Approximation of Minimum Triangulation for Polyhedron with Bounded Degrees

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Abstract. Finding minimum triangulations of convex polyhedra is NP-hard. The best approximation algorithms only give a ratio 2 for this problem, and for combinatorial algorithms it is shown to be the best possible asymptotically. In this paper we improve the approximation ratio of finding minimum triangulations for some special classes of 3-dimensional convex polyhedra. (1) For polyhedra without 3-cycles and degree-4 vertices we achieve a tight approximation ratio 3/2. (2) For polyhedra with vertices of degree-5 or above, we achieve an upper bound $2 - \frac{1}{12}$ on the approximation ratio. (3) For polyhedra with n vertices and vertex degrees bounded by a constant Δ we achieve an asymptotic tight ratio $2 - \Omega(1/\Delta) - \Omega(1/n)$.

1 Introduction

A *triangulation* of a d-dimensional polyhedron is its subdivision into a set of simplices, such that the simplices do not overlap and intersect only at common faces. We are interested in three-dimensional polyhedron triangulations (also called *tetrahedralizations*), which have important applications in computer graphics, finite element analysis, computer-aided design, etc. as well as having fundamental theoretical significance. In particular, we want to find triangulations consisting of a small number of tetrahedra.

The problem of polyhedron triangulation has been studied extensively. Convex polyhedra can always be triangulated, but different triangulations may contain different numbers of tetrahedra, i.e., the *size* of triangulations can be different. It is shown in [2] that finding a minimum triangulation, i.e., a triangulation with the minimum possible size, is NP-hard. There are several algorithms to triangulate a polyhedron, but not specifically addressing the problem of minimum triangulation. For example, the simple *pulling* heuristic [7], which picks a vertex and connects it to all other non-adjacent faces of the polyhedron, gives an approximation ratio 2 for finding minimum triangulations. Though simple, no better triangulation algorithms were known for a long time.

In [3] a new triangulation algorithm is given, by making use of the properties of 3-cycles. A *3-cycle* is a cycle of length three on the surface graph of a polyhedron such that both sides contain some other vertices (i.e. the triangular faces of

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the polyhedron are not regarded as 3-cycles). A 3-cycle separates a polyhedron into two parts. The idea of the algorithm (which we call **CutPull** in this paper) is very simple: we partition the polyhedron along all the 3-cycles into subpolyhedra, each is free of 3-cycles. We then apply the pulling heuristic to each resulting subpolyhedra. It was shown that such an algorithm gives an approximation ratio of $2 - \Omega(\frac{1}{\sqrt{n}})$ where n is the number of vertices of the polyhedron.

Although the above bound seems to be a slight improvement only, it was proved in the same paper that this approximation ratio is the best possible, for algorithms that only consider the combinatorial structure of the polyhedra. This lower bound is proved by utilizing a property of vertex-edge chain structures (VECSs), first introduced in [2]. A VECS of size m consists of the vertices $(a, b, q_0, q_1, ..., q_{m+1})$, forming the set of triangular faces $\{aq_iq_{i+1}, bq_iq_{i+1}(0 \leq i \leq m)\}$ (Fig. 1(a)). It consists of a chain of degree-4 vertices. An important property of the VECS is [3]: if a polyhedron contains a VECS as a substructure, and the interior edge ab (called the *main diagonal*) is not present in a triangulation of the polyhedron, then in this triangulation at least $2m$ tetrahedra are 'incident' to the VECS. On the other hand, if ab is present, $m+1$ incident tetrahedra may be sufficient for the triangulation.

In view of these results, the following question is raised in that paper: can the approximation ratio be improved when the maximum degree is constant? Another interesting question is that, are there special types of polyhedra that have optimal triangulations or with better approximation ratios using **CutPull**? In this paper we give some results about these problems.

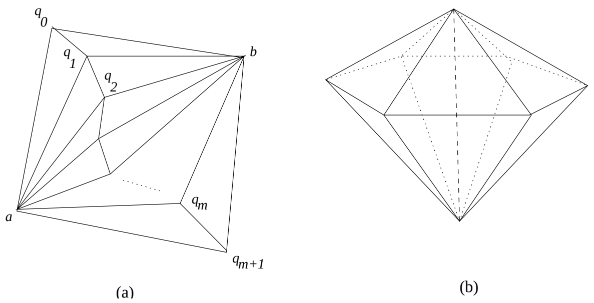

Fig. 1. (a) A VECS of size m. (b) A bipyramid with $n-2$ vertices in the middle chain; here $n = 8$.

The rest of this paper is organized as follows:

- In Sect. 2, we give new bounds on the relationship between the size of minimum triangulation, the maximum vertex degree, and the number of 3-cycles of a polyhedron, improving our previous results.
- In Sect. 3, we show that when a polyhedron has no 3-cycles and no degree-4 vertices, **CutPull** gives an improved approximation ratio, 3/2 instead of 2,

and this bound is tight. For polyhedra with vertices of degree-5 or above, an upper bound $2 - \frac{1}{12}$ on the approximation ratio is proved.
– In Sect. 4, we give an improved analysis of the **CutPull** algorithm for polyhedra having vertex degrees bounded by a constant Δ. The analysis gives an asymptotically tight approximation ratio which is better than that of the general-degree case. In particular, the ratio is 12/7 for $\Delta = 6$ and 7/4 for $\Delta = 7$.

2 Preliminaries

Throughout this paper, let P be a convex polyhedron with n vertices, Δ the maximum vertex degree, and k the number of 3-cycles. We only consider polyhedra with vertices in *general position*, i.e., no four vertices are coplanar. Let also t be the size of minimum triangulation of P, and e_m the number of interior edges in a minimum triangulation of P. It is known that $t = e_m + n - 3$ [1]. It is also shown in [3] that e_m is related to Δ under the restriction that the polyhedron has no 3-cycles, by the formula $2e_m(\Delta + 1) \geq n$ which is tight to within a constant multiplicative factor. In this section we first give an alternative but very simple proof to tighten the inequality by almost a factor of 2 (the constant-factor improvement is important when we come to Sect. 4), then extend it to the case with 3-cycles.

Lemma 1. *For a polyhedron P with no 3-cycles and $n > 4$, $e_m\Delta \geq n - 2$, and this is tight.*

Proof. Consider a face $v_0v_1v_2$ in P. We claim that each face must be incident by at least one interior edge. Assume this is not so. Then there is a face $v_0v_1v_2$ of P that has no incident interior edges. It is in some tetrahedron with fourth vertex v_3, and v_0v_3, v_1v_3, v_2v_3 have to be surface edges of P. Therefore the three triangles $v_0v_1v_3, v_1v_2v_3, v_2v_0v_3$ are either 3-cycles or faces. But 3-cycles are forbidden. If all three triangles are faces, then P is simply a tetrahedron with $n = 4$. Therefore our claim holds. Since there are $2n - 4$ faces, there are at least $2n - 4$ interior edges, but each is counted at most 2Δ times since each of the endpoints can be incident to at most Δ faces. Thus $e_m(2\Delta) \geq 2n - 4$, i.e. $e_m\Delta \geq n - 2$. This bound can be achieved by considering a bipyramid [9] (Fig. 1(b)), in which $e_m = 1, \Delta = n - 2$. □

We can generalize Lemma 1 to polyhedra having k 3-cycles:

Lemma 2. *For a polyhedron with k 3-cycles and $n > 4$, $e_m\Delta \geq n - 2 - 3k$.*

Proof. As in Lemma 1, for each of the $2n - 4$ faces, there should be at least one incident interior edge, unless, among the three bounding edges of the face, at least one is on a 3-cycle. Each 3-cycle can share an edge with at most six faces (on both sides of the three edges). Thus there remain at least $2n - 4 - 6k$ faces having incident interior edges. With the same argument as in Lemma 1, $e_m(2\Delta) \geq 2n - 4 - 6k$, and the result follows. □

We next prove a lemma relating the size of triangulations produced by **CutPull** and the number of 3-cycles of a polyhedron.

Lemma 3. *For a polyhedron with k 3-cycles, the* **CutPull** *algorithm produces triangulation of size at most $2n - 7 - k$.*

Proof. We prove this by induction on k. When $k = 0$, **CutPull** reduces to pulling, which gives $2n - 4 - \Delta \leq 2n - 7$ tetrahedra, so the claim holds. When $k > 0$, **CutPull** picks a 3-cycle, partitions the polyhedron into two subpolyhedra, with n_1 and n_2 vertices respectively, and having k_1 and k_2 3-cycles respectively. We have $n = n_1 + n_2 - 3, 0 \leq k_1 < k, 0 \leq k_2 < k, k = k_1 + k_2 + 1$. By induction assumption on the two subpolyhedra, their triangulations have sizes at most $2n_1 - 7 - k_1$ and $2n_2 - 7 - k_2$ respectively. So the size of triangulation is at most $(2n_1 - 7 - k_1) + (2n_2 - 7 - k_2) = 2(n+3) - 14 - (k_1 + k_2) = 2n - 7 - (k_1 + k_2 + 1) = 2n - 7 - k$. Thus the claim holds. □

3 Analysis for a Special Class of Polyhedra

From the results in [2] and [3], it can be seen that the major problems in finding minimum triangulations appear in 3-cycles and VECSs. In this section we first analyze the special case in which the polyhedra concerned have no 3-cycles and no degree-4 vertices (thus no VECSs). Note that the non-existence of 3-cycles implies that there are no degree-3 vertices, and thus all vertices have degrees at least five. We show that in this case the approximation ratio of the **CutPull** algorithm is at most $3/2$, better than the general case ratio $2 - \Omega(\frac{1}{\sqrt{n}})$. Moreover this is tight: we construct polyhedra having an approximation ratio no better than $3/2 - \epsilon$ using **CutPull** for any $\epsilon > 0$. We then consider the case when 3-cycles are present.

Empirically, it has been observed that 3-cycles are not very common in polyhedra, in particular those not induced by degree-3 vertices (every degree-3 vertex induces a 3-cycle); but polyhedra usually have some degree-3 and degree-4 vertices. However our results still have the following significance:

(i) as far as we know this is one of the very few classes of polyhedra that is known to have approximation ratio $2 - \epsilon$ for constant $\epsilon > 0$. For example, 'stacked polyhedra' [7] can be triangulated optimally in linear time, or the 'k-opt polyhedra' [8].
(ii) the existence of 3-cycles and degree-4 vertices can be checked in linear time, in contrast to k-opt polyhedra where no algorithm is known to check whether a polyhedron is k-opt.
(iii) they may arise as intermediate polyhedra in the processing of other triangulation algorithms, e.g. peeling [6]. If there are few degree-4 vertices, we might peel those degree-4 vertices first and may be left behind a polyhedron with degrees at least five (but note that new degree-4 vertices may appear in peeling).

(iv) Certain classes of polyhedra, such as prisms, antiprisms, etc. [5] have no 3-cycles and degrees at least five (provided that the coplanar points are perturbed so that the faces are suitably triangulated, and if necessary with simple modification/replication).

We classify all vertices of a polyhedron P into two types (w.r.t. a particular triangulation): a vertex is called *type-I* if some interior edge is directly incident to it. Otherwise it is called *type-II*. For any vertex v, we define the *neighborhood* $N(v)$ of v to be the set of vertices directly connected to v on the surface graph, i.e. $N(v) = \{u|(u, v) \in \text{surface edges of } P\}$. $N(v)$ forms a 3-dimensional polygon. Consider any triangulation of the polygon $N(v)$. (Note that this is slightly different from the definition of 'dome' or 'cap' [4] [3] in that a triangulation of $N(v)$ may not yield a convex patch of triangular faces.) Triangles with two edges on the polygon $N(v)$ are called *'ear' triangles*, and all others are called *'internal' triangles* (Fig. 2).

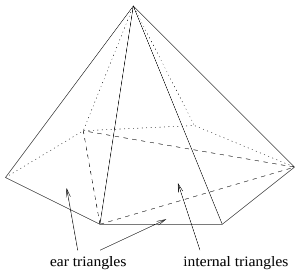

Fig. 2. ear and internal triangles.

We present some observations about type-II vertices in the lemma below, which we shall skip the easy proof:

Lemma 4. *Suppose v is a type-II vertex of degree d in a polyhedron P with respect to a triangulation.*
(i) All tetrahedra incident to v form a triangulation of the region bounded by the 3-D polygon $N(v)$ and the faces of P around v. There are $d - 2$ tetrahedra in this part of the triangulation. Their bases triangulate the polygon $N(v)$.
(ii) For any triangulation of $N(v)$, if $d \geq 5$ and v is not lying on any 3-cycles, there is at least one vertex in $N(v)$ having two or more incident interior edges. The triangulation of $N(v)$ consists of at least two 'ear' triangles and at least one 'internal' triangle.

Lemma 5. *For a polyhedron P having no 3-cycles and with vertex degrees at least five, there are at least $4n/3 - 3$ tetrahedra in any triangulation of P.*

Proof. Suppose we have n_1 type-I vertices and n_2 type-II vertices, and $n_1 + n_2 = n$. We consider the two cases, each using a different method to bound the size of triangulation:

Case 1: $n_1 \geq \alpha n$, where α ($0 \leq \alpha \leq 1$) is a constant we will specify later. We want to count the number of interior edge endpoints incident to these vertices (each interior edge having two endpoints). By definition, for each type-I vertex there is at least one interior edge endpoint incident to it. This gives n_1 edge endpoints. In addition, for each of the n_2 type-II vertices, there is at least one type-I vertex in the neighborhood that has two or more edge endpoints incident to it (Lemma 4). But the previous step did not count the extra endpoints (only one endpoint was counted for each type-I vertex). Thus there are at least n_2 additional edge endpoints, if all of them are distinct. Note that these must be interior edges, otherwise there is a 3-cycle. It can be shown that at most two type-II vertices share such an additional endpoint; the worst case is as in Fig. 3 where two type-II vertices (v_1 and v_9) sharing a type-I vertex (v_3) that only has two interior edge endpoints. Thus at least $n_2/2$ edge endpoints are added. Since each interior edge has two endpoints to be counted,

$$e_m \geq \frac{1}{2}(n_1 + \frac{n_2}{2}) = \frac{n_1}{2} + \frac{n_2}{4} = \frac{n + n_1}{4} \geq \frac{(1+\alpha)n}{4}$$

Thus

$$t = e_m + n - 3 \geq \frac{(1+\alpha)n}{4} + n - 3 = \frac{(5+\alpha)n}{4} - 3$$

Case 2: $n_1 < \alpha n$. Then $n_2 > (1-\alpha)n$. For each type-II vertex v, all tetrahedra incident to it constitute a triangulation of $N(v)$ (Lemma 4)(Fig. 2). Consider any triangulation of the 3-D polygon $N(v)$, with each triangle corresponding to a tetrahedron having v as a vertex. We count the number of tetrahedra incident to the type-II $N(v)$'s. All 'internal' tetrahedra of a type-II vertex v will not be counted by other type-II vertices (since the other three vertices of the tetrahedron are type-I). The 'ear' tetrahedra may be counted twice. For example in Fig. 3 tetrahedra $v_1v_2v_3v_4$ and $v_1v_3v_5v_6$ are 'ear' tetrahedra of a type-II vertex v_1, but the tetrahedra $v_1v_2v_3v_4$ (resp. $v_1v_3v_5v_6$) may also be counted by v_2 (resp. v_6) if v_2 (resp. v_6) are type-II. It cannot be counted more than twice since the other two vertices of the tetrahedron are type-I. Since $d \geq 5$, there is at least one 'internal' tetrahedron and at least two 'ear' tetrahedra, giving a total of at least two tetrahedra (each 'ear' counted as 0.5 for this vertex to avoid double counting). Thus the total number of tetrahedra incident to these type-II vertices is at least $2n_2 \geq 2(1-\alpha)n > 2(1-\alpha)n - 3$.

Considering both cases, the number of tetrahedra is at least $\min(\frac{(5+\alpha)n}{4} - 3, 2(1-\alpha)n - 3)$. It is easy to show that this is maximized when $\alpha = 1/3$, thus $t \geq 4n/3 - 3$. □

The above bound is tight (up to a constant additive factor) as shown below:

Lemma 6. *There exist polyhedra, without 3-cycles and degree-4 vertices, whose sizes of minimum triangulation are at most* $4n/3 + 8/3$.

Proof. The construction proceeds as follows.

Step 1. First consider the 9 vertices $v_1, ..., v_9$ as in Fig. 3; $N(v_1) = \{v_2, v_3, v_4, v_5, v_6\}$, $N(v_9) = \{v_3, v_4, v_5, v_7, v_8\}$. $N(v_1)$ and $N(v_9)$ share the triangle $v_3v_4v_5$.

This may be regarded as a (non-convex) polyhedron with v_3v_4, v_3v_5 being the nonconvex edges. They will become interior edges when we further construct the polyhedron. We now have a (partial) triangulation consisting of six tetrahedra $v_1v_2v_3v_4, v_1v_3v_4v_5$, $v_1v_3v_5v_6$, $v_9v_7v_3v_4$, $v_9v_3v_4v_5$ and $v_9v_3v_5v_8$.

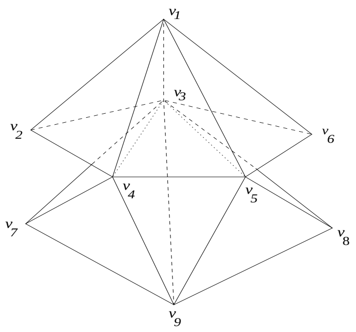

Fig. 3. Step 1 of the construction.

Step 2. We patch each of the two non-convex 'gaps' by the following structure. For the right-side gap, we add three vertices u_1, u_2 and u_3 and form the polyhedron as in Fig. 4, producing a new non-convex gap. The u_i's are placed carefully so that convexity is maintained (except at the new non-convex gap). The new region can be triangulated by the tetrahedra $v_5v_6u_1u_2, v_3v_5v_6u_2, u_2u_3v_5v_8$ and $v_3v_5u_2v_8$. The left-side gap is treated similarly. Each application of step 2 replaces the two old gaps by two new ones, using six more vertices and eight more tetrahedra. Note that the new gap and the old gap share a vertex. Step 2 is recursively applied to the new gaps. This step can be applied any number of times. We can keep the maximum degree of the polyhedron constant by alternating which vertex the original gap touches with the new gap.

Step 3. Finally, we patch each of the two gaps by the structure shown in Fig. 5. This structure has 12 vertices (8 of them are new vertices when patched), all are of degree at least five, and need at most $2n - 4 - \Delta = 2(12) - 9 = 15$ additional tetrahedra to triangulate (by pulling). The structure is patched so that the final polyhedron is convex.

It can be seen that this convex polyhedron has no 3-cycles, and all vertices have degrees at least five.

Note that we constructed a triangulation of this polyhedron along with the above steps. This may not be a minimum triangulation, but a minimum triangulation must have same or smaller size. If Step 2 is applied p times, then $n = 9 + 6p + 16$ and $t \leq 6 + 8p + 30$, giving $t \leq 4n/3 + 8/3$. □

The tight bound on the size of triangulations gives a tight bound on the approximation ratio of **CutPull**:

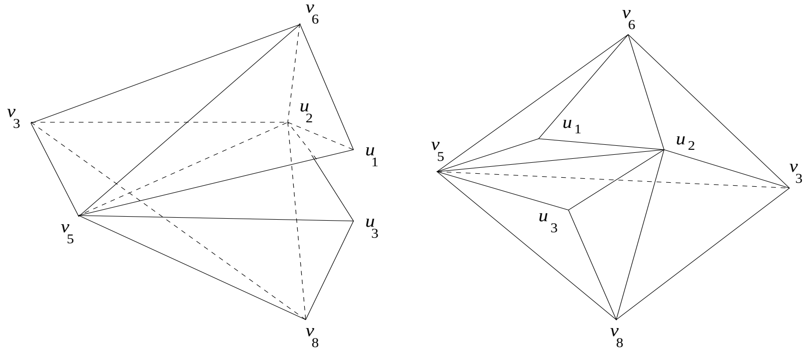

Fig. 4. Step 2, showing 3-D view (left) and view from right (right).

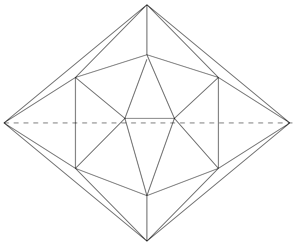

Fig. 5. Final patch for Step 3. The two triangles at back (as indicated by the dashed line) are to be attached to the gap.

Theorem 1. *The approximation ratio of* **CutPull** *algorithm for polyhedron without 3-cycles and all vertices having degree at least five is at most 3/2, and this is tight.*

Proof. The approximation ratio follows from Lemmas 3 and 5: $r \leq \frac{2n-7}{4n/3-3} < 3/2$. That the bound is tight follows from the constructed polyhedron in Lemma 6, having constant degrees, no 3-cycles, and $t \leq 4n/3 + 8/3$. Thus for those polyhedra, **CutPull** gives $r \geq \frac{2n-4-\Delta}{4n/3+8/3} = 3/2 - \epsilon$ where $\epsilon = \Theta(1/n)$ tends to 0 as n tends to infinity. $\square$

With the presence of 3-cycles (but still without degree-3 and degree-4 vertices), the argument in Lemma 5 works for vertices not lying on any 3-cycles. Thus if there are n' vertices not lying on 3-cycles, we have $t \geq 4n'/3 - 3$. Note that we have $n' \geq n - 3k$, thus with Lemma 3 we have:

Theorem 2. *For polyhedra with k 3-cycles and having degrees at least five, the approximation ratio $r \leq \frac{2n-7-k}{max(n-3,4n/3-4k-3)}$. In particular, it can be shown to be less than $2 - \frac{1}{12} = 1.9166...$ for any k.*

4 Analysis for Bounded-degree Polyhedra

In this section, we turn our attention to another class of convex polyhedra in which the vertex degrees are bounded by a constant. This occurs frequently for randomly-generated polyhedra. We show that in this case the **CutPull** algorithm can be applied with improved approximation ratio, and the ratio is tight up to combinatorial considerations.

4.1 Upper Bound

We first prove the upper bound, using Lemmas 2 and 3:

Theorem 3. *When the maximum degree Δ of a convex polyhedron is bounded by a constant, the* **CutPull** *algorithm gives an approximation ratio $2-\frac{2}{\Delta+1}-\Omega(\frac{1}{n})$.*

Proof. Suppose there are k 3-cycles in the polyhedron. By Lemma 3 we have

$$r \leq \frac{2n-7-k}{n-3+e_m} < \frac{2n-1-k}{n+e_m}$$

By Lemma 2, $e_m\Delta + 3k \geq n-2$, thus $e_m\Delta + (\frac{\Delta+1}{2})k \geq n-2$, and

$$k \geq \frac{2}{\Delta+1}n - \frac{4}{\Delta+1} - \frac{2\Delta}{\Delta+1}e_m$$

Substituting into the previous inequality gives

$$r \leq \frac{2n-1-\frac{2}{\Delta+1}n+\frac{4}{\Delta+1}+\frac{2\Delta}{\Delta+1}e_m}{n+e_m} = \frac{(2-\frac{2}{\Delta+1})n+(2-\frac{2}{\Delta+1})e_m-(1-\frac{4}{\Delta+1})}{n+e_m}$$
$$= 2-\frac{2}{\Delta+1}-\frac{1-\frac{4}{\Delta+1}}{n+e_m} = 2-\frac{2}{\Delta+1}-\Omega(\frac{1}{n})$$

since $e_m = O(n)$. □

For general (non-constant) Δ, we can actually prove that the approximation ratio is bounded by $r \leq 2-\Omega(\frac{1}{\Delta})-\Omega(\frac{\Delta}{n})$. The worst case is when $\Delta = \Theta(\sqrt{n})$ in which the the bound reduces to the $2-\Omega(\frac{1}{\sqrt{n}})$ given in [3].

4.2 Lower Bound

It is proved in [3] that no combinatorial algorithms can give an approximation ratio better than $2-O(\frac{1}{\sqrt{n}})$ for the minimum triangulation problem. The proof is based on constructing two polyhedra P1 and P2 with the same combinatorial structure but having different sizes in their minimum triangulations. In this subsection we prove similar results for the constant degree case. This shows that our upper bound in the previous subsection is asymptotically tight when only combinatorial information is considered.

For convenience we first review very briefly the construction for the general-degree case. First, a set of m VECSs each of size m are placed as in Fig. 6(a). Wedge W_i has vertices (a_i, b_i, c_i, d_i) with $a_i b_i$ being the main diagonal. All faces $a_i c_i d_i$ lie on the vertical plane $y = -1$ while all faces $b_i c_i d_i$ lie on the horizontal plane $z = 1$. All main diagonals pass through the origin. Vertices $q_i^1, ..., q_i^m$ are added between c_i and d_i for each wedge. The a_i's form a convex chain w.r.t. $(0, -1, -\infty)$, and the b_i's form a convex chain w.r.t. $(\infty, 0, 1)$. We have $n = m^2 + 4m$.

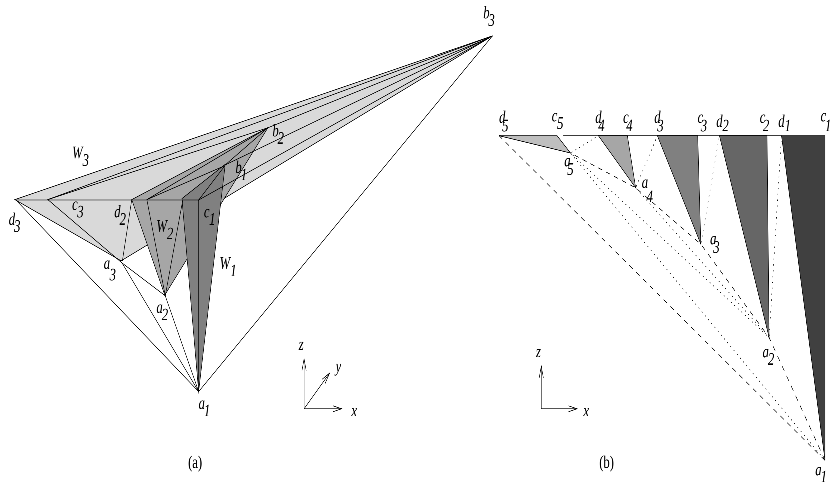

Fig. 6. (a) Construction of P1 and P2, showing 3 wedges. q_i's are not shown. (b) The wedges with $m = 5$, showing the zig-zag paths on the vertical plane.

Second, notice that all vertices lie on two planes, violating the general position assumption. We remove this degeneracy by perturbing the vertices slightly, so that the polyhedron has the following set of edges:

$a_k c_k, a_k d_k, b_k c_k, b_k d_k (1 \leq k \leq m)$;
$d_k a_{k+1}, d_k c_{k+1}, b_k c_{k+1}, a_k a_{k+1}, b_k b_{k+1} (1 \leq k \leq m-1)$;
$a_1 a_k, b_1 b_k (3 \leq k \leq m)$;
$q_k^i q_k^{i+1}, q_k^1 c_k, q_k^m d_k (1 \leq i \leq m-1, 1 \leq k \leq m)$;
$q_k^i a_k, q_k^i b_k (1 \leq i \leq m, 1 \leq k \leq m)$;
$c_1 b_m, a_1 b_m, a_1 d_m$.

Now the main diagonals all intersect at the origin. In the third step for P1, we 'push' the wedges towards each other slightly so that all wedges intersect each other. For P2, we move the wedges slightly apart so that they do not intersect. In this way, P1 will have a large size of triangulation because the wedges are 'interlocked' (i.e. penetrating each other), while P2 can have a small triangulation, although the two have the same combinatorial structure.

Our construction for the constant-degree case is very similar to the construction above. The two polyhedra P1 and P2 consists of m wedges each, with each wedge being a VECS of size s, where s is constant. Thus the number of vertices $n = m(s+4)$. The placement of wedges are almost identical to that as described above. There is only one thing we need to change. In the second step, we perturb the vertices into general position, resulting in edges a_1a_k and b_1b_k ($1 \leq k \leq m$). Since we fixed s constant, m is no longer constant and the above set of edges causes the degrees of a_1 and b_1 to be non-constant. To cope for this, our constant-degree construction perturbs the vertices in another way, so that zig-zags paths, e.g. $a_1a_na_2a_{n-1}...$, $b_1b_nb_2b_{n-1}...$ appears (Fig. 6(b)). Such triangulation is always possible by applying sufficiently small perturbations to the vertices. The maximum degree $\Delta = s+7$ (attained at e.g. a_2 in Fig. 6(b)) is thus constant.

Theorem 4. *When the maximum degree Δ of a convex polyhedron is bounded by a constant, any algorithm that only considers the combinatorial structure cannot give an approximation ratio better than $2 - O(\frac{1}{\Delta+1}) - O(\frac{1}{n})$.*

Proof. We first show that any minimum triangulation of P1 has at least $(\frac{2\Delta-14}{\Delta-3})n - \Delta$ tetrahedra, while any minimum triangulation of P2 has at most $(\frac{\Delta+1}{\Delta-3})n$ tetrahedra.

As discussed in the introduction, the wedges have the property that they admit triangulation of size either at most $s+1$ or at least $2s$, depending on the presence of their 'main diagonal' in the triangulation. For P1, at most one main diagonal of these m wedges can be present in any triangulation. Thus

$$t_{P1} \geq (m-1)(2s) + (s+1) = (\frac{n}{\Delta-3} - 1)(2\Delta-14) + (\Delta-6) > (\frac{2\Delta-14}{\Delta-3})n - \Delta.$$

For P2, each wedge can be triangulated into $s+1$ tetrahedra using their main diagonals. Removing these wedges leaves a non-convex region. This can be triangulated into $4(m-1)+3(m-2)+2 = 7m-8$ tetrahedra, using the 'shielding' argument same as that in [3]; due to space limitation we do not repeat it here. Note that the a_i's and b_i's have to be 'zig-zagged' in a matching manner for the proof to work. So

$$t_{P2} \leq m(s+1) + 7m - 8 = (\frac{n}{s+4})(s+1+7) - 8 < (\frac{\Delta+1}{\Delta-3})n$$

A combinatorial algorithm cannot distinguish P1 and P2, and always has to give the triangulation of larger size. With the above bounds, we thus have

$$r \geq \frac{(\frac{2\Delta-14}{\Delta-3})n - \Delta}{(\frac{\Delta+1}{\Delta-3})n} = \frac{2\Delta-14}{\Delta+1} - \frac{\Delta(\Delta-3)}{n(\Delta+1)} = 2 - O(\frac{1}{\Delta+1}) - O(\frac{1}{n})$$

□

5 Conclusion

We gave improved approximation ratios for the minimum polyhedron triangulation problem for two special classes of polyhedra: one having no 3-cycles and no degree-4 vertices, and one with constant degrees. For the case without 3-cycles and degree-4 vertices, our algorithm gives a ratio of $3/2$. This seems to be a rather restricted class of polyhedra; can it be optimally triangulated in polynomial time? Can we identify the (more restricted?) class of polyhedra which our algorithm will give the optimal triangulation? Stacked polyhedra are one known type. Can we identify classes of polyhedra that can be triangulated optimally or near-optimally in polynomial time, using perhaps other algorithms? The results may also be generalized to polyhedra having few (but nonzero) degree-4 vertices.

For the constant degree case, we get an asymptotically tight approximation ratio $r = 2 - \Omega(\frac{1}{\Delta}) - \Omega(\frac{1}{n})$, the lower bound being established if only combinatorial structure is considered. It is actually not known whether the constant-degree case is NP-hard (like the general-degree case), and what happens when non-combinatorial information is considered.

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