Category Theory

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Introductory Remarks

- Category Theory is a theory of abstraction (of algebraic structure).
- ▶ It had its origins in Algebraic Topology with the work of Eilenberg and Mac Lane (1942-45).
- ▶ It provides tools and techniques which allow the formulation and analysis of common features amongst apparently different mathematical/computational theories.
- We can discover new relationships between things that are seemingly unconnected.
- Category theory concentrates on how things behave and not on internal details.
- As such, category theory can clarify and simplify our ideas—and indeed lead to new ideas and new results.

Introductory Remarks

- ► Connections with Computer Science were first made in the 1980s, and the subject has played a central role ever since.
- ► Some contributions (chosen by me . . . there are many many more) are
 - Cartesian closed categories as models of pure functional languages.
 - ► The use of strong monads to model notions of computation (well incorporated into Haskell).
 - Precise correspondences between categorical structures and type theories (as internal languages).
 - The categorical solution of domain equations as models of recursive types.
 - Categories for concurrent computation.
 - Nominal categories as models of variable binding.

Seminar Outline

Categories

Functors

Natural Transformations

Isomorphisms, Products, Coproducts

Algebras

Case Study: Modelling (Haskell) Algebraic Datatypes

Adjunctions

Colimits and Applications to Initial Algebras

Examples of Categories

- ► The collection of all sets and all functions
 - ► Each set has an identity function; functions compose; composition is associative.
- ► The collection of all elements of a preorder and all instances of the order relation (relationships) ≤
 - ► Each element has an identity relationship (reflexivity); relationships compose (transitivity); composition is associative.
- ► The collection of all elements of a singleton $\{*\}$ (!) and any collection of algebraic terms with just one variable x_0
 - * has an identity term x_0 ; terms compose (substitution); composition is associative.

Definition of A Category

A **category** \mathcal{C} is specified by the following data:

- ▶ A collection $ob \ C$ of entities called **objects**. An object will often be denoted by a capital letter such as A, B, C...
- ► For any two objects A and B, a collection C(A, B) of entities called **morphisms**. A morphism in C(A, B) will often be denoted by a small letter such as f, g, h....
- ▶ If $f \in \mathcal{C}(A, B)$ then A is called the **source** of f, and B is the **target** of f and we write (equivalently) $f : A \to B$.

Definition of A Category

A **category** \mathcal{C} is specified by the following data (continued):

- ▶ There is an operation assigning to each object A of C an identity morphism $id_A: A \rightarrow A$.
- ► There is an operation

$$C(B,C) \times C(A,B) \longrightarrow C(A,C)$$

assigning to each pair of morphisms $f: A \to B$ and $g: B \to C$ their **composition** which is a morphism denoted by $g \circ f: A \to C$ or just $gf: A \to C$.

▶ Such morphisms f and g, with a common source and target B, are said to be **composable**.

Definition of A Category

A **category** \mathcal{C} is specified by the following data (continued):

► These operations are unitary

$$id_B \circ f = f: A \to B$$

 $f \circ id_A = f: A \to B$

▶ and associative, that is given morphisms $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$ then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

If we say "f is a morphism" we implicitly assume that the source and target are recoverable, that is, we can work out $f \in \mathcal{C}(A, B)$ for some A and B.

More Examples

- ► The category $\mathcal{P}art$ with $ob \ \mathcal{P}art$ all sets and morphisms $\mathcal{P}art(A,B)$ the partial functions $A \to B$.
 - ▶ The identity function id_A is a partial function!
 - Figure $f: A \to B$, $g: B \to C$, then for each element a of A, $(g \circ f)(a)$ is defined with value g(f(a)) if and only if both f(a) and g(f(a)) are defined.
- ▶ Given a category C, the opposite category C^{op} has
 - ▶ $ob \ \mathcal{C}^{op} \stackrel{\text{def}}{=} ob \ \mathcal{C}$ and $\mathcal{C}^{op}(A,B) = \{ f^{op} \mid f \in \mathcal{C}(B,A) \}.$
 - ► The identity on an object A in C^{op} is defined to be id_A^{op} .
 - If $f^{op}: A \to B$ and $g^{op}: B \to C$ are morphisms in C^{op} , then $f: B \to A$ and $g: C \to B$ are composable morphisms in C. We define $g^{op} \circ f^{op} \stackrel{\text{def}}{=} (f \circ g)^{op}: A \to C$.

More Examples

- ► A **discrete** category is one for which the only morphisms are identities.
- A semigroup (S,b) is a set S together with an associative binary operation $b\colon S\times S\to S$, $(s,s')\mapsto s\cdot s'$. An identity element for a semigroup S is some (necessarily unique) element e of S such that for all $s\in S$ we have $e\cdot s=s\cdot e=s$. A monoid (M,b,e) is a semigroup (M,b) with identity element e. Any monoid is a single object category C with $C(*,*)\stackrel{\mathrm{def}}{=} M$. Concrete examples are
 - Addition on the natural numbers, $(\mathbb{N}, +, 0)$.
 - ▶ Concatenation of finite lists over a set A, (list(A), ++, []).

More Examples

- ▶ Mon has objects monoids and morphisms monoid homomorphisms: $h: M \to M'$ is a homomorphism if h(e) = e and $h(m_1 \cdot m_2) = h(m_1) \cdot h(m_2)$ for all $m_i \in M$.
- ► *PreSet* has objects preorders and morphisms the monotone functions; and *ParSet* has objects partially ordered sets and morphisms the monotone functions.
- ► The category of relations *Rel* has objects sets and morphisms binary relations on sets; composition is relation-composition.
- ► The category of lattices *Lat* has objects lattices and morphisms the lattice homomorphisms.
- ► The category *CLat* has objects the complete lattices and morphisms the complete lattice homomorphisms.
- ► The category *Grp* of groups and homomorphisms.

Examples of Functors

- ▶ Let \mathcal{C} be a category. The **identity** functor $id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ is defined by $id_{\mathcal{C}}(A) \stackrel{\text{def}}{=} A$ on objects and $id_{\mathcal{C}}(f) \stackrel{\text{def}}{=} f$ on morphisms; so $f: A \to B \Longrightarrow id_{\mathcal{C}}(f): id_{\mathcal{C}}(A) \to id_{\mathcal{C}}(B)$.
- ▶ Let (X, \leq_X) and (Y, \leq_Y) be categories and $m: X \to Y$ a monotone function. Then m gives rise to a functor

$$M: (X, \leq_X) \to (Y, \leq_Y)$$

defined by $M(x) \stackrel{\text{def}}{=} m(x)$ on objects $x \in X$ and by $M(\leq_X) = \leq_Y$ on morphisms; since m is monotone, $\leq_X : x \to x' \Longrightarrow M(\leq_X) : M(x) \to M(x')$.

Examples of Functors

▶ We may define a functor $F: Set \to Mon$ by $FA \stackrel{\text{def}}{=} list(A)$ and $Ff \stackrel{\text{def}}{=} map(f)$, where $map(f): list(A) \to list(B)$ is defined by

$$map(f)([]) \stackrel{\text{def}}{=} []$$

$$map(f)([a_1,...,a_n]) \stackrel{\text{def}}{=} [f(a_1),...,f(a_n)]$$

It is easy to see that map(f) is a homomorphism of monoids.

▶ Note that $F(id_A) = id_{FA}$

$$F(id_A)([a_1,...,a_n]) \stackrel{\text{def}}{=} map(id_A)([a_1,...,a_n])$$

$$= id_{list(A)}([a_1,...,a_n])$$

$$\stackrel{\text{def}}{=} id_{FA}([a_1,...,a_n])$$

Examples of Functors

ightharpoonup ... and note that $F(g \circ f) = Fg \circ Ff$

$$F(g \circ f)([a_1,...,a_n]) \stackrel{\text{def}}{=} map(g \circ f)([a_1,...,a_n])$$

$$= [(g \circ f)(a_1),...,(g \circ f)(a_n)]$$

$$= [g(f(a_1)),...,g(f(a_n))]$$

$$= map(g)([f(a_1),...,f(a_n)])$$

$$= map(g)(map(f)([a_1,...,a_n]))$$

$$= (Fg \circ Ff)([a_1,...,a_n]).$$

Definition of a Functor

A functor $F: \mathcal{C} \to \mathcal{D}$ is specified by

- \triangleright an operation taking objects A in C to objects FA in D, and
- ▶ an operation sending morphisms $f: A \to B$ in \mathcal{C} to morphisms $Ff: FA \to FB$ in \mathcal{D} , such that
 - $F(id_A) = id_{FA}$, and
 - ► $F(g \circ f) = Fg \circ Ff$ provided $g \circ f$ is defined.

More Functor Examples

▶ Given a set A, recall that the powerset $\mathcal{P}(A)$ is the set of subsets of A. We can define the **covariant powerset** functor $\mathcal{P} \colon \mathcal{S}et \to \mathcal{S}et$ which is given by

$$f: A \to B \quad \mapsto \quad \mathcal{P}(f) \equiv f_*: \mathcal{P}(A) \to \mathcal{P}(B),$$

where $f: A \to B$ is a function and f_* is defined by

$$f_*(A') \stackrel{\text{def}}{=} \{ f(a') \mid a' \in A' \} \text{ where } A' \in \mathcal{P}(A).$$

• f_* is sometimes called the **direct image** of f.

More Functor Examples

▶ We can define a **contravariant powerset** functor $\mathcal{P} \colon \mathcal{S}et^{op} \to \mathcal{S}et$ by setting

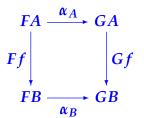
$$f^{op} \colon B \to A \quad \mapsto \quad f^{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A),$$

where $f: A \to B$ is a function in $\mathcal{S}et$, and the function f^{-1} is defined by $f^{-1}(B') \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in B'\}$ where $B' \in \mathcal{P}(B)$.

• f^* is sometimes called the **inverse image** of f.

Definition of a Natural Transformation

Let $F,G:\mathcal{C}\to\mathcal{D}$ be functors. Then a **natural transformation** α from F to G, written $\alpha\colon F\to G$, is specified by giving a morphism $\alpha_A\colon FA\to GA$ in \mathcal{D} for each object A in \mathcal{C} , such that for any $f\colon A\to B$ in \mathcal{C} , we have a commutative diagram



The α_A are the **components** of the natural transformation.

Examples of Natural Transformations

▶ Recall $F: Set \to Mon$ where $FA \stackrel{\text{def}}{=} list(A)$ and $F(f: A \to B) \stackrel{\text{def}}{=} map(f): list(A) \to list(B)$. Define a natural transformation $rev: F \to F$, by specifying functions $rev_A: list(A) \to list(A)$ where

$$rev_A([]) \stackrel{\text{def}}{=} [] \qquad rev_A([a_1,\ldots,a_n]) \stackrel{\text{def}}{=} [a_n,\ldots,a_1]$$

We check

$$(Ff \circ rev_A)([a_1,\ldots,a_n]) = [f(a_n),\ldots,f(a_1)]$$

= $(rev_B \circ Ff)([a_1,\ldots,a_n]).$

Examples of Natural Transformations

▶ Let \mathcal{C} and \mathcal{D} be categories and let F, G, H be functors from \mathcal{C} to \mathcal{D} . Also let $\alpha \colon F \to G$ and $\beta \colon G \to H$ be natural transformations. We can define a natural transformation $\beta \circ \alpha \colon F \to H$ by setting the components to be

$$(\beta \circ \alpha)_A \stackrel{\mathrm{def}}{=} \beta_A \circ \alpha_A.$$

This yields a category $\mathcal{D}^{\mathcal{C}}$ with objects functors from \mathcal{C} to \mathcal{D} , morphisms natural transformations between such functors, and composition as given above.

Examples of Natural Transformations

- ▶ Define a functor F_X : Set o Set by
 - (! Products) $F_X(A) \stackrel{\text{def}}{=} (X \Rightarrow A) \times X$ on objects
 - (! Products) $F_X(f) \stackrel{\text{def}}{=} (f \circ -) \times id_X$ on morphisms

Then define a natural transformation $ev: F_X \to id_{Set}$ with components $ev_A: (X \Rightarrow A) \times X \to A$ by

 $ev_A(g,x) \stackrel{\mathrm{def}}{=} g(x)$ where $(g,x) \in (X \Rightarrow A) \times X$. To see that we have defined a natural transformation let $f: A \to B$ and note that

$$(id_{Set}(f) \circ ev_A)(g,x) = f(ev_A(g,x))$$

= ...(ev_B \circ F_X(f))(g,x).

Isomorphisms and Equivalences

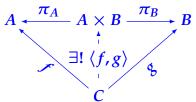
- ▶ A morphism $f: A \to B$ is an **isomorphism** if there is some $g: B \to A$ for which $f \circ g = id_B$ and $g \circ f = id_A$.
- ightharpoonup g is an **inverse** for f and vise versa.
- ▶ A is isomorphic to B, $A \cong B$, if such a mutually inverse pair of morphisms exists.
- ▶ Bijections in *Set* are isomorphisms. There are typically many isomorphisms witnessing that two sets are bijective.
- ▶ In the category determined by a partially ordered set, the only isomorphisms are the identities, and in a preorder X with $x,y \in X$ we have $x \cong y$ iff $x \leq y$ and $y \leq x$. Note that in this case there can be only one pair of mutually inverse morphisms witnessing the fact that $x \cong y$.

Definition of Binary Products

A binary product of objects A and B in C is specified by

- ▶ an object $A \times B$ of C, together with
- ▶ two projection morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$,

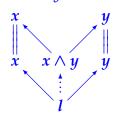
for which given any object C and morphisms $f\colon C\to A$, $g\colon C\to B$, there exists a unique morphism $\langle f,g\rangle\colon C\to A\times B$ for which



 $\langle f,g\rangle\colon C\to A\times B$ is called the **mediating** morphism for f and g.

Examples of Binary Products

- ▶ Let (X, \leq) be a preorder. $l \in X$ is a lower bound of $x, y \in X$ just in case $l \leq x, y$. $u \in X$ is a upper bound of $x, y \in X$ just in case $x, y \leq u$.
- ▶ $x \in S \subseteq X$ is greatest in S if $(\forall s \in S)(s \le x)$ and is least in S if $(\forall s \in S)(x \le s)$.
- ▶ In a preorder a greatest lower bound $x \land y$ of x and y (if it exists) is a binary product $x \times y$ of the category determined by (X, \leq) with projections $x \land y \leq x$ and $x \land y \leq y$. $x \land y$ is also called the **meet** of x and y.



Examples of Binary Products

▶ The binary product of A and B in Set has

$$A \times B \stackrel{\mathrm{def}}{=} \{ (a,b) \mid A \in A, b \in B \}$$

with projection functions $\pi_A(a,b) \stackrel{\text{def}}{=} a$ and $\pi_B(a,b) \stackrel{\text{def}}{=} b$. The mediating function for any $f \colon C \to A$ and $g \colon C \to B$ is

$$\langle f,g\rangle(c)\stackrel{\mathrm{def}}{=} (f(c),g(c)).$$

In any \mathcal{C} , if $p_i\colon P\to A_i$ is any product of A_1 and A_2 then $A_1\times A_2\cong P$. All binary products are determined up to isomorphism: Existence yields mediating morphisms $\phi\colon A_1\times A_2\to P$ and $\psi\colon P\to A_1\times A_2$; uniqueness means that ϕ and ψ witness an isomorphism.

Definition of Finite Products

A product of a non-empty finite family of objects $(A_i \mid i \in I)$ in C, where $I \stackrel{\text{def}}{=} \{1, ..., n\}$, is specified by

- ▶ an object $A_1 \times ... \times A_n$ (or $\Pi_{i \in I} A_i$) in \mathcal{C} , and
- ▶ for every $j \in I$, a morphism $\pi_j : A_1 \times ... \times A_n \to A_j$ in \mathcal{C} called the jth product projection

such that for any object C and family of morphisms $(f_i\colon C\to A_i\mid i\in I)$ there is a unique morphism

$$\langle f_1,\ldots,f_n\rangle\colon C\to A_1\times\ldots\times A_n$$

for which given any $j \in I$, we have $\pi_i \circ \langle f_1, \dots, f_n \rangle = f_i$.

Note: We get binary products when $I \stackrel{\text{def}}{=} \{1,2\}!$

Examples of Finite Products

▶ A finite product of $(A_1, ..., A_n) \equiv (A_i \mid i \in I)$ in **Set** is given by the cartesian product $A_1 \times ... \times A_n$ with the obvious projection functions. Given functions $(f: C \rightarrow A_i \mid i \in I)$ then

$$\langle f_1,\ldots,f_n\rangle(c)\stackrel{\mathrm{def}}{=}(f_1(c),\ldots,f_n(c))$$

- ▶ In a preorder (X, \leq) , a finite product $x_1 \times \ldots \times x_n$, if it exists, is a meet (greatest lower bound) of (x_1, \ldots, x_n) .
- ▶ A **terminal** object **1** in a category \mathcal{C} has the property that there is a unique morphism $!_A : A \to \mathbf{1}$ for every $A \in ob \, \mathcal{C}$. It is the finite product of an **empty** family of morphisms (check this!). Such a **1** may not exist, but is unique up to isomorphism if it does.

Definition of Finite Coproducts

A **coproduct** of a non-empty family of objects $(A_i \mid i \in I)$ in C, where $I = \{1, ..., n\}$, is specified by

- ▶ an object $A_1 + ... + A_n$ ($\sum_{i \in I} A_i$), together with
- ▶ insertion morphisms $\iota_i : A_i \to A_1 + \ldots + A_n$,

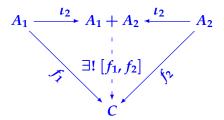
such that for any C and any family of morphisms $(f_i \colon A_i \to C \mid i \in I)$ there is a unique morphism

$$[f_1,\ldots,f_n]:A_1+\ldots+A_n\to C$$

for which given any $j \in I$, we have $[f_1, \ldots, f_n] \circ \iota_j = f_j$.

Definition of Finite Coproducts

In the case that $I \stackrel{\text{def}}{=} \{1,2\}$ we have



(Compare to the diagrams for colimits later on.)

Examples of (Co)Products

▶ In **Set** the binary coproduct of sets A_1 and A_2 is given by their disjoint union $A_1 \uplus A_2$, defined as the union $(A_1 \times \{1\}) \cup (A_2 \times \{2\})$ with the insertion functions

$$\iota_{A_1}: A_1 \to A_1 \uplus A_2 \leftarrow A_2: \iota_{A_2}$$

where ι_{A_1} is defined by $a_1 \mapsto (a_1, 1)$ for all $a_1 \in A_1$, and ι_{A_2} by $a_2 \mapsto (a_2, 2)$ for all $a_2 \in A_2$.

Let preorder (X, \leq) have top and bottom elements and all finite meets and joins (least upper bounds). Then the top of X is terminal, the bottom of X initial, and finite meets and joins are finite products and coproducts respectively.

Examples of (Co)Products

- Given (X, \leq) and (Y, \leq) in $\mathcal{P}ar\mathcal{S}et$, the binary product is the cartesian product $X \times Y$ in $\mathcal{S}et$, with the pointwize order $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$, together with the (monotone) set-theoretic projection functions. The binary coproduct is $X \uplus Y$, with $(z, \delta) \leq (z', \delta')$ iff $\delta = \delta'$ $(\delta, \delta' \in \{1, 2\})$, and $z \leq z'$ (either in X or in Y).
- An **initial** object 0 in a category \mathcal{C} has the property that there is a unique morphism $!_A \colon 0 \to A$ for every $A \in ob \mathcal{C}$. It is the finite coproduct of an empty family of morphisms (check this!). Such a 0 may not exist, but is unique if it does.

Useful "Fact" for (Co)Products

- ▶ Suppose that we have $(f_i: C \to A_i \mid i \in \{1,2\})$ and $\theta: C \to A_1 \times A_2$. In order to prove that $\theta = \langle f_1, f_2 \rangle$ it is sufficient to show that $\pi_{A_i} \circ \theta = f_i$ for each i.
- ▶ Suppose that we have $(f_i: A_i \to C \mid i \in \{1,2\})$ and $\theta: A_1 + A_2 \to C$. In order to prove that $\theta = [f_1, f_2]$ it is sufficient to show that $\theta \circ \iota_{A_i} = f_i$ for each i.

Note: this "fact" is simply a consequence of uniqueness of mediating morphisms. It is crucial to the proof that (co)products are unique up to isomorphism, where both $\phi \circ \psi$ and id (from an earlier slide) are shown to be mediating, and hence equal.

Notation for Finite (Co)Products

▶ Suppose that $f_1: A_1 \to B_1$ and $f_2: A_2 \to B_2$. Then

$$f_1 \times f_2 \stackrel{\text{def}}{=} \langle f_1 \circ \pi_{A_1}, f_2 \circ \pi_{A_2} \rangle \colon A_1 \times A_2 \to B_1 \times B_2$$

$$f_1 + f_2 \stackrel{\text{def}}{=} [\iota_{B_1} \circ f_1, \iota_{B_2} \circ f_2] \colon A_1 + A_2 \to B_1 + B_2$$

and hence one can prove that

$$\pi_{B_i} \circ (f_1 \times f_2) = f_i \circ \pi_{A_i} (f_1 + f_2) \circ \iota_{A_i} = \iota_{B_i} \circ f_i$$

▶ This notation is easily extended to finite families $(A_i \mid i \in \{1, ..., n\})$ and $(B_i \mid i \in \{1, ..., n\})$... or indeed infinite families $(A_i \mid i \in I)$ and $(B_i \mid i \in I)$ where I is any set.

More Examples of (Co)Products

▶ Suppose that \mathcal{C} has binary (co)products. The functors $B \times (-)$, B + (-): $\mathcal{C} \to \mathcal{C}$ are defined by

$$f: A \longrightarrow A' \mapsto id_B \times f: B \times A \longrightarrow B \times A'$$

 $f: A \longrightarrow A' \mapsto id_B + f: B + A \longrightarrow B + A'$

Note that it is common to write $f \times B$ instead of $f \times id_B$; ditto +.

More Examples of (Co)Products

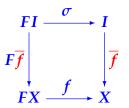
▶ Suppose that F_1 and F_2 are objects (that is, functors) of $\mathcal{D}^{\mathcal{C}}$ and that \mathcal{D} has finite (co)products. Then both $F_1 \times F_2$ and $F_1 + F_2$ exist and are defined pointwize. For products this means

$$(F_1 \times F_2)(\xi) \stackrel{\text{def}}{=} F_1 \xi \times F_2 \xi$$

where ξ is either an object or morphism of C. The projections $\pi^i \colon F_1 \times F_2 \to F_i$ are defined with pointwize components $\pi^i_A \colon F_1A \times F_2A \to F_iA$. These projections π^i are indeed natural transformations.

Algebras for $F: \mathcal{C} \to \mathcal{C}$

▶ An algebra for F is a morphism $\sigma \colon FI \to I$ in C. The algebra is initial if given any $f \colon FX \to X$ there is a homomorphism $\overline{f} \colon I \to X$, meaning that (\overline{f}) is a morphism and)



and such a \overline{f} is unique.

▶ There is a category C^F of algebras and algebra homomorphisms (details omitted) in which initial algebras are initial objects.

Algebras for $F: \mathcal{C} \to \mathcal{C}$

▶ 1+(-): Set \rightarrow Set has an initial algebra

$$[z,s]:1+\mathbb{N}\to\mathbb{N}$$

where $z\colon 1 \to \mathbb{N}$ maps * to 0 and $s\colon \mathbb{N} \to \mathbb{N}$ adds 1. If

$$f: 1+X \to X$$

the function $\overline{f}: \mathbb{N} \to X$ is uniquely defined by

$$\overline{f}(0) \stackrel{\text{def}}{=} \widehat{x}(*) \stackrel{\text{def}}{=} x$$

$$\overline{f}(n+1) \stackrel{\text{def}}{=} \phi^{n+1}(x) = \phi(\overline{f}(n))$$

where $\hat{x} \stackrel{\text{def}}{=} f \circ \iota_1 \colon 1 \to 1 + X$ and

$$\phi \stackrel{\text{def}}{=} f \circ \iota_X \colon X \to 1 + X \text{ (and hence } f = [\widehat{x}, \phi]).$$

Algebras for $F: \mathcal{C} \to \mathcal{C}$

▶ EXAMPLE: the function $(+n): \mathbb{N} \to \mathbb{N}$ which adds n, for any $n \in \mathbb{N}$, is definable as $[\widehat{n}, s]$ where

$$1+\mathbb{N}\xrightarrow{\left[\widehat{n},s\right]}\mathbb{N}$$

and also $(*n) \stackrel{\text{def}}{=} \overline{[z, (+n)]} : \mathbb{N} \to \mathbb{N}$.

 \blacktriangleright A monoid (M, b, e) is an algebra

$$1 + (M \times M) \xrightarrow{[\widehat{e}, b]} M$$

plus the relevant equations.

Case Study: (Haskell) Algebraic Datatypes

We shall

- ▶ Define a Haskell (recursive) datatype grammar.
- ▶ Show that any datatype declaration **D** gives rise to a functor $F \equiv F_{\mathbf{D}} : \mathcal{S}et \rightarrow \mathcal{S}et$.
- ▶ Demonstrate that D can be modelled by an initial algebra $\sigma\colon FI \to I$, where I is the set Exp_D of expressions of type D (up to isomorphism).

Later on we will

- ▶ Show that the functor F preserves colimits of diagrams of the form $D: \omega \to Set$, and such colimits exist . . .
- ▶ and (hence) that *F* must have an initial algebra for purely categorical reasons.

A Recursive Datatype

► A set of **type patterns** *T* is defined by

$$T := D \mid \text{Unit} \mid \text{Int} \mid T \times T$$

► A datatype is specified by the statement

$$\mathbf{D} = \mathbf{K}_1 \; T_1 \; | \; \dots \; | \; \mathbf{K}_m \; T_m$$

➤ A collection of type assignments is defined inductively by the following rules

$$\frac{z \in \mathbb{Z}}{\text{()} :: \texttt{Unit}} \qquad \frac{z \in \mathbb{Z}}{\underline{z} :: \texttt{Int}} \qquad \frac{E :: T_i}{\texttt{K}_i \; E :: \mathsf{D}} \qquad \frac{E_1 :: T_1 \quad E_2 :: T_2}{(E_1, E_2) :: T_1 \times T_2}$$

and
$$Exp_T \stackrel{\text{def}}{=} \{ E \mid E :: T \}.$$

Defining *F* from **D**

▶ The functor F is defined (as a coproduct in Set^{Set}) by

$$F \stackrel{\mathrm{def}}{=} F_{T_1} + \ldots + F_{T_m}$$

where each $F_{T_i} : \mathcal{S}et \to \mathcal{S}et$.

- ▶ Functors F_T : Set o Set are defined by recursion on the structure of T by setting
 - $ightharpoonup F_{D} \stackrel{\text{def}}{=} id_{\mathcal{S}et}$
 - $ightharpoonup F_{\mathrm{Unit}}(g\colon U \to V) \stackrel{\mathrm{def}}{=} id_1\colon 1 \to 1$ where 1 is terminal in Set
 - $ightharpoonup F_{\operatorname{Int}}(g\colon U\to V)\stackrel{\mathrm{def}}{=} id_{\mathbb{Z}}\colon \mathbb{Z}\to \mathbb{Z}$
 - $\vdash F_{T_1 \times T_2} \stackrel{\text{def}}{=} F_{T_1} \times F_{T_2}$

Defining An Initial Algebra $\sigma: FI \to I$

▶ We set $I \stackrel{\text{def}}{=} Exp_D$ and we define

$$\sigma \stackrel{\mathrm{def}}{=} [\widehat{\mathsf{K}_1} \circ \sigma_{T_1} \dots \widehat{\mathsf{K}_m} \circ \sigma_{T_m}] \colon F\mathsf{I} \stackrel{\mathrm{def}}{=} F_{T_1}\mathsf{I} + \dots + F_{T_m}\mathsf{I} \longrightarrow \mathsf{I}$$

where the function $\widehat{K}_i \colon Exp_{T_i} \to I$ applies the constructor and we define functions $\sigma_T \colon F_T I \to Exp_T$ by recursion over T as follows

- $ightharpoonup \sigma_{\mathbf{D}}(E \in \mathbf{I}) \stackrel{\text{def}}{=} E \in Exp_{\mathbf{D}}$
- $ightharpoonup \sigma_{ ext{Unit}}(* \in 1) \stackrel{ ext{def}}{=} () \in \mathit{Exp}_{ ext{Unit}}.$
- $\qquad \qquad \bullet \quad \sigma_{\rm Int}(z \in \mathbb{Z}) \stackrel{\rm def}{=} \underline{z} \in Exp_{\rm Int}.$
- $\overset{\boldsymbol{\sigma}_{T_1 \times T_2}}{\stackrel{\boldsymbol{\sigma}_{T_1 \times T_2}}{=}} ((e_1, e_2) \in F_{T_1} \mathsf{I} \times F_{T_2} \mathsf{I}) \overset{\text{def}}{=} (\boldsymbol{\sigma}_{T_1}(e_1), \boldsymbol{\sigma}_{T_2}(e_2)) \in Exp_{T_1 \times T_2}$
- ▶ It may be useful to note that $\sigma(\iota_i(e_i \in F_{T_i} \mathsf{I})) = \mathsf{K}_i \ \sigma_{T_i}(e_i)$.

▶ Suppose that $f: FX \to X$ in **Set**. We have to prove that there is a unique \overline{f} such that

$$F_{T_1}I + \dots + F_{T_m}I = FI \xrightarrow{\sigma} I$$

$$F\overline{f} \downarrow \qquad \qquad \downarrow \overline{f}$$

$$F_{T_1}X + \dots + F_{T_m}X = FX \xrightarrow{f} X$$

▶ Note $\overline{f} : Exp_D \to F_D X$; we will define $\overline{f} \stackrel{\text{def}}{=} \theta_D$ and functions

$$\theta_T \colon Exp_T \to F_T X$$

by recursion on T:

- $\bullet \theta_{\mathsf{D}}(\mathbb{K}_i \ E_i \in Exp_{\mathsf{D}}) \stackrel{\mathsf{def}}{=} f(\iota_i(\theta_{T_i}(E_i))) \in X.$
- $\bullet \; \theta_{\text{Unit}}(() \in Exp_{\text{Unit}}) \stackrel{\text{def}}{=} * \in 1.$
- $\bullet \ \theta_{\text{Int}}(\underline{z} \in Exp_{\text{Int}}) \stackrel{\text{def}}{=} z \in \mathbb{Z}.$
- $\theta_{T_1 \times T_2}((E_1, E_2) \in Exp_{T_1 \times T_2}) \stackrel{\text{def}}{=} (\theta_{T_1}(E_1), \theta_{T_2}(E_2)) \in F_{T_1} \mathbb{I} \times F_{T_2} \mathbb{I}.$

▶ Observe that for any T we have $\theta_T \circ \sigma_T = F_T \theta_D$, which follows from an easy induction.

Note that by universality of coproducts $\overline{f}\circ\sigma=f\circ F\overline{f}$ iff

$$\overline{f} \circ \sigma \circ \iota_i = f \circ F\overline{f} \circ \iota_i$$

Then for any $e_i \in F_{T_i}$

$$(\theta_{\mathsf{D}} \circ \sigma \circ \iota_{i})(e_{i}) = \theta_{\mathsf{D}}(\mathsf{K}_{i} \ \sigma_{\mathsf{T}_{i}}(e_{i}))$$

$$\stackrel{\text{def}}{=}_{\theta_{\mathsf{D}}} f(\iota_{i}(\theta_{T_{i}}(\sigma_{T_{i}}(e_{i})))$$

$$= f(\iota_{i}((F_{T_{i}}\theta_{\mathsf{D}})(e_{i})))$$

$$= f((F_{T_{1}}\theta_{\mathsf{D}} + \ldots + F_{T_{m}}\theta_{\mathsf{D}})(\iota_{i}(e_{i})))$$

$$\stackrel{\text{def}}{=}_{F} (f \circ F\theta_{\mathsf{D}} \circ \iota_{i})(e_{i})$$

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Adjunctions between Preorders

► A pair of monotone functions

$$X \stackrel{l}{\longleftarrow} Y$$

is said to be an adjunction if for all $x \in X$ and $y \in Y$,

$$l(x) \le y \iff x \le r(y)$$

We say that *l* is **left adjoint** to *r* and that *r* is right adjoint to *l*. We write *l* ⊢ *r*.

Examples

Let 1 def (*) be the one element preorder. Then there are adjunctions (⊥ ¬! ¬ ⊤)

$$X \xrightarrow{!} 1 \qquad X \xrightarrow{!} 1$$

provided that X has both top and bottom elements. For example, for any $x \in X$,

$$!(x) \stackrel{\text{def}}{=} * \le * \iff x \le \top(*) \stackrel{\text{def}}{=} \top$$

Examples

▶ Define $\Delta: X \to X \times X$ by $\Delta(x) \stackrel{\text{def}}{=} (x, x)$. Then there are adjoints $(\lor \dashv \Delta \dashv \land)$

$$X \xrightarrow{\Delta} X \times X \qquad X \xrightarrow{\Delta} X \times X$$

just in case X has all binary meets and joins: for any $l \in X$,

$$\Delta(l) \stackrel{\text{def}}{=} (l, l) \le (x, x') \Longleftrightarrow l \le \wedge (x, x') \stackrel{\text{def}}{=} x \wedge x'$$

► This structure corresponds to *X* having binary products and coproducts.

Adjunctions between Categories

- ▶ Let $L: \mathcal{C} \to \mathcal{D}$ and $R: \mathcal{D} \to \mathcal{C}$ be functors. L is **left adjoint** to R, written $L \dashv R$, if given any objects A of \mathcal{C} and B of \mathcal{D} we have
 - ▶ a bijection between morphisms $LA \to B$ in \mathcal{D} and $A \to RB$ in \mathcal{C} , that is, between $\mathcal{C}(LA,B)$ and $\mathcal{D}(A,RB)$,

$$\frac{f \colon LA \to B}{\overline{f} \colon A \to RB} \qquad \qquad \frac{g \colon A \to RB}{\widehat{g} \colon LA \to B}$$

▶ this bijection is *natural in A and B*: given morphisms $\phi: A' \to A$ in \mathcal{C} and $\psi: B \to B'$ in \mathcal{D} we have

$$\overline{\psi \circ f \circ \mathbf{L} \phi} = \mathbf{R} \psi \circ \overline{f} \circ \phi \quad \text{and/or} \quad (\mathbf{R} \psi \circ g \circ \phi)^{\wedge} = \psi \circ \widehat{g} \circ \mathbf{L} \phi.$$

The forgetful functor *U*: Mon → Set taking a monoid to its underlying set, and the functor list(-): Set → Mon taking a set to finite lists over the set, are adjoints:

$$list(-) \dashv U$$

So there is a natural bijection between $\mathcal{M}on(list(A), M)$ and $\mathcal{S}et(A, UM)$

$$\frac{f \colon list(A) \to M}{\overline{f} \colon A \to UM}$$

$$\frac{g \colon A \to UM}{\widehat{g} \colon list(A) \to M}$$

► This is given by

$$g: A \longrightarrow UM \mapsto$$

$$\widehat{g}: list(A) \xrightarrow{[a_1, \ldots, a_n] \mapsto g(a_1) \ldots g(a_n)} M,$$

and

$$f: list(A) \longrightarrow M \quad \longmapsto \quad \overline{f}: A \xrightarrow{a \mapsto f([a])} UM.$$

Note that

$$\widehat{f}[a_1,...,a_n] = \overline{f}(a_1)...\overline{f}(a_n)
= f([a_1])...f([a_n]) = f([a_1]++...++[a_n])$$

It is an exercise to verify that $\overline{\widehat{g}} = g$ and that this bijection is natural.

▶ For a fixed set A, the functor $(-) \times B : Set \to Set$ has a right adjoint $B \Rightarrow (-) : Set \to Set$. If $c : C \to C'$ then

$$B \Rightarrow c : B \Rightarrow C \longrightarrow B \Rightarrow C' \text{ where } (B \Rightarrow c)(\theta) \stackrel{\text{def}}{=} c \circ \theta$$

$$f : A \times B \to C$$

$$\overline{f} \stackrel{\text{def}}{=} \lambda a.\lambda b. f(a,b) : A \to B \Rightarrow C$$

$$g : A \to B \Rightarrow C$$

$$g : A \to B \Rightarrow C$$

$$\widehat{g} \stackrel{\text{def}}{=} \lambda (a,b).g(a)(b) : A \times B \to C$$

It is immediate that we have a bijection; naturality is an exercise. Having products and such a "function structure" is known as cartesian closure.

► The diagonal functor $\Delta \colon \mathcal{S}et \to \mathcal{S}et \times \mathcal{S}et$ taking a function $f \colon A \to B$ to $(f,f) \colon (A,A) \to (B,B)$ has right and left adjoints Π and Σ taking any morphism $(f_1,f_2) \colon (A_1,A_2) \to (B_1,B_2)$ of $\mathcal{S}et \times \mathcal{S}et$ to $f_1 \times f_2 \stackrel{\text{def}}{=} \langle f_1 \circ \pi_{A_1}, f_2 \circ \pi_{A_2} \rangle \colon A_1 \times A_2 \to B_1 \times B_2$ and

$$f_1 imes f_2 \stackrel{\mathrm{def}}{=} \langle f_1 \circ \pi_{A_1}, f_2 \circ \pi_{A_2} \rangle \colon A_1 imes A_2 o B_1 imes B_2$$
 and $f_1 + f_2 \stackrel{\mathrm{def}}{=} [\iota_{B_1} \circ f_1, \iota_{B_2} \circ f_2] \colon A_1 + A_2 o B_1 + B_2$ respectively, where the bijection for Π is

$$\frac{(f,g) \quad \widehat{m} \stackrel{\text{def}}{=} (\pi_A \circ m, \pi_B \circ m) : \Delta C \longrightarrow (A,B)}{\overline{(f,g)} \stackrel{\text{def}}{=} \langle f,g \rangle \qquad m : C \longrightarrow \Pi(A,B)}$$

▶ If we replace **Set** by any category **C** with (co)products, defining $\Delta: \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ analogously, everything still works.

Let \mathcal{C} be a category with finite products. Existence of a right adjoint R to the functor $(-) \times B \colon \mathcal{C} \to \mathcal{C}$ for each object B of \mathcal{C} , is equivalent to \mathcal{C} being cartesian closed.

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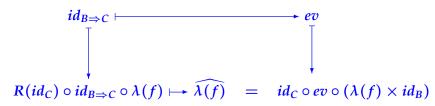
 (\Rightarrow) Given an object B of C set $B\Rightarrow C\stackrel{\mathrm{def}}{=}R(C)$ for any object C of C. Given a morphism $f\colon A\times B\to C$ we define $\lambda(f)\colon A\to (B\Rightarrow C)$ to be the mate of f across the given adjunction. The morphism

$$ev: (B \Rightarrow C) \times B \rightarrow C$$

is the mate $(\widehat{id_{B\Rightarrow C}})$ of the identity $id_{B\Rightarrow C}: (B\Rightarrow C) \rightarrow (B\Rightarrow C)$.

Let $\mathcal C$ be a category with finite products. Existence of a right adjoint R to the functor $(-) \times B \colon \mathcal C \to \mathcal C$ for each object B of $\mathcal C$, is equivalent to $\mathcal C$ being cartesian closed.

Next, we need to show that $ev \circ (\lambda(f) \times id_B) = f$. This follows directly from the naturality of the adjunction; we consider naturality in A and C at the morphisms $\lambda(f) \colon A \to (B \Rightarrow C)$ and $id_C \colon C \to C$:



We let the reader show that $\lambda(f)$ is the unique morphism satisfying the latter equation.

(\Leftarrow) Conversely, let B be an object of C. We define a right adjoint to $(-) \times B$ denoted by $B \Rightarrow (-)$, by setting

$$c: C \longrightarrow C' \mapsto B \Rightarrow c \stackrel{\text{def}}{=} \lambda(c \circ ev) \colon (B \Rightarrow C) \to (B \Rightarrow C')$$

for each morphism $c\colon C\to C'$ of $\mathcal C$ (this matches our earlier definition – check). We define a bijection by declaring the mate of $f\colon A\times B\to C$ to be $\lambda(f)\colon A\to (B\Rightarrow C)$ and the mate of $g\colon A\to (B\Rightarrow C)$ to be

$$\widehat{g} \stackrel{\text{def}}{=} ev \circ (g \times id_B) \colon A \times B \to C.$$

It remains to verify that we have defined a bijection which is natural in the required sense. We only check one part of naturality. Let $a: A' \to A$ and $c: C \to C'$ be morphisms of C. Then

$$ev \circ ((\lambda(c \circ ev) \circ \lambda(f) \circ a) \times id) =$$

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$$c \circ ev \circ (\lambda(f) \times id) \circ (a \times id) =$$

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implying that $\lambda(c \circ f \circ (a \times id)) = (B \Rightarrow c) \circ \lambda(f) \circ a$ since C is a CCC.

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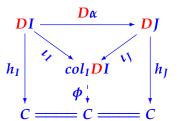
$$c \circ ev \circ (\lambda(f) \times id) \circ (a \times id) =$$

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implying that $\lambda(c \circ f \circ (a \times id)) = (B \Rightarrow c) \circ \lambda(f) \circ a$ since C is a CCC.

Colimits

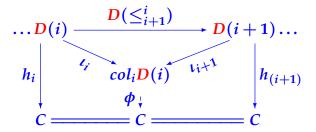
▶ Given a diagram $D: \mathbb{I} \to \mathcal{C}$, a colimit for D is given by an object col_IDI of \mathcal{C} together with a family of morphisms $(\iota_I: DI \to col_IDI \mid I \in \mathbb{I})$ such that for any $\alpha: I \to J$ in \mathbb{I} we have $\iota_J \circ D\alpha = \iota_I$. This data satisfies: given any family $(h_I: DI \to C \mid I \in \mathbb{I})$ such that $h_J \circ D\alpha = h_I$, there is a unique morphism $\phi: col_IDI \to C$ satisfying $\phi \circ \iota_I = h_I$ for each object I of \mathbb{I} (and hence $\phi = [h_I \mid I \in \mathbb{I}]$)



▶ Binary coproducts arise from the discrete category $\mathbb{I} \stackrel{\text{def}}{=} \{1,2\}$.

Colimits

▶ Let $D: \omega \to \mathcal{C}$; suppose that $i \leq i+1$ is a typical morphism in ω . Then a colimit diagram, if it exists, can be taken as



where for any given functions $h_i: D(i) \to C$ commuting with the functions $D(\leq_{i+1}^i)$, a unique such ϕ exists.

This fact follows, since $h_j \circ D(\leq_j^i) = h_i$ for a general morphism \leq_j^i (where $i \leq j$ in ω) is immediate.

Colimits

- ▶ It is a fact that *Set* has all (small) colimits.
- ▶ It is a fact that a colimit for $\Delta : \omega \times \omega \to \mathcal{C}$ exists if and only if a colimit for $\Delta' : \omega \to \mathcal{C}$ where $\Delta'(i \in \omega) \stackrel{\text{def}}{=} \Delta(i,i)$ exists, and when they (both) exist they are isomorphic, that is

$$col_k \Delta'(k) \cong col_{(i,j)} \Delta(i,j)$$

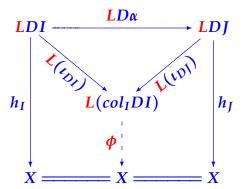
Further (exercise: define the diagrams that give rise to the colimits below ...)

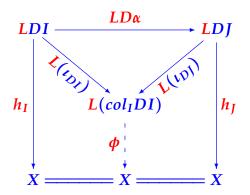
$$col_i(col_j\Delta(i,j)) \cong col_j(col_i\Delta(j,i))$$

and all of the above colimits are isomorphic.

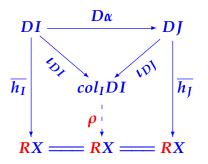
Let $D\colon \mathbb{I} o \mathcal{C}$, and $L\colon \mathcal{C} o \mathcal{D}$ and $L\dashv R$ for some R. Then $L(col_IDI)\cong col_ILDI$

and is witnessed by $[L(\iota_{DI}) \mid I \in \mathbb{I}]: col_I LDI \to L(col_I DI)$. It suffices to show that $L(col_I DI)$ is a colimit for $LD: \mathbb{I} \to \mathcal{D}$.



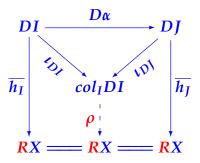


Suppose that $h_I = h_J \circ LD\alpha$. We need to show there is a unique ϕ as above.



But

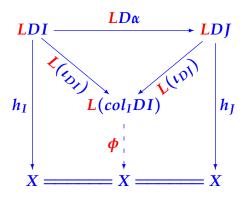
$$h_I=h_J\circ LDlpha \Longrightarrow \overline{h_I}=\overline{h_J\circ LDlpha}=\overline{h_J}\circ Dlpha$$
 where the final equality follows by naturality.



Therefore there is ρ with $\rho \circ \iota_{DI} = \overline{h_I}$. Define

$$\phi \stackrel{\text{def}}{=} \widehat{\rho} \colon L(col_IDI) \to X$$

Left Adjoints Preserve Colimits



Hence, again using naturality,

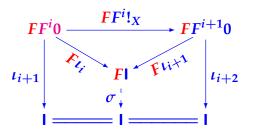
$$\boldsymbol{\phi} \circ \boldsymbol{L}(\iota_{DI}) \stackrel{\text{def}}{=} \widehat{\boldsymbol{\rho}} \circ \boldsymbol{L}(\iota_{DI}) = \widehat{\boldsymbol{\rho} \circ \iota_{DI}} = \widehat{\overline{h_I}} = h_I$$

Existence of Initial Algebras

Suppose that F preserves colimits of the form $D:\omega\to\mathcal{C}$ and that \mathcal{C} has an initial object 0. Define

$$D(i \le i+1) \stackrel{\text{def}}{=} F^i!_X : F^i 0 \to F^{i+1} 0 \text{ for } i \in \omega.$$
 Then $I \stackrel{\text{def}}{=} col_i Di$ (if it exists) is an initial algebra for F .

Since F preserves colimits and $I \stackrel{\text{def}}{=} col_i Di$ we can define $\sigma: FI \to I$



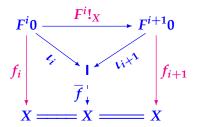
where $\sigma \circ F \iota_i = \iota_{i+1}$.

Existence of Initial Algebras

Let $f: FX \to X$. Define $f_0 \stackrel{\text{def}}{=} !_X \colon 0 \to X$ and $f_{i+1} \stackrel{\text{def}}{=} f \circ Ff_i$. Certainly $f_1 \circ F^0 !_X \equiv f_1 \circ !_X = f_0$ and for $i \geq 1$ we have inductively

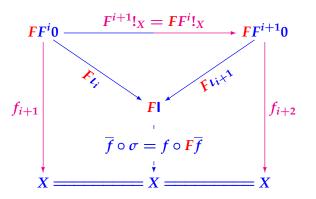
$$f_{i+1} \circ F^i!_X \stackrel{\text{def}}{=} f \circ Ff_i \circ F^i!_X = f \circ F(f_i \circ F^{i-1}!_X) = f \circ Ff_{i-1} \stackrel{\text{def}}{=} f_i$$

and hence \overline{f} exists where $\overline{f} \circ \iota_i = f_i$.



Existence of Initial Algebras

We now have $\sigma \circ F\iota_i = \iota_{i+1}$; and $f_{i+1} \stackrel{\text{def}}{=} f \circ Ff_i$ (which implied $f_{i+1} = f_{i+2} \circ F^{i+1}!_X$) yielding $\overline{f} \circ \iota_i = f_i$



The equality follows since

$$\overline{f} \circ \sigma \circ \mathbf{F}\iota_i = f_{i+1} \quad f \circ \mathbf{F}\overline{f} \circ \mathbf{F}\iota_i = f \circ \mathbf{F}(\overline{f} \circ \iota_i) = f \circ \mathbf{F}f_i = f_{i+1}$$

Suppose that a functor $F \colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F ::= P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D \colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$(F \times G)(col_{i}Di) \stackrel{\text{def}}{=} (Fcol_{i}Di) \times (Gcol_{i}Di)$$

$$\cong (col_{j}FDj) \times (col_{i}GDi)$$

$$\cong col_{i}((col_{j}DFj) \times DGi)$$

$$\cong col_{i}(col_{j}(DFi \times DGj))$$

$$\cong col_{k}(DFk \times DGk)$$

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Suppose that a functor $F \colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F := P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D \colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$(F+G)(col_iDi) \stackrel{\text{def}}{=} (Fcol_iDi) + (Gcol_iDi)$$

 $\cong (col_iFDi) + (col_iGDi)$
 $\cong col_i(DFi + DGi)$

The first step follows by induction on F and G; the second step can be proven directly from the definition of a colimit (coproduct).

Hence any such F preserves $D: \omega \to Set$ colimits.

It follows from this, plus the fact that identity functors and constant functors preserve colimits of diagrams $D\colon\omega\to\mathcal{C}$ for any \mathcal{C} , that the datatype functor

$$F \stackrel{\text{def}}{=} F_{T_1} + \ldots + F_{T_m} \colon \mathcal{S}et \longrightarrow \mathcal{S}et$$

preserves colimits of shape $D: \omega \longrightarrow Set$. Since in fact Set has all colimits, by purely categorical reasoning it has an initial algebra $\sigma: FI \longrightarrow I$.

Mini Project

Find out what nominal sets are, and learn the basic properties of the category **Nom** (of nominal sets and finitely supported functions) such as finite products and coproducts. Follow this up by learning what a nominal algebraic datatype is. Then see if you can construct an initial algebra model of expressions for such a datatype, proving the relevant properties, and further show that initial algebras exist for purely categorical reasons, much as we did in these slides for (ordinary) algebraic datatypes.

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