) cospans of term graphs (typed over $\Sigma$)

- You abstract the identity of nodes not in the interface

- ...but this way graphs get a “standard” notion of sentence

- arrows in $GTh(\Sigma)$ are conditioned terms $s \vdash D$ (over $\Sigma$)

- $s$ a term (the functional)

- $D$ a sub-term closed set of terms (the domain restriction)
functorial characterizations

Partial algebras with \( \bot \)-preserving operators, tight homomorphisms and conditioned Kleene equations

Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “term graph” equations

\[
\left[ GT h(\Sigma), \text{Set}_\bot \right]_E^\times
\]

\[
\left[ GST h(\Sigma), 2^{\text{Set}} \right]_E^\times
\]
completeness & entailment

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Completeness for partial algebras

\[ \forall s, t \in GTh(\Sigma). \{ s \equiv t \iff \forall M \in [GTh(\Sigma), \text{Set}_\bot]^E. M(s) = M(t) \} \]

---

Complete entailment system for partial algebras

\[ \begin{align*}
    | & D_s \equiv t | D_t \\
    s | D_s \cup D \equiv t | D_t \cup D
\end{align*} \]

\[ \begin{align*}
    | & D_u \\
    u_i | D_u \\
\end{align*} \]

\[ \begin{align*}
    (s | D_s, t | D_t) \in E \\
    s[u/x] | D_s[u/x] \cup D_u \equiv t[u/x] | D_t[u/x] \cup D_u
\end{align*} \]

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Claim: no finite complete entailment system for multi-algebras

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(more of) running example...

"garbage equivalent": same as multialgebra terms, different as [linear cospans of] term graphs

let $y$ be $c(x)$ in $f(y, y)$

let $\langle y_1, y_2 \rangle$ be $\langle c(x), c(x) \rangle$ in $f(y_1, y_1)$
a semiring-based functor

How to further generalize multialgebras?

First, generalize the finite power-set monad...

...then, take the Kleisli category!!

\[ S = \langle S, 0, 1, \oplus, \otimes \rangle \]

\[ S^X = [X, S] \]

[finite support]

\[ S^f : X \rightarrow Y (g)(y) = \bigoplus_{x \in f^{-1}(y)} g(x) \]
A monoidal monadic detour

The construction presented on semi-modules is an instance of the monoidal structure induced by monoidal monads!!

First step: symmetric monoidal functor

\[ F : (C, \otimes_C, e_C, \rho_C) \rightarrow (D, \otimes_D, e_D, \rho_D) \]

\[ m_{a,b} : F(a) \otimes_D F(b) \rightarrow F(a \otimes_C b) \]

\[ m_e : e_D \rightarrow F(e_C) \]

natural transformations plus coherence axioms

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some more coherence

Second step: symmetric monoidal monad \( \langle T, \eta, \mu \rangle \)

\[\eta_a : a \to T(a)\]
\[\mu_a : T(T(a)) \to T(a)\]

\[\begin{array}{ccc}
T(a) & \xrightarrow{T(\eta_a)} & T(T(a)) \\
\downarrow{\eta_{T(a)}} & & \downarrow{\mu_a} \\
T(T(a)) & \xrightarrow{\mu_a} & T(a)
\end{array}\]
\[\begin{array}{ccc}
T(T(T(a))) & \xrightarrow{T(\mu_a)} & T(T(a)) \\
\downarrow{\mu_{T(a)}} & & \downarrow{\mu_a} \\
T(T(a)) & \xrightarrow{\mu_a} & T(a)
\end{array}\]
Second step: symmetric monoidal monad $\langle T, \eta, \mu \rangle$

\[ \eta_a : a \rightarrow T(a) \]
\[ \mu_a : T(T(a)) \rightarrow T(a) \]

\[ a \otimes b \xrightarrow{id_{a,b}} a \otimes b \]
\[ a \otimes b \xrightarrow{\eta_{a,b}} T(a) \otimes T(b) \]
\[ T(a) \otimes T(b) \xrightarrow{\eta_a \otimes \eta_b} T(a \otimes b) \]

\[ e \xrightarrow{id_e} e \]
\[ e \xrightarrow{\eta_e} T(e) \]

\[ e \xrightarrow{\mu_a} T(T(a)) \]
\[ e \xrightarrow{id_e} T(e) \]

\[ m_{a,b} : T(a) \otimes T(b) \rightarrow T(a \otimes b) \]

Some more coherence
Kleisli category of a monad [aka, free algebras of a monad]

\[ f : a \rightarrow b \in \mathcal{K}_T \iff f : a \rightarrow T(b) \in C \]

arrows are substitutions

\[
\begin{array}{ccccccc}
T(b) & \xrightarrow{T(g)} & T(T(c)) & \xrightarrow{\mu_c} & T(c) \\
& & \downarrow{g} & & \\
& a & \xrightarrow{f} & b & \xrightarrow{g \circ f} & \\
\end{array}
\]

Known fact: if \( T \) is a (symmetric) monoidal monad, so is \( \mathcal{K}_T \)

Less known fact: if the product is cartesian, \( \mathcal{K}_T \) is gs-monoidal

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How to further generalize multialgebras?

First, generalize the finite power-set monad...

...then, take the Kleisli category!!

\[ S = \langle S, 0, 1, \oplus, \otimes \rangle \]
\[ S^X = [X, S] \]
\[ S^f: X \rightarrow Y (g)(y) = \bigoplus_{x \in f^{-1}(y)} g(x) \]
looking at the monad structure

Recasting finite power-sets via boolean algebra \( \langle \{tt, ff\}, ff, tt, \lor, \land \rangle \) [free dioid]

\[ \eta_X : X \to [X, S] \]

\[ \eta_X(x) = \iota_x \] [the injection function: 1 for x, 0 elsewhere]

\[ \mu_X : [[X, S], S] \to [X, S] \]

\[ [\mu_X(\lambda : [X, S] \to S')] (x) = \bigoplus_{f : X \to S} \lambda(f) \otimes f(x) \]
a monoidal monad

\[ m_{X,Y} : S^X \times S^Y \rightarrow S^{X \times Y} \]

\[ [m_{X,Y}(\langle f, g \rangle)](\langle x, y \rangle) \mapsto f(x) \otimes g(y) \]

\[ \begin{array}{ccc}
X \times Y & \xrightarrow{i_- \times i_-} & S^X \times S^Y \\
\downarrow id_{X,Y} & & \downarrow - \otimes = \\
X \times Y & \xrightarrow{i_{\langle -,- \rangle}} & S^{X \times Y} \\
\end{array} \]

\[ m_\emptyset : \emptyset \rightarrow S^\emptyset \quad m_\emptyset = \eta_\emptyset = id_\emptyset \]
the associated Kleisli cat

\[ [X, S] = \{ n_1 \cdot x_1 \oplus \ldots \oplus n_k \cdot x_k \mid n_i \in S, x_i \in X \} \]

Kleisli cat: sets as objects, multiset relations as arrows!!

\[ f : X \rightarrow [Y, S] \]

\[ \forall x \in X. g \circ f(x) = \bigoplus_{n \cdot y \in f(x)} n \cdot g(y) \]

\[ \forall \langle x, w \rangle \in X \times W. h \otimes k(\langle x, w \rangle) = \bigoplus_{n \cdot y \in h(x), m \cdot z \in k(w)} nm \cdot \langle y, z \rangle \]

the Kleisli category is gs-monoidal
(use of) running example...

\[
f(y_i, y_j) = z_{ij}
\]
\[
c(x) = \bigoplus_{i=1,2} n_i \cdot y_i
\]
\[
(c \otimes c)(\langle x, x \rangle) = \bigoplus_{i,j=1,2} n_i n_j \cdot \langle y_i, y_j \rangle
\]
\[
f \circ (c \otimes c) \circ \nabla_X
\]

\[
Z = \{ z_{ij} \mid i, j = 1, 2 \}
\]
\[
Y = \{ y_1, y_2 \}
\]
\[
X = \{ x \}
\]
(use of) running example...

\[
\begin{align*}
  f(y_i, y_j) &= z_{ij} \\
  c(x) &= \bigoplus_{i=1,2} n_i \cdot y_i \\
  (c \otimes c)(\langle x, x \rangle) &= \bigoplus_{i,j=1,2} n_i n_j \cdot \langle y_i, y_j \rangle \\
  Z &= \{z_{ij} \mid i, j = 1, 2\} \\
  Y &= \{y_1, y_2\} \\
  X &= \{x\} \\
  f \circ \nabla_Y \circ c \\
  x &\mapsto \bigoplus_{i=1,2} n_i \cdot z_{ii}
\end{align*}
\]
(use of) running example...

\[ f(y_i, y_j) = z_{ij} \]

\[ c(x) = \bigoplus_{i=1,2} n_i \cdot y_i \]

\[ (c \otimes c)(\langle x, x \rangle) = \bigoplus_{i,j=1,2} n_i n_j \cdot \langle y_i, y_j \rangle \]

\[ Z = \{ z_{ij} \mid i, j = 1, 2 \} \]

\[ Y = \{ y_1, y_2 \} \]

\[ X = \{ x \} \]
functorial characterizations

Multiset-algebras with tight point-to-multiset operators, tight point-to-point homomorphisms and “term graph” equations

\[ \left[ GSTh(\Sigma), S^{Set} \right]_E \]

Multiset-algebras: each operator is a multiset relation (hence, in relational algebras jargon, tight and point-to-multiset)...

Multiset-homomorphisms: since natural transformations must preserve \( \nabla_X \) and \( !X \), homs are just functions (again, tight and point-to-point)
power-sets vs. natural numbers

If the semiring is the free dioid \( \langle \{ tt, ff \}, ff, tt, \lor, \land \rangle \)

\[
\begin{align*}
x & \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii} \quad \text{and} \quad x & \mapsto \bigoplus_{i=1,2} n_i \cdot z_{ii}
\end{align*}
\]

coincide...

Stronger claim: no finitary complete entailment system for multi-algebras
a novel completeness

If the semiring is the free semiring $\langle\{0, 1, 2, \ldots\}, 0, 1, +, \times\rangle$

and at most one coefficient (among $\{n_1, n_2\}$) is not zero

$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii}$ and $x \mapsto \bigoplus_{i=1,2} n_i \cdot z_{ii}$ do not coincide...

$x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ii}$ and $x \mapsto \bigoplus_{i,j=1,2} n_i n_j \cdot z_{ij}$ coincide...

[recovering partial functions]
a new completeness result

If the semiring is the free semiring \( \langle \{0, 1, 2, \ldots \}, 0, 1, +, \times \rangle \)

the three multiset relations are different

\( \forall s, t \in GSTh(\Sigma).\{ s \equiv t \iff \forall M \in [GSTh(\Sigma), \mathbb{N}^{-}]_E^\times. M(s) = M(t) \} \)
to be addressed...

Algebraic issues
- tackling hyper-graphs and hyper-signatures
- (singular vs plural) interpretation for hyper-operators
- considering cospans of cocomplete categories
- free construction for suitable algebraic varieties

Coalgebraic issues
- analyzing trace equivalence [via Hasuo-Jacobs]