Graph Transformation with Topological Constraints

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Graph transformation

- GT: declarative modelling technique that extends term rewriting
- states as graphs, transitions as application of transformation rules (transformation steps), indeterministic character
- structured graphs
- use of preorders and derived topologies to define higher-level views of graphs

Algebraic GT

- category of graphs — sets of edges and vertexes with source and target functions
- transformation rule $L \Longrightarrow R$:
  - partial injective morphism — single-pushout (SPO)
  - spans of total injective morphisms (from an interface) — double-pushout (DPO)
- graph G, rule match $m : L \to G$:
  - total morphism that does not identify preserved elements and deleted ones
  - extra requirement (DPO): leaves no dangling edges

Rules and actions

- suggested intuition: rules as actions, rule application as action event, led by an agent, in a context
- dangling condition (DPO): all the effects of the transformation step are included in the match (modularity wrt rule application)
- interface — subgraph of the match that is preserved
- locality of formulation (SPO): the specification of an action event depends on the agent, not on the context
- boundary — includes the edges that may be deleted as side-effect

Distributed systems

- DS: structure made of multiple components and connections
- GT approach based on two-layer hierarchy (local/global) and DPO (Taentzer *et al.*)
  - global network architecture and reconfiguration
  - local components and their behaviour
  - interfaces between components

Loose distributed approach

- network architecture — multiple levels of connections
- architecture preservation — graphs with constraints
  - local: disabling rule matches
  - global: invalidating transformation steps
- simple representation: transitive associations — a preorder
- looser than a hierarchy (e.g. containment, a poset)

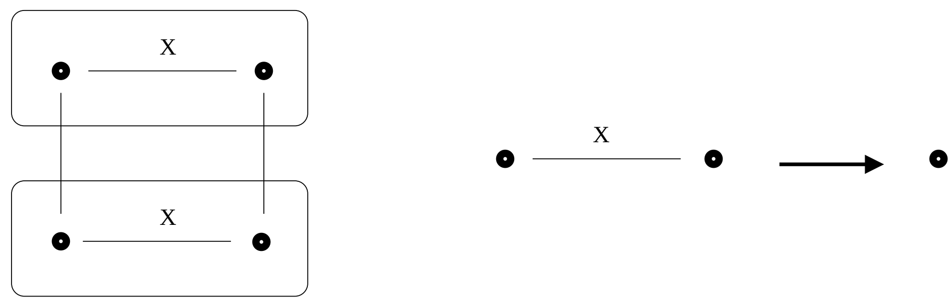
LD systems

- behaviour modelled using SPO — dangling condition dropped locally
- network architecture as preorder of places
- underlying graph elements mapped to places
- preorder-derived network topology
- region of application, implied by the topology, including the boundary — modularity recovered at the architectural level, in a looser sense wrt DPO

summarising

- SPO rule: specification of the action of an agent
- region of application: affected subgraph
- association as abstract connection relation
- modelling of granular systems
- possible use: high-level search for connection paths by pattern-matching

GT example



- SPO approach, initial graph (with two components) and transformation rule
- two applications of the rule, one to each component, may lead to disconnected components
- different ways to rule disconnection out — e.g. by negative application condition
- here we look at a direct one — an invariant stating that the two components stay connected

Spatial graphs

- structured graph G (decoupled presentation)
  - underlying graph U_G
  - network architecture (spatial type graph): distinguished directed graph P_G in which nodes are places and edges express association.
    stronger requirement (proper hierarchy): P_G is acyclic (a containment hierarchy)
  - total surjective *location* map (loc) from elements of U_G to places, represented as coupling edges
  - coupling as containment, no empty places
  - algebraic characterisation

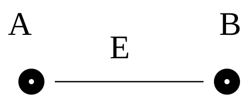
Location

- minimal requirements on P_G
  - refl-trans closure of association gives $\leq$ as a preorder
  - loc is a function
  - for each edge e between n_1, n_2 in U_G, it holds $loc(n_1), loc(n_2) \leq loc(e)$
  - for each n_3 s.t. $loc(n_3) \leq loc(e)$, either $loc(n_3) \leq loc(n_1)$ or $loc(n_3) \leq loc(n_2)$
- stronger requirements:
  - $\leq$ is a poset
  - there is a (unique) prime join in the preorder for each pair of places that are connected by an edge

Places and components

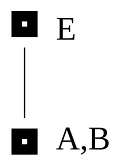
- $\leq$ induces a subcomponent relation
- can be used to express pre/post-conditions in rules (using non-emptiness)
- $ctl(p) = \{x : loc(x) = p\}$ is not generally a subgraph (e.g. an edge)
- hierarchy of components (vertical)
- given $p_1 < p_2$, then $ctl(p_2)$ is a subcomponent of $ctl(p_1)$
- connections (horizontal)
- given $p_1 \leq q, p_2 \leq q$, then $ctl(q)$ is a connection (common connecting subcomponent) between $ctl(p_1)$ and $ctl(p_2)$

shapes

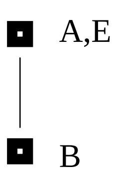


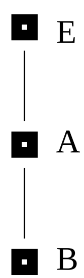
E is embedded in A

E is independent



A and B are associated to the same component



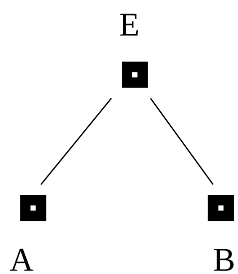


A is in the interface of the component associated to B

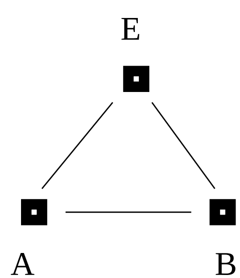
shapes (continued)

E is embedded in A

E is independent



A and B are associated to components that share a connector



A and B are in mirror components

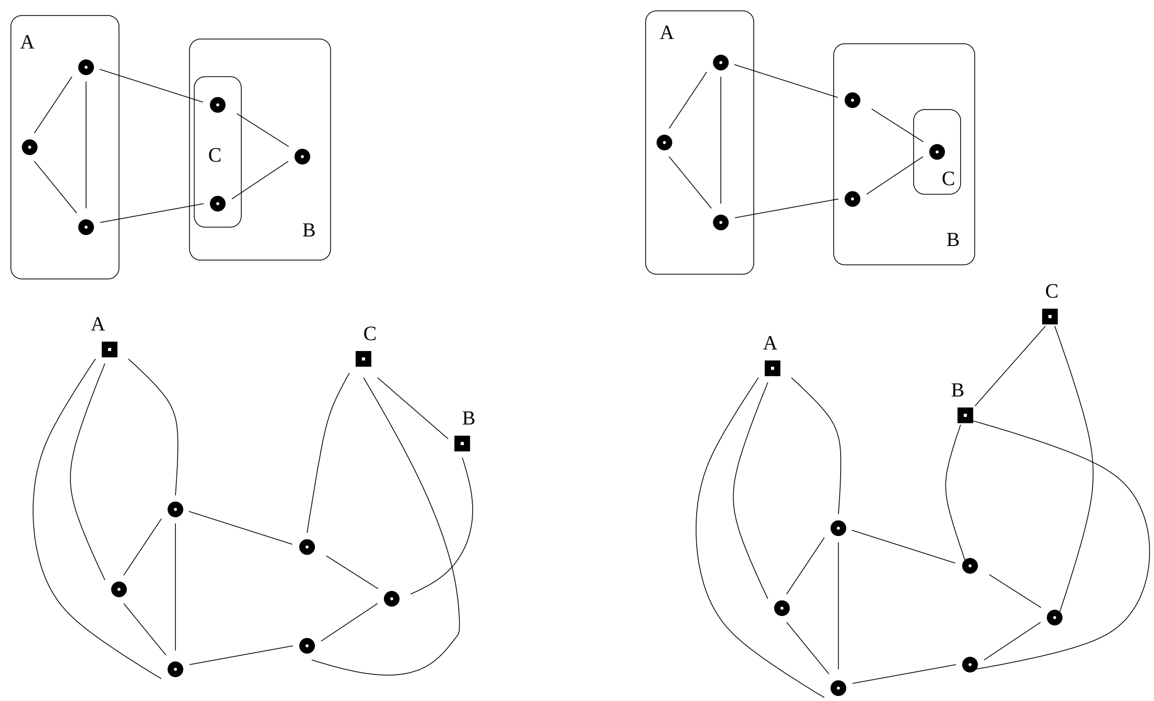
Derived topology

- $\leq$ is the specialisation preorder of an Alexandroff topology
- upper-closed sets of places are the open sets
- downward-closed sets of places are the closed sets
- the topology of the network (abstract regions) can be mapped back to the underlying graph (concrete regions)
- closed concrete regions are subgraphs
  closed view — gluing aspect
- open concrete regions are complements of subgraphs
  open view — connecting aspect

Regions and matches

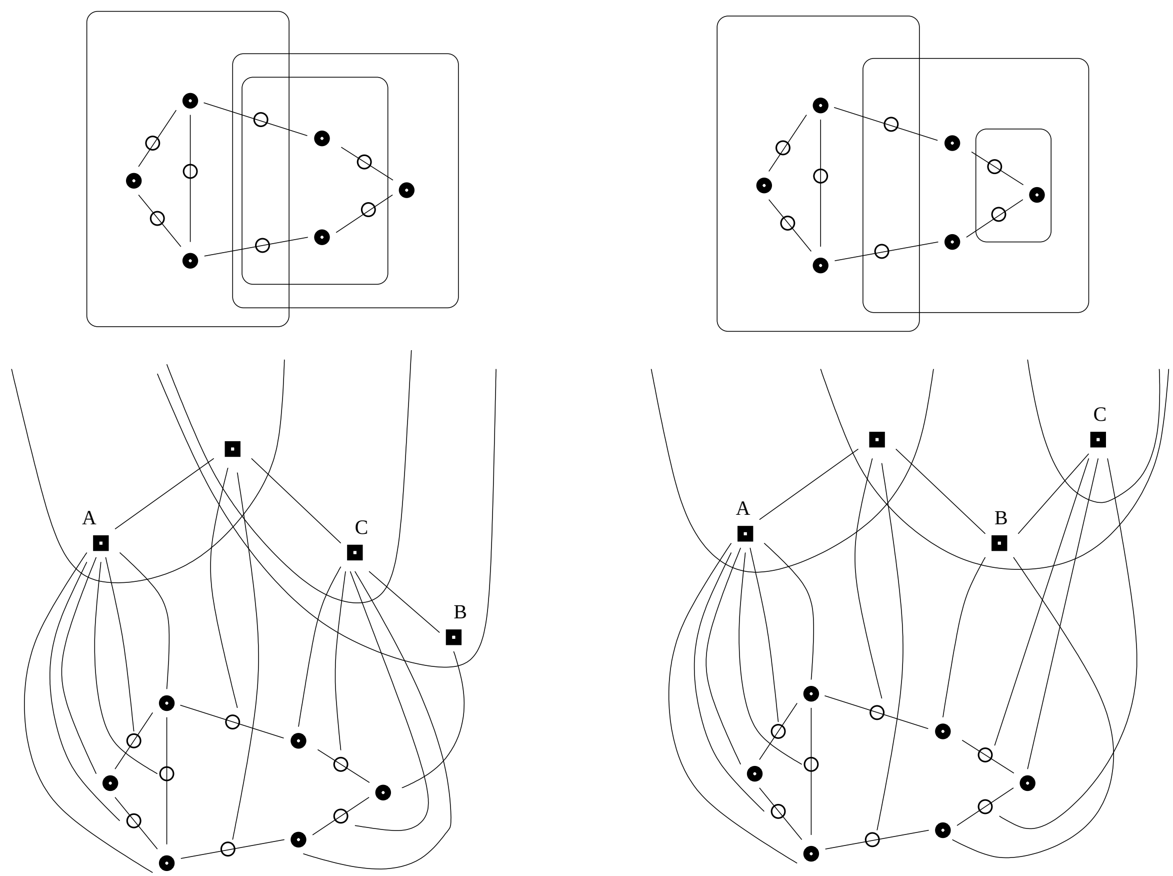
- each rule match gives an open abstract region $reg(m) = \bigcup_{x \in m}\{p : loc(x) \leq p\}$ (subbase)
- the open concrete region $ctl(m) = \{x : loc(x) \in reg(m)\}$ includes all the dangling edges
- the closed concrete region $proj(m)$ (closure of $ctl(m)$) — the smallest subgraph wrt the topology that includes all the side-effects (the boundary)

examples: incomplete view



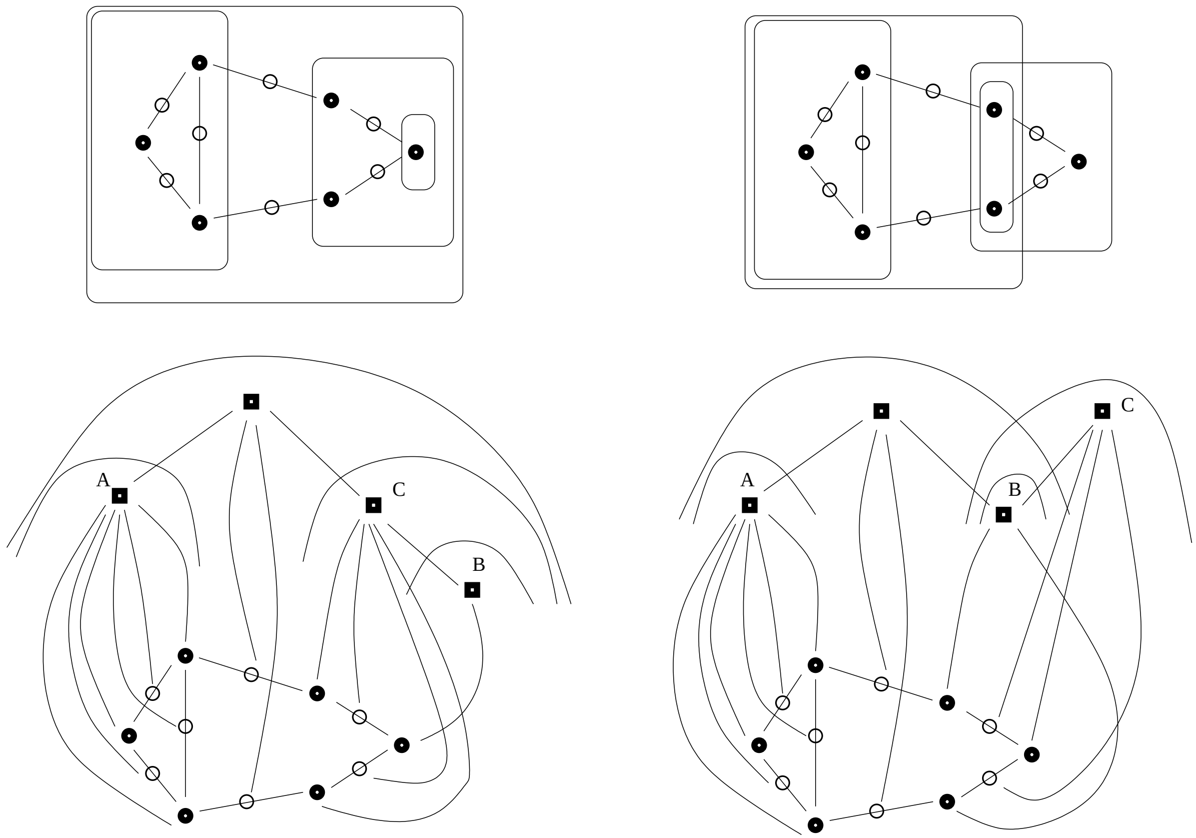
- (Left) component A connected to B through C
- (Right) component A connected to B that includes C
- boxes as places, inclusion as reverse order

examples: open view



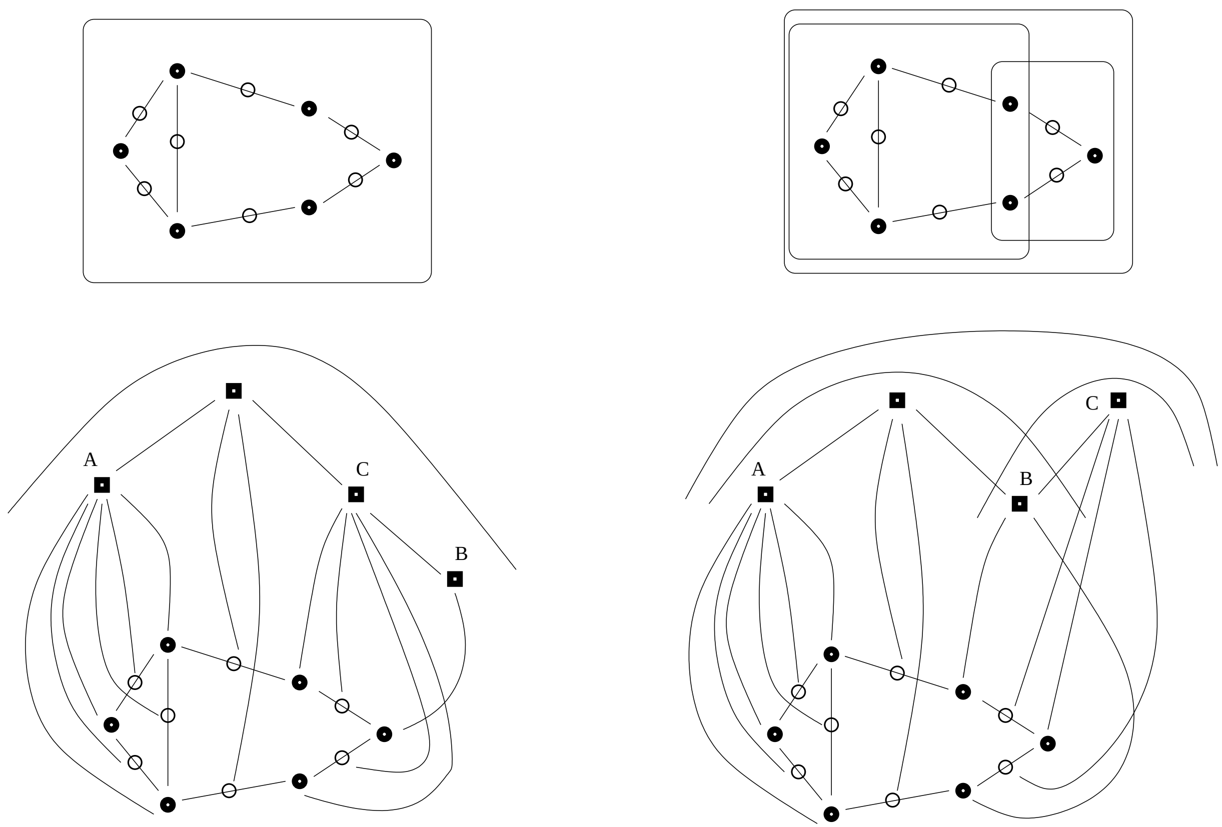
- prime join added, map extended to edges
- boxes as upper-closed sets (open sets)

examples: closed view



- boxes as downward-closed sets (closed sets)

examples: closure view

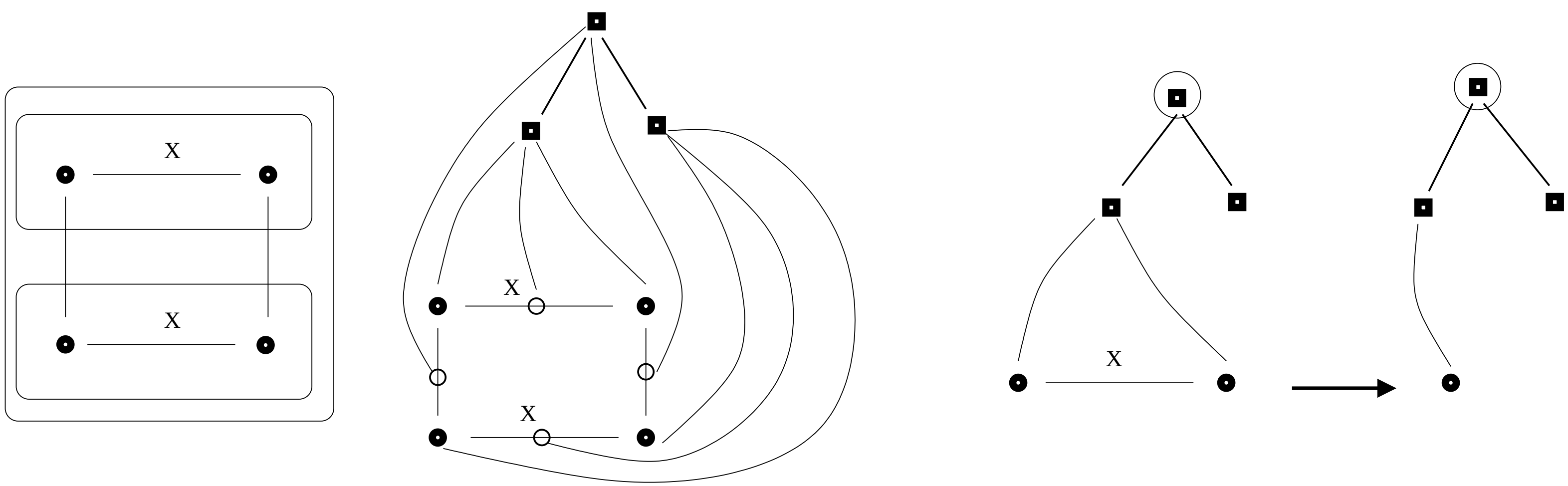


- boxes as non-point-like closed sets

Relations over regions

- relations between regions to express constraints on the model
  - inclusion between abstract regions: abstract subcomponent relation
  - intersection of open abstract regions (hence of concrete regions) is not empty: components are connected — they share a connector, e.g. a boundary edge
  - previous one false, intersection of closure of open abstract regions is not empty: components are adjacent — there is a component (concrete) that connects with both

GT example - revised

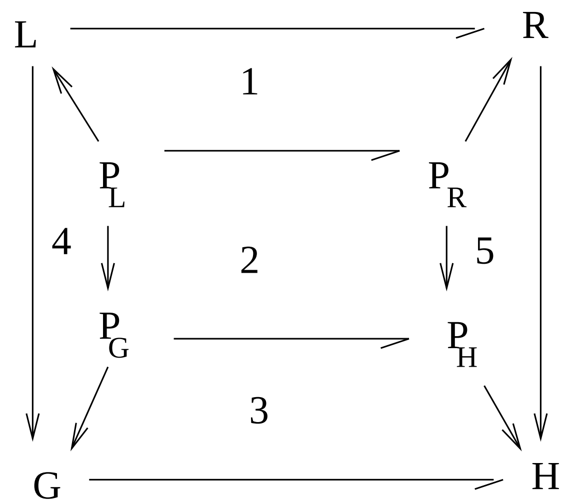


- initial graph with closed view and transformation rule
- prime join added to the original hierarchy
- mapping extended to edges
- constraint as rule invariant (places are non-empty)

SPO and DPO

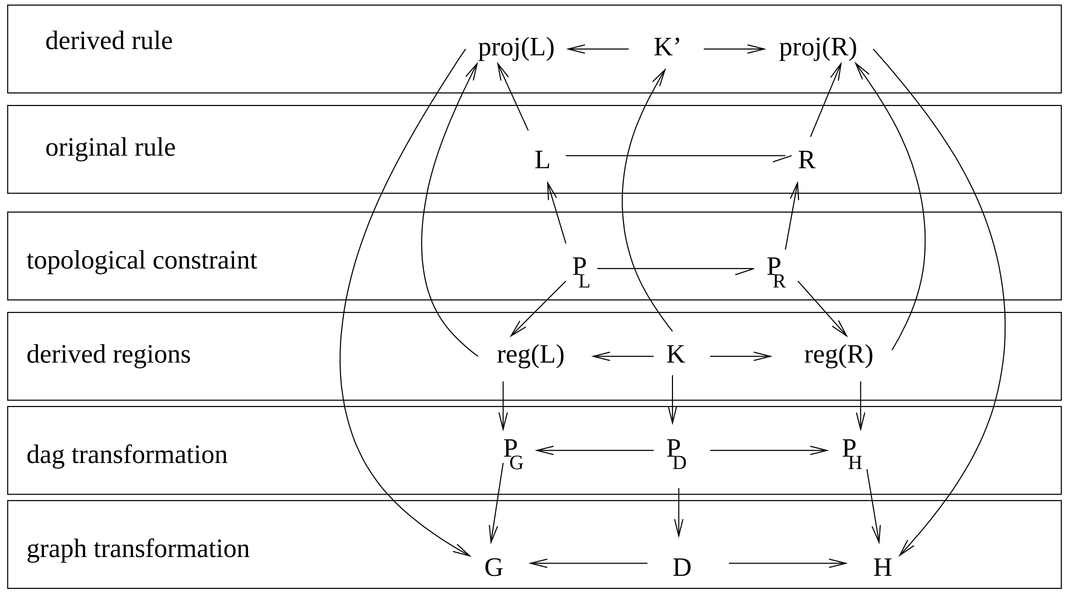
- from a formal point of view
  - each match can be completed with an open abstract region
  - therefore with a concrete region, which includes all the side effects
  - possible to extract a DPO rule (minimal wrt the topology) that corresponds to the application of the SPO one — the closure region gives its match
- this can be shown diagrammatically

SPO diagram



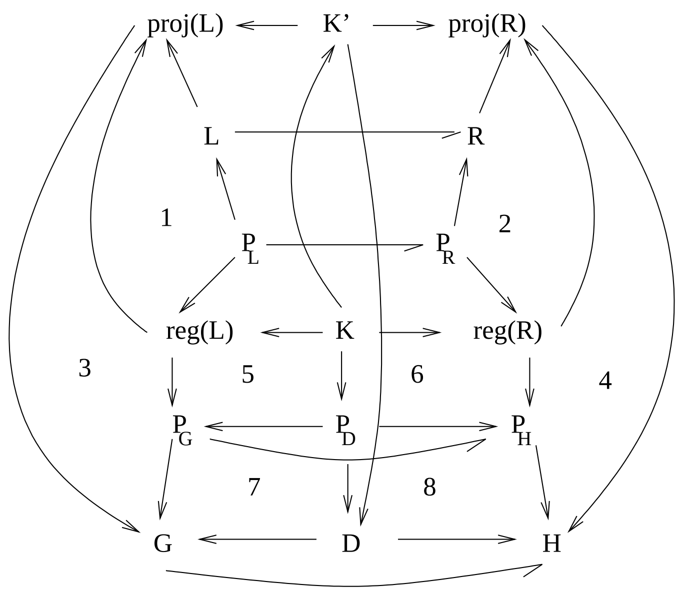
- SPO rule application diagram with hierarchy
- pushouts 2, $[L, R, G, H]$
- pullbacks 1,3,4,5

Diagram scheme



- SPO match and derived DPO rule

SPO-DPO diagram



- embedded SPO diagram
- pullbacks 1–4, 7, 8, $[K, K', reg(L), proj(L)]$, $[K, K', reg(R), proj(R)]$
- pushouts 5, 6, $[K', proj(L), D, G]$, $[K', proj(R), D, H]$

Limit examples

- trivial underlying graph: a set of isolated nodes, connected by the network
- trivial network:
  - converting SPO matches into DPO ones:
  - let loc defined as an injection on nodes, such that for each pair of distinct n_1, n_2, $loc(n_1)$ and $loc(n_2)$ are $\leq$-incomparable, and edges are mapped to prime joins
  - then for each SPO match, its closed concrete region is the smallest DPO one
  - in general, they are only the smallest wrt the topology

Conclusion

- possibility to extend notion of distributed graph transformation, allowing for more levels
- SPO with a weak notion of DPO-style compositionality
- local dangling condition dropped to allow for simpler rules
- network architecture, global dangling condition
- algebraic characterisation
- work to be done on the modelling as well as on the formalisation