General Aims

- To teach basics of *category theory*.
- To study *programming language syntax with binding*.

- We only cover the category theory we need.
- Some categorical machinery is simplified – you read the abstract stuff *after* these lectures.
- We study syntax by examples – we omit the general theory of binding syntax.
- Syntax with binding is a hot research topic . . .

Basics of Algebraic and Binding Syntax

See OHP for Examples

- Algebraic syntax specified by constructor symbols C_i.
- Each symbol has an arity $a \in \mathbb{N}$.
- These generate (*finite*) expressions such as

$$\mathsf{C}_3\, e_0\ \ldots\ e_{a-1}$$

- … from datatypes of the form

$$\mathsf{datatype}\quad Exp \quad = \quad \ldots \quad \mathsf{C}_3\ \underbrace{Exp\ \ldots\ Exp}_{length\ a} \quad \ldots$$

- *Binding* syntax subsumes algebraic syntax.

- Binding syntax is specified by giving some constructor symbols C_i where each symbol has an arity $a \in \mathbb{N}$ and a binding depth $(b_0, \ldots, b_{a-1}) \in \mathbb{N}^a$.

- These generate (*finite*) expressions such as

$$\mathsf{C}_3 \ \underbrace{\ldots (v^0,\ \ldots,\ v^{b_j - 1}\ , e_j) \ldots}_{\textit{length } a}$$

- … from datatypes of the form

$$\mathsf{datatype}\quad \mathit{Exp} \quad = \quad \ldots \quad \mathsf{C}_3\ \underbrace{(\underbrace{\mathbb{V},\ \ldots\ ,\mathbb{V}}_{\textit{length } b},\mathit{Exp})\ \ldots\ (\underbrace{\mathbb{V},\ \ldots\ ,\mathbb{V}}_{\textit{length } b},\mathit{Exp})}_{\textit{length } a} \quad \ldots$$

Learning Outcomes: *You Should*

- know how examples of programming language syntax with binding can be specified inductively;
- be able to define basic categorical structures;
- know, by example, how to compute simple initial algebras;
- understand simple *abstract* models of syntax and know how to *manufacture* categorical models *from* syntax;
- be able to prove these models are essentially the same;
- understand current issues concerning variable binding and read the literature.

Definition of a Category

A category $\mathcal{C}$ is specified by:

- A collection *ob* $\mathcal{C}$ of objects; $A, B, C \ldots$
- A collection *mor* $\mathcal{C}$ of morphisms; $f, g, h \ldots$
- For each f a source $src(f)$ in *ob* $\mathcal{C}$ and a target $tar(f)$ in *ob* $\mathcal{C}$. Write

$$f : src(f) \longrightarrow tar(f) \qquad or \qquad f : A \to B$$

- f and g composable if $tar(f) = src(g)$.

- If $f{:}A \to B$ and $g{:}B \to C$ then there is $g \circ f{:}A \to C$, called the composition.

- For any object A there is an identity morphism $id_A{:}A \to A$. For any f

$$\begin{aligned} id_{tar(f)} \circ f &= f \\ f \circ id_{src(f)} &= f \end{aligned}$$

- $\circ$ is associative: given $f{:}A \to B$, $g{:}B \to C$ and $h{:}C \to D$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Examples of Categories

- Consider $Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{S}\ Exp \mid \mathsf{A}\ Exp\ Exp$ with typical elements

$$\mathsf{V}\ v^0 \qquad \mathsf{V}\ v^{45} \qquad \mathsf{A}\ (\mathsf{S}\ (\mathsf{V}\ v^3))\ (\mathsf{V}\ v^2)$$

- There is a category with typical morphisms

$$6 \xrightarrow{[\mathsf{V}\ v^4,\ \mathsf{V}\ v^2,\ \mathsf{V}\ v^1,\ \mathsf{S}\ (\mathsf{V}\ v^5)]} 4$$

$$2 \xrightarrow{[\mathsf{A}\ (\mathsf{A}\ v^0\ v^0)\ v^1,\ \mathsf{A}\ v^1\ v^0,\ \mathsf{A}\ v^0\ (\mathsf{S}\ v^0)]} 3$$

If

$$1 \xrightarrow{[\mathsf{S}\ v^0, \mathsf{A}\ v^0\ v^0]} 2 \xrightarrow{[\mathsf{A}\ (\mathsf{A}\ v^0\ v^1)\ v^1, \mathsf{A}\ v^1\ v^0, \mathsf{A}\ v^0\ (\mathsf{S}\ v^1)]} 3$$

the composition is

$$1 \xrightarrow{\begin{array}{l}[\mathsf{A}\ (\mathsf{A}\ (\mathsf{S}\ v^0)\ (\mathsf{A}\ v^0\ v^0))\ (\mathsf{A}\ v^0\ v^0), \\ \quad \mathsf{A}\ (\mathsf{A}\ v^0\ v^0)\ (\mathsf{S}\ v^0), \\ \quad\quad \mathsf{A}\ (\mathsf{S}\ v^0)\ (\mathsf{S}\ (\mathsf{A}\ v^0\ v^0))]\end{array}} 3$$

Set

- The objects are sets.

- Morphisms are triples (A, f, B) where $f \subseteq A \times B$ is a *graph* of a function:

$$(\forall a \in A)(\exists! b \in B)((a, b) \in f)$$

- Composition is given by

$$(B, g, C) \circ (A, f, B) \stackrel{\text{def}}{=} (A, g \circ f, C)$$

- id_A is (A, id, A).

$(X, \leq)$

- $(X, \leq)$ is a preordered set: $\leq$ is *reflexive* and *transitive*.
- The collection of objects is the set X.
- The collection of morphisms is the set $\leq$. Typical morphism (x, x').
- Composition is given by $(y,z) \circ (x,y) \stackrel{\text{def}}{=} (x,z)$.
- $id_x \stackrel{\text{def}}{=} (x,x)$.

Preset

- The objects are the preordered sets.
- The morphisms are the monotone functions.

A morphism $(X, \leq_X) \longrightarrow (Y, \leq_Y)$ is specified by a function $f : X \to Y$ such that

$$x \leq_X x' \quad \Longrightarrow \quad f(x) \leq_Y f(x')$$

$\mathbb{F}$

- The set of objects of $\mathbb{F}$ is $\mathbb{N}$.
  - We regard $n \in \mathbb{N}$ as the set $\{0, \ldots, n-1\}$ for $n \geq 1$, and 0 is the empty set $\varnothing$.
- A morphism $\rho : n \to n'$ is any set-theoretic function.

Isomorphisms and Equivalences

- A morphism $f : A \to B$ is an isomorphism if there is some $g : B \to A$ for which

$$f \circ g = id_B \qquad \wedge \qquad g \circ f = id_A$$

- We say g is an inverse for f and vise versa.
- We say A is isomorphic to B,

$$f \quad : \quad A \cong B \quad : \quad g$$

if such a mutually inverse pair of morphisms exists.

- f and g witness the isomorphism.

Examples of Isomorphisms

- Bijections in $\mathcal{S}et$ are isomorphisms.
- In $(X, \leq)$
  - if $\leq$ is a partial order, the only isomorphisms are the identities, *or*
  - if $\leq$ is a preorder and $x, y \in X$ we have $x \cong y$ iff $x \leq y$ and $y \leq x$, with only one witness:

$$(x,y) \quad : \quad x \cong y \quad : \quad (y,x)$$

Definition of a Functor

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is specified by

- assigning an object FA in $\mathcal{D}$ to any object A in $\mathcal{C}$, and
- assigning a morphism $F f : FA \rightarrow FB$ in $\mathcal{D}$, to any morphism $f : A \rightarrow B$ in $\mathcal{C}$,

for which

- $F(id_A) = id_{FA}$
- $F(g \circ f) = F g \circ F f$

An Example of a Functor

Define $F : \mathcal{S}et \to \mathcal{S}et$ by

- $FA \stackrel{\text{def}}{=} [A]$, the *finite lists* over A
- $F\,f \stackrel{\text{def}}{=} map(f)$ where

$map(f) : [A] \to [B]$ is defined by

$$map(f)(as) \stackrel{\text{def}}{=} \text{case } as \text{ of}$$
$$\varepsilon \to \varepsilon$$
$$[a_0, \ldots, a_{l-1}] \to [f(a_0), \ldots, f(a_{l-1})]$$

To see that $F(g \circ f) = Fg \circ Ff$ note that

$$
\begin{aligned}
F(g \circ f)([a_0, \ldots, a_{l-1}]) &\stackrel{\text{def}}{=} map(g \circ f)([a_0, \ldots, a_{l-1}]) \\
&= [g(f(a_0)), \ldots, g(f(a_{l-1}))] \\
&= map(g)([f(a_0), \ldots, f(a_{l-1})]) \\
&= map(g)(map(f)([a_0, \ldots, a_{l-1}])) \\
&= Fg \circ Ff([a_0, \ldots, a_{l-1}]).
\end{aligned}
$$

More Examples

- The functors between two preorders A and B are precisely the *monotone functions* from A to B.
- We can define a functor $\mathcal{P}:\mathcal{S}et \longrightarrow \mathcal{S}et$ by setting

$$f:B \longrightarrow A \quad \mapsto \quad \mathcal{P}\,f:\mathcal{P}(A) \longrightarrow \mathcal{P}(B),$$

where the function $\mathcal{P}\,f$ is defined by

$$\mathcal{P}\,f(A') \stackrel{\text{def}}{=} \{f(a) \in B \mid a \in A'\}$$

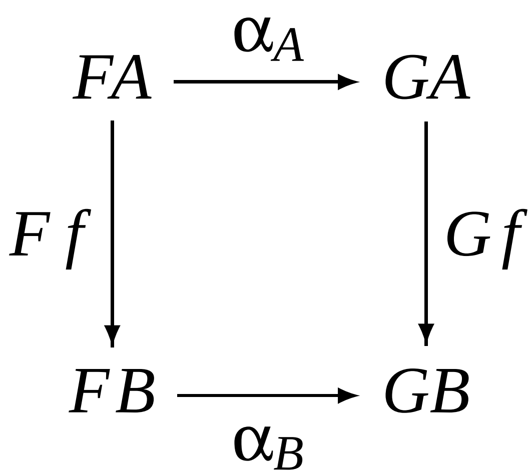
where $A' \in \mathcal{P}(A)$.

Definition of a Natural Transformation

Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. Then a natural transformation

$$\alpha : F \longrightarrow G \qquad \text{is} \qquad (\ \alpha_A : FA \longrightarrow GA \quad | \quad A \ \text{in} \ ob\ \mathcal{C}\ )$$

such that for any $f : A \longrightarrow B$ in $\mathcal{C}$,



An Example of a Natural Transformation

- Recall $F: \mathcal{S}et \rightarrow \mathcal{S}et$ where $FA \stackrel{\text{def}}{=} [A]$ and $F\,f \stackrel{\text{def}}{=} map(f)$.
- There is a natural transformation $rev\!: F \rightarrow F$ with components $rev_A\!: [A] \rightarrow [A]$ defined by

$$rev_A(as) \quad \stackrel{\text{def}}{=} \quad \text{case } as \text{ of } \begin{cases} \varepsilon \rightarrow \varepsilon \\ [a_0, \ldots, a_{l-1}] \rightarrow [a_{l-1}, \ldots, a_0] \end{cases}$$

- Naturality is

$$\begin{aligned} F\,f \circ rev_A([a_0, \ldots, a_{l-1}]) &= [f(a_{l-1}), \ldots, f(a_0)] \\ &= rev_B \circ F\,f([a_0, \ldots, a_{l-1}]) \end{aligned}$$

Another Example

- Define $F_X : \mathcal{S}et \to \mathcal{S}et$ by
  - $F_X(A) \stackrel{\text{def}}{=} (X \to A) \times X$
  - $F_X(f) : (X \to A) \times X \longrightarrow (X \to B) \times X$ where $(g,x) \mapsto (f \circ g, x)$
- Then $ev : F_X \to id_{\mathcal{S}et}$ defined by $ev_A(g,x) \stackrel{\text{def}}{=} g(x)$ is natural

$$\begin{aligned}
(id_{\mathcal{S}et}(f) \circ ev_A)(g,x) &= f(g(x)) \\
&= ev_B(f \circ g, x) \\
&= ev_B(F_X(f)(g,x)) \\
&= (ev_B \circ F_X(f))(g,x).
\end{aligned}$$

Definition of Functor Category

- Let F, G, H be functors $\mathcal{C} \to \mathcal{D}$ and $\alpha : F \to G$ and $\beta : G \to H$ be natural transformations.

- Define $\beta \circ \alpha : F \to H$ by

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A$$

- Then $\mathcal{D}^{\mathcal{C}}$ is the functor category of $\mathcal{C}$ and $\mathcal{D}$, where
  - objects are *functors* $\mathcal{C} \to \mathcal{D}$,
  - morphisms are *natural trans* $\alpha : F \to G : \mathcal{C} \to \mathcal{D}$

- An isomorphism in a functor category is referred to as a natural isomorphism.

- If there is a natural isomorphism between the functors F and G, then we say that F and G are naturally isomorphic, written

$$\phi : F \cong G : \psi$$

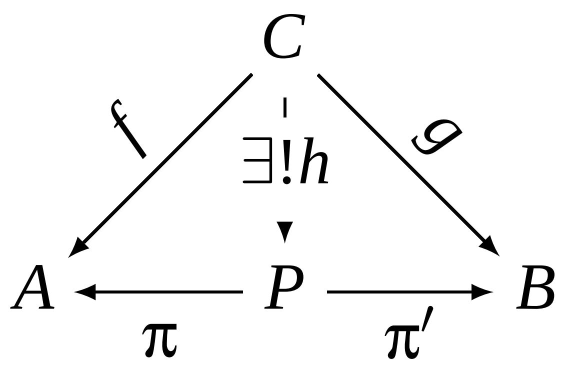
with witnesses the natural transformations ϕ and ψ.

Motivating Binary Products

(*Property* $\Phi(P)$)

- *Given* any two sets A and B,
- *there are* functions $\pi : P \to A$, $\pi' : P \to B$ *such that*:

given any $f : C \to A$, $g : C \to B$ *there is a unique* $h : C \to P$ s.t.



- Suppose that $A \stackrel{\text{def}}{=} \{a, b\}$ and $B \stackrel{\text{def}}{=} \{c, d, e\}$.

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  - Let P be $A \times B \stackrel{\text{def}}{=} \{(x, y) \mid x \in A, y \in B\}$ and
  - π and π' be coordinate projections.

- Let $f{:}C \to A$ and $g{:}C \to B$ be any two functions. Define

$$h{:}C \to P \qquad z \mapsto (f(z), g(z))$$

- We can check (*Property* $\Phi(P)$) ...

- Now define $P' \stackrel{\text{def}}{=} \{1,2,3,4,5,6\}$ and

- $p{:}P' \to A$ and $q{:}P' \to B$ where

$$p(1),\quad p(2),\quad p(3) = a \qquad\qquad q(1),\quad q(4) = c$$
$$p(4),\quad p(5),\quad p(6) = b \qquad\qquad q(2),\quad q(5) = d$$
$$q(3),\quad q(6) = e$$

- We can check (*Property* $\Phi(P')$) ...

- ... the required function $h{:}C \to P'$ exists and is unique: for example, $x \in C$ and $f(x) = a$ and $g(x) = d$ forces $h(x) = 2$

- Note $P' \cong \{(a,c),(a,d),(a,e),(b,c),(b,d),(b,e)\} = P$

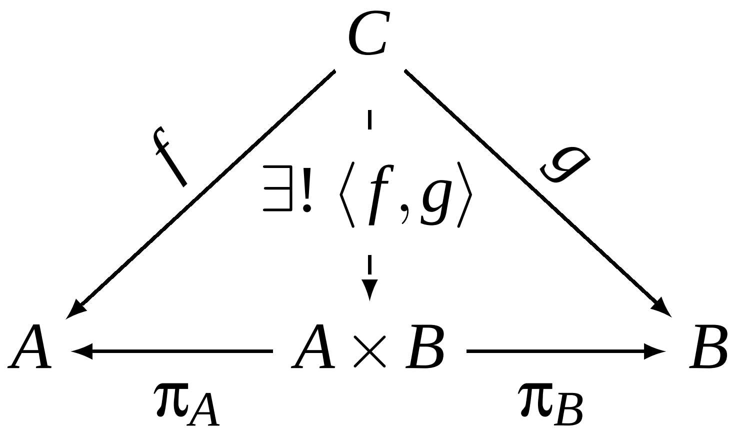
Definition of Binary Products

A binary product of objects A and B in a category $\mathcal{C}$ is specified by

- an object $A \times B$ of $\mathcal{C}$, together with
- two projection morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$,

for which given any object C and morphisms $f : C \to A$, $g : C \to B$, there is a unique morphism $\langle f, g \rangle : C \to A \times B$ for which $\pi_A \circ \langle f, g \rangle = f$ and $\pi_B \circ \langle f, g \rangle = g$.

- Diagrams are helpful



- The unique morphism $\langle f,g\rangle : C \rightarrow A \times B$ is called the mediating morphism

- A property involving *existence of a unique morphism* leading to a *structure determined up to isomorphism* is a universal property.

- Call $\langle f, g \rangle$ the pair of f and g.

- $\mathcal{C}$ has binary products if there is $A \times B$ for any A and B

-
  - $\mathcal{C}$ has specified binary products if there is a *canonical choice*.
  - In $\mathcal{S}et$ take $A \times B \stackrel{\text{def}}{=} \{ (a, b) \mid a \in A, b \in B \}$ with standard projections.

Examples of Binary Products

- *Preset* Given $A \stackrel{\text{def}}{=} (X, \leq_X)$ and $B \stackrel{\text{def}}{=} (Y, \leq_Y)$,

$$A \times B \stackrel{\text{def}}{=} (X \times Y, \leq_{X \times Y})$$

where $X \times Y$ is cartesian product, and

$$(x,y) \leq_{X \times Y} (x',y') \iff x \leq_X x' \wedge y \leq_Y y'$$

The projection

$$\pi_A : (X \times Y, \leq_{X \times Y}) \longrightarrow (X, \leq_X)$$

is given by $(x,y) \mapsto x$, and is monotone

- *Part* Given A and B,

$$P \stackrel{\text{def}}{=} (A \times B) \cup (A \times \{*_A\}) \cup (B \times \{*_B\})$$

  - $\pi_A\colon (A \times B) \cup (A \times \{*_A\}) \cup (B \times \{*_B\}) \longrightarrow A$

    is undefined on $B \times \{*_B\}$, π_B on $A \times \{*_A\}$

  - $\pi_A(a, *_A) = a$ for all $a \in A$, ...

- $\mathbb{F}$ The product of n and m is written $n \times m$ and is given by $n * m$, that is, the set $\{0, \ldots, (n * m) - 1\}$.

Additional Notation

- Can define $A \times B \times C$ and $\langle f, g, h \rangle$
- Take $f : A \to B$ and $f' : A' \to B'$. We write

$$f \times f' \quad \stackrel{\text{def}}{=} \quad \langle f \circ \pi, f' \circ \pi' \rangle \quad : \quad A \times A' \to B \times B'$$

- Universal property means

$$id_A \times id_{A'} = id_{A \times A'} \quad \text{and} \quad (g \times g') \circ (f \times f') = g \circ f \times g' \circ f'$$

where $g : B \to C$ and $g' : B' \to C'$.

- Write A^2 or f^2 for $A \times A$ and $f \times f$

Another Example – Presheaves on $\mathbb{F}$

$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{S}et^{\mathbb{F}}$ If F and F' are presheaves, $F \times F' : \mathbb{F} \longrightarrow \mathcal{S}et$ defined by

$$(F \times F')(n) \quad \stackrel{\text{def}}{=} \quad (Fn) \times (F'n)$$

for n in $\mathbb{F}$ and if $\rho : n \longrightarrow n'$

$$(F \times F')(\rho) \quad \stackrel{\text{def}}{=} \quad (F\rho) \times (F'\rho)$$

Also

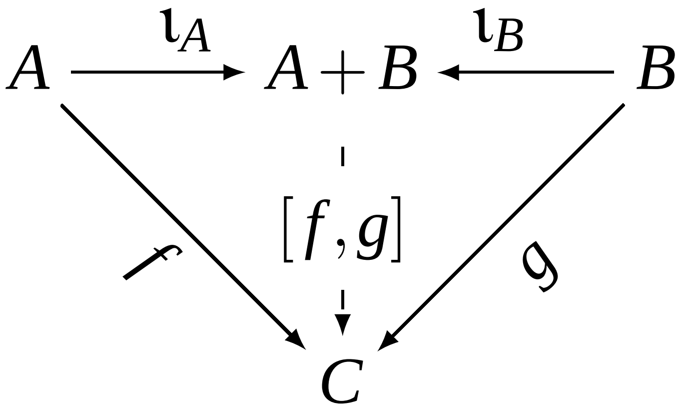
$$\pi_F : F \times F' \longrightarrow F \qquad (\pi_F)_n \stackrel{\text{def}}{=} \pi_{Fn}$$

Definition of Binary Coproducts

A binary coproduct of A and B is specified by

- an object $A+B$, together with
- two insertion morphisms $\iota_A : A \to A+B$ and $\iota_B : B \to A+B$,

such that there is a unique $[f,g]$ for which



for all such f and g

Example of Binary Coproducts

- *Set* For sets A and B define

$$A + B \stackrel{\text{def}}{=} (A \times \{1\}) \cup (B \times \{2\})$$

and

$$\iota_A : A \longrightarrow A + B \qquad a \mapsto (a, 1)$$

Given $f{:}A \longrightarrow C$ and $g{:}B \longrightarrow C$, then $[f,g]{:}A + B \longrightarrow C$ is defined by

$$[f,g](\xi) \stackrel{\text{def}}{=} \text{case } \xi \text{ of}$$

$$\iota_A(\xi_A) = (\xi_A, 1) \mapsto f(\xi_A)$$

$$\iota_B(\xi_B) = (\xi_B, 2) \mapsto f(\xi_B)$$

Additional Notation

- Can define $A+B+C$ with the cotupling $[f,g,h]$
- Take morphisms $f:A \to B$ and $f':A' \to B'$. We write

$$f+f' \stackrel{\text{def}}{=} [\iota_B \circ f, \iota_{B'} \circ f'] \quad : \quad A+A' \to B+B'$$

- Universality means

$$id_A + id_{A'} = id_{A+A'} \quad \text{and} \quad (g+g') \circ (f+f') = g \circ f + g' \circ f'$$

where $g:B \to C$ and $g':B' \to C'$.

- If $l:C \to D$ then $l \circ [f,g] = [l \circ f, l \circ g]$

More Examples

- $\mathbb{F}$ The coproduct of n and m is $n+m$ where we interpret $+$ as addition on $\mathbb{N}$.

- $\mathcal{F}$ If F and F' are presheaves then $F+F'$ is defined by

$$(F+F')\xi \stackrel{\text{def}}{=} (F\xi)+(F'\xi)$$

for any object or morphism ξ in $\mathbb{F}$, and

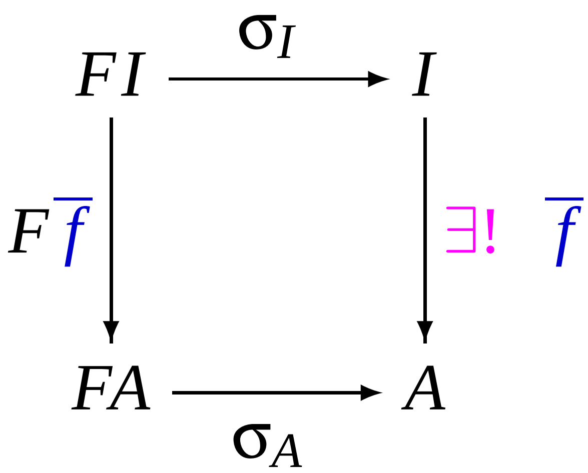
$$\iota_F : F+F' \to F \qquad (\iota_F)_n \stackrel{\text{def}}{=} \iota_{Fn} : (Fn)+(F'n) \to Fn$$

Sometimes say $+$ is defined pointwize.

Definition of Algebras

- Let $F : \mathcal{C} \to \mathcal{C}$. An algebra for the functor F is a pair (A, σ_A) where $\sigma_A : FA \to A$.

- An initial F-algebra (I, σ_I) is an algebra for which given any other (A, σ_A),



Motivation for Initial Algebras

- (Some) *Datatypes* are *initial algebras*

- The datatype

$$Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{S}\ Exp \mid \mathsf{A}\ Exp\ Exp$$

is modeled by an object E such that

$$E \cong \mathbb{V} + E + (E \times E) \qquad\qquad \dagger$$

- We show how to solve $\dagger$ in $\mathcal{S}et$.

- If $\Sigma : \mathcal{S}et \to \mathcal{S}et$ is $\Sigma\xi \stackrel{\text{def}}{=} \mathbb{V} + \xi + (\xi \times \xi)$, then the solution we construct is an initial algebra (σ_E, E).

An Initial Algebra for $1+(-):\mathcal{S}et \longrightarrow \mathcal{S}et$

- $1:\mathcal{S}et \to \mathcal{S}et$ is defined by

$$f:A \to B \quad \mapsto \quad id_{\{*\}}:\{*\} \to \{*\}$$

- $1+(-)$ is defined by

$$f:A \to B \quad \mapsto \quad id_1 + f:1+A \to 1+B$$

- The initial algebra is $\mathbb{N}$ up to isomorphism.

- We set $S_0 \stackrel{\text{def}}{=} \varnothing$ and $S_{r+1} \stackrel{\text{def}}{=} 1 + S_r$.
- Note there is an insertion $\iota_{S_r}\colon S_r \to S_{r+1}$.
- Note also that $i_r\colon S_r \hookrightarrow S_{r+1}$ where $i_0 \stackrel{\text{def}}{=} \varnothing\colon S_0 \to S_1$, and $i_{r+1} \stackrel{\text{def}}{=} id_1 + i_r$.
- We also write $i'_r\colon S_r \hookrightarrow T$ where $T \stackrel{\text{def}}{=} \cup_r S_r$
- T is the object part of an initial algebra for $1 + (-)$.

- As $\sigma_T : 1 + T \to T$ then σ_T must be a copair.
- We set $\sigma_T \stackrel{\text{def}}{=} [k, k']$ where $k : 1 \to T$ and $k' : T \to T$
- Note that

$$1 \xrightarrow{\ \iota_1\ } 1 + \varnothing = S_1 \xrightarrow{\ i'_1\ } T$$

and we set $k \stackrel{\text{def}}{=} i'_1 \circ \iota_1$.

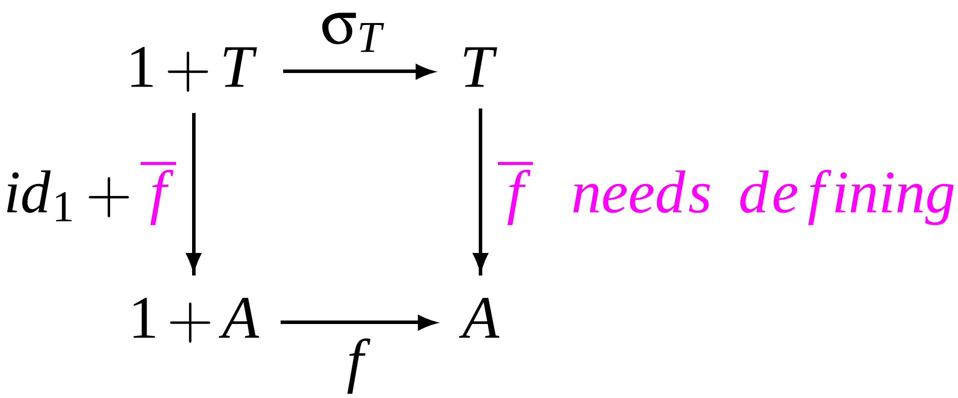
- Note that

$$S_r \xrightarrow{\iota_{S_r}} 1 + S_r = S_{r+1} \xrightarrow{i'_{r+1}} T$$

and we set $k'_r \stackrel{\text{def}}{=} i'_{r+1} \circ \iota_{S_r}$.

-
  - In fact $k'_{r+1} \circ i_r = k'_r$ by induction on r.
  - Hence can legitimately define $k' : T \to T$ by setting $k'(\xi) \stackrel{\text{def}}{=} k'_r(\xi)$ for any r such that $\xi \in S_r$.

- We check initiality



- We define a family of functions $\overline{f}_r : S_r \longrightarrow A$

$$\overline{f}_0 \stackrel{\text{def}}{=} \varnothing : S_0 \longrightarrow A \quad \wedge \quad \overline{f}_{r+1} \stackrel{\text{def}}{=} [f \circ \iota_1, f \circ \iota_A \circ \overline{f}_r]$$

-
  - In fact $\overline{f}_{r+1} \circ i_r = \overline{f}_r$.
  - Hence we can legitimately define $\overline{f} : T \longrightarrow A$ by $\overline{f}(\xi) \stackrel{\text{def}}{=} \overline{f}_r(\xi)$ for any r where $\xi \in S_r$.

- To check that the diagram commutes, we have to prove that

$$\overline{f} \circ [k, k'] = f \circ (id_1 + \overline{f})$$

- By the universal property of coproducts, this is equivalent to showing

$$[\overline{f} \circ k, \overline{f} \circ k'] = [f \circ \iota_1, f \circ \iota_A \circ \overline{f}]$$

which we can do by checking that the respective components are equal.

- We give details for $\overline{f} \circ k' = f \circ \iota_A \circ \overline{f}$.

- $\overline{f} \circ k' = f \circ \iota_A \circ \overline{f}$. Take any element $\xi \in T$. Then we have

$$\begin{aligned}
\overline{f}(k'(\xi)) &= \overline{f}(\iota_{S_r}(\xi)) \\
&= \overline{f}_{r+1}(\iota_{S_r}(\xi)) \\
&= [f \circ \iota_1, f \circ \iota_A \circ \overline{f}_r](\iota_{S_r}(\xi)) \\
&= f(\iota_A(\overline{f}_r(\xi))) \\
&= f(\iota_A(\overline{f}(\xi)))
\end{aligned}$$

The first equality is by definition of k' and k'_r; the second by definition of $\overline{f}$; the third by definition of $\overline{f}_{r+1}$.

- You check that $T \cong N$.

Some Results for Use in Modelling Syntax

- Let F and F' be two presheaves in $\mathcal{F}$. Suppose for any n in $\mathbb{F}$, $F'n \subset Fn$, and

$$\begin{array}{ccc} F'n & \subset & Fn \\ {\scriptstyle F'\rho}\downarrow & & \downarrow{\scriptstyle F\rho} \\ F'n' & \subset & Fn' \end{array}$$

commutes for any $\rho : n \to n'$.

- There is a natural transformation

$$i : F' \hookrightarrow F$$

- We define

$$\delta \colon \mathcal{F} \to \mathcal{F}$$

Suppose that F is an object in $\mathcal{F}$. Then δF is defined by

$$\rho \colon n \to n' \quad \mapsto \quad F(\rho + id_1) \colon F(n+1) \longrightarrow F(n'+1)$$

- If $\alpha \colon F \to F'$ in $\mathcal{F}$, then the components of $\delta\,\alpha$ are given by

$$(\delta\,\alpha)_n \stackrel{\mathrm{def}}{=} \alpha_{n+1}$$

- $(S_r \mid r \geq 0)$ is a family of presheaves in $\mathcal{F}$, with $i_r : S_r \hookrightarrow S_{r+1}$. Then there is a union presheaf T in $\mathcal{F}$, such that $i'_r : S_r \hookrightarrow T$. We sometimes write $\cup_r S_r$ for T.

- Let $\rho : n \to n'$. Then

$$Tn \stackrel{\text{def}}{=} \bigcup_r S_r n$$

and $T\rho : Tn \to Tn'$ is defined by

$$(T\rho)(\xi) \stackrel{\text{def}}{=} (S_r \rho)(\xi)$$

where $\xi \in Tn$, and $\xi \in S_r(n)$ for some r.

- Let $(\phi_r : S_r \to A \mid r \geq 0)$ be natural transformations in $\mathcal{F}$, the S_r as before, and such that $\phi_{r+1} \circ i_r = \phi_r$. Then there is a *unique natural transformation*

$$\phi : T \to A$$

such that $\phi \circ i'_r = \phi_r$.

- The functions $\phi_n : Tn \to An$ defined by

$$\phi_n(\xi) \stackrel{\text{def}}{=} (\phi_r)_n(\xi) \qquad \xi \in S_r n$$

yield the required natural transformation.

Syntax with Distinguished Variables and without Binding

- The set of expressions *Exp* is inductively defined by

$$Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{S}\ Exp \mid \mathsf{A}\ Exp\ Exp$$

- v^i occurs in e is written $v^i \in e$.
- The set of (free) variables of any e is denoted by $fv(e)$.
- We will want to consider expressions e for which

$$fv(e) \subset \{v^0, \ldots, v^{n-1}\}$$

and we give an inductive definition of such expressions.

- First we define inductively a set of judgements $\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} e$ where $n \geq 1$, $\Gamma^n \stackrel{\text{def}}{=} v^0, \ldots, v^{n-1}$ is a list, and of course e is an expression.

- We refer to Γ^n as an environment of variables.

$$\frac{0 \leq i < n}{\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} v^i} \qquad \frac{\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} e}{\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} \mathsf{S}\ e} \qquad \frac{\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} e \quad \Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} e'}{\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} \mathsf{A}\ e\ e'}$$

- One can then prove by rule induction that if $\Gamma^n \vdash^{\mathsf{d}\overline{\mathsf{b}}} e$ then $fv(e) \subset \Gamma^n$. We prove by Rule Induction

$$(\forall\, (\Gamma^n, e) \in \vdash^{\mathsf{d}\overline{\mathsf{b}}}) \ \boxed{(fv(e) \subset \Gamma^n)}$$

Syntax with Distinguished Variables and Binding

- Consider

$$Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{L}\ \mathbb{V}\ Exp \mid \mathsf{E}\ Exp\ Exp$$

- We inductively define a set of judgements $\Gamma^n \vdash^{\mathsf{db}} e$ where $n \geq 1$.

$$\frac{0 \leq i < n}{\Gamma^n \vdash^{\mathsf{db}} v^i} \qquad \frac{\Gamma^{n+1} \vdash^{\mathsf{db}} e}{\Gamma^n \vdash^{\mathsf{db}} \mathsf{L}\ v^n\ e} \qquad \frac{\Gamma^n \vdash^{\mathsf{db}} e \quad \Gamma^n \vdash^{\mathsf{db}} e'}{\Gamma^n \vdash^{\mathsf{db}} \mathsf{E}\ e\ e'}$$

- One can then prove by rule induction that if $\Gamma^n \vdash^{\mathsf{db}} e$ then $fv(e) \subset \Gamma^n$.

- Notice that the rule for introducing abstractions $\mathsf{L}\ v^n\ e$ forces a *distinguished* choice of binding variable.
- The advantage of *distinguished binding* is that the expressions correspond exactly to the terms of the λ-calculus, without the need to define α-equivalence.
- In essence, we are forced to pick a representative of each α-equivalence class.

Syntax with Arbitrary Variables and Binding

- Expressions are still defined by

$$Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{L}\ \mathbb{V}\ Exp \mid \mathsf{E}\ Exp\ Exp$$

- Now let Δ range over *all non-empty finite lists* of variables *which have distinct elements*. Thus a typical non-empty Δ is $v^1, v^8, v^{100}, v^2 \in [\mathbb{V}]$. Let $x, y, \ldots$ range over $\mathbb{V}$.

- Define $\Delta \vdash^{\mathsf{ab}} e$ by

$$\frac{x \in \Delta}{\Delta \vdash^{\mathsf{ab}} x} \qquad \frac{\Delta, x \vdash^{\mathsf{ab}} e}{\Delta \vdash^{\mathsf{ab}} \mathsf{L}\ x\ e} \qquad \frac{\Delta \vdash^{\mathsf{ab}} e \quad \Delta \vdash^{\mathsf{ab}} e'}{\Delta \vdash^{\mathsf{ab}} \mathsf{E}\ e\ e'}$$

- We define simultaneous substitution – used to define α-equivalence, and to construct mathematical models.

- $\mathsf{el}_p(\Delta)$ is the pth element of Δ, with position 0 the "first" element.

- We will define by recursion over expressions e, new expressions $e\{\varepsilon/\varepsilon\}$ and $e\{\Delta'/\Delta\}$, where $\mathsf{len}(\Delta) = \mathsf{len}(\Delta')$.

- For example,

$$(\mathsf{L}\ v^8\ (\mathsf{A}\ v^{10}\ v^2))\{v^3, v^8/v^8, v^2\} = \mathsf{L}\ v^{11}\ (\mathsf{A}\ v^{10}\ v^8)$$

$$(\mathsf{V}\,x)\{\Delta'/\Delta\} \stackrel{\text{def}}{=} \begin{cases} x & \text{if} \quad (\forall p)(\mathsf{el}_p(\Delta) \neq x) \\ \mathsf{el}_p(\Delta') & \text{if} \quad (\exists p)(\mathsf{el}_p(\Delta) = x) \end{cases}$$

$$(\mathsf{L}\,x\,e)\{\Delta'/\Delta\} \stackrel{\text{def}}{=} \begin{cases} \mathsf{L}\,x\,e\{\overline{\Delta'}/\overline{\Delta}\} \\ \quad \text{if} \quad (\forall p)(\mathsf{el}_p(\Delta') \neq x \ \vee\ \mathsf{el}_p(\Delta) \notin \mathit{fv}(e)) \\ \mathsf{L}\,x'\,e\{\overline{\Delta'},x'/\overline{\Delta},x\} \\ \quad \text{if} \quad (\exists p)(\mathsf{el}_p(\Delta') = x \wedge \mathsf{el}_p(\Delta) \in \mathit{fv}(e)) \end{cases}$$

$$(\mathsf{E}\,e\,e')\{\Delta'/\Delta\} \stackrel{\text{def}}{=} \mathsf{E}\,e\{\Delta'/\Delta\}\,e'\{\Delta'/\Delta\}$$

where

- $\overline{\Delta}$ is Δ with x deleted (from position p, if it occurs) and, *if* x *does occur*, $\overline{\Delta'}$ is Δ' with the element in position p deleted, and is otherwise Δ'; and
- x' is the variable v^w where w is 1 plus the maximum of the indices appearing in $\overline{\Delta'}$ and $fv(e)$.

- We inductively define the relation $\sim_\alpha$ of α-equivalence
  - Single axiom (schema) $\mathsf{L}\, x\, e \sim_\alpha \mathsf{L}\, x'\, e\{x'/x\}$ with $x' \notin fv(e)$
  - Rules such as

$$\frac{e \sim_\alpha e' \qquad e' \sim_\alpha e''}{e \sim_\alpha e''} \qquad\qquad \frac{e \sim_\alpha e'}{\mathsf{L}\, x\, e \sim_\alpha \mathsf{L}\, x\, e'}$$

- Note that the terms of the λ-calculus are given by the

$$[e]_\alpha \stackrel{\text{def}}{=} \{e' \mid e' \sim_\alpha e\}$$

A Programme for Modelling Syntax

Step 1 define an *abstract endofunctor* $\Sigma_{\mathbb{V}}$ on $\mathcal{F} \stackrel{\text{def}}{=} \mathcal{S}et^{\mathbb{F}}$ (similar to the datatype in question);

Step 2 construct an *initial algebra* T for $\Sigma_{\mathbb{V}}$;

Step 3 show that the *syntax yields a functor* $Exp : \mathbb{F} \to \mathcal{S}et$;

Step 4 show that $T \cong Exp$

Modelling $Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{S}\ Exp \mid \mathsf{A}\ Exp\ Exp$

Step 1

- First, we define the functor $\mathbb{V} : \mathbb{F} \to \mathcal{S}et$. Let $\rho : m \to n$ in $\mathbb{F}$. Then we set

$$\mathbb{V}m \stackrel{\text{def}}{=} \{v^0, \ldots, v^{m-1}\} \qquad \wedge \qquad \mathbb{V}\rho(v^i) \stackrel{\text{def}}{=} v^{\rho i}$$

- Define a functor $\Sigma_{\mathbb{V}} : \mathcal{S}et^{\mathbb{F}} \to \mathcal{S}et^{\mathbb{F}}$ by setting

$$\Sigma_{\mathbb{V}}\xi \stackrel{\text{def}}{=} \mathbb{V} + \xi + \xi^2$$

Step 2

- $T \stackrel{\text{def}}{=} \bigcup(S_r \mid r \geq 0)$.
- $S_0 \stackrel{\text{def}}{=} \varnothing$, the empty presheaf, and

$$S_{r+1} \stackrel{\text{def}}{=} \Sigma_{\mathbb{V}} S_r = \mathbb{V} + S_r + S_r^2$$

- Need to check $i_r : S_r \hookrightarrow S_{r+1}$ for all $r \geq 0$. We use induction over r.
- It is immediate that $i_0 : S_0 \hookrightarrow S_1$.

- Now suppose that $i_r : S_r \hookrightarrow S_{r+1}$. We are required to show that $i_{r+1} : S_{r+1} \hookrightarrow S_{r+2}$, that is,

$$
\begin{array}{ccc}
\mathbb{V}n + S_r n + (S_r n)^2 & \subset & \mathbb{V}n + S_{r+1} n + (S_{r+1} n)^2 \\
{\scriptstyle \mathbb{V}\rho + S_r \rho + (S_r \rho)^2}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbb{V}\rho + S_{r+1} \rho + (S_{r+1} \rho)^2} \\
\mathbb{V}n' + S_r n' + (S_r n')^2 & \subset & \mathbb{V}n' + S_{r+1} n' + (S_{r+1} n')^2
\end{array}
$$

- $\Sigma_{\mathbb{V}} i_r = id_{\mathbb{V}} + i_r + i_r^2$. Thus we have $i_{r+1} = \Sigma_{\mathbb{V}} i_r$.

- We define the structure map $\sigma_T \stackrel{\text{def}}{=} [\kappa, \kappa', \kappa'']: \mathbb{V} + T + T^2 \to T$

- $\mathbb{V} \cong S_1$, so that $\kappa: \mathbb{V} \cong S_1 \hookrightarrow T$. $S_1 = \mathbb{V} + \varnothing + \varnothing^2$, and so $S_1 n = \mathbb{V} n \times \{1\}$.

- We define κ' by

$$\kappa'_r : S_r \xrightarrow{\iota_{S_r}} \mathbb{V} + S_r + S_r^2 = S_{r+1} \hookrightarrow T$$

- check that $\kappa'_{r+1} \circ i_r = \kappa'_r$, ie

$$\begin{array}{ccccc} S_{r+1} & \xrightarrow{\iota_{S_{r+1}}} & S_{r+2} & \xrightarrow{\hookrightarrow} & T \\ {\scriptstyle i_r}\uparrow & & {\scriptstyle i_{r+1}}\uparrow & & \Vert \\ S_r & \xrightarrow[\iota_{S_r}]{} & S_{r+1} & \xrightarrow[\hookrightarrow]{} & T \end{array} \qquad i_{r+1} = id_{\mathbb{V}} + i_r + i_r^2$$

- Write $S'_r \stackrel{\text{def}}{=} S_r^2$. Consider the family of morphisms

$$\kappa''_r : S'_r = S_r^2 \xrightarrow{\iota_{S_r^2}} \mathbb{V} + S_r + S_r^2 = S_{r+1} \hookrightarrow T$$

- κ''_r satisfy the union conditions ...

- Hence they define $\kappa'' : U \rightarrow T$ where $U \stackrel{\text{def}}{=} \cup_r S'_r$. But note that

$$U n = \cup_r S'_r n = \cup_r (S_r n)^2 = (\cup_r S_r n)^2 = (T n)^2 = T^2 n$$

and also $U \rho = T^2 \rho$. Hence $U = T^2$.

- We check initiality

$$\begin{array}{ccc} \mathbb{V}+T+T^2 & \xrightarrow{\sigma_T} & T \\ {\scriptstyle \mathbb{V}+\overline{\alpha}+\overline{\alpha}^2}\big\downarrow & (*) & \big\downarrow{\scriptstyle \overline{\alpha}} \\ \mathbb{V}+A+A^2 & \xrightarrow{\alpha} & A \end{array}$$

- To define $\overline{\alpha}\colon T \to A$ we specify a family $\overline{\alpha}_r\colon S_r \to A$.

- Note that $\overline{\alpha}_0\colon \varnothing \to A$ and thus we define

$$(\overline{\alpha}_0)_n \stackrel{\text{def}}{=} \varnothing\colon \varnothing \to An$$

- Note that $\overline{\alpha}_{r+1}\colon S_{r+1} = \mathbb{V}+S_r+S_r^2 \to A$ and hence

$$\overline{\alpha}_{r+1} \stackrel{\text{def}}{=} [\alpha\circ\iota_{\mathbb{V}}, \alpha\circ\iota_A\circ\overline{\alpha}_r, \alpha\circ\iota_{A^2}\circ\overline{\alpha}_r^2]$$

- Need to verify $\overline{\alpha}_{r+1}\circ i_r = \overline{\alpha}_r$ for all $r \geq 0$.

- Proving that the diagram $(*)$ commutes is equivalent to proving

$$[\overline{\alpha} \circ \kappa, \overline{\alpha} \circ \kappa', \overline{\alpha} \circ \kappa'] = [\alpha \circ \iota_{\mathbb{V}}, \alpha \circ \iota_A \circ \overline{\alpha}, \alpha \ \circ \iota_{A^2} \circ \overline{\alpha}^2]$$

- We prove that $\overline{\alpha}_n \circ \kappa'_n = \alpha_n \circ \iota_{An} \circ \overline{\alpha}_n : Tn \to An$.

- Suppose that ξ is an arbitrary element of Tn, where $\xi \in S_r n$.

$$\begin{aligned}
\overline{\alpha}_n(\kappa'_n(\xi)) &= \overline{\alpha}_n(\iota_{S_r n}(\xi)) \\
&= (\overline{\alpha}_{r+1})_n(\iota_{S_r n}(\xi)) \\
&= [\alpha_n \circ \iota_{\mathbb{V}n}, \alpha_n \circ \iota_{An} \circ (\overline{\alpha}_r)_n, \alpha_n \circ \iota_{(An)^2} \circ (\overline{\alpha}_r)_n^2](\iota_{S_r n}(\xi)) \\
&= \alpha_n(\iota_{An}((\overline{\alpha}_r)_n(\xi))) \\
&= \alpha_n(\iota_{An}(\overline{\alpha}_n(\xi)))
\end{aligned}$$

Step 3

- Suppose that $\rho : n \to n'$ is any function. We define

$$Exp_{\mathsf{d\bar{b}}}\, n \stackrel{\text{def}}{=} \{\, e \mid \Gamma^n \vdash^{\mathsf{d\bar{b}}} e \,\}$$

- We can define $(Exp_{\mathsf{d\bar{b}}}\, \rho)e$ by recursion over e, by setting
  - $(Exp_{\mathsf{d\bar{b}}}\, \rho)(\mathsf{V}\, v^i) \stackrel{\text{def}}{=} \mathsf{V}\, \rho i$
  - $(Exp_{\mathsf{d\bar{b}}}\, \rho)(\mathsf{S}\, e) \stackrel{\text{def}}{=} \mathsf{S}\, (Exp_{\mathsf{d\bar{b}}}\, \rho)e$
  - $(Exp_{\mathsf{d\bar{b}}}\, \rho)(\mathsf{A}\, e\, e') \stackrel{\text{def}}{=} \mathsf{A}\, (Exp_{\mathsf{d\bar{b}}}\, \rho)e\, (Exp_{\mathsf{d\bar{b}}}\, \rho)e'$

- … and then showing that if $e \in \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}\, n$, then $(\mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}\, \rho)e \in \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}\, n'$.

- Thus we have a function

$$\mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}\, \rho : \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}\, n \longrightarrow \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}\, n'$$

for any $\rho : n \to n'$.

- Note that there are natural transformations

$$\mathsf{S} : \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}} \longrightarrow \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}} \qquad \wedge \qquad \mathsf{A} : {\mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}}^{2} \longrightarrow \mathit{Exp}_{\mathsf{d}\bar{\mathsf{b}}}$$

Step 4

- We now show that $T \cong Exp_{\mathsf{d\bar{b}}}$ in $\mathcal{F}$
- We define $\phi : T \to Exp_{\mathsf{d\bar{b}}}$ and $\psi : Exp_{\mathsf{d\bar{b}}} \to T$, such that

$$\phi_n \quad : \quad T n \cong Exp_{\mathsf{d\bar{b}}}\, n \quad : \quad \psi_n$$

- To specify $\phi : T \to Exp_{\mathsf{d\bar{b}}}$ define a family $\phi_r : S_r \to Exp_{\mathsf{d\bar{b}}}$.
  - $\phi_0 : S_0 = \varnothing \to Exp_{\mathsf{d\bar{b}}}$ has components $(\phi_0)_n : \varnothing \to Exp_{\mathsf{d\bar{b}}}\, n$
  - Recursively we define

$$\phi_{r+1} \quad \stackrel{\text{def}}{=} \quad [\mathsf{V}, \mathsf{S} \circ \phi_r, \mathsf{A} \circ \phi_r^2] \quad : \quad S_{r+1} = \mathbb{V} + S_r + S_r^2 \to Exp_{\mathsf{d\bar{b}}}$$

- To specify $\psi : Exp_{\mathsf{d\bar{b}}} \to T$, for any n in $\mathbb{F}$ we define functions

$$\psi_n : Exp_{\mathsf{d\bar{b}}}\, n \to T n$$

as follows.

- $\psi_n(\mathsf{V}\ v^i) \stackrel{\text{def}}{=} (v^i, 1) \in S_1 n$
- $\psi_n(\mathsf{S}\ e) \stackrel{\text{def}}{=} \iota_{S_r n}(\psi_n(e))$ where $r \geq 1$ is the height of the deduction of $\mathsf{S}\ e$
- $\psi_n(\mathsf{A}\ e\ e') \stackrel{\text{def}}{=} \iota_{(S_r n)^2}((\psi_n(e), \psi_n(e')))$ where $r \geq 1$ is the height of the deduction of $\mathsf{A}\ e\ e'$.

- We next check that for any n in $\mathbb{F}$,

$$Tn \underset{\psi_n}{\overset{\phi_n}{\rightleftarrows}} Exp_{\mathsf{d\bar{b}}}\, n \qquad (\cong)$$

- Suppose $\xi \in S_r n \subset Tn$ for some r. Then by definition,

$$\psi_n(\phi_n(\xi)) = \psi_n((\phi_r)_n(\xi))$$

- By induction, for all $r \geq 0$, if $\xi \in S_r n$ and n any object of $\mathbb{F}$, then

$$\psi_n((\phi_r)_n(\xi)) = \xi$$

- Let $\xi \in S_{r+1}n = \mathbb{V}n + S_r n + S_r n^2$. Then we have

$$\psi_n((\phi_{r+1})_n(\xi)) = \psi_n([\mathsf{V}_n, \mathsf{S}_n \circ (\phi_r)_n, \mathsf{A}_n \circ (\phi_r)_n^2](\xi))$$

- Consider the case when $\xi = \iota_{S_r n}(\xi')$ for some $\xi' \in S_r n$. We have

$$\begin{aligned}
\psi_n((\phi_{r+1})_n(\xi)) &= \psi_n((\mathsf{S}_n \circ (\phi_r)_n)(\xi')) \\
&= \psi_n(\mathsf{S}\ (\phi_r)_n(\xi')) \\
&= \iota_{S_r n}(\psi_n((\phi_r)_n(\xi'))) \\
&= \iota_{S_r n}(\xi') \\
&= \xi
\end{aligned}$$

Modelling $Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{L}\ \mathbb{V}\ Exp \mid \mathsf{E}\ Exp\ Exp$

Case $\Gamma^n \vdash^{\mathsf{db}} e$ with *Distinguished Binding*

- *Step 1* The abstract endofunctor $\Sigma_{\mathbb{V}} : \mathcal{F} \rightarrow \mathcal{F}$ is

$$\Sigma_{\mathbb{V}}\xi \stackrel{\text{def}}{=} \mathbb{V} + \delta\,\xi + \xi^2$$

Motto: Any constructor with 1 *argument and which binds* b *variables is modelled by* $\delta^b\xi$. Thus

$$\mathsf{Split}\ P\ \mathsf{as}\ \langle x, y\rangle\ \mathsf{in}\ E$$

would be modelled by $\xi \mapsto \xi \times \delta\,\delta\,\xi$

- *Step 2* We can show that the functor $\Sigma_{\mathbb{V}}$ has an initial algebra $\sigma_T : \Sigma_{\mathbb{V}} T \to T$, by adapting the previous methods.

- Have to define

$$\sigma_T \stackrel{\text{def}}{=} [\kappa, \kappa', \kappa''] \stackrel{\text{def}}{=} \mathbb{V} + \delta\, T + T \times T \to T$$

via

$$\kappa'_r : \delta\, S_r \xrightarrow{\iota_{S_r}} \mathbb{V} + \delta\, S_r + S_r^2 = S_{r+1} \hookrightarrow T$$

as

$$(\delta\, T)n \stackrel{\text{def}}{=} T(n+1) = \bigcup_r S_r(n+1) = \bigcup_r (\delta\, S_r)n = (\bigcup_r \delta\, S_r)n$$

- *Step 3* Suppose $\rho : n \to n'$. Define

$$Exp_{\mathsf{db}}\ n \stackrel{\text{def}}{=} \{\, e \mid \Gamma^n \vdash^{\mathsf{db}} e \,\}$$

- Let $\rho\{n'/n\} : n+1 \to n'+1$ be

$$\rho\{n'/n\}(j) \stackrel{\text{def}}{=} \begin{cases} j & \text{if} \quad 0 \le j \le n-1 \\ n' & \text{if} \quad j = n \end{cases}$$

Consider

- $(Exp_{\mathsf{db}}\ \rho)(\mathsf{L}\ v^n\ e) \stackrel{\text{def}}{=} \mathsf{L}\ v^{n'}\ (Exp_{\mathsf{db}}\ \rho\{n'/n\})(e)$ and
- $(Exp_{\mathsf{db}}\ \rho)(\mathsf{E}\ e\ e') \stackrel{\text{def}}{=} \mathsf{E}\ ((Exp_{\mathsf{db}}\ \rho)e)\ ((Exp_{\mathsf{db}}\ \rho)e')$

- If $\Gamma^n \vdash^{\mathsf{db}} e$ and $\rho : n \to n'$, then $\Gamma^{n'} \vdash^{\mathsf{db}} (Exp_{\mathsf{db}}\ \rho)e$ yielding a functor Exp_{db} in $\mathcal{F}$.

- There are natural transformations

$$\mathsf{L} : \delta\, Exp_{\mathsf{db}} \rightarrow Exp \quad \wedge \quad \mathsf{E} : Exp^2 \rightarrow Exp$$

- The components are functions

$$\mathsf{L}_n : Exp_{\mathsf{db}}\,(n+1) \rightarrow Exp_{\mathsf{db}}\,n \quad \mapsto \quad e \mapsto \mathsf{L}\; v^n\; e$$

- Naturality is

$$\begin{array}{rcl}
(\delta\, Exp_{\mathsf{db}})n = Exp_{\mathsf{db}}\,(n+1) & \xrightarrow{\mathsf{L}_n} & Exp_{\mathsf{db}}\,n \\
(\delta\, Exp_{\mathsf{db}})\rho = Exp_{\mathsf{db}}\,(\rho + id_1)\ \big\downarrow & & \big\downarrow\ Exp_{\mathsf{db}}\,\rho \\
(\delta\, Exp_{\mathsf{db}})n' = Exp_{\mathsf{db}}\,(n'+1) & \xrightarrow[\mathsf{L}_{n'}]{} & Exp_{\mathsf{db}}\,n'
\end{array}$$

- Note that at the element e, this requires that

$$\mathsf{L}\, v^{n'}\ (Exp_{\mathsf{db}}\ \rho\{n'/n\})e = \mathsf{L}\, v^{n'}\ ((Exp_{\mathsf{db}}\ (\rho + id_1))e)$$

- This equality holds if and only if

$$\rho\{n'/n\} = \rho + id_1$$

- ...which is true if and only if in $\mathbb{F}$

$$\iota_1 : 1 \longrightarrow m+1 \quad * \mapsto m \qquad \iota_m : m \longrightarrow m+1 \quad i \mapsto \rho i$$

- *Step 4* A routine calculation that $T \cong Exp_{\mathsf{db}}$

Modelling $Exp ::= \mathsf{V}\ \mathbb{V} \mid \mathsf{L}\ \mathbb{V}\ Exp \mid \mathsf{E}\ Exp\ Exp$

Case $\Delta \vdash^{\mathsf{ab}} e$ **with** ***Arbitrary Binding***

- *Step 1* The abstract endofunctor $\Sigma_{\mathbb{V}} : \mathcal{F} \to \mathcal{F}$ is

$$\Sigma_{\mathbb{V}}\xi \stackrel{\text{def}}{=} \mathbb{V} + \delta\,\xi + \xi^2$$

Note: The functor is the SAME as before

- *Step 2* Thus solving for the initial algebra is the same as before!

- *Step 3* We define Exp_{ab} . For n in $\mathbb{F}$ we set

$$Exp_{\mathsf{ab}}\, n \stackrel{\text{def}}{=} \{\ [e]_\alpha \ \mid\ \Gamma^n \vdash^{\mathsf{ab}} e\ \}$$

- Now let $\rho : n \to n'$. We define

$$(Exp_{\mathsf{ab}}\, \rho)([e]_\alpha) \stackrel{\text{def}}{=} [e\{v^{\rho 0}, \ldots, v^{\rho(n-1)} / v^0, \ldots, v^{n-1}\}]_\alpha$$

- One has to check that this is well defined . . . see the notes.

- *Step 4* Note that current *Step 2* was same as before. Rather than prove $Exp_{\mathsf{ab}} \cong T$ as a final step, we could in fact make use of the previous work, which proved that $Exp_{\mathsf{db}} \cong T$. Thus we omit *Step 2*, and instead show

$$\phi : Exp_{\mathsf{ab}} \cong Exp_{\mathsf{db}} : \psi$$

- The components of ψ are functions $\psi_n : Exp_{db}\ n \longrightarrow Exp_{ab}\ n$ given by $\psi_n(e) \stackrel{\text{def}}{=} [e]_\alpha$.

- We consider the naturality of ψ at a morphism $\rho : n \longrightarrow n'$, computed at an element ξ of $Exp_{db}\ n$. We show naturality for the case $\xi = \mathsf{L}\ v^n\ e$.

$$
\begin{aligned}
(Exp_{ab}\ \rho) \circ \psi_n(\xi) &= (Exp_{ab}\ \rho)[\mathsf{L}\ v^n\ e]_\alpha \\
&= [(\mathsf{L}\ v^n\ e)\{v^{\rho 0}, \ldots, v^{\rho(n-1)} / v^0, \ldots, v^{n-1}\}]_\alpha \\
&\stackrel{\text{def}}{=} \square
\end{aligned}
$$

Let us consider the case when renaming takes place.

- Suppose that there is a j for which $\rho(j) = n$ and $v^j \in fv(e)$.

- Then

$$(\mathsf{L}\ v^n\ e)\{v^{\rho(0)},\ldots,v^{\rho(n-1)}/v^0,\ldots,v^{n-1}\} =$$

$$\mathsf{L}\ v^w\ e\{v^{\rho(0)},\ldots v^{\rho(n-1)},v^w/v^0,\ldots,v^{n-1},v^n\}$$

- $w = 1 + \mathit{MaxIndex}(e\ ;\ \rho(0),\ldots,\rho(n-1))$ thus $\rho(i) < w$ for all $0 \leq i \leq n-1$.
- But $fv(e) \subset v^0,\ldots,v^n$ and $n = \rho(j) \in \rho(0),\ldots,\rho(n-1)$.
- Also $\rho(i) < n'$, and so we must have $w \leq n'$.
- If $w < n'$, then $v^{n'}$ is not free in $e\{v^{\rho(0)},\ldots v^{\rho(n-1)},v^w/v^0,\ldots,v^{n-1},v^n\}$ and otherwise $w = n'$.

Either way (why!?),

$$\mathsf{L}\, v^{w}\; e\{v^{\rho(0)}, \ldots v^{\rho(n-1)}, v^{w}/v^{0}, \ldots, v^{n-1}, v^{n}\}$$
$$\sim_{\alpha} \mathsf{L}\, v^{n'}\; e\{v^{\rho(0)}, \ldots v^{\rho(n-1)}, v^{n'}/v^{0}, \ldots, v^{n-1}, v^{n}\}$$

and so

$$\begin{aligned} \square &= [\mathsf{L}\, v^{n'}\; e\{v^{\rho 0}, \ldots, v^{\rho(n-1)}, v^{n'}/v^{0}, \ldots, v^{n-1}, v^{n}\}]_{\alpha} \\ &= [\mathsf{L}\, v^{n'}\; (Exp_{\mathsf{db}}\, \rho\{n'/n\})e]_{\alpha} \\ &= \psi_{n'} \circ (Exp_{\mathsf{db}}\, \rho)(\xi) \end{aligned}$$

- Next we define $\phi_n : Exp_{\mathsf{ab}}\ n \to Exp_{\mathsf{db}}\ n$ by setting $\phi_n([e]_\alpha) \stackrel{\text{def}}{=} R^n(e)$ where
  - $R^m(\mathsf{V}\ x) \stackrel{\text{def}}{=} \mathsf{V}\ x$
  - $R^m(\mathsf{L}\ x\ e) \stackrel{\text{def}}{=} \mathsf{L}\ v^m\ R^{m+1}(e\{v^m/x\})$
  - $R^m(\mathsf{E}\ e\ e') \stackrel{\text{def}}{=} \mathsf{E}\ R^m(e)\ R^m(e')$
- This is best understood by a simple example . . .
- The verification that

$$\phi : Exp_{\mathsf{ab}} \cong Exp_{\mathsf{db}} : \psi$$

is omitted from the lectures. See the notes.

$$R^3(\mathsf{L}\ v^7\ (\mathsf{L}\ v^3\ (\mathsf{E}\ v^7\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^6\ (\mathsf{E}\ v^2\ v^3))))))$$

$$= \mathsf{L}\ v^3\ R^4(\mathsf{L}\ v^3\ (\mathsf{E}\ v^7\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^6\ (\mathsf{E}\ v^2\ v^3))))))\{v^3/v^7\}$$

$$= \mathsf{L}\ v^3\ R^4(\mathsf{L}\ v^4\ (\mathsf{E}\ v^3\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^6\ (\mathsf{E}\ v^2\ v^4)))))$$

$$= \mathsf{L}\ v^3\ (\mathsf{L}\ v^4\ R^5(\mathsf{E}\ v^3\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^6\ (\mathsf{E}\ v^2\ v^4))))\{v^4/v^4\})$$

$$= \mathsf{L}\ v^3\ (\mathsf{L}\ v^4\ (\mathsf{E}\ v^3\ (\mathsf{E}\ v^0\ (R^5(\mathsf{L}\ v^6\ (\mathsf{E}\ v^2\ v^4))))))$$

$$= \mathsf{L}\ v^3\ (\mathsf{L}\ v^4\ (\mathsf{E}\ v^3\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^5\ R^5(\mathsf{E}\ v^2\ v^4)\{v^5/v^6\}))))$$

$$= \mathsf{L}\ v^3\ (\mathsf{L}\ v^4\ (\mathsf{E}\ v^3\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^5\ (R^5(\mathsf{E}\ v^2\ v^4))))))$$

$$= \mathsf{L}\ v^3\ (\mathsf{L}\ v^4\ (\mathsf{E}\ v^3\ (\mathsf{E}\ v^0\ (\mathsf{L}\ v^5\ (\mathsf{E}\ v^2\ v^4)))))$$

Where to Now? You might

- learn more *Category Theory*;
- learn more *Type Theory*;
- learn more *Categorical Type Theory*;
- spend some time trying to *understand the key problems* and issues concerning modelling and reasoning about *binding syntax*; and
- read the *current research literature* on modelling and implementing *binding syntax*.