Categorical Logic

and

Type Theory

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Overview

- We shall study the connections between *logic* and *category theory*; and *type theory* and *category theory*.
- We shall assume an undergraduate knowledge of basic logic, and an appreciation of type theory from programming and from Mark's lectures.
- We shall cover concisely the aspects of order theory and category theory required for the course ...

High Level Topics

- *Order Theory:* We shall study properties of orders, and use these properties to model logics.

- *Induction:* We assume familiarity with this topic, but review some notation and basic ideas.

- *Category Theory:* We shall study some simple category theory, and use this to model type theories.

High Level Topics

- *Logic:* We will define a simple logic, derive an order theoretic semantics from first principles, and show how a theory in the logic "corresponds" to a special ordered structure.
- *Type Theory:* We will define a simple type system, derive a categorical semantics from first principles, and show how a theory in the type system "corresponds" to a special category.
- *Applications:* We apply the correspondences to prove a result about logic, and a result about type theory.

Order Theory

- An order makes precise our intuitions about the relation *less than*.
- We shall review basic structure of orders, such as bounds, greatest and least elements, meets and joins, and the (possibly unfamiliar) Heyting implication.
- *Why?* We shall see later on that very similar structure can be found in simple logics.
- We shall also define functions between orders which preserve structure, and use such functions to define when two structures have "the same" properties.

Preordered Sets

- A **preorder** on a set X is a binary relation $\leq$ on X which is reflexive and transitive.
- A **preordered** set $(X, \leq)$ is a set X, equipped with a preorder $\leq$ on the set X.
- If $x \leq y$ and $y \leq x$ then we shall write $x \cong y$ and say that x and y are **isomorphic** elements. Note that we can regard $\cong$ as a relation on X, which is in fact an equivalence relation.

Partially Ordered Sets

- A **partial order** on a set X is a binary relation $\leq$ which is reflexive, transitive and anti-symmetric.
- A **partially ordered set** (**poset**) $(X, \leq)$ is a set X equipped with a partial order $\leq$ on the set X.

Examples of Ordered Sets

- The set $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is called the **powerset** of X. The powerset is a poset with order given by inclusion of subsets, $A \subseteq B$.

- Given preorders X and Y, their **cartesian product** has underlying set

$$X \times Y \stackrel{\text{def}}{=} \{(x,y) \mid x \in X, y \in Y\}$$

with order given **pointwise**, that is $(x,y) \leq (x',y')$ iff $x \leq x'$ and $y \leq y'$.

Properties in Ordered Sets

- Suppose that X is a preorder and A is a subset of X. An element $x \in X$ is an **upper bound** for A if for every $a \in A$ we have $a \leq x$ (written $A \leq x$).

- An element $x \in X$ is a **greatest element** of A if it is an upper bound of A which belongs to A;

- Lower bounds and least elements are defined analogously.

Meets and Joins

- Let X be a preordered set and $A \subseteq X$. A **join** of A, if such exists, is a least element in the set of upper bounds for A.

- A **meet** of A, if it exists, is a greatest element in the set of lower bounds for A.

- If A has at least one join, we write $\bigvee A$ for a choice of one of the joins of A. Write also $x \vee x'$ for $\bigvee \{x, x'\}$.

- $\bigwedge A$ is a choice of one of the meets of A.

Functions between Ordered Sets

- Let $f : X \to Y$ be a function, with X and Y equipped with orders. f is **monotone** if for $x, x' \in X$ we have $x \leq x'$ implies $f(x) \leq f(x')$; f is also called a **homomorphism of preorders**.

- The posets X and Y are **isomorphic** if there are monotone functions $f : X \to Y$ and $g : Y \to X$ for which $gf = id_X$ and $fg = id_Y$.

- The monotone function g is an **inverse** for f.

Prelattices

- A **prelattice** is a preordered set which has finite meets and joins, that is, meets and joins of finite subsets.

- A **homomorphism of prelattices** is a function $f : X \to Y$ (with X and Y prelattices) which preserves finite meets and joins, that is

$$f(\bigwedge\{x_1, \ldots, x_n\}) \cong \bigwedge\{f(x_1), \ldots, f(x_n)\}$$

(similarly for joins) and also $f(\top) \cong \top$ and $f(\bot) \cong \bot$.

Heyting Prelattices

- A **Heyting prelattice** X is a prelattice in which for each pair of elements $y, z \in X$ there is an element $y \Rightarrow z \in X$ such that

$$x \leq y \Rightarrow z \qquad \text{iff} \qquad x \wedge y \leq z.$$

We call $y \Rightarrow z$ the **Heyting implication** of y and z.

- In a Heyting prelattice X, the Heyting implication of y and z is unique.

Suppose that a and a' are two candidates for the element $y \Rightarrow z \in X$. Then $a \leq a$ implies $a \wedge y \leq z$ implies $a \leq a'$; the converse is similar.

Distributive Prelattices and Examples

Let X be a prelattice. Then X is **distributive** if it satisfies $x \wedge (y \vee z) \cong (x \wedge y) \vee (x \wedge z)$ for all x, y, z in X.

- $\mathcal{P}(X)$ is a Heyting (pre)lattice where $A \Rightarrow A' \stackrel{\text{def}}{=} (X \setminus A) \cup A'$.
- In fact every Boolean (pre)lattice is a Heyting lattice.
- Any finite distributive lattice X is a Heyting lattice.

Induction and Recursion

- Induction is a method for constructing sets.
- One begins with certain (base) elements which must be in the set, and then defines *rules* which explain how new elements are constructed from old elements.
- Recursion is a method for defining functions over inductively defined sets.
- Define the function on the base elements, and then define the function on a new element e in terms of how the function acts on the "old elements" from which e is constructed.

Rules

Given a set U, a **base rule** is a pair $(\varnothing, b)$ in $\mathcal{P}(U) \times U$

$$\frac{}{b}\,(R)$$

and an **inductive rule** is a pair $(H, c) = (\{h_1, \ldots, h_k\}, c)$ in $\mathcal{P}(U) \times U$

$$\frac{h_1 \quad h_2 \quad \ldots \quad h_k}{c}\,(R)$$

Inductively Defined Sets

Given a set U and a set $\mathcal{R}$, a **derivation** is a finite tree with nodes labelled by elements of U such that

- each leaf node label b arises as a base rule $(\varnothing, b) \in \mathcal{R}$
- for any non-leaf node label c, if H is the set of children of c then $(H, c) \in \mathcal{R}$ is an inductive rule.

Then the set **inductively defined** by $\mathcal{R}$ consists of those elements $u \in U$ which have a derivation with root node labelled by u.

Examples

- Let U be the set $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ and let $\mathcal{R}$ be the set of rules

$$\{R_1 = (\varnothing, u_1), R_2 = (\varnothing, u_3), R_3 = (\{u_1, u_3\}, u_4), R_4 = (\{u_1, u_3, u_4\}, u_5)\}$$

Then a derivation for u_5 is given by the tree

$$\begin{tikzcd}
& u_5 \arrow[dl] \arrow[d] \arrow[dr] & & \\
u_1 & u_3 & u_4 \arrow[dl] \arrow[dr] & \\
& u_1 & & u_3
\end{tikzcd}$$

The tree is usually written up-side down and in the following style

$$\dfrac{\dfrac{}{u_1}\,R_1 \qquad \dfrac{}{u_3}\,R_2 \qquad \dfrac{\dfrac{}{u_1}\,R_1 \qquad \dfrac{}{u_3}\,R_2}{u_4}\,R_3}{u_5}\,R_4$$

Rule Induction

Let I be inductively defined by $\mathcal{R}$. Suppose we wish to show

$$\forall i \in I. \quad \boxed{\phi(i)}.$$

Then all we need to do is

- for every base rule $\overline{b} \in \mathcal{R}$ prove that $\phi(b)$ holds; and
- for every inductive rule $\frac{h_1 \ldots h_k}{c} \in \mathcal{R}$ prove that whenever $h_i \in I$,

$$(\phi(h_1) \text{ and } \phi(h_2) \text{ and } \ldots \text{ and } \phi(h_k)) \quad \text{implies} \quad \phi(c)$$

We call the $\phi(h_j)$ **inductive hypotheses**. We refer to proving the • tasks as "verifying **property closure**".

Recursively Defined Functions

Let I be inductively defined by a set of rules $\mathcal{R}$, and A any set. A function $f : I \to A$ can be defined by

- specifying an element $f(b) \in A$ for every base rule $\overline{b} \in \mathcal{R}$; and
- specifying $f(c) \in A$ in terms of $f(h_1) \in A$ and $f(h_2) \in A$ .... and $f(h_k) \in A$ for every inductive rule $\frac{h_1 \ldots, h_k}{c} \in \mathcal{R}$,

Example

- The factorial function $F : \mathbb{N} \to \mathbb{N}$ is usually defined recursively. We set
  - $F(0) \stackrel{\text{def}}{=} 1$ and
  - $\forall n \in \mathbb{N}. F(n+1) \stackrel{\text{def}}{=} (n+1) * F(n)$.

Category Theory

- Category Theory can be thought of as a “theory of functions”. A category embodies the basic ideas of a source, target, identity functions, composition, and properties of composition.

- *Sets and functions* is an example of a category.

- Structures found in a category are usually defined by stating what properties they have, rather than giving a description of how the structure can be built up. The properties define the structure “uniquely”, and are often called *universal* properties.

- Given sets A and B, we can form $A \times B$ (cartesian product), $A + B$ (disjoint union) and $A \Rightarrow B$ (function space). These sets have universal properties which define them uniquely up to bijection. We shall show that these properties can be described in any category; they will be used to model the types in a type theory.

Definition of a Category

A **category** $\mathcal{C}$ is specified by the following data:

- A collection $ob\,\mathcal{C}$ of entities called **objects**, written A, B, $C \ldots$
- A collection $mor\,\mathcal{C}$ of entities called **morphisms** written f, g, $h \ldots$
- For each morphism f a **source** $src(f)$ which is an object of $\mathcal{C}$ and a **target** $tar(f)$ also an object of $\mathcal{C}$. We shall write $f : src(f) \longrightarrow tar(f)$ or perhaps $f : A \rightarrow B$.

- Morphisms f and g are **composable** if $tar(f) = src(g)$. If $f : A \to B$ and $g : B \to C$, then there is a morphism $gf : A \to C$.

- For each object A of $\mathcal{C}$ there is an **identity** morphism $id_A : A \to A$, where

$$
\begin{aligned}
id_{tar(f)} \circ f &= f \\
f \circ id_{src(f)} &= f
\end{aligned}
$$

- Composition is **associative**, that is given morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$ then

$$(hg)f = h(gf).$$

Examples of Categories

- Sets and total functions, $\mathcal{S}et$. The objects are sets and morphisms are (A, f, B) where $f \subseteq A \times B$ is a function. Composition is given by

$$(B, g, C) \circ (A, f, B) = (A, gf, C)$$

Finally, if A is any set, the identity is (A, id_A, A).

- Any preordered set $(X, \leq)$ is a category. The objects are elements of X. The collection of morphisms is the set of pairs (x, y) where $x \leq y$. Composition is $(y, z) \circ (x, y) \stackrel{\text{def}}{=} (x, z)$ (because $\leq$ is transitive).

Definition of a Functor

A **functor** $F : \mathcal{C} \to \mathcal{D}$ is specified by

- an operation taking objects A in $\mathcal{C}$ to objects FA in $\mathcal{D}$, and

- an operation sending morphisms $f : A \to B$ in $\mathcal{C}$ to morphisms $Ff : FA \to FB$ in $\mathcal{D}$,

for which $F(id_A) = id_{FA}$, and whenever the composition of morphisms gf is defined in $\mathcal{C}$ we have $F(gf) = Fg \circ Ff$.

Examples of Functors

- The set $[A]$ of finite lists over a set A gives a monoid via list concatenation.

Hence we may define $F : \mathcal{S}et \to \mathcal{M}on$ by $FA \stackrel{\text{def}}{=} [A]$ and $F f \stackrel{\text{def}}{=} map(f)$, where $map(f) : [A] \to [B]$ is defined by

$$map(f)([a_1, \dots, a_n]) = [f(a_1), \dots, f(a_n)],$$

with $[a_1, \dots, a_n]$ any element of $[A]$.

To see that $F(gf) = Fg \circ Ff$ where $A \xrightarrow{f} B \xrightarrow{g} C$ note that

$$
\begin{aligned}
F(gf)([a_1,\ldots,a_n]) &\stackrel{\text{def}}{=} map(gf)([a_1,\ldots,a_n]) \\
&= [gf(a_1),\ldots,gf(a_n)] \\
&= map(g)([f(a_1),\ldots,f(a_n)]) \\
&= map(g)(map(f)([a_1,\ldots,a_n])) \\
&= Fg \circ Ff([a_1,\ldots,a_n]).
\end{aligned}
$$

■ The functors between two preorders A and B are precisely the monotone functions from A to B.

Definition of a Natural Transformation

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a **natural transformation** α from F to G, written $\alpha : F \to G$, is specified by giving a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$ for each object A in $\mathcal{C}$, such that for any $f : A \to B$ in $\mathcal{C}$, we have

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

Examples of Natural Transformations

- Recall $F : \mathcal{S}et \to \mathcal{M}on$ where $FA \stackrel{\text{def}}{=} [A]$ and $F f \stackrel{\text{def}}{=} map(f)$ and $map(f) : [A] \to [B]$. We can define a natural transformation $rev : F \to F$ by

$$rev_A([a_1, \ldots, a_n]) \stackrel{\text{def}}{=} [a_n, \ldots, a_1]$$

We check

$$F f \circ rev_A([a_1, \ldots, a_n]) = [f(a_n), \ldots, f(a_1)] = rev_B \circ F f([a_1, \ldots, a_n]).$$

- The **functor category** $[\mathcal{C}, \mathcal{D}]$ has objects functors and morphisms natural transformations.

Isomorphisms and Equivalences

- A morphism $f : A \to B$ is an **isomorphism** if there is some $g : B \to A$ for which $fg = id_B$ and $gf = id_A$.
- We shall say g is an **inverse** for f and vise versa.
- We say that A is **isomorphic** to B, $A \cong B$, if such a mutually inverse pair of morphisms exists.
- An isomorphism in a functor category is referred to as a **natural isomorphism**.

Examples

- Bijections in $\mathcal{S}et$ are isomorphisms.

- In the category determined by a partially ordered set, the only isomorphisms are the identities, and in a preorder X with $x, y \in X$ we have $x \cong y$ iff $x \leq y$ and $y \leq x$. Note that in this case there can be only one pair of mutually inverse morphisms witnessing the fact that $x \cong y$.

Definition of Binary Products

A **binary product** of objects A and B in a category $\mathcal{C}$ is specified by

- an object $A \times B$ of $\mathcal{C}$, together with
- two **projection** morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$,

for which given any object C and morphisms $f : C \to A$, $g : C \to B$, there is a unique morphism $\langle f, g \rangle : C \to A \times B$ for which $\pi_A \langle f, g \rangle = f$ and $\pi_B \langle f, g \rangle = g$.

- The data for a binary product is more readily understood as a commutative diagram,

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\exists!\ \langle f,g\rangle"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

The unique morphism $\langle f, g\rangle : C \to A \times B$ is called the **mediating** morphism for f and g.

- The definition can be extended to *families* of objects $(A_i \mid i \in I)$.

Functors Preserving Products

- The functor $F : \mathcal{C} \to \mathcal{D}$ **preserves finite products** if for any finite family of objects $(A_1, \ldots, A_n)$ in $\mathcal{C}$ the morphism

$$m \stackrel{\text{def}}{=} \langle F\pi_i \mid i \in I \rangle : F(A_1 \times \ldots \times A_n) \to FA_1 \times \ldots \times FA_n$$

is an isomorphism.

- We refer to m as the **canonical** isomorphism.
- F is **strict** if the above isomorphisms are identities.

Examples

- A binary product of x and y in a preordered set X is given by $x \wedge y$ with projections $x \wedge y \leq x$ and $x \wedge y \leq y$.

- A (non-empty) finite product of $(A_i \mid i \in I)$ in $\mathcal{S}et$ is given by the cartesian product $\Pi A_{i \in I}$. The product of the empty family is a **terminal** object 1, with the property that there is a unique morphism $!_A : A \to 1$ for every A.

- The functor $\mathcal{C}(C, -)$ preserves finite products.

Definition of Binary Coproducts

A **binary coproduct** of A and B is specified by

- an object $A+B$, together with
- two **insertion** morphisms $\iota_A : A \to A+B$ and $\iota_B : B \to A+B$,

such that there is a unique $[f,g]$ for which

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "f"'] & A+B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\iota_B"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

Definition of Cartesian Closed Categories

- $\mathcal{C}$ is **cartesian closed** if it has finite products, and for any B and C there is $B \Rightarrow C$ and morphism

$$ev : (B \Rightarrow C) \times B \to C$$

such that for any $f : A \times B \to C$ there is a unique morphism $\lambda(f) : A \to (B \Rightarrow C)$ such that $f = ev \circ (\lambda(f) \times id_B)$.

- $B \Rightarrow C$ is called the **exponential** of B and C

- $\lambda(f)$ is the **exponential mate** of f.

Examples

- The category $\mathcal{S}et$.

  – The terminal object is $\{\varnothing\}$ and binary products are given by cartesian product.

  – $B \Rightarrow C$ is the set of functions from B to C.

  – The function $ev : (B \Rightarrow C) \times B \to C$ is given by $ev(h,b) = h(b)$, where $b \in B$ and $h : B \to C$ is a function.

  – Given $f : A \times B \to C$ we define $\lambda(f) : A \to (B \Rightarrow C)$ by $\lambda(f)(a)(b) = f(a,b)$.

- A Heyting prelattice viewed as a category is indeed cartesian closed, with Heyting implications as exponentials. In fact such a prelattice also has finite coproducts.

Definition of Bicartesian Closed Categories

- A category $\mathcal{C}$ is a **bicartesian closed category** if it is a cartesian closed category which has finite coproducts.

- A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be **bicartesian closed** if it preserves finite (co)products and exponentials.

- We shall also call such a functor a **morphism** of bicartesian closed categories.

Distributive Categories

A category with finite products and coproducts is said to be **distributive** if the mediating morphisms

$$[id_A \times i, id_A \times j] : (A \times B) + (A \times C) \xrightarrow{\cong} A \times (B + C)$$

and $!_{A \times 0} : 0 \xrightarrow{\cong} A \times 0$ are isomorphisms.

- The category $\mathcal{S}et$, and any category $[\mathcal{C}, \mathcal{S}et]$; categorical structure is defined pointwise.
- Any Heyting prelattice which is regarded as a category.
- In fact any bicartesian closed category is automatically distributive.

Categorical Logic

- We shall define *intuitionistic propositional logic*, and *theories* in the logic.

- Working from first principles, we shall derive a semantics for the logic—we examine each of the rules for deriving theorems, and extract constraints on our semantic model which guarantee soundness.

- We show how structure preserving functions can transform one model into another . . .

- and use this to show how theories correspond to order theoretic structures with a universal property.

Logical Propositions

The set of (first order) propositions *Prop* is inductively defined by the rules below

$$\frac{}{p}\ [p \in Gnd] \qquad \frac{}{\mathsf{true}} \qquad \frac{}{\mathsf{false}} \qquad \frac{\phi \quad \psi}{\phi \wedge \psi} \qquad \frac{\phi \quad \psi}{\phi \vee \psi} \qquad \frac{\phi \quad \psi}{\phi \rightarrow \psi}$$

Signatures and Theories

- An *IpL* **sequent** takes the form $\Delta \vdash \phi$, where Δ is a finite list of propositions.

- An *IpL*-**signature** *Sg* is specified by a set of ground propositions *Gnd*.

- An *IpL*-**theory** *Th* is a pair (Sg, Ax) where *Ax* is a set of sequents. Each such sequent is called an **axiom** of *Th*.

- Given *Th*, the **theorems** are inductively generated by the rules on the next slide. If $\Delta \vdash \phi$ is a theorem we shall sometimes write $Th \ \triangleright \ \Delta \vdash \phi$.

$$\frac{}{\Delta \vdash \phi}\,[\Delta \vdash \phi \in Ax] \qquad \frac{\Delta, \phi, \phi', \Delta' \vdash \psi}{\Delta, \phi', \phi, \Delta' \vdash \psi}\,\text{EXCH} \qquad \frac{\Delta, \phi, \phi, \Delta' \vdash \psi}{\Delta, \phi, \Delta' \vdash \psi}\,\text{CTRN}$$

$$\frac{}{\Delta', \phi, \Delta \vdash \phi}\,\text{ID} \qquad \frac{\Delta \vdash \phi \quad \phi, \Delta' \vdash \psi}{\Delta', \Delta \vdash \psi}\,\text{CUT}$$

$$\frac{\Delta \vdash \phi}{\Delta \vdash \phi \vee \psi}\,\text{OR-I}_l \qquad \frac{\Delta \vdash \psi}{\Delta \vdash \phi \vee \psi}\,\text{OR-I}_l$$

$$\frac{\Delta, \phi \vdash \theta \quad \Delta, \psi \vdash \theta \quad \Delta \vdash \phi \vee \psi}{\Delta \vdash \theta}\,\text{OR-E}$$

Deriving a Semantics – Preliminaries

- $\Delta \vdash \phi$ means that if all of the propositions in Δ are valid, then ϕ is valid. Recall $[\![\phi]\!] \in \mathbb{B} = \{\bot, \top\}$.

- We look for a mathematical "space" H in which we can model the *IpL* propositions.

- We shall assume that H has some notion of "element".

- We model ϕ as an element $[\![\phi]\!] \in H$. We call $[\![\phi]\!]$ the **denotation** of ϕ.

- A sequent $\phi \vdash \psi$ is a "relationship". So we capture this with binary relations $\leq$ over H.

- The minimal requirement is that $[\![\phi]\!] \leq [\![\psi]\!]$ whenever $Th \ \triangleright \ \phi \vdash \psi$. We say that the theorem is **satisfied** by the semantics.

- What about $Th \ \triangleright \ \Delta \vdash \phi$? We define $[\![\Delta]\!] \stackrel{\text{def}}{=} \Box(\overline{\Delta})$ where $\overline{\Delta}$ is the finite list of denotations of the propositions in Δ, and there is a function $[H] \longrightarrow H$ written $L \mapsto \Box(L)$.

- If $\Delta = \phi$ we expect $[\![\Delta]\!] = [\![\phi]\!]$. Thus we set $\Box(h) \stackrel{\text{def}}{=} h$.

We return to the denotation of propositions. We define $- \mapsto [\![-]\!]$ by recursion. Write $\boxed{\#} : H \times H \to H$ for a function which gives the semantics of a logical operator #, so that

- $[\![\mathsf{true}]\!] \stackrel{\text{def}}{=} \top \in H$, an element to be determined;
- $[\![\phi \wedge \psi]\!] \stackrel{\text{def}}{=} [\![\psi]\!] \boxed{\wedge} [\![\psi]\!]$;
- $[\![\phi \to \psi]\!] \stackrel{\text{def}}{=} [\![\psi]\!] \boxed{\to} [\![\psi]\!]$,

where the denotation $[\![p]\!]$ may be chosen to be any element of H.

Deriving a Semantics of Theories

- A **structure M** in H for a signature is given by specifying an element $[\![p]\!]_{\mathbf{M}} \in H$ for each $p \in \mathit{Gnd}$.

- One can then define $[\![\phi]\!]_{\mathbf{M}}$ by recursion, and a **model** of *Th* is a structure **M** which satisfies each of the axioms of *Th*.

- We shall now look for conditions on $(H, \leq, \Box(-), \boxed{\#}, \bot, \top)$ which *ensure that for any theory Th and for any model* **M**, *the theorems are all satisfied*.

We shall attempt to discover necessary and sufficient conditions. A typical rule for deducing theorems looks like

$$\frac{\Delta_1 \vdash \phi_1 \qquad \ldots \qquad \Delta_n \vdash \phi_n}{\Delta \vdash \phi}\ R$$

In order to ensure that all theorems are satisfied, we want to find necessary and sufficient conditions on all such rules R to ensure that for all Δ_i, ϕ_i, Δ and ϕ

$$(\Box(\overline{\Delta_1}) \leq [\![\phi_1]\!]) \text{ and } \ldots \text{ and } (\Box(\overline{\Delta_n}) \leq [\![\phi_n]\!]) \text{ implies } \Box(\overline{\Delta}) \leq [\![\phi]\!]$$

If this holds, we shall say that the semantics is sound for the rule.

It is clearly sufficient to require

$$(\Box(L_1) \leq h_1) \text{ and } \ldots \text{ and } (\Box(L_n) \leq h_n) \text{ implies } \Box(L) \leq h \quad (*)$$

However, $(*)$ is also necessary. Each sequent $\Delta_j \vdash \phi_j$ can take the form

$$Ax \ \rhd \ p_1, \ldots, p_m \vdash p$$

The images $([\![p_1]\!], \ldots, [\![p_m]\!], [\![p]\!])$ must be onto $[H] \times H$. Thus in fact it is necessary that $(*)$ holds.

- Note that from ID we have $\phi \vdash \phi$. As ϕ could be any ground proposition, it is necessary that for any $h \in H$ we have $h \leq h$, that is $\leq$ must be *reflexive*.

- From CUT we have

$$\frac{\delta \vdash \phi \quad \phi \vdash \psi}{\delta \vdash \psi}$$

Axioms could take the form $p \vdash q$. Thus it is necessary that $\leq$ be *transitive*.

- Thus $(H, \leq)$ is a preordered set.

$\boxed{\text{ID}}$ The NSC for soundness are that $\Box(L,h,L') \leq h$ for all $L,L' \in [H]$ and $h \in H$. Equivalently, the NSC are that $\Box(L)$ is a *lower bound* of the set $Set(L)$ for any $L \in [H]$.

$\boxed{\text{EXCH}}$

$$\Box(L,h,h',L') \cong \Box(L,h',h,L')$$

$\boxed{\text{CTRN}}$

$$\Box(L,h,h,L') \cong \Box(L,h,L')$$

CUT

- The NSC are that for any $L, L' \in [H]$ and $k, k' \in H$, if $\Box(L) \leq k$ and $\Box(k, L') \leq k'$ then $\Box(L, L') \leq k'$.

- Taking $L \stackrel{\text{def}}{=} h$ for any $h \in H$ and $k' \stackrel{\text{def}}{=} \Box(k, L')$, we can deduce that if $h \leq k$ then $\Box(h, L') \leq \Box(k, L')$.

- Let $L \stackrel{\text{def}}{=} k_1, \ldots, k_n$ and for each i, $h \leq k_i$. Then

$$\begin{array}{rcl} h = \Box(h) & \cong & \Box(h, \ldots, h, h) \\ & \leq & \Box(k_1, \ldots, k_{n-1}, k_n) \end{array}$$

Hence we see that $\Box(L)$ is a *greatest* lower bound (meet) of the finite set $Set(L)$.

$\boxed{\text{OR-I}_l \text{ and } \text{OR-I}_r \text{ and } \text{OR-E}}$ NSC are

$$\Box(L) \leq h \text{ implies } \Box(L) \leq h \boxed{\vee} k \quad (1)$$

$$\Box(L) \leq k \text{ implies } \Box(L) \leq h \boxed{\vee} k \quad (2)$$

$$(\Box(L,h) \leq l \text{ and } \Box(L,k) \leq l \text{ and } \Box(L) \leq h \boxed{\vee} k)$$
$$\text{implies } \Box(L) \leq l \quad (3)$$

- By taking L to be h and k in (1) and (2) we see $h \vee k \leq h \boxed{\vee} k$.

- Now take L to be $h \boxed{\vee} k$ and l to be $h \vee k$ in (3). Note that $\Box(h \boxed{\vee} k, h) \leq h \leq h \vee k$, similarly $\Box(h \boxed{\vee} k, k) \leq k \leq h \vee k$, and hence $h \boxed{\vee} k \leq h \vee k$.

This is not sufficient to ensure (3) holds. In fact the joins in H must be distributive for meets,

$$\Box(L',h' \boxed{\vee} k') \leq \Box(L',h') \boxed{\vee} \Box(L',k') \qquad (*)$$

To see sufficiency for (3), note that $\Box(L) \leq \Box(L,h \boxed{\vee} k)$ if $\Box(L) \leq h \boxed{\vee} k$, and thus using $(*)$ we have $\Box(L) \leq l \boxed{\vee} l \leq l$.

$\vdots$

We conclude that $\boxed{(H,\leq) \text{ is a Heyting prelattice.}}$

Semantics of Propositions

Let *Gnd* be a set of ground propositions, and **M** a structure for *Gnd* in a Heyting prelattice H. Then the semantics is given by recursion:

$$\frac{}{[\![p]\!] \textit{ is specified}}$$

$$\frac{}{[\![\mathsf{false}]\!] \stackrel{\text{def}}{=} \bot} \quad (\text{where } \bot \in H \text{ is the bottom element})$$

$$\frac{[\![\phi]\!] = h \quad [\![\psi]\!] = k}{[\![\phi \wedge \psi]\!] = h \wedge k} \qquad \frac{[\![\phi]\!] = h \quad [\![\psi]\!] = k}{[\![\phi \vee \psi]\!] = h \vee k} \qquad \frac{[\![\phi]\!] = h \quad [\![\psi]\!] = k}{[\![\phi \rightarrow \psi]\!] = h \Rightarrow k}$$

Soundness

The structure **M satisfies** a sequent $\Delta \vdash \phi$ if

$$[\![\Delta]\!] \stackrel{\text{def}}{=} \bigwedge Set(\overline{\Delta}) = \bigwedge \{ \, [\![\delta]\!] \mid \delta \in \Delta \, \} \leq [\![\phi]\!]$$

Let $Th = (Sg, Ax)$ be an *IpL*-theory and **M** a model of *Th* in a Heyting prelattice. Then **M** satisfies each of the theorems of *Th*.

Transporting Models

- Suppose $f : H \to K$ is a homomorphism of Heyting prelattices, $\mathbf{M}$ a model of $Th = (Sg, Ax)$ in H. We define a new model $f_*\mathbf{M}$, of Th in K.

- We can define a structure $f_*\mathbf{M}$ for Sg in K by $[\![p]\!]_{f_*\mathbf{M}} \stackrel{\text{def}}{=} f([\![p]\!]_{\mathbf{M}}) \in K$.

- In fact this is a model. First, we can prove by rule induction

$$\forall \phi \in Prop. \quad \boxed{[\![\phi]\!]_{f_*\mathbf{M}} \cong f([\![\phi]\!]_{\mathbf{M}})}$$

Now suppose that $Ax \;\rhd\; \Delta \vdash \phi$. Then

$$[\![\Delta]\!]_{f_* \mathbf{M}} \stackrel{\text{def}}{=} \bigwedge\{\, [\![\delta]\!]_{f_* \mathbf{M}} \mid \delta \in \Delta \,\} \cong$$

$$\bigwedge\{\, f([\![\delta]\!]_{\mathbf{M}}) \mid \delta \in \Delta \,\} \cong'$$

$$f(\bigwedge\{\, [\![\delta]\!]_{\mathbf{M}} \mid \delta \in \Delta \,\}) \leq f([\![\phi]\!]_{\mathbf{M}}) \cong [\![\phi]\!]_{f_* \mathbf{M}}$$

Thus $f_* \mathbf{M}$ satisfies the axioms of *Th* too, and is thus a model of *Th* in *K*.

Classifying Prelattices

A Heyting prelattice $Cl(Th)$ is called the **classifying** prelattice of Th if there is a model $\mathbf{G}$ of Th in $Cl(Th)$ for which given any Heyting prelattice K, and a model $\mathbf{M}$ of Th in K, then there is a homomorphism of Heyting prelattices $m : Cl(Th) \to K$ such that

$$\begin{tikzcd} Th \arrow[r, dashed, "\mathbf{M}"] \arrow[d, dashed, "\mathbf{G}"'] & K \\ Cl(Th) \arrow[ur, "m"'] & \end{tikzcd} \qquad m_* \mathbf{G} = \mathbf{M}.$$

Adjoint Rules

The class of *IpL* theorems previously defined is exactly the same as that defined by

$$\frac{}{\Delta \vdash \phi}\ [\Delta \vdash \phi \in Ax] \qquad \frac{}{\Delta, \mathsf{false} \vdash \phi} \dashv \text{FALSE-E} \qquad \frac{}{\Delta \vdash \mathsf{true}} \dashv \text{TRUE-I}$$

$$\frac{\Delta \vdash \phi \quad \Delta \vdash \psi}{\Delta \vdash \phi \wedge \psi} \dashv \text{AND-I} \qquad \frac{\Delta, \phi \vdash \theta \quad \Delta, \psi \vdash \theta}{\Delta, \phi \vee \psi \vdash \theta} \dashv \text{OR-E}$$

Constructing Classifiers

Each *IpL*-theory *Th* has a classifying Heyting prelattice $Cl(Th)$. In fact we can construct a **canonical** classifier using the syntax of *Th*, where $m_* \mathbf{G} = \mathbf{M}$.

Proof: Given $Th = (Sg, Ax)$, define a relation $\leq$ on propositions by

$$\phi \leq \psi \text{ if and only if } Th \;\rhd\; \phi \vdash \psi.$$

Then $(Prop, \leq)$ is a Heyting prelattice.

Categorical Type Theory

- We shall define an equational type theory with products, sums, and functions.

- Working from first principles, we shall derive a semantics.

  – First we examine the rules for deriving type assignments, and show that basic properties lead naturally to categorical models.

  – Second, we examine each of the rules for deriving equations, and extract constraints on our models which guarantee soundness.

Categorical Type Theory

- We show how structure preserving functors can transform one model into another ...
- and use this to show how theories correspond to categories with a universal property.

Signatures

A $\lambda\times+$-**signature**, *Sg*, is given by :

- A collection of **ground types**. The collection of *types* is inductively defined:

$$\frac{-}{\gamma} \qquad \frac{}{\mathsf{unit}} \qquad \frac{}{\mathsf{null}} \qquad \frac{\sigma \quad \tau}{\sigma\times\tau} \qquad \frac{\sigma \quad \tau}{\sigma+\tau} \qquad \frac{\sigma \quad \tau}{\sigma\Rightarrow\tau}$$

- A collection of **function symbols** $f : \sigma_1 \ldots \sigma_a \to \sigma$ which may be **constants** $k : \sigma$.

Raw Terms

We define the **raw terms** generated by a $\lambda\times+$-signature:

$$\frac{}{x} \qquad \frac{}{k} \qquad \frac{M_1 \quad \dots \quad M_a}{f(M_1,\dots,M_a)} \qquad \frac{}{\langle\rangle} \qquad \frac{M \quad N}{\langle M,N\rangle}$$

$$\frac{P}{\mathsf{Fst}(P)} \qquad \frac{M}{\mathsf{Inl}_\tau(M)} \qquad \frac{S \quad E \quad F}{\mathsf{Case}(S, x.E \mid y.F)} \qquad \frac{M}{\lambda x:\sigma.M} \qquad \frac{F \quad A}{F\,A}$$

- We shall soon make use of *simultaneous substitution* of raw terms for free variables, $T[\vec{U}/\vec{v}]$. For example, $\langle x,y\rangle[\mathsf{Inl}(y),x/x,y] = \langle \mathsf{Inl}(y),x\rangle$.

Proved Terms

- A **context** is a finite list of (variable, type) pairs, usually written as $\Gamma = [x_1 : \sigma_1, \ldots, x_n : \sigma_n]$, where the variables are required to be distinct.

- A **term-in-context** is a judgement of the form $\Gamma \vdash M : \sigma$

- Given a signature *Sg*, the **proved terms** are those terms-in-context which are inductively generated by the following rules.

$$\frac{}{\Gamma, x:\sigma, \Gamma' \vdash x:\sigma} \qquad \frac{}{\Gamma \vdash k:\sigma} \qquad \frac{\Gamma \vdash M_1:\sigma_1 \quad \dots \quad \Gamma \vdash M_a:\sigma_a}{\Gamma \vdash f(M_1,\dots,M_a):\tau}$$

$$\frac{\Gamma \vdash S:\sigma+\tau \quad \Gamma, x:\sigma \vdash E:\delta \quad \Gamma, y:\tau \vdash F:\delta}{\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F):\delta}$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x:\sigma.M:\sigma \Rightarrow \tau} \qquad \frac{\Gamma \vdash F:\sigma \Rightarrow \tau \quad \Gamma \vdash A:\sigma}{\Gamma \vdash F\,A:\tau}$$

Admissible Properties

Whenever $Sg \ \triangleright \ \Gamma \vdash M : \sigma$, we have $Sg \ \triangleright \ \pi\Gamma \vdash M : \sigma$.

We use rule induction. More precisely we prove

$$\forall \ Sg \ \triangleright \ \Gamma \vdash M : \sigma. \qquad \boxed{Sg \ \triangleright \ \pi\Gamma \vdash M : \sigma}$$

We give some examples of property closure.

$$\frac{\Gamma \vdash M_1 : \sigma_1 \quad \dots \quad \Gamma \vdash M_a : \sigma_a}{\Gamma \vdash f(M_1, \dots, M_a) : \sigma} \quad (f : \sigma_1, \dots, \sigma_a \to \sigma)$$

(*Property Closure for the inductive rule for function symbols*): The inductive hypotheses are $Sg \ \triangleright \ \pi\Gamma \vdash M_i : \sigma_i$ for each i, that is, there is a derivation for each term-in-context. But now we can just apply an instance of the rule to these derivations to deduce that $Sg \ \triangleright \ \pi\Gamma \vdash f(M_1, \dots, M_a) : \sigma$, as required.

Theories

- A $\lambda\times+$**-theory**, Th, is a pair (Sg, Ax) where Ax is a collection of equations-in-context for Sg.

- An **equation-in-context** is a judgement $\Gamma \vdash M = M' : \sigma$ where $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M' : \sigma$ are proved terms.

- The **theorems** of Th consist of the judgements of the form $\Gamma \vdash M = M' : \sigma$ inductively generated by the rules on the following slides—it is a consequence of the rules that $Sg \ \triangleright\ \Gamma \vdash M : \sigma$ and $Sg \ \triangleright\ \Gamma \vdash M' : \sigma$.

$$\frac{Ax \;\rhd\; \Gamma \vdash M = M' : \sigma}{\Gamma \vdash M = M' : \sigma}$$

$$\frac{\Gamma \vdash M = M' : \sigma}{\pi\Gamma \vdash M = M' : \sigma} \quad \text{(where } \pi \text{ is a permutation)}$$

$$\frac{\Gamma \vdash M = M' : \sigma}{\Gamma' \vdash M = M' : \sigma} \quad \text{(where } \Gamma \subseteq \Gamma'\text{)}$$

$$\frac{\Gamma, x : \sigma \vdash N = N' : \tau \quad \Gamma \vdash M = M' : \sigma}{\Gamma \vdash N[M/x] = N'[M'/x] : \tau}$$

$$\frac{Sg \;\triangleright\; \Gamma \vdash M : \mathsf{unit}}{\Gamma \vdash M = \langle\rangle : \mathsf{unit}} \qquad \frac{Sg \;\triangleright\; \Gamma \vdash M : \sigma \quad Sg \;\triangleright\; \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Fst}(\langle M, N\rangle) = M : \sigma}$$

$$\frac{Sg \;\triangleright\; \Gamma \vdash P : \sigma \times \tau}{\Gamma \vdash \langle \mathsf{Fst}(P), \mathsf{Snd}(P)\rangle = P : \sigma \times \tau}$$

$$\frac{Sg \;\triangleright\; \Gamma \vdash S : \mathsf{null} \quad Sg \;\triangleright\; \Gamma, x : \mathsf{null} \vdash M : \sigma}{\Gamma \vdash \mathsf{Emp}_{\sigma}(S) = M[S/x] : \sigma}$$

$$\frac{Sg \;\rhd\; \Gamma \vdash M : \sigma \quad Sg \;\rhd\; \Gamma, x : \sigma \vdash E : \delta \quad Sg \;\rhd\; \Gamma, y : \tau \vdash F : \delta}{\Gamma \vdash \mathsf{Case}(\mathsf{Inl}_\tau(M), x.E \mid y.F) = E[M/x] : \delta}$$

$$\frac{Sg \;\rhd\; \Gamma, x : \sigma \vdash M : \tau \quad Sg \;\rhd\; \Gamma \vdash A : \sigma}{\Gamma \vdash (\lambda x : \sigma.M)A = M[A/x] : \tau}$$

$$\frac{Sg \;\rhd\; \Gamma \vdash F : \sigma \Rightarrow \tau}{\Gamma \vdash \lambda x : \sigma.(F x) = F : \sigma \Rightarrow \tau} \quad (\text{provided } x \notin fv(F))$$

$$\frac{\Gamma, x : \sigma \vdash M = M' : \tau}{\Gamma \vdash \lambda x : \sigma.M = \lambda x : \sigma.M' : \sigma \Rightarrow \tau}$$

Deriving a Semantics For Proved Terms

- Suppose we model (or interpret) σ and τ by "objects" A and B. Let us model $x:\sigma \vdash M:\tau$ as a "relationship" $A \xrightarrow{m} B$.

- We first think about the process of substitution. Let

$$[\![x:\sigma \vdash M:\tau]\!] = A \xrightarrow{m} B \qquad [\![y:\tau \vdash N:\gamma]\!] = B \xrightarrow{n} C$$

Then

$$[\![x:\sigma \vdash N[M/y]:\gamma]\!] = A \xrightarrow{\Box(n,m)} C$$

- Let $z:\gamma \vdash L:\delta$ be a further proved term. Note that we shall identify the semantics of the proved terms

$$x:\sigma \vdash (L[N/z])[M/y]:\delta \quad \text{and} \quad x:\sigma \vdash L[N[M/y]/z]:\delta$$

Thus

$$\Box(\Box(l,n),m) = \Box(l,\Box(n,m))$$

- We will have to model $x:\sigma \vdash x:\sigma$ as a relationship $A \xrightarrow{\star_A} A$. We can deduce that if $E \xrightarrow{e} A$, then $\Box(\star_A, e) = e$ because $x[E/x] = E$.

We summarise our deductions, writing $n \circ m$ for $\Box(n,m)$ and id_A for $\star_A$, which *amount to the definition of a category*:

– Types are interpreted by "objects," say A, $B\ldots$ and proved terms are interpreted by "relationships," say $A \xrightarrow{m} B\ldots$

– For each object A there is a relationship id_A.

– Given relationships $A \xrightarrow{m} B$ and $B \xrightarrow{n} C$, there is a relationship $A \xrightarrow{n \circ m} C$.

– Given relationships $E \xrightarrow{e} A$ and $A \xrightarrow{m} B$, then we have $id_A \circ e = e$ and $m \circ id_A = m$.

– For any $A \xrightarrow{m} B$, $B \xrightarrow{n} C$ and $C \xrightarrow{l} D$, we have $l \circ (n \circ m) = (l \circ n) \circ m$.

Summary

- We will model a proved term $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau$ in a category with *finite products* as a morphism of the form

$$[\![\Gamma \vdash M : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

where $\Gamma \stackrel{\text{def}}{=} x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $[\![\Gamma]\!]$ stands for $[\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!]$.

- Substitution of terms will be modelled by categorical composition ...

Deriving a Semantics for Theories

- First we consider the types of *Sg*. We have to give an object $[\![\gamma]\!]$ of $\mathcal{C}$ to interpret each of the ground types γ, $[\![\mathsf{unit}]\!]$ to interpret unit, and $[\![\mathsf{null}]\!]$ to interpret null.

- We define $[\![\sigma \times \tau]\!] \stackrel{\text{def}}{=} [\![\sigma]\!] \square [\![\tau]\!]$, etc

- We choose a morphism $[\![f]\!] : [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!] \to [\![\sigma]\!]$ in $\mathcal{C}$ for each function symbol.

- Recall that the interpretation of $\Gamma \vdash M : \sigma$ is given by $[\![\Gamma \vdash M : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$. *By looking at how to soundly interpret the theorems of Th we will deduce what the interpretation must be.*

A typical rule looks like

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathsf{R}(M) : \tau}\ (\mathsf{R})$$

Now suppose that $m \stackrel{\text{def}}{=} [\![\Gamma \vdash M : \sigma]\!]$ which is an element of $\mathcal{C}([\![\Gamma]\!], [\![\sigma]\!])$. How do we model $[\![\Gamma \vdash \mathsf{R}(M) : \tau]\!] \in \mathcal{C}([\![\Gamma]\!], [\![\tau]\!])$? All we can say at the moment is that this will depend on m, and we can model this idea by having a function

$$\Phi : \mathcal{C}([\![\Gamma]\!], [\![\sigma]\!]) \longrightarrow \mathcal{C}([\![\Gamma]\!], [\![\tau]\!])$$

and setting $[\![\Gamma \vdash \mathsf{R}(M) : \tau]\!] \stackrel{\text{def}}{=} \Phi(m)$.

Suppose that $x:\gamma \vdash M:\sigma$ and $y:\gamma' \vdash N:\gamma$ are any two given proved terms. If $m \stackrel{\text{def}}{=} [\![x:\gamma \vdash M:\sigma]\!]$ and $n \stackrel{\text{def}}{=} [\![y:\gamma' \vdash N:\gamma]\!]$ then $[\![y:\gamma' \vdash M[N/x]:\sigma]\!] = m \circ n$. Note that there are "equal" proved terms

$$y:\gamma' \vdash \mathsf{R}(M)[N/x]:\tau \qquad \text{and} \qquad y:\gamma' \vdash \mathsf{R}(M[N/x]):\tau.$$

and so

$$\Phi(m) \circ n = \Phi(m \circ n). \tag{$*$}$$

$(*)$ will hold if there are natural transformations

$$\Phi : \mathcal{C}(-,A) \longrightarrow \mathcal{C}(-,B) : \mathcal{C}^{op} \longrightarrow \mathcal{S}et.$$

Recall that the rule for introducing product terms is

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau}$$

In order to soundly interpret this rule we shall need a natural transformation

$$\Phi : \mathcal{C}(-, A) \times \mathcal{C}(-, B) \longrightarrow \mathcal{C}(-, A \Box B)$$

for all objects A and B of $\mathcal{C}$.

Now let $m : C \to A$ and $n : C \to B$ be morphisms of $\mathcal{C}$. Applying naturality in C at the morphism $\langle m, n \rangle : C \to A \times B$ we deduce

Then we can make the definition

$$[\![\Gamma \vdash \langle M, N \rangle : A \times B]\!] \stackrel{\text{def}}{=}$$

$$[\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash M : \sigma]\!], [\![\Gamma \vdash N : \sigma]\!] \rangle} [\![\sigma]\!] \times [\![\tau]\!] \xrightarrow{q_{[\![\sigma]\!],[\![\tau]\!]}} [\![\sigma]\!] \Box [\![\tau]\!].$$

$$\frac{\Gamma \vdash H : \sigma \times \tau}{\Gamma \vdash \mathsf{Fst}(H) : \sigma}$$

To model this rule we shall need a natural transformation $\Phi : \mathcal{C}(-, A \Box B) \longrightarrow \mathcal{C}(-, A)$. Using the Yoneda lemma (see notes), the components of Φ are given by $\theta \mapsto p \circ \theta$ for some $p : A \Box B \to A$. So now we can define

$$[\![\Gamma \vdash \mathsf{Fst}(H) : \sigma]\!] \stackrel{\text{def}}{=} [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash H : \sigma \times \tau]\!]} [\![\sigma]\!] \Box [\![\tau]\!] \xrightarrow{p_{[\![\sigma]\!],[\![\tau]\!]}} [\![\sigma]\!].$$

Now we think about the equations

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Fst}(\langle M, N \rangle) = M : \sigma} \quad (1)$$

$$\frac{\Gamma \vdash H : \sigma \times \tau}{\Gamma \vdash \langle \mathsf{Fst}(H), \mathsf{Snd}(H) \rangle = H : \sigma \times \tau} \quad (3)$$

If we put $h \stackrel{\text{def}}{=} [\![\Gamma \vdash H : \sigma \times \tau]\!] : C \to A \Box B$, $m \stackrel{\text{def}}{=} [\![\Gamma \vdash M : \sigma]\!] : C \to A$ and $n \stackrel{\text{def}}{=} [\![\Gamma \vdash N : \tau]\!] : C \to B$, and our categorical interpretation satisfies the equations-in-context, this forces

$$
\begin{aligned}
p_{A,B} \circ q_{A,B} \circ \langle m, n \rangle &= m && (1) \\
p'_{A,B} \circ q_{A,B} \circ \langle m, n \rangle &= n && (2) \\
q_{A,B} \circ \langle p_{A,B} \circ h, p'_{A,B} \circ h \rangle &= h && (3)
\end{aligned}
$$

These equations imply that, up to isomorphism, $A \Box B$ and $A \times B$ are the same. Thus we may *soundly interpret binary product types by binary categorical product*.

To soundly interpret the rule

$$\frac{\Gamma \vdash S : \mathsf{null}}{\Gamma \vdash \mathsf{Emp}_\sigma(S) : \sigma}$$

we shall need a natural transformation $\Phi : \mathcal{C}(-,N) \longrightarrow \mathcal{C}(-,A)$, where $N = [\![\mathsf{null}]\!]$. The Yoneda Lemma tells us that the components of Φ are given by $\theta \mapsto n_A \circ \theta$ where $n_A : N \to A$ is a morphism, one for each A. So now we can define

$$[\![\Gamma \vdash \mathsf{Emp}_\sigma(S) : \sigma]\!] \stackrel{\text{def}}{=} [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash S : \mathsf{null}]\!]} N \xrightarrow{n_{[\![\sigma]\!]}} [\![\sigma]\!].$$

If we write $s \stackrel{\text{def}}{=} [\![\Gamma \vdash S : \mathsf{null}]\!] : C \to N$, and $m \stackrel{\text{def}}{=} [\![\Gamma, x : \mathsf{null} \vdash M : \sigma]\!] : C \times N \to A$ then

$$\Gamma \vdash \mathsf{Emp}_{\sigma}(S) = M[S/x] : \sigma$$

will be soundly modelled providing that

$$n_A \circ s = m \circ \langle id_C, s \rangle \qquad (\dagger)$$

holds for any such morphisms. Suppose that $t : N \to A$. Taking s to be id_N and m to be $t \circ \pi_N$, then

$$n_A = t \circ \pi_N \circ \langle id_N, id_N \rangle = t$$

Thus N is *an initial object in the category $\mathcal{C}$*. (In fact $(\dagger)$ forces N to be *distributive*, that is $\pi_N : C \times N \to N$ is an isomorphism for every C.)

Formal Semantics of Proved Terms

Let $\mathcal{C}$ be a BCC. Then a **structure**, **M**, for some *Sg* in $\mathcal{C}$ is specified by:

- For every ground type γ an object $[\![\gamma]\!]$ of $\mathcal{C}$,

- for every function symbol $f : \sigma_1 \ldots \sigma_n \to \tau$ a morphism $[\![f]\!] : [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!] \to [\![\tau]\!]$, where we define $[\![\sigma]\!]$ by recursion, setting $[\![\mathsf{unit}]\!] \stackrel{\text{def}}{=} 1$, $[\![\sigma \times \tau]\!] \stackrel{\text{def}}{=} [\![\sigma]\!] \times [\![\tau]\!]$ etc.

Then for every proved term $\Gamma \vdash M : \sigma$ we specify a morphism

$$[\![\Gamma \vdash M : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$$

by recursion.

$$\frac{}{[\![\Gamma, x : \sigma, \Gamma' \vdash x : \sigma]\!] \stackrel{\text{def}}{=} \pi : [\![\Gamma]\!] \times [\![\sigma]\!] \times [\![\Gamma']\!] \to [\![\sigma]\!]}$$

$$\frac{}{[\![\Gamma \vdash k : \sigma]\!] \stackrel{\text{def}}{=} [\![k]\!] \circ ! : [\![\Gamma]\!] \to 1 \to [\![\sigma]\!]} \quad (k : \sigma)$$

$$\frac{[\![\Gamma \vdash M_1 : \sigma_1]\!] = m_1 : [\![\Gamma]\!] \to [\![\sigma_1]\!] \quad \dots}{[\![\Gamma \vdash f(\vec{M}) : \tau]\!] = [\![f]\!] \circ \langle m_1, \dots, m_n \rangle : [\![\Gamma]\!] \to ([\![\sigma_1]\!] \times \dots \times [\![\sigma_n]\!]) \to [\![\tau]\!]}$$

$$\frac{[\![\Gamma \vdash P : \sigma \times \tau]\!] = p : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!])}{[\![\Gamma \vdash \mathsf{Fst}(P) : \sigma]\!] = \pi_1 \circ p : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!]) \to [\![\sigma]\!]}$$

$$\frac{\begin{cases} [\![\Gamma \vdash S : \sigma + \tau]\!] = s : [\![\Gamma]\!] \to [\![\sigma]\!] + [\![\tau]\!] \\ [\![\Gamma, x : \sigma \vdash E : \delta]\!] = e : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\delta]\!] \\ [\![\Gamma, y : \sigma \vdash F : \delta]\!] = f : [\![\Gamma]\!] \times [\![\tau]\!] \to [\![\delta]\!] \end{cases}}{\begin{array}{l} [\![\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F) : \delta]\!] = \\ [e, f] \circ \cong \circ \langle id_{[\![\Gamma]\!]}, s \rangle : [\![\Gamma]\!] \to [\![\Gamma]\!] \times ([\![\sigma]\!] + [\![\tau]\!]) \\ \qquad\qquad\qquad \cong ([\![\Gamma]\!] \times [\![\sigma]\!]) + ([\![\Gamma]\!] \times [\![\tau]\!]) \to [\![\delta]\!] \end{array}}$$

$$\frac{[\![\Gamma, x : \sigma \vdash M : \tau]\!] = m : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\tau]\!]}{[\![\Gamma \vdash \lambda x : \sigma . M : \sigma \Rightarrow \tau]\!] = \lambda(m) : [\![\Gamma]\!] \to [\![\sigma]\!] \Rightarrow [\![\tau]\!]}$$

$$\frac{[\![\Gamma \vdash F : \sigma \Rightarrow \tau]\!] = f : [\![\Gamma]\!] \to ([\![\sigma]\!] \Rightarrow [\![\tau]\!]) \quad [\![\Gamma \vdash A : \sigma]\!] = a : [\![\Gamma]\!] \to [\![\sigma]\!]}{[\![\Gamma \vdash F\,A : \tau]\!] \stackrel{\text{def}}{=} ev \circ \langle f, a \rangle : [\![\Gamma]\!] \to ([\![\sigma]\!] \Rightarrow [\![\tau]\!]) \times [\![\sigma]\!] \to [\![\tau]\!]}$$

Soundness

Let **M** be a structure for a $\lambda\times+$-signature in a bicartesian closed category $\mathcal{C}$. **M** **satisfies** the equation-in-context $\Gamma \vdash M = M' : \sigma$ if $[\![\Gamma \vdash M : \sigma]\!]$ and $[\![\Gamma \vdash M' : \sigma]\!]$ are equal. We say that **M** is a **model** of a $\lambda\times+$-theory $Th = (Sg, Ax)$ if **M** satisfies the axioms.

Then **M** satisfies any equation-in-context which is a theorem of *Th*.

Proof: This can be shown by rule induction using the rules for deriving theorems.

Let

$$m \stackrel{\text{def}}{=} [\![\Gamma, x : \sigma \vdash M : \tau]\!] : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\tau]\!]$$

and $a \stackrel{\text{def}}{=} [\![\Gamma \vdash A : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$. Then we have

(*Property Closure for the (base) rule*):

$$\frac{Sg \;\rhd\; \Gamma, x : \sigma \vdash M : \tau \qquad Sg \;\rhd\; \Gamma \vdash A : \sigma}{\Gamma \vdash (\lambda x : \sigma.M)A = M[A/x] : \tau}$$

$$\begin{array}{rcl}
[\![\Gamma \vdash (\lambda x : \sigma.M)A : \tau]\!] & = & ev\langle [\![\Gamma \vdash \lambda x : \sigma.M : \tau]\!], [\![\Gamma \vdash A : \sigma]\!]\rangle \\
& = & ev\langle \lambda(m), a\rangle \\
& = & ev(\lambda(m) \times id)\langle id, a\rangle \\
& = & m\langle id, a\rangle \\
& = & [\![\Gamma \vdash M[A/x] : \tau]\!]
\end{array}$$

Transporting Models

Suppose that we are given a morphism of bicartesian closed categories $F : \mathcal{C} \to \mathcal{D}$. Let $\mathbf{M}$ be a model of *Th* in $\mathcal{C}$. We shall show how to define a new model, of *Th* in $\mathcal{D}$, denoted by $F_*\mathbf{M}$. We shall need a lemma:

If we set $[\![\gamma]\!]_{F_*\mathbf{M}} \stackrel{\text{def}}{=} F[\![\gamma]\!]_{\mathbf{M}}$ where γ is a ground type of *Th*, then it follows from this that there is a canonical isomorphism $[\![\sigma]\!]_{F_*\mathbf{M}} \cong F[\![\sigma]\!]_{\mathbf{M}}$ where σ is any type of *Th*.

A structure $F_*\mathbf{M}$ is given by $[\![\gamma]\!]_{F_*\mathbf{M}} \stackrel{\text{def}}{=} F[\![\gamma]\!]_{\mathbf{M}}$ on ground types and $[\![f]\!]_{F_*\mathbf{M}}$ is given by the composition

$$[\![\sigma_1]\!]_{F_*\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{F_*\mathbf{M}} \cong F[\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times F[\![\sigma_n]\!]_{\mathbf{M}} \cong'$$
$$F([\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{\mathbf{M}}) \xrightarrow{F[\![f]\!]_{\mathbf{M}}} F[\![\tau]\!]_{\mathbf{M}} \cong [\![\tau]\!]_{F_*\mathbf{M}}$$

where $f : \sigma_1, \ldots, \sigma_n \to \tau$ is a function symbol of *Th*, the isomorphims $\cong$ exist because of the lemma, and $\cong'$ arises from F preserving finite products.

In fact $F_* \mathbf{M}$ is a model of *Th*.

Given a proved term $\Gamma \vdash M : \sigma$ one can show by induction that the morphism $[\![\Gamma \vdash M : \sigma]\!]_{F_* \mathbf{M}}$ is given by the composition

$$[\![\sigma_1]\!]_{F_* \mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{F_* \mathbf{M}} \cong F([\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{\mathbf{M}}) \xrightarrow{F[\![\Gamma \vdash M:\sigma]\!]_{\mathbf{M}}} F[\![\sigma]\!]_{\mathbf{M}}.$$

If we are given proved terms $\Gamma \vdash M : \sigma$ and $\Gamma \vdash N : \sigma$ for which $[\![\Gamma \vdash M : \sigma]\!]_{\mathbf{M}} = [\![\Gamma \vdash N : \sigma]\!]_{\mathbf{M}}$ then certainly $[\![\Gamma \vdash M : \sigma]\!]_{F_* \mathbf{M}} = [\![\Gamma \vdash N : \sigma]\!]_{F_* \mathbf{M}}$. Thus if $\mathbf{M}$ is a model of *Th* in $\mathcal{C}$ then $F_* \mathbf{M}$ is a model of *Th* in $\mathcal{D}$.

Classifying Categories

Let Th be a $\lambda \times +$-theory. A bicartesian closed category $Cl(Th)$ is called the **classifying** category of Th if there is a model $\mathbf{G}$ of Th in $Cl(Th)$ for which given any category $\mathcal{D}$ with finite products, and a model $\mathbf{M}$ of Th in $\mathcal{D}$, then there is a functor $M : Cl(Th) \to \mathcal{D}$ such that

$$\begin{tikzcd} Th \arrow[r, dashed, "\mathbf{M}"] \arrow[d, dashed, "\mathbf{G}"'] & \mathcal{D} \\ Cl(Th) \arrow[ur, "M"'] & \end{tikzcd}$$

where $M_* \mathbf{G} = \mathbf{M}$.

Constructing Classifiers

Every $\lambda\times+$-theory Th has a classifying category $Cl(Th)$. We can construct a **canonical** classifying category using the syntax of Th.

Proof:

- The objects of $Cl(Th)$ are the types of Th.
- A morphism $\sigma \to \tau$ is an equivalence class $(x:\sigma \mid M)$ of pairs $(x:\sigma, M)$ where $Sg \ \triangleright\ x:\sigma \vdash M:\tau$, with equivalence relation

$$(x:\sigma, M) \sim (x':\sigma, M') \qquad \text{iff} \qquad Th \ \triangleright\ x:\sigma \vdash M = M'[x/x']:\tau.$$

- Given σ and τ, the binary product is $\sigma \times \tau$ with projection $\pi_\sigma : \sigma \times \tau \to \sigma$ given by $(z : \sigma \times \tau \mid \mathsf{Fst}(z))$. If $(x : \gamma \mid M) : \gamma \to \sigma$ and $(y : \gamma \mid N) : \gamma \to \tau$, then the mediating morphism is

$$(z : \gamma \mid \langle M[z/x], N[z/y] \rangle) : \gamma \to \sigma \times \tau.$$

- $(x : \sigma \mid \langle\rangle)$ is the unique morphism $\sigma \to \mathsf{unit}$ so that unit is a terminal object for $Cl(Th)$.

- We define a structure **G** for *Sg* in $Cl(Th)$. $[\![\gamma]\!]_{\mathbf{G}} \stackrel{\text{def}}{=} \gamma$ (and hence it follows that $[\![\sigma]\!]_{\mathbf{G}} = \sigma$ for any type σ).

- Also define for $f : \sigma_1, \sigma_2 \to \tau$

$$[\![f]\!]_{\mathbf{G}} \stackrel{\text{def}}{=} (z : \sigma_1 \times \sigma_2 \mid f(\mathsf{Fst}(z), \mathsf{Snd}(z)))$$

Certainly we have

$$Sg \;\triangleright\; z : \sigma_1 \times \sigma_2 \vdash f(\mathsf{Fst}(z), \mathsf{Snd}(z)) : \tau$$

- If $k : \sigma$ then $[\![k]\!]_{\mathbf{G}} \stackrel{\text{def}}{=} (x : \mathsf{unit} \mid k)$.

- It is easy to check that **G** is indeed a model of $Th = (Sg, Ax)$.

- Now let **M** be a model of Th in $\mathcal{D}$. We define $M : Cl(Th) \to \mathcal{D}$ by

$$(x : \sigma \mid M) : \sigma \longrightarrow \tau \quad \longmapsto \quad [\![x : \sigma \vdash M : \tau]\!]_{\mathbf{M}} : [\![\sigma]\!]_{\mathbf{M}} \longrightarrow [\![\tau]\!]_{\mathbf{M}}$$

The soundness theorem says that the definition makes sense. It is easy to see that M is a bicartesian closed functor.

Some Applications

- We show that given a proof of $\phi \vee \psi$, there is a proof of either ϕ or of ψ. This reinforces the fact that the logic *IpL* is *constructive*. We prove the result by setting up a direct connection between a classifying Heyting prelattice, and the Booleans. The property is trivial in a Boolean model, and can be reflected onto the classifier.

- We show that by starting with a very simple type theory, the expressive power (in a sense to be made precise) is not increased by adding products, sums and functions. This is proved by establishing an equivalent categorical problem, and solving it using categorical methods.

The Disjunction Property

- Suppose that $Th = (Sg, \varnothing)$ is an *IpL* theory. If $Th \;\rhd \vdash \phi \vee \psi$, then $Th \;\rhd \vdash \phi$ *or* $Th \;\rhd \vdash \psi$. This is known as the **disjunction property**, DP.

- The **categorical disjunction property**, CDP, for a Heyting prelattice, states that if we have $\top \cong h \vee k$ then either $\top \cong h$ or $\top \cong k$.

- We shall prove DP by showing that $Cl(Th)$ satisfies CDP.

The Gluing Lemma For Heyting Prelattices

Let $\Gamma : H \to K$ be a function which *preserves finite meets*. Define

$$\mathcal{GL}(\Gamma) \stackrel{\text{def}}{=} \{\ (k,h) \in K \times H \ \mid\ k \leq \Gamma(h)\ \}$$

with pointwise order. Then $\mathcal{GL}(\Gamma)$ is a Heyting prelattice:

- the top element is $(\top_K, \top_H)$
- $(k,h) \wedge (k',h') \stackrel{\text{def}}{=} (k \wedge k', h \wedge h')$
- $\ldots$
- $(k,h) \Rightarrow (k',h') \stackrel{\text{def}}{=} ((k \Rightarrow k') \wedge \Gamma(h \Rightarrow h'), h \Rightarrow h')$

and $\pi_2 : \mathcal{GL}(\Gamma) \to H$ is a homomorphism of Heyting prelattices.

Gluing Classifiers to the Booleans

Set $\mathbb{B} \stackrel{\text{def}}{=} \{\bot, \top\}$ where $\bot \leq \top$. Let $\Gamma_{\mathbb{B}} : Cl(Th) \to \mathbb{B}$ be defined by

$$\phi \mapsto \begin{cases} \top & \text{if } \phi \cong \mathsf{true} \\ \bot & \text{otherwise} \end{cases}$$

Then $\Gamma_{\mathbb{B}}$ is a finite meet preserving function, and hence $\mathcal{GL}(\Gamma_{\mathbb{B}})$ is a Heyting prelattice. Further, $\mathcal{GL}(\Gamma_{\mathbb{B}})$ satisfies CDP.

Proof: Note that $Th \ \triangleright \ \mathsf{true} \vdash \phi \wedge \psi$ if and only if $Th \ \triangleright \ \mathsf{true} \vdash \phi$ and $Th \ \triangleright \ \mathsf{true} \vdash \psi$. Hence $\Gamma_{\mathbb{B}}$ preserves binary meets, and further $\Gamma_{\mathbb{B}}(\mathsf{true}) = \top$. Then apply the gluing lemma. Now suppose that

$$(b,\phi) \vee (b',\phi') = (b \vee b', \phi \vee \phi') \cong (\top, \mathsf{true}) \in \mathcal{GL}(\Gamma_{\mathbb{B}}).$$

Hence $b \vee b' \cong \top \in \mathbb{B}$ and hence either $b = \top$ or $b' = \top$. In the former case we must have $\top \leq \Gamma_{\mathbb{B}}(\phi)$ hence $\top = \Gamma_{\mathbb{B}}(\phi)$ implying $\phi \cong \mathsf{true}$. Thus CDP holds for $\mathcal{GL}(\Gamma_{\mathbb{B}})$.

Proving the CDP by Gluing

The logic *IpL* satisfies DP.

Proof: We show CDP for $Cl(Th)$. Let us define a structure **M** for *Th* in $\mathcal{GL}(\Gamma_{\mathbb{B}})$. We set

$$[\![p]\!]_{\mathbf{M}} \stackrel{\text{def}}{=} (\Gamma_{\mathbb{B}}(p), [\![p]\!]_{\mathbf{G}}) \in \mathcal{GL}(\Gamma_{\mathbb{B}})$$

(because $[\![p]\!]_{\mathbf{G}} = p$). This is trivially a model of *Th* because $Ax = \varnothing$.

Hence there is a homomorphism m such that the upper left triangle commutes up to equality (of structures)

$$
\begin{tikzcd}
 & Cl(Th) \arrow[r, equal] \arrow[d, "m"] & Cl(Th) \arrow[dd, "id_{Cl(Th)}"] \\
Th \arrow[ur, "\mathbf{G}"] \arrow[r, "\mathbf{M}"] \arrow[dr, "\mathbf{G}"'] & \mathcal{G}l(\Gamma) \arrow[d, "\pi_2"] & \\
 & Cl(Th) \arrow[r, equal] & Cl(Th)
\end{tikzcd}
$$

as does the lower one given the definition of $\mathbf{M}$. Hence from the universal property of $Cl(Th)$ we have $\pi_2 \circ m \cong id_{Cl(Th)}$.

Now let $\phi \vee \psi \cong \mathsf{true}$ in $Cl(Th)$. We must have

$$m(\phi \vee \psi) \cong m(\phi) \vee m(\psi) \cong (\top, \mathsf{true})$$

in $\mathcal{GL}(\Gamma_{\mathbb{B}})$. By CDP for $\mathcal{GL}(\Gamma_{\mathbb{B}})$ we have either

(i) $m(\phi) \cong (\top, \mathsf{true})$ in which case
$\pi_2(m(\phi)) = \mathsf{true} \cong id(\phi) = \phi$, or

(ii) $m(\psi) \cong (\top, \mathsf{true})$ so that $\psi \cong \mathsf{true}$ similarly.

Algebraic Theories

An **algebraic theory** is a $\lambda \times +$-theory in which there are no product, sum, and function types. More precisely, an algebraic theory $Th = (Sg, Ax)$ consists of

- a collection of **types** and **function symbols**;
- raw terms generated from these data, using only the rules

$$\frac{}{x} \qquad \frac{}{k} \qquad \frac{M_1 \quad \ldots \quad M_a}{f(M_1, \ldots, M_a)}$$

- proved terms, generated as expected; and
- theorems, generated by the rules of equality.

A Conservative Extension

Let $Th = (Sg, Ax)$ be an algebraic theory. Let $Th' = (Sg', Ax')$ be the $\lambda\times+$-theory with ground types and function symbols those of Sg, and $Ax' \stackrel{\text{def}}{=} Ax$. Let $\Gamma \stackrel{\text{def}}{=} [x_1 : \gamma_1, \ldots, x_n : \gamma_n]$. Suppose that

$$Sg' \;\rhd\; [x_1 : \gamma_1, \ldots, x_n : \gamma_n] \vdash E : \gamma$$

Then there exists M for which

$$Sg \;\rhd\; \Gamma \vdash M : \gamma \qquad \text{and} \qquad Th' \;\rhd\; \Gamma \vdash E = M : \gamma.$$

Moreover, if there is M' for which $Sg \;\rhd\; \Gamma \vdash M' : \gamma$ and also $Th' \;\rhd\; \Gamma \vdash E = M' : \gamma$ then we have $Th \;\rhd\; \Gamma \vdash M = M' : \gamma$.

Free Bicartesian Closed Cateories

Let $\mathcal{C}$ be a category with finite products. Then $\mathcal{F}\,\mathcal{C}$ is the **relatively free** BCCC generated by $\mathcal{C}$ if there is a finite product preserving functor $I : \mathcal{C} \longrightarrow \mathcal{F}\,\mathcal{C}$ such that if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is finite product preserving and $\mathcal{D}$ is a BCCC then there is a BCCC functor $\overline{F} : \mathcal{F}\,\mathcal{C} \longrightarrow \mathcal{D}$ for which $\phi : \overline{F}I \cong F$ and $\overline{F}$ is unique up to isomorphism.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "I"] \arrow[dr, "F"'] & \mathcal{F}\,\mathcal{C} \arrow[d, "\overline{F}"] \\
 & \mathcal{D}
\end{tikzcd}$$

Relating Th and Th' Categorically

We define a functor $I : Cl(Th) \to Cl(Th')$. *Very roughly*, if

$$(x : \gamma \mid M)_{Th} : \gamma \to \gamma'$$

then we set

$$I(x : \gamma \mid M)_{Th} \stackrel{\text{def}}{=} (x : \gamma \mid M)_{Th'}$$

Warning: the objects of $Cl(Th)$ are in fact *lists* of types (in the example above the source γ and target γ' are lists of length one) and you should consult my notes for the precise definition of I.

Full and Faithful Functors

- $F : \mathcal{C} \to \mathcal{D}$ is **faithful** if given a parallel pair of morphisms $f, g : A \to B$ in $\mathcal{C}$ for which $Ff = Fg$, then $f = g$. Thus

$$\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$$

is *1-1*.

- F is **full** if given objects A and B in $\mathcal{C}$ and a morphism $g : FA \to FB$ in $\mathcal{D}$, then there is some $f : A \to B$ in $\mathcal{C}$ for which $Ff = g$. Thus

$$\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$$

is *onto*.

Outlining a Proof of the Con. Extension

1. Show that $I : Cl(Th) \to Cl(Th')$ yields a *free BCCC*.
2. Prove a purely categorical result called the “logical relations” *gluing lemma*.
3. Apply the gluing lemma and the free BCCC property, to show that I is full and faithful ...

$$Cl(Th)(\gamma,\gamma') \xrightarrow{\cong} Cl(Th')(I\gamma, I\gamma')$$

Existence: Suppose that $Sg' \;\triangleright\; x:\gamma \vdash E:\gamma'$. Then we certainly have

$$e \stackrel{\mathrm{def}}{=} (x:\gamma \mid E)_{Th'} : I\gamma \to I\gamma'$$

in $Cl(Th')$. Using the fullness of I, there is a morphism $(x:\gamma \mid M)_{Th} : \gamma \to \gamma'$ which is taken to e by I. But this implies

$$Th' \;\triangleright\; x:\gamma \vdash M = E:\gamma'$$

as required.

A Free BCCC

The functor $I : Cl(Th) \to Cl(Th')$ presents $Cl(Th')$ as the relatively free BCC generated by $Cl(Th)$.

Proof: Let $F : Cl(Th) \to \mathcal{C}$ preserve finite products where $\mathcal{C}$ is a BCCC. We shall define a functor $\overline{F} : Cl(Th') \to \mathcal{C}$ by recursion over the syntactic structure of $Cl(Th')$. For example

- $\overline{F}\gamma \stackrel{\text{def}}{=} F[\gamma]$ where γ is a ground type of Sg',
- $\overline{F}(\sigma \times \tau) \stackrel{\text{def}}{=} \overline{F}\sigma \times \overline{F}\tau$,
- $\overline{F}(z : \delta \mid \langle\rangle) \stackrel{\text{def}}{=} ! : \overline{F}\delta \to 1_{\mathcal{C}}$,
- $\overline{F}(z : \delta \mid \mathsf{Fst}(P)) \stackrel{\text{def}}{=} \pi_1 \overline{F}(z : \delta \mid P)$ where $\pi_1 : \overline{F}\sigma \times \overline{F}\tau \to \overline{F}\sigma$,

Gluing Lemma by Logical Relations

Let $\mathcal{D}$ be a BCC and let $I : \mathcal{C} \to \mathcal{D}$ preserve finite products. We define a category $\mathcal{G}l(\Gamma)$ as follows:

- Objects of $\mathcal{G}l(\Gamma)$ are $(F, \lhd, D)$ where $F : \mathcal{C}^{op} \to \mathcal{S}et$ is a functor, D is an object of $\mathcal{D}$, and for each object C of $\mathcal{C}$, $\lhd_C \subseteq FC \times \mathcal{D}(IC, D)$.
- A morphism $(\alpha, d) : (F, \lhd, D) \to (F', \lhd', D')$ is given by a natural transformation $\alpha : F \to F'$ and a morphism $d : D \to D'$ in $\mathcal{D}$ for which if $x \lhd_C u$ then $\alpha_C(x) \lhd'_C d \circ u$, where of course $x \in FC$ and $u \in \mathcal{D}(IC, D)$.

Then $\mathcal{G}l(\Gamma)$ is a bicartesian closed category and the obvious functor $\pi_2 : \mathcal{G}l(\Gamma) \to \mathcal{D}$ is a morphism of BCCCs.

Freeness Implies Full and Faithful

Let $\mathcal{C}$ be a locally small category, and $\mathcal{F}\,\mathcal{C}$ the freely generated bicartesian closed category. Then the canonical functor $I : \mathcal{C} \to \mathcal{F}\,\mathcal{C}$ is full and faithful.

Proof We apply the gluing lemma to I. We define a functor $J : \mathcal{C} \to \mathcal{G}l(\Gamma)$: on objects C of $\mathcal{C}$ define JC by $(\mathcal{C}(-,C), \lhd^{C}, IC)$ where the subset

$$\lhd^{C}_{C'} \;\subseteq\; \mathcal{C}(C',C) \times \mathcal{D}(IC',IC)$$

is defined by just requiring $c \lhd^{C}_{C'} Ic$ for each morphism $c : C' \to C$ in $\mathcal{C}$. On morphisms c of $\mathcal{C}$ we set $Jc \stackrel{\text{def}}{=} (\mathcal{C}(-,c), Ic)$.

Consider the following diagram

$$
\begin{tikzcd}
 & Cl(Th') \arrow[r, equal] \arrow[d, "\overline{J}"] & Cl(Th') \arrow[dd, "id_{Cl(Th')}"] \\
Cl(Th) \arrow[ur, "I"] \arrow[r, "J"] \arrow[dr, "I"'] & \mathcal{G}l(\Gamma) \arrow[d, "P_2"] & \\
 & Cl(Th') \arrow[r, equal] & Cl(Th')
\end{tikzcd}
$$