Category Theory

Roy L. Crole

University of Leicester, UK

April 2015

Introductory Remarks

- Category Theory is a theory of abstraction (of algebraic structure).
- It had its origins in Algebraic Topology with the work of Eilenberg and Mac Lane (1942-45).
- It provides tools and techniques which allow the formulation and analysis of common features amongst apparently different mathematical/computational theories.
- We can discover new relationships between things that are seemingly unconnected.
- Category theory concentrates on how things behave and not on internal details.
- As such, category theory can clarify and simplify our ideas—and indeed lead to new ideas and new results.

Introductory Remarks

- Connections with Computer Science were first made in the 1980s, and the subject has played a central role ever since.
- Some contributions (chosen by me . . . there are many many more) are
  - Cartesian closed categories as models of pure functional languages.
  - The use of strong monads to model notions of computation (well incorporated into Haskell).
  - Precise correspondences between categorical structures and type theories (as internal languages).
  - The categorical solution of domain equations as models of recursive types.
  - Categories for concurrent computation.
  - Nominal categories as models of variable binding.

Seminar Outline

Categories

Functors

Natural Transformations

Isomorphisms, Products, Coproducts

Algebras

Case Study: Modelling (Haskell) Algebraic Datatypes

Adjunctions

Colimits and Applications to Initial Algebras

Examples of Categories

- The collection of all sets and all functions
  - Each set has an identity function; functions compose; composition is associative.
- The collection of all elements of a preorder and all instances of the order relation (relationships) $\leq$
  - Each element has an identity relationship (reflexivity); relationships compose (transitivity); composition is associative.
- The collection of all elements of a singleton $\{ * \}$ (!) and any collection of algebraic terms with just one variable x_0
  - $*$ has an identity term x_0; terms compose (substitution); composition is associative.

Definition of A Category

A **category** $\mathcal{C}$ is specified by the following data:

- A collection $ob\,\mathcal{C}$ of entities called **objects**. An object will often be denoted by a capital letter such as A, B, $C\ldots$
- For any two objects A and B, a collection $\mathcal{C}(A,B)$ of entities called **morphisms**. A morphism in $\mathcal{C}(A,B)$ will often be denoted by a small letter such as f, g, $h\ldots$.
- If $f \in \mathcal{C}(A,B)$ then A is called the **source** of f, and B is the **target** of f and we write (equivalently) $f\colon A \to B$.

Definition of A Category

A **category** $\mathcal{C}$ is specified by the following data (continued):

- There is an operation assigning to each object A of $\mathcal{C}$ an **identity** morphism $id_A \colon A \to A$.
- There is an operation

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

  assigning to each pair of morphisms $f \colon A \to B$ and $g \colon B \to C$ their **composition** which is a morphism denoted by $g \circ f \colon A \to C$ or just $gf \colon A \to C$.
- Such morphisms f and g, with a common source and target B, are said to be **composable**.

Definition of A Category

A **category** $\mathcal{C}$ is specified by the following data (continued):

- These operations are **unitary**

$$\begin{aligned} id_B \circ f &= f \colon A \to B \\ f \circ id_A &= f \colon A \to B \end{aligned}$$

- and **associative**, that is given morphisms $f \colon A \to B$, $g \colon B \to C$ and $h \colon C \to D$ then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

If we say "f is a morphism" we implicitly assume that the source and target are recoverable, that is, we can work out $f \in \mathcal{C}(A, B)$ for some A and B.

More Examples

- The category $\mathcal{Part}$ with $ob\,\mathcal{Part}$ all sets and morphisms $\mathcal{Part}(A,B)$ the partial functions $A \longrightarrow B$.
  - The identity function id_A is a partial function!
  - Given $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$, then for each element a of A, $(g \circ f)(a)$ is defined with value $g(f(a))$ if and only if both $f(a)$ and $g(f(a))$ are defined.
- Given a category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{op}$ has
  - $ob\,\mathcal{C}^{op} \stackrel{\text{def}}{=} ob\,\mathcal{C}$ and $\mathcal{C}^{op}(A,B) = \{\, f^{op} \mid f \in \mathcal{C}(B,A) \,\}$.
  - The identity on an object A in $\mathcal{C}^{op}$ is defined to be id_A^{op}.
  - If $f^{op}\colon A \longrightarrow B$ and $g^{op}\colon B \longrightarrow C$ are morphisms in $\mathcal{C}^{op}$, then $f\colon B \longrightarrow A$ and $g\colon C \longrightarrow B$ are composable morphisms in $\mathcal{C}$.

    We define $g^{op} \circ f^{op} \stackrel{\text{def}}{=} (f \circ g)^{op}\colon A \longrightarrow C$.

More Examples

- A **discrete** category is one for which the only morphisms are identities.
- A **semigroup** (S,b) is a set S together with an associative binary operation $b\colon S \times S \to S$, $(s,s') \mapsto s \cdot s'$. An **identity element** for a semigroup S is some (necessarily unique) element e of S such that for all $s \in S$ we have $e \cdot s = s \cdot e = s$. A **monoid** (M,b,e) is a semigroup (M,b) with identity element e. Any monoid is a **single object category** $\mathcal{C}$ with $\mathcal{C}(*,*) \stackrel{\mathrm{def}}{=} M$. Concrete examples are
  - Addition on the natural numbers, $(\mathbb{N},+,0)$.
  - Concatenation of finite lists over a set A, $(list(A),{+\!+},[\,])$.

More Examples

- $\mathcal{M}on$ has objects monoids and morphisms monoid homomorphisms: $h \colon M \to M'$ is a **homomorphism** if $h(e) = e$ and $h(m_1 \cdot m_2) = h(m_1) \cdot h(m_2)$ for all $m_i \in M$.
- $\mathcal{P}re\mathcal{S}et$ has objects preorders and morphisms the monotone functions; and $\mathcal{P}ar\mathcal{S}et$ has objects partially ordered sets and morphisms the monotone functions.
- The category of relations $\mathcal{R}el$ has objects sets and morphisms binary relations on sets; composition is relation-composition.
- The category of lattices $\mathcal{L}at$ has objects lattices and morphisms the lattice homomorphisms.
- The category $\mathcal{CL}at$ has objects the complete lattices and morphisms the complete lattice homomorphisms.
- The category $\mathcal{G}rp$ of groups and homomorphisms.

Examples of Functors

- Let $\mathcal{C}$ be a category. The **identity** functor $id_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is defined by $id_{\mathcal{C}}(A) \stackrel{\text{def}}{=} A$ on objects and $id_{\mathcal{C}}(f) \stackrel{\text{def}}{=} f$ on morphisms; so $f\colon A \to B \Longrightarrow id_{\mathcal{C}}(f)\colon id_{\mathcal{C}}(A) \to id_{\mathcal{C}}(B)$.
- Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be categories and $m\colon X \to Y$ a monotone function. Then m gives rise to a functor

$$M\colon (X, \leq_X) \to (Y, \leq_Y)$$

  defined by $M(x) \stackrel{\text{def}}{=} m(x)$ on objects $x \in X$ and by $M(\leq_X) = \leq_Y$ on morphisms; since m is monotone,

$$\leq_X\colon x \to x' \Longrightarrow M(\leq_X)\colon M(x) \to M(x').$$

Examples of Functors

- We may define a functor $F\colon \mathcal{S}et \to \mathcal{M}on$ by $FA \stackrel{\text{def}}{=} list(A)$ and $Ff \stackrel{\text{def}}{=} map(f)$, where $map(f)\colon list(A) \to list(B)$ is defined by

$$\begin{aligned} map(f)([]) &\stackrel{\text{def}}{=} [] \\ map(f)([a_1,\ldots,a_n]) &\stackrel{\text{def}}{=} [f(a_1),\ldots,f(a_n)] \end{aligned}$$

  It is easy to see that $map(f)$ is a homomorphism of monoids.
- Note that $F(id_A) = id_{FA}$

$$\begin{aligned} F(id_A)([a_1,\ldots,a_n]) &\stackrel{\text{def}}{=} map(id_A)([a_1,\ldots,a_n]) \\ &= id_{list(A)}([a_1,\ldots,a_n]) \\ &\stackrel{\text{def}}{=} id_{FA}([a_1,\ldots,a_n]) \end{aligned}$$

Examples of Functors

- ...and note that $F(g \circ f) = Fg \circ Ff$

$$
\begin{aligned}
F(g \circ f)([a_1, \ldots, a_n]) &\stackrel{\textbf{def}}{=} map(g \circ f)([a_1, \ldots, a_n]) \\
&= [(g \circ f)(a_1), \ldots, (g \circ f)(a_n)] \\
&= [g(f(a_1)), \ldots, g(f(a_n))] \\
&= map(g)([f(a_1), \ldots, f(a_n)]) \\
&= map(g)(map(f)([a_1, \ldots, a_n])) \\
&= (Fg \circ Ff)([a_1, \ldots, a_n]).
\end{aligned}
$$

Definition of a Functor

A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ is specified by

- an operation taking objects A in $\mathcal{C}$ to objects FA in $\mathcal{D}$, and
- an operation sending morphisms $f\colon A \to B$ in $\mathcal{C}$ to morphisms $Ff\colon FA \to FB$ in $\mathcal{D}$, such that
  - $F(id_A) = id_{FA}$, and
  - $F(g \circ f) = Fg \circ Ff$ provided $g \circ f$ is defined.

More Functor Examples

- Given a set A, recall that the powerset $\mathcal{P}(A)$ is the set of subsets of A. We can define the **covariant powerset** functor $\mathcal{P}\colon \mathcal{S}et \to \mathcal{S}et$ which is given by

$$f\colon A \to B \quad \mapsto \quad \mathcal{P}(f) \equiv f_*\colon \mathcal{P}(A) \to \mathcal{P}(B),$$

  where $f\colon A \to B$ is a function and f_* is defined by $f_*(A') \stackrel{\mathbf{def}}{=} \{f(a') \mid a' \in A'\}$ where $A' \in \mathcal{P}(A)$.
- f_* is sometimes called the **direct image** of f.

More Functor Examples

- We can define a **contravariant powerset** functor $\mathcal{P}\colon \mathcal{S}et^{op} \to \mathcal{S}et$ by setting

$$f^{op}\colon B \to A \quad \mapsto \quad f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A),$$

  where $f\colon A \to B$ is a function in $\mathcal{S}et$, and the function f^{-1} is defined by $f^{-1}(B') \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in B'\}$ where $B' \in \mathcal{P}(B)$.
- f^* is sometimes called the **inverse image** of f.

Definition of a Natural Transformation

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. Then a **natural transformation** α from F to G, written $\alpha\colon F \to G$, is specified by giving a morphism $\alpha_A\colon FA \to GA$ in $\mathcal{D}$ for each object A in $\mathcal{C}$, such that for any $f\colon A \to B$ in $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

The α_A are the **components** of the natural transformation.

Examples of Natural Transformations

- Recall $F\colon \mathcal{Set} \to \mathcal{Mon}$ where $FA \stackrel{\textbf{def}}{=} list(A)$ and $F(f\colon A \to B) \stackrel{\textbf{def}}{=} map(f)\colon list(A) \to list(B)$. Define a natural transformation $rev\colon F \to F$, by specifying functions $rev_A\colon list(A) \to list(A)$ where

  $$rev_A([\,]) \stackrel{\textbf{def}}{=} [\,] \qquad rev_A([a_1,\dots,a_n]) \stackrel{\textbf{def}}{=} [a_n,\dots,a_1]$$

  We check

  $$\begin{aligned}(Ff \circ rev_A)([a_1,\dots,a_n]) &= [f(a_n),\dots,f(a_1)] \\ &= (rev_B \circ Ff)([a_1,\dots,a_n]).\end{aligned}$$

Examples of Natural Transformations

- Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let F, G, H be functors from $\mathcal{C}$ to $\mathcal{D}$. Also let $\alpha\colon F \longrightarrow G$ and $\beta\colon G \longrightarrow H$ be natural transformations. We can define a natural transformation $\beta \circ \alpha\colon F \longrightarrow H$ by setting the components to be

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A.$$

  This yields a category $\mathcal{D}^{\mathcal{C}}$ with objects functors from $\mathcal{C}$ to $\mathcal{D}$, morphisms natural transformations between such functors, and composition as given above.

Examples of Natural Transformations

- Define a functor $F_X\colon \mathcal{S}et \to \mathcal{S}et$ by
  - (! Products) $F_X(A) \stackrel{\mathbf{def}}{=} (X \Rightarrow A) \times X$ on objects
  - (! Products) $F_X(f) \stackrel{\mathbf{def}}{=} (f \circ -) \times id_X$ on morphisms

  Then define a natural transformation $ev\colon F_X \to id_{\mathcal{S}et}$ with components $ev_A\colon (X \Rightarrow A) \times X \to A$ by $ev_A(g,x) \stackrel{\mathbf{def}}{=} g(x)$ where $(g,x) \in (X \Rightarrow A) \times X$. To see that we have defined a natural transformation let $f\colon A \to B$ and note that

$$
\begin{aligned}
(id_{\mathcal{S}et}(f) \circ ev_A)(g,x) &= f(ev_A(g,x)) \\
&= \ldots (ev_B \circ F_X(f))(g,x).
\end{aligned}
$$

Isomorphisms and Equivalences

- A morphism $f: A \to B$ is an **isomorphism** if there is some $g: B \to A$ for which $f \circ g = id_B$ and $g \circ f = id_A$.
- g is an **inverse** for f and vise versa.
- A is **isomorphic** to B, $A \cong B$, if such a mutually inverse pair of morphisms exists.
- Bijections in $\mathcal{S}et$ are isomorphisms. There are typically many isomorphisms witnessing that two sets are bijective.
- In the category determined by a partially ordered set, the only isomorphisms are the identities, and in a preorder X with $x, y \in X$ we have $x \cong y$ iff $x \leq y$ and $y \leq x$. Note that in this case there can be only one pair of mutually inverse morphisms witnessing the fact that $x \cong y$.

Definition of Binary Products

A **binary product** of objects A and B in $\mathcal{C}$ is specified by

- an object $A \times B$ of $\mathcal{C}$, together with
- two **projection** morphisms $\pi_A \colon A \times B \to A$ and $\pi_B \colon A \times B \to B$,

for which given any object C and morphisms $f \colon C \to A$, $g \colon C \to B$, there exists a unique morphism $\langle f, g \rangle \colon C \to A \times B$ for which

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\ & C \arrow[ul, "f"] \arrow[u, dashed, "\exists! \langle f, g \rangle"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\langle f, g \rangle \colon C \to A \times B$ is called the **mediating** morphism for f and g.

Examples of Binary Products

- Let $(X, \leq)$ be a preorder. $l \in X$ is a **lower bound** of $x, y \in X$ just in case $l \leq x, y$. $u \in X$ is a **upper bound** of $x, y \in X$ just in case $x, y \leq u$.
- $x \in S \subseteq X$ is **greatest in S** if $(\forall s \in S)(s \leq x)$ and is **least in S** if $(\forall s \in S)(x \leq s)$.
- In a preorder a greatest lower bound $x \wedge y$ of x and y (if it exists) is a binary product $x \times y$ of the category determined by $(X, \leq)$ with projections $x \wedge y \leq x$ and $x \wedge y \leq y$. $x \wedge y$ is also called the **meet** of x and y.

$$\begin{tikzcd}
x & & y \\
x \arrow[u, equal] & x \wedge y \arrow[ul] \arrow[ur] & y \arrow[u, equal] \\
& l \arrow[ul] \arrow[ur] \arrow[u, dotted] &
\end{tikzcd}$$

Examples of Binary Products

- The binary product of A and B in $\mathcal{S}et$ has

$$A \times B \stackrel{\text{def}}{=} \{ (a,b) \mid A \in A, b \in B \}$$

with projection functions $\pi_A(a,b) \stackrel{\text{def}}{=} a$ and $\pi_B(a,b) \stackrel{\text{def}}{=} b$. The mediating function for any $f\colon C \to A$ and $g\colon C \to B$ is

$$\langle f,g \rangle(c) \stackrel{\text{def}}{=} (f(c), g(c)).$$

- In any $\mathcal{C}$, if $p_i\colon P \to A_i$ is any product of A_1 and A_2 then $A_1 \times A_2 \cong P$. All binary products are determined up to isomorphism: Existence yields mediating morphisms $\phi\colon A_1 \times A_2 \to P$ and $\psi\colon P \to A_1 \times A_2$; uniqueness means that ϕ and ψ witness an isomorphism.

Definition of Finite Products

A **product** of a non-empty finite family of objects $(A_i \mid i \in I)$ in $\mathcal{C}$, where $I \stackrel{\text{def}}{=} \{1, \ldots, n\}$, is specified by

- an **object** $A_1 \times \ldots \times A_n$ (or $\Pi_{i \in I} A_i$) in $\mathcal{C}$, and
- for every $j \in I$, a morphism $\pi_j \colon A_1 \times \ldots \times A_n \to A_j$ in $\mathcal{C}$ called the jth **product projection**

such that for any object C and family of morphisms $(f_i \colon C \to A_i \mid i \in I)$ there is a unique morphism

$$\langle f_1, \ldots, f_n \rangle \colon C \to A_1 \times \ldots \times A_n$$

for which given any $j \in I$, we have $\pi_j \circ \langle f_1, \ldots, f_n \rangle = f_j$.

Note: We get binary products when $I \stackrel{\text{def}}{=} \{1, 2\}$!

Examples of Finite Products

- A finite product of $(A_1, \ldots, A_n) \equiv (A_i \mid i \in I)$ in $\mathcal{S}et$ is given by the cartesian product $A_1 \times \ldots \times A_n$ with the obvious projection functions. Given functions $(f\colon C \to A_i \mid i \in I)$ then
$$\langle f_1, \ldots, f_n \rangle(c) \stackrel{\text{def}}{=} (f_1(c), \ldots, f_n(c))$$
- In a preorder $(X, \leq)$, a finite product $x_1 \times \ldots \times x_n$, if it exists, is a meet (greatest lower bound) of $(x_1, \ldots, x_n)$.
- A **terminal** object $\mathbf{1}$ in a category $\mathcal{C}$ has the property that there is a unique morphism $!_A\colon A \to \mathbf{1}$ for every $A \in ob\,\mathcal{C}$. It is the finite product of an empty family of morphisms (check this!). Such a $\mathbf{1}$ may not exist, but is unique up to isomorphism if it does.

Definition of Finite Coproducts

A **coproduct** of a non-empty family of objects $(A_i \mid i \in I)$ in $\mathcal{C}$, where $I = \{1, \dots, n\}$, is specified by

- an object $A_1 + \dots + A_n$ $(\Sigma_{i \in I} A_i)$, together with
- **insertion** morphisms $\iota_j \colon A_j \to A_1 + \dots + A_n$,

such that for any C and any family of morphisms $(f_i \colon A_i \to C \mid i \in I)$ there is a unique morphism

$$[f_1, \dots, f_n] \colon A_1 + \dots + A_n \to C$$

for which given any $j \in I$, we have $[f_1, \dots, f_n] \circ \iota_j = f_j$.

Definition of Finite Coproducts

In the case that $I \stackrel{\text{def}}{=} \{ 1, 2 \}$ we have

$$
\begin{tikzcd}
A_1 \arrow[r, "\iota_2"] \arrow[dr, "f_1"'] & A_1 + A_2 \arrow[d, dashed, "\exists!\ [f_1, f_2]"] & A_2 \arrow[l, "\iota_2"'] \arrow[dl, "f_2"] \\
& C &
\end{tikzcd}
$$

(Compare to the diagrams for colimits later on.)

Examples of (Co)Products

- In $\mathcal{S}et$ the binary coproduct of sets A_1 and A_2 is given by their disjoint union $A_1 \uplus A_2$, defined as the union $(A_1 \times \{1\}) \bigcup (A_2 \times \{2\})$ with the insertion functions

$$\iota_{A_1} : A_1 \to A_1 \uplus A_2 \leftarrow A_2 : \iota_{A_2}$$

where ι_{A_1} is defined by $a_1 \mapsto (a_1, 1)$ for all $a_1 \in A_1$, and ι_{A_2} by $a_2 \mapsto (a_2, 2)$ for all $a_2 \in A_2$.

- Let preorder $(X, \leq)$ have top and bottom elements and all finite meets and joins (least upper bounds). Then the top of X is terminal, the bottom of X initial, and finite meets and joins are finite products and coproducts respectively.

Examples of (Co)Products

- Given $(X, \leq)$ and $(Y, \leq)$ in $\mathcal{ParSet}$, the binary product is the cartesian product $X \times Y$ in $\mathcal{Set}$, with the pointwise order $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$, together with the (monotone) set-theoretic projection functions. The binary coproduct is $X \uplus Y$, with $(z, \delta) \leq (z', \delta')$ iff $\delta = \delta'$ ($\delta, \delta' \in \{1, 2\}$), and $z \leq z'$ (either in X or in Y).
- An **initial** object $\mathbf{0}$ in a category $\mathcal{C}$ has the property that there is a unique morphism $!_A : \mathbf{0} \to A$ for every $A \in ob\,\mathcal{C}$. It is the finite coproduct of an empty family of morphisms (check this!). Such a $\mathbf{0}$ may not exist, but is unique if it does.

Useful “Fact” for (Co)Products

- Suppose that we have $(f_i : C \to A_i \mid i \in \{1,2\})$ and $\theta : C \to A_1 \times A_2$. In order to prove that $\theta = \langle f_1, f_2 \rangle$ it is sufficient to show that $\pi_{A_i} \circ \theta = f_i$ for each i.
- Suppose that we have $(f_i : A_i \to C \mid i \in \{1,2\})$and $\theta : A_1 + A_2 \to C$. In order to prove that $\theta = [f_1, f_2]$ it is sufficient to show that $\theta \circ \iota_{A_i} = f_i$ for each i.

Note: this “fact” is simply a consequence of uniqueness of mediating morphisms. It is crucial to the proof that (co)products are unique up to isomorphism, where both $\phi \circ \psi$ and id (from an earlier slide) are shown to be mediating, and hence equal.

Notation for Finite (Co)Products

- Suppose that $f_1\colon A_1 \to B_1$ and $f_2\colon A_2 \to B_2$. Then

$$
\begin{aligned}
f_1 \times f_2 &\stackrel{\text{def}}{=} \langle f_1 \circ \pi_{A_1}, f_2 \circ \pi_{A_2} \rangle \colon A_1 \times A_2 \to B_1 \times B_2 \\
f_1 + f_2 &\stackrel{\text{def}}{=} [\iota_{B_1} \circ f_1, \iota_{B_2} \circ f_2] \colon A_1 + A_2 \to B_1 + B_2
\end{aligned}
$$

  and hence one can prove that

$$
\begin{aligned}
\pi_{B_i} \circ (f_1 \times f_2) &= f_i \circ \pi_{A_i} \\
(f_1 + f_2) \circ \iota_{A_i} &= \iota_{B_i} \circ f_i
\end{aligned}
$$

- This notation is easily extended to finite families $(A_i \mid i \in \{1, \ldots, n\})$ and $(B_i \mid i \in \{1, \ldots, n\})$ ... or indeed infinite families $(A_i \mid i \in I)$ and $(B_i \mid i \in I)$ where I is any set.

More Examples of (Co)Products

- Suppose that $\mathcal{C}$ has binary (co)products. The functors $B \times (-)$, $B + (-) \colon \mathcal{C} \to \mathcal{C}$ are defined by

$$f \colon A \longrightarrow A' \quad \mapsto \quad id_B \times f \colon B \times A \longrightarrow B \times A'$$

$$f \colon A \longrightarrow A' \quad \mapsto \quad id_B + f \colon B + A \longrightarrow B + A'$$

Note that it is common to write $f \times B$ instead of $f \times id_B$; ditto $+$.

More Examples of (Co)Products

- Suppose that F_1 and F_2 are objects (that is, functors) of $\mathcal{D}^{\mathcal{C}}$ and that $\mathcal{D}$ has finite (co)products. Then both $F_1 \times F_2$ and $F_1 + F_2$ exist and are defined pointwize. For products this means

  $$(F_1 \times F_2)(\xi) \stackrel{\mathbf{def}}{=} F_1\xi \times F_2\xi$$

  where ξ is either an object or morphism of $\mathcal{C}$. The projections $\pi^i \colon F_1 \times F_2 \longrightarrow F_i$ are defined with pointwize components $\pi^i_A \colon F_1A \times F_2A \longrightarrow F_iA$. These projections π^i are indeed natural transformations.

Algebras for $F \colon \mathcal{C} \to \mathcal{C}$

- An **algebra** for F is a morphism $\sigma \colon FI \to I$ in $\mathcal{C}$. The algebra is **initial** if given any $f \colon FX \to X$ there is a **homomorphism** $\overline{f} \colon I \to X$, meaning that ($\overline{f}$ is a morphism and)

$$\begin{tikzcd} FI \arrow[r, "\sigma"] \arrow[d, "F\overline{f}"'] & I \arrow[d, "\overline{f}"] \\ FX \arrow[r, "f"] & X \end{tikzcd}$$

  and such a $\overline{f}$ is unique.
- There is a category $\mathcal{C}^F$ of algebras and algebra homomorphisms (details omitted) in which initial algebras are initial objects.

Algebras for $F\colon \mathcal{C} \to \mathcal{C}$

- $1 + (-)\colon \mathcal{S}et \to \mathcal{S}et$ has an initial algebra

  $$[z, s]\colon 1 + \mathbb{N} \to \mathbb{N}$$

  where $z\colon 1 \to \mathbb{N}$ maps $*$ to 0 and $s\colon \mathbb{N} \to \mathbb{N}$ adds 1. If

  $$f\colon 1 + X \to X$$

  the function $\overline{f}\colon \mathbb{N} \to X$ is uniquely defined by

  $$\begin{aligned} \overline{f}(0) &\stackrel{\text{def}}{=} \widehat{x}(*) \stackrel{\text{def}}{=} x \\ \overline{f}(n+1) &\stackrel{\text{def}}{=} \phi^{n+1}(x) = \phi(\overline{f}(n)) \end{aligned}$$

  where $\widehat{x} \stackrel{\text{def}}{=} f \circ \iota_1\colon 1 \to 1 + X$ and
  $\phi \stackrel{\text{def}}{=} f \circ \iota_X\colon X \to 1 + X$ (and hence $f = [\widehat{x}, \phi]$).

Algebras for $F\colon \mathcal{C} \to \mathcal{C}$

- EXAMPLE: the function $(+n)\colon \mathbb{N} \to \mathbb{N}$ which adds n, for any $n \in \mathbb{N}$, is definable as $\overline{[\widehat{n}, s]}$ where

$$\mathbf{1} + \mathbb{N} \xrightarrow{[\widehat{n}, s]} \mathbb{N}$$

and also $(*n) \stackrel{\mathbf{def}}{=} \overline{[z, (+n)]}\colon \mathbb{N} \to \mathbb{N}$.

- A monoid (M, b, e) is an algebra

$$\mathbf{1} + (M \times M) \xrightarrow{[\widehat{e}, b]} M$$

plus the relevant equations.

Case Study: (Haskell) Algebraic Datatypes

We shall

- Define a Haskell (recursive) datatype grammar.
- Show that any datatype declaration **D** gives rise to a functor $F \equiv F_{\mathbf{D}} \colon \mathcal{S}et \to \mathcal{S}et$.
- Demonstrate that **D** can be modelled by an initial algebra $\sigma \colon FI \to I$, where I is the set $Exp_{\mathbf{D}}$ of expressions of type **D** (up to isomorphism).

Later on we will

- Show that the functor F preserves colimits of diagrams of the form $D \colon \omega \to \mathcal{S}et$, and such colimits exist ...
- and (hence) that F must have an initial algebra for purely categorical reasons.

A Recursive Datatype

- A set of **type patterns** T is defined by

$$T ::= \mathbf{D} \mid \mathtt{Unit} \mid \mathtt{Int} \mid T \times T$$

- A **datatype** is specified by the statement

$$\mathbf{D} = \mathtt{K}_1\ T_1 \mid \ldots \mid \mathtt{K}_m\ T_m$$

- A collection of **type assignments** is defined inductively by the following rules

$$\frac{}{() :: \mathtt{Unit}} \qquad \frac{z \in \mathbb{Z}}{\underline{z} :: \mathtt{Int}} \qquad \frac{E :: T_i}{\mathtt{K}_i\ E :: \mathbf{D}} \qquad \frac{E_1 :: T_1 \quad E_2 :: T_2}{(E_1, E_2) :: T_1 \times T_2}$$

and $Exp_T \stackrel{\text{def}}{=} \{\, E \mid E :: T \,\}$.

Defining F from D

- The functor F is defined (as a coproduct in $\mathcal{S}et^{\mathcal{S}et}$) by

$$F \stackrel{\text{def}}{=} F_{T_1} + \ldots + F_{T_m}$$

  where each $F_{T_i} \colon \mathcal{S}et \to \mathcal{S}et$.

- Functors $F_T \colon \mathcal{S}et \to \mathcal{S}et$ are defined by recursion on the structure of T by setting
  - $F_{\mathsf{D}} \stackrel{\text{def}}{=} id_{\mathcal{S}et}$
  - $F_{\texttt{Unit}}(g \colon U \to V) \stackrel{\text{def}}{=} id_{\mathbf{1}} \colon \mathbf{1} \to \mathbf{1}$ where $\mathbf{1}$ is terminal in $\mathcal{S}et$
  - $F_{\texttt{Int}}(g \colon U \to V) \stackrel{\text{def}}{=} id_{\mathbb{Z}} \colon \mathbb{Z} \to \mathbb{Z}$
  - $F_{T_1 \times T_2} \stackrel{\text{def}}{=} F_{T_1} \times F_{T_2}$

Defining An Initial Algebra $\sigma\colon F\mathsf{I} \to \mathsf{I}$

- We set $\mathsf{I} \stackrel{\text{def}}{=} Exp_{\mathsf{D}}$ and we define

$$\sigma \stackrel{\text{def}}{=} [\widehat{\mathtt{K}_1} \circ \sigma_{T_1} \ldots \widehat{\mathtt{K}_m} \circ \sigma_{T_m}]\colon F\mathsf{I} \stackrel{\text{def}}{=} F_{T_1}\mathsf{I} + \ldots + F_{T_m}\mathsf{I} \longrightarrow \mathsf{I}$$

  where the function $\widehat{\mathtt{K}_i}\colon Exp_{T_i} \to \mathsf{I}$ applies the constructor and we define functions $\sigma_T\colon F_T\mathsf{I} \to Exp_T$ by recursion over T as follows
  - $\sigma_{\mathsf{D}}(E \in \mathsf{I}) \stackrel{\text{def}}{=} E \in Exp_{\mathsf{D}}$
  - $\sigma_{\mathtt{Unit}}(* \in \mathbf{1}) \stackrel{\text{def}}{=} () \in Exp_{\mathtt{Unit}}$.
  - $\sigma_{\mathtt{Int}}(z \in \mathbb{Z}) \stackrel{\text{def}}{=} \underline{z} \in Exp_{\mathtt{Int}}$.
  - $\sigma_{T_1 \times T_2}((e_1, e_2) \in F_{T_1}\mathsf{I} \times F_{T_2}\mathsf{I}) \stackrel{\text{def}}{=} (\sigma_{T_1}(e_1), \sigma_{T_2}(e_2)) \in Exp_{T_1 \times T_2}$
- It may be useful to note that $\sigma(\iota_i(e_i \in F_{T_i}\mathsf{I})) = \mathtt{K}_i\ \sigma_{T_i}(e_i)$.

Verifying Initiality

- Suppose that $f\colon FX \to X$ in $\mathcal{S}et$. We have to prove that there is a unique $\overline{f}$ such that

$$\begin{tikzcd}
F_{T_1}\mathbf{I} + \ldots + F_{T_m}\mathbf{I} = F\mathbf{I} \arrow[r, "\sigma"] \arrow[d, "F\overline{f}"'] & \mathbf{I} \arrow[d, "\overline{f}"] \\
F_{T_1}X + \ldots + F_{T_m}X = FX \arrow[r, "f"] & X
\end{tikzcd}$$

Verifying Initiality

- Note $\overline{f}\colon \mathit{Exp}_{\mathsf{D}} \to F_{\mathsf{D}}X$; we will define $\overline{f} \stackrel{\mathbf{def}}{=} \theta_{\mathsf{D}}$ and functions

$$\theta_T\colon \mathit{Exp}_T \to F_T X$$

by recursion on T:

  - $\theta_{\mathsf{D}}(\mathtt{K}_i\ E_i \in \mathit{Exp}_{\mathsf{D}}) \stackrel{\mathbf{def}}{=} f(\iota_i(\theta_{T_i}(E_i))) \in X$.
  - $\theta_{\mathtt{Unit}}(() \in \mathit{Exp}_{\mathtt{Unit}}) \stackrel{\mathbf{def}}{=} * \in \mathbf{1}$.
  - $\theta_{\mathtt{Int}}(\underline{z} \in \mathit{Exp}_{\mathtt{Int}}) \stackrel{\mathbf{def}}{=} z \in \mathbb{Z}$.
  - $\theta_{T_1 \times T_2}((E_1, E_2) \in \mathit{Exp}_{T_1 \times T_2}) \stackrel{\mathbf{def}}{=} (\theta_{T_1}(E_1), \theta_{T_2}(E_2)) \in F_{T_1}\mathsf{I} \times F_{T_2}\mathsf{I}$.

Verifying Initiality

- Observe that for any T we have $\theta_T \circ \sigma_T = F_T\theta_{\mathsf{D}}$, which follows from an easy induction.
  Note that by universality of coproducts $\overline{f} \circ \sigma = f \circ F\overline{f}$ iff

$$\overline{f} \circ \sigma \circ \iota_i = f \circ F\overline{f} \circ \iota_i$$

Then for any $e_i \in F_{T_i}\mathsf{I}$

$$
\begin{array}{rcl}
(\theta_{\mathsf{D}} \circ \sigma \circ \iota_i)(e_i) & = & \theta_{\mathsf{D}}(\mathtt{K}_i\ \sigma_{T_i}(e_i)) \\
& \overset{\mathbf{def}}{=}_{\theta_{\mathsf{D}}} & f(\iota_i(\theta_{T_i}(\sigma_{T_i}(e_i)))) \\
& = & f(\iota_i((F_{T_i}\theta_{\mathsf{D}})(e_i))) \\
& = & f((F_{T_1}\theta_{\mathsf{D}} + \ldots + F_{T_m}\theta_{\mathsf{D}})(\iota_i(e_i))) \\
& \overset{\mathbf{def}}{=}_{F} & (f \circ F\theta_{\mathsf{D}} \circ \iota_i)(e_i)
\end{array}
$$

The steps follow by: definition of σ; definition of θ_{D}; the observation; properties of $+$; the definition of F.

Verifying Initiality

- Observe that for any T we have $\theta_T \circ \sigma_T = F_T \theta_{\mathsf{D}}$, which follows from an easy induction.
  Note that by universality of coproducts $\overline{f} \circ \sigma = f \circ F\overline{f}$ iff

  $$\overline{f} \circ \sigma \circ \iota_i = f \circ F\overline{f} \circ \iota_i$$

  Then for any $e_i \in F_{T_i}\mathsf{I}$

  $$\begin{array}{rcl} (\theta_{\mathsf{D}} \circ \sigma \circ \iota_i)(e_i) & = & \theta_{\mathsf{D}}(\mathtt{K}_i\ \sigma_{T_i}(e_i)) \\ & \overset{\text{def}}{=}_{\theta_{\mathsf{D}}} & f(\iota_i(\theta_{T_i}(\sigma_{T_i}(e_i)))) \\ & = & f(\iota_i((F_{T_i}\theta_{\mathsf{D}})(e_i))) \\ & = & f((F_{T_1}\theta_{\mathsf{D}} + \ldots + F_{T_m}\theta_{\mathsf{D}})(\iota_i(e_i))) \\ & \overset{\text{def}}{=}_{F} & (f \circ F\theta_{\mathsf{D}} \circ \iota_i)(e_i) \end{array}$$

  The steps follow by: definition of σ; definition of θ_{D}; the observation; properties of $+$; the definition of F.

Verifying Initiality

- Observe that for any T we have $\theta_T \circ \sigma_T = F_T \theta_{\mathsf{D}}$, which follows from an easy induction.
  Note that by universality of coproducts $\overline{f} \circ \sigma = f \circ F\overline{f}$ iff

$$\overline{f} \circ \sigma \circ \iota_i = f \circ F\overline{f} \circ \iota_i$$

Then for any $e_i \in F_{T_i}\mathsf{I}$

$$\begin{array}{rcl}
(\theta_{\mathsf{D}} \circ \sigma \circ \iota_i)(e_i) & = & \theta_{\mathsf{D}}(\mathtt{K}_i\ \sigma_{T_i}(e_i)) \\
& \stackrel{\mathbf{def}}{=}_{\theta_{\mathsf{D}}} & f(\iota_i(\theta_{T_i}(\sigma_{T_i}(e_i)))) \\
& = & f(\iota_i((F_{T_i}\theta_{\mathsf{D}})(e_i))) \\
& = & f((F_{T_1}\theta_{\mathsf{D}} + \ldots + F_{T_m}\theta_{\mathsf{D}})(\iota_i(e_i))) \\
& \stackrel{\mathbf{def}}{=}_{F} & (f \circ F\theta_{\mathsf{D}} \circ \iota_i)(e_i)
\end{array}$$

The steps follow by: definition of σ; definition of θ_{D}; the observation; properties of $+$; the definition of F.

Verifying Initiality

- Observe that for any T we have $\theta_T \circ \sigma_T = F_T\theta_{\mathsf{D}}$, which follows from an easy induction.
  Note that by universality of coproducts $\overline{f} \circ \sigma = f \circ F\overline{f}$ iff

$$\overline{f} \circ \sigma \circ \iota_i = f \circ F\overline{f} \circ \iota_i$$

  Then for any $e_i \in F_{T_i}\mathsf{I}$

$$\begin{array}{rcl}
(\theta_{\mathsf{D}} \circ \sigma \circ \iota_i)(e_i) & = & \theta_{\mathsf{D}}(\mathtt{K}_i\ \sigma_{T_i}(e_i)) \\
& \overset{\mathbf{def}}{=}_{\theta_{\mathsf{D}}} & f(\iota_i(\theta_{T_i}(\sigma_{T_i}(e_i)))) \\
& = & f(\iota_i((F_{T_i}\theta_{\mathsf{D}})(e_i))) \\
& = & f((F_{T_1}\theta_{\mathsf{D}} + \ldots + F_{T_m}\theta_{\mathsf{D}})(\iota_i(e_i))) \\
& \overset{\mathbf{def}}{=}_{F} & (f \circ F\theta_{\mathsf{D}} \circ \iota_i)(e_i)
\end{array}$$

  The steps follow by: definition of σ; definition of θ_{D}; the observation; properties of $+$; the definition of F.

Verifying Initiality

- Observe that for any T we have $\theta_T \circ \sigma_T = F_T \theta_{\mathsf{D}}$, which follows from an easy induction.
  Note that by universality of coproducts $\overline{f} \circ \sigma = f \circ F\overline{f}$ iff

$$\overline{f} \circ \sigma \circ \iota_i = f \circ F\overline{f} \circ \iota_i$$

Then for any $e_i \in F_{T_i}\mathsf{I}$

$$\begin{array}{rcl}
(\theta_{\mathsf{D}} \circ \sigma \circ \iota_i)(e_i) & = & \theta_{\mathsf{D}}(\mathtt{K}_i\ \sigma_{T_i}(e_i)) \\
& \stackrel{\mathbf{def}}{=}_{\theta_{\mathsf{D}}} & f(\iota_i(\theta_{T_i}(\sigma_{T_i}(e_i)))) \\
& = & f(\iota_i((F_{T_i}\theta_{\mathsf{D}})(e_i))) \\
& = & f((F_{T_1}\theta_{\mathsf{D}} + \ldots + F_{T_m}\theta_{\mathsf{D}})(\iota_i(e_i))) \\
& \stackrel{\mathbf{def}}{=}_{F} & (f \circ F\theta_{\mathsf{D}} \circ \iota_i)(e_i)
\end{array}$$

The steps follow by: definition of σ; definition of θ_{D}; the observation; properties of $+$; the definition of F.

Adjunctions between Preorders

- A pair of monotone functions

$$X \underset{r}{\overset{l}{\rightleftarrows}} Y$$

is said to be an **adjunction** if for all $x \in X$ and $y \in Y$,

$$l(x) \leq y \iff x \leq r(y)$$

- We say that l is **left adjoint** to r and that r is right adjoint to l. We write $l \dashv r$.

Examples

- Let $\mathbf{1} \stackrel{\text{def}}{=} \{ * \}$ be the one element preorder. Then there are adjunctions $(\bot \dashv ! \dashv \top)$

$$\begin{tikzcd} X \arrow[r, shift left, "!"] & \mathbf{1} \arrow[l, shift left, "\bot"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, shift left, "!"] & \mathbf{1} \arrow[l, shift left, "\top"] \end{tikzcd}$$

provided that X has both top and bottom elements. For example, for any $x \in X$,

$$!(x) \stackrel{\text{def}}{=} * \leq * \iff x \leq \top(*) \stackrel{\text{def}}{=} \top$$

Examples

- Define $\Delta\colon X \to X \times X$ by $\Delta(x) \stackrel{\text{def}}{=} (x,x)$. Then there are adjoints $(\vee \dashv \Delta \dashv \wedge)$

$$X \underset{\vee}{\overset{\Delta}{\rightleftarrows}} X \times X \qquad X \underset{\wedge}{\overset{\Delta}{\rightleftarrows}} X \times X$$

  just in case X has all binary meets and joins: for any $l \in X$,

$$\Delta(l) \stackrel{\text{def}}{=} (l,l) \leq (x,x') \iff l \leq \wedge(x,x') \stackrel{\text{def}}{=} x \wedge x'$$

- This structure corresponds to X having binary products and coproducts.

Adjunctions between Categories

- Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors. L is **left adjoint** to R, written $L \dashv R$, if given any objects A of $\mathcal{C}$ and B of $\mathcal{D}$ we have
  - a bijection between morphisms $LA \to B$ in $\mathcal{D}$ and $A \to RB$ in $\mathcal{C}$, that is, between $\mathcal{C}(LA, B)$ and $\mathcal{D}(A, RB)$,

$$\frac{f\colon LA \to B}{\overline{f}\colon A \to RB} \qquad\qquad \frac{g\colon A \to RB}{\widehat{g}\colon LA \to B}$$

  - this bijection is *natural in A and B*: given morphisms $\phi\colon A' \to A$ in $\mathcal{C}$ and $\psi\colon B \to B'$ in $\mathcal{D}$ we have

$$\overline{\psi \circ f \circ L\phi} = R\psi \circ \overline{f} \circ \phi \quad \text{and/or} \quad (R\psi \circ g \circ \phi)^{\wedge} = \psi \circ \widehat{g} \circ L\phi.$$

Examples of Adjunctions

- The **forgetful** functor $U\colon \mathcal{M}on \to \mathcal{S}et$ taking a monoid to its underlying set, and the functor $list(-)\colon \mathcal{S}et \to \mathcal{M}on$ taking a set to finite lists over the set, are adjoints:

$$list(-) \dashv U$$

So there is a natural bijection between $\mathcal{M}on(list(A), M)$ and $\mathcal{S}et(A, UM)$

$$\frac{f\colon list(A) \to M}{\overline{f}\colon A \to UM} \qquad\qquad \frac{g\colon A \to UM}{\widehat{g}\colon list(A) \to M}$$

Examples of Adjunctions

- This is given by

$$g : A \longrightarrow UM \quad \mapsto$$

$$\widehat{g} : list(A) \xrightarrow[{[\,] \mapsto e}]{[a_1, \ldots, a_n] \mapsto g(a_1) \ldots g(a_n)} M,$$

and

$$f : list(A) \longrightarrow M \quad \longmapsto \quad \overline{f} : A \xrightarrow{a \mapsto f([a])} UM.$$

- Note that

$$\begin{aligned} \widehat{\overline{f}}[a_1, \ldots, a_n] &= \overline{f}(a_1) \ldots \overline{f}(a_n) \\ &= f([a_1]) \ldots f([a_n]) = f([a_1] +\!\!+ \ldots +\!\!+ [a_n]) \end{aligned}$$

It is an exercise to verify that $\overline{\widehat{g}} = g$ and that this bijection is natural.

Examples of Adjunctions

- For a fixed set A, the functor $(-) \times B \colon \mathcal{S}et \to \mathcal{S}et$ has a right adjoint $B \Rightarrow (-) \colon \mathcal{S}et \to \mathcal{S}et$. If $c \colon C \to C'$ then

$$B \Rightarrow c \colon B \Rightarrow C \longrightarrow B \Rightarrow C' \text{ where } (B \Rightarrow c)(\theta) \stackrel{\text{def}}{=} c \circ \theta$$

$$\frac{f \colon A \times B \to C}{\overline{f} \stackrel{\text{def}}{=} \lambda a.\lambda b.f(a,b) \colon A \to B \Rightarrow C}$$

$$\frac{g \colon A \to B \Rightarrow C}{\widehat{g} \stackrel{\text{def}}{=} \lambda(a,b).g(a)(b) \colon A \times B \to C}$$

It is immediate that we have a bijection; naturality is an exercise. Having products and such a "function structure" is known as cartesian closure.

Examples of Adjunctions

- The **diagonal functor** $\Delta\colon \mathcal{S}et \to \mathcal{S}et \times \mathcal{S}et$ taking a function $f\colon A \to B$ to $(f, f)\colon (A, A) \to (B, B)$ has right and left adjoints Π and Σ taking any morphism $(f_1, f_2)\colon (A_1, A_2) \to (B_1, B_2)$ of $\mathcal{S}et \times \mathcal{S}et$ to
  $f_1 \times f_2 \stackrel{\text{def}}{=} \langle f_1 \circ \pi_{A_1}, f_2 \circ \pi_{A_2} \rangle\colon A_1 \times A_2 \to B_1 \times B_2$ and
  $f_1 + f_2 \stackrel{\text{def}}{=} [\iota_{B_1} \circ f_1, \iota_{B_2} \circ f_2]\colon A_1 + A_2 \to B_1 + B_2$ respectively, where the bijection for Π is

$$\frac{(f, g) \qquad \hat{m} \stackrel{\text{def}}{=} (\pi_A \circ m, \pi_B \circ m) \quad : \Delta C \longrightarrow (A, B)}{\overline{(f, g)} \stackrel{\text{def}}{=} \langle f, g \rangle \qquad m \quad : C \longrightarrow \Pi(A, B)}$$

- If we replace $\mathcal{S}et$ by any category $\mathcal{C}$ with (co)products, defining $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ analogously, everything still works.

CCCs via Adjunctions

Let $\mathcal{C}$ be a category with finite products. Existence of a right adjoint R to the functor $(-) \times B \colon \mathcal{C} \to \mathcal{C}$ for each object B of $\mathcal{C}$, is equivalent to $\mathcal{C}$ being cartesian closed.

CCCs via Adjunctions

Let $\mathcal{C}$ be a category with finite products. Existence of a right adjoint R to the functor $(-) \times B \colon \mathcal{C} \to \mathcal{C}$ for each object B of $\mathcal{C}$, is equivalent to $\mathcal{C}$ being cartesian closed.

($\Rightarrow$) Given an object B of $\mathcal{C}$ set $B \Rightarrow C \stackrel{\mathbf{def}}{=} R(C)$ for any object C of $\mathcal{C}$. Given a morphism $f \colon A \times B \to C$ we define $\lambda(f) \colon A \to (B \Rightarrow C)$ to be the mate of f across the given adjunction. The morphism

$$ev \colon (B \Rightarrow C) \times B \to C$$

is the mate $(\widehat{id_{B \Rightarrow C}})$ of the identity $id_{B \Rightarrow C} \colon (B \Rightarrow C) \to (B \Rightarrow C)$.

CCCs via Adjunctions

Let $\mathcal{C}$ be a category with finite products. Existence of a right adjoint R to the functor $(-) \times B \colon \mathcal{C} \to \mathcal{C}$ for each object B of $\mathcal{C}$, is equivalent to $\mathcal{C}$ being cartesian closed.

Next, we need to show that $ev \circ (\lambda(f) \times id_B) = f$. This follows directly from the naturality of the adjunction; we consider naturality in A and C at the morphisms $\lambda(f) \colon A \to (B \Rightarrow C)$ and $id_C \colon C \to C$:

$$
\begin{tikzcd}
id_{B\Rightarrow C} \arrow[rrr, mapsto] \arrow[d, mapsto] & & & ev \arrow[d, mapsto] \\
R(id_C) \circ id_{B\Rightarrow C} \circ \lambda(f) \arrow[r, mapsto] & \widehat{\lambda(f)} & = & id_C \circ ev \circ (\lambda(f) \times id_B)
\end{tikzcd}
$$

We let the reader show that $\lambda(f)$ is the unique morphism satisfying the latter equation.

CCCs via Adjunctions

($\Leftarrow$) Conversely, let B be an object of $\mathcal{C}$. We define a right adjoint to $(-) \times B$ denoted by $B \Rightarrow (-)$, by setting

$$c : C \longrightarrow C' \quad \mapsto \quad B \Rightarrow c \stackrel{\text{def}}{=} \lambda(c \circ ev) \colon (B \Rightarrow C) \to (B \Rightarrow C')$$

for each morphism $c \colon C \to C'$ of $\mathcal{C}$ (this matches our earlier definition – check). We define a bijection by declaring the mate of $f \colon A \times B \to C$ to be $\lambda(f) \colon A \to (B \Rightarrow C)$ and the mate of $g \colon A \to (B \Rightarrow C)$ to be

$$\widehat{g} \stackrel{\text{def}}{=} ev \circ (g \times id_B) \colon A \times B \to C.$$

CCCs via Adjunctions

It remains to verify that we have defined a bijection which is natural in the required sense. We only check one part of naturality. Let $a\colon A' \to A$ and $c\colon C \to C'$ be morphisms of $\mathcal{C}$. Then

$$
\begin{aligned}
ev \circ ((\lambda(c \circ ev) \circ \lambda(f) \circ a) \times id) \quad &= \\
ev \circ (\lambda(c \circ ev) \times id) \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ ev \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ f \circ (a \times id)
\end{aligned}
$$

implying that $\lambda(c \circ f \circ (a \times id)) = (B \Rightarrow c) \circ \lambda(f) \circ a$ since $\mathcal{C}$ is a CCC.

The steps above are: categorical properties of $\times$; cartesian closure of $\mathcal{C}$; cartesian closure again.

CCCs via Adjunctions

It remains to verify that we have defined a bijection which is natural in the required sense. We only check one part of naturality. Let $a\colon A' \to A$ and $c\colon C \to C'$ be morphisms of $\mathcal{C}$. Then

$$
\begin{aligned}
ev \circ ((\lambda(c \circ ev) \circ \lambda(f) \circ a) \times id) &= \\
ev \circ (\lambda(c \circ ev) \times id) \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ ev \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ f \circ (a \times id)
\end{aligned}
$$

implying that $\lambda(c \circ f \circ (a \times id)) = (B \Rightarrow c) \circ \lambda(f) \circ a$ since $\mathcal{C}$ is a CCC.

The steps above are: categorical properties of $\times$; cartesian closure of $\mathcal{C}$; cartesian closure again.

CCCs via Adjunctions

It remains to verify that we have defined a bijection which is natural in the required sense. We only check one part of naturality. Let $a\colon A' \to A$ and $c\colon C \to C'$ be morphisms of $\mathcal{C}$. Then

$$\begin{aligned}
ev \circ ((\lambda(c \circ ev) \circ \lambda(f) \circ a) \times id) &= \\
ev \circ (\lambda(c \circ ev) \times id) \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ ev \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ f \circ (a \times id)
\end{aligned}$$

implying that $\lambda(c \circ f \circ (a \times id)) = (B \Rightarrow c) \circ \lambda(f) \circ a$ since $\mathcal{C}$ is a CCC.

The steps above are: categorical properties of $\times$; cartesian closure of $\mathcal{C}$; cartesian closure again.

CCCs via Adjunctions

It remains to verify that we have defined a bijection which is natural in the required sense. We only check one part of naturality. Let $a\colon A' \to A$ and $c\colon C \to C'$ be morphisms of $\mathcal{C}$. Then

$$\begin{aligned}
ev \circ ((\lambda(c \circ ev) \circ \lambda(f) \circ a) \times id) &= \\
ev \circ (\lambda(c \circ ev) \times id) \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ ev \circ (\lambda(f) \times id) \circ (a \times id) &= \\
c \circ f \circ (a \times id)
\end{aligned}$$

implying that $\lambda(c \circ f \circ (a \times id)) = (B \Rightarrow c) \circ \lambda(f) \circ a$ since $\mathcal{C}$ is a CCC.

The steps above are: categorical properties of $\times$; cartesian closure of $\mathcal{C}$; cartesian closure again.

Colimits

- Given a **diagram** $D\colon \mathbb{I} \to \mathcal{C}$, a **colimit** for D is given by an object $col_I DI$ of $\mathcal{C}$ together with a family of morphisms $(\iota_I\colon DI \to col_I DI \mid I \in \mathbb{I})$ such that for any $\alpha\colon I \to J$ in $\mathbb{I}$ we have $\iota_J \circ D\alpha = \iota_I$. This data satisfies: given any family $(h_I\colon DI \to C \mid I \in \mathbb{I})$ such that $h_J \circ D\alpha = h_I$, there is a unique morphism $\phi\colon col_I DI \to C$ satisfying $\phi \circ \iota_I = h_I$ for each object I of $\mathbb{I}$ (and hence $\phi = [h_I \mid I \in \mathbb{I}]$)

$$\begin{tikzcd}
DI \arrow[rr, "D\alpha"] \arrow[dr, "\iota_I"'] \arrow[dd, "h_I"'] & & DJ \arrow[dl, "\iota_J"] \arrow[dd, "h_J"] \\
& col_I DI \arrow[d, dashed, "\phi"'] & \\
C \arrow[r, equal] & C \arrow[r, equal] & C
\end{tikzcd}$$

- Binary coproducts arise from the discrete category $\mathbb{I} \stackrel{\text{def}}{=} \{1, 2\}$.

Colimits

- Let $D\colon \omega \to \mathcal{C}$; suppose that $i \leq i+1$ is a typical morphism in ω. Then a colimit diagram, if it exists, can be taken as

$$
\begin{tikzcd}
\dots D(i) \arrow[rr, "D(\leq^i_{i+1})"] \arrow[dr, "\iota_i"'] \arrow[dd, "h_i"'] & & D(i+1)\dots \arrow[dl, "\iota_{i+1}"] \arrow[dd, "h_{(i+1)}"] \\
& col_i D(i) \arrow[d, dashed, "\phi"'] & \\
C \arrow[r, equal] & C \arrow[r, equal] & C
\end{tikzcd}
$$

where for any given functions $h_i\colon D(i) \to C$ commuting with the functions $D(\leq^i_{i+1})$, a unique such ϕ exists.
This fact follows, since $h_j \circ D(\leq^i_j) = h_i$ for a general morphism $\leq^i_j$ (where $i \leq j$ in ω) is immediate.

Colimits

- It is a fact that $\mathcal{S}et$ has all (small) colimits.
- It is a fact that a colimit for $\Delta\colon \omega \times \omega \to \mathcal{C}$ exists if and only if a colimit for $\Delta'\colon \omega \to \mathcal{C}$ where $\Delta'(i \in \omega) \stackrel{\text{def}}{=} \Delta(i, i)$ exists, and when they (both) exist they are isomorphic, that is

$$col_k \Delta'(k) \cong col_{(i,j)} \Delta(i, j)$$

Further (exercise: define the diagrams that give rise to the colimits below ...)

$$col_i(col_j \Delta(i, j)) \cong col_j(col_i \Delta(j, i))$$

and *all* of the above colimits are isomorphic.

Left Adjoints Preserve Colimits

Let $D\colon \mathbb{I} \to \mathcal{C}$, and $L\colon \mathcal{C} \to \mathcal{D}$ and $L \dashv R$ for some R. Then

$$L(col_I DI) \cong col_I LDI$$

and is witnessed by $[L(\iota_{DI}) \mid I \in \mathbb{I}]\colon col_I LDI \to L(col_I DI)$. It suffices to show that $L(col_I DI)$ is a colimit for $LD\colon \mathbb{I} \to \mathcal{D}$.

$$\begin{tikzcd}
LDI \arrow[rr, "LD\alpha"] \arrow[dr, "L(\iota_{DI})"'] \arrow[ddd, "h_I"'] & & LDJ \arrow[dl, "L(\iota_{DJ})"] \arrow[ddd, "h_J"] \\
& L(col_I DI) \arrow[dd, dashed, "\phi"'] & \\
& & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

Left Adjoints Preserve Colimits

$$
\begin{tikzcd}
LDI \arrow[rr, "LD\alpha"] \arrow[dr, "L(\iota_{DI})"] \arrow[dd, "h_I"'] & & LDJ \arrow[dl, "L(\iota_{DJ})"'] \arrow[dd, "h_J"] \\
& L(col_I DI) \arrow[d, dashed, "\phi"'] & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
$$

Suppose that $h_I = h_J \circ LD\alpha$. We need to show there is a unique ϕ as above.

Left Adjoints Preserve Colimits

$$\begin{tikzcd}
DI \arrow[rr, "D\alpha"] \arrow[dr, "\iota_{DI}"] \arrow[dd, "\overline{h_I}"'] & & DJ \arrow[dl, "\iota_{DJ}"'] \arrow[dd, "\overline{h_J}"] \\
& col_I DI \arrow[d, dashed, "\rho"'] & \\
RX \arrow[r, equal] & RX \arrow[r, equal] & RX
\end{tikzcd}$$

But

$$h_I = h_J \circ LD\alpha \Longrightarrow \overline{h_I} = \overline{h_J \circ LD\alpha} = \overline{h_J} \circ D\alpha$$

where the final equality follows by naturality.

Left Adjoints Preserve Colimits

$$\begin{tikzcd}
DI \arrow[rr, "D\alpha"] \arrow[dr, "\iota_{DI}"] \arrow[dd, "\overline{h_I}"'] & & DJ \arrow[dl, "\iota_{DJ}"'] \arrow[dd, "\overline{h_J}"] \\
& col_I DI \arrow[d, dashed, "\rho"'] & \\
RX \arrow[r, equal] & RX \arrow[r, equal] & RX
\end{tikzcd}$$

Therefore there is ρ with $\rho \circ \iota_{DI} = \overline{h_I}$. Define

$$\phi \stackrel{\text{def}}{=} \widehat{\rho} \colon L(col_I DI) \to X$$

Left Adjoints Preserve Colimits

$$
\begin{tikzcd}
LDI \arrow[rr, "LD\alpha"] \arrow[dr, "L(\iota_{DI})"] \arrow[dd, "h_I"'] & & LDJ \arrow[dl, "L(\iota_{DJ})"'] \arrow[dd, "h_J"] \\
& L(col_I DI) \arrow[d, dashed, "\phi"'] & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
$$

Hence, again using naturality,

$$\phi \circ L(\iota_{DI}) \stackrel{\textbf{def}}{=} \widehat{\rho} \circ L(\iota_{DI}) = \widehat{\rho \circ \iota_{DI}} = \widehat{\overline{h_I}} = h_I$$

Existence of Initial Algebras

Suppose that F preserves colimits of the form $D\colon \omega \to \mathcal{C}$ and that $\mathcal{C}$ has an initial object $\mathbf{0}$. Define $D(i \leq i+1) \stackrel{\mathbf{def}}{=} F^i!_X\colon F^i\mathbf{0} \to F^{i+1}\mathbf{0}$ for $i \in \omega$. Then $\mathsf{I} \stackrel{\mathbf{def}}{=} col_i Di$ (if it exists) is an initial algebra for F.

Since F preserves colimits and $\mathsf{I} \stackrel{\mathbf{def}}{=} col_i Di$ we can define $\sigma\colon F\mathsf{I} \to \mathsf{I}$

$$\begin{tikzcd}
FF^i\mathbf{0} \arrow[rr, "FF^i!_X"] \arrow[dr, "F\iota_i"] \arrow[dd, "\iota_{i+1}"'] & & FF^{i+1}\mathbf{0} \arrow[dl, "F\iota_{i+1}"'] \arrow[dd, "\iota_{i+2}"] \\
& F\mathsf{I} \arrow[d, dashed, "\sigma"'] & \\
\mathsf{I} \arrow[r, equal] & \mathsf{I} \arrow[r, equal] & \mathsf{I}
\end{tikzcd}$$

where $\sigma \circ F\iota_i = \iota_{i+1}$.

Existence of Initial Algebras

Let $f\colon FX \to X$. Define $f_0 \stackrel{\mathbf{def}}{=} !_X\colon \mathbf{0} \to X$ and $f_{i+1} \stackrel{\mathbf{def}}{=} f \circ Ff_i$.
Certainly $f_1 \circ F^0!_X \equiv f_1 \circ !_X = f_0$ and for $i \geq 1$ we have inductively

$$f_{i+1} \circ F^i!_X \stackrel{\mathbf{def}}{=} f \circ Ff_i \circ F^i!_X = f \circ F(f_i \circ F^{i-1}!_X) = f \circ Ff_{i-1} \stackrel{\mathbf{def}}{=} f_i$$

and hence $\overline{f}$ exists where $\overline{f} \circ \iota_i = f_i$.

$$\begin{tikzcd}
F^i\mathbf{0} \arrow[rr, "F^i!_X"] \arrow[dr, "\iota_i"'] \arrow[dd, "f_i"'] & & F^{i+1}\mathbf{0} \arrow[dl, "\iota_{i+1}"] \arrow[dd, "f_{i+1}"] \\
& \mathbf{I} \arrow[d, dashed, "\overline{f}"'] & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

Existence of Initial Algebras

We now have $\sigma \circ F\iota_i = \iota_{i+1}$; and $f_{i+1} \stackrel{\text{def}}{=} f \circ Ff_i$ (which implied $f_{i+1} = f_{i+2} \circ F^{i+1}!_X$) yielding $\overline{f} \circ \iota_i = f_i$

$$
\begin{tikzcd}
FF^i0 \arrow[rr, "F^{i+1}!_X = FF^i!_X"] \arrow[dr, "F\iota_i"'] \arrow[dd, "f_{i+1}"'] & & FF^{i+1}0 \arrow[dl, "F\iota_{i+1}"] \arrow[dd, "f_{i+2}"] \\
& F\mathsf{I} \arrow[d, dashed, "\overline{f} \circ \sigma = f \circ F\overline{f}"] & \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}
$$

The equality follows since

$$\overline{f} \circ \sigma \circ F\iota_i = f_{i+1} \quad f \circ F\overline{f} \circ F\iota_i = f \circ F(\overline{f} \circ \iota_i) = f \circ Ff_i = f_{i+1}$$

Datatype Initial Algebra, Categorically

Suppose that a functor $F\colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F ::= P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D\colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$
\begin{array}{rcl}
(F \times G)(col_i Di) & \stackrel{\text{def}}{=} & (F col_i Di) \times (G col_i Di) \\
& \cong & (col_j FDj) \times (col_i GDi) \\
& \cong & col_i((col_j DFj) \times DGi) \\
& \cong & col_i(col_j(DFi \times DGj)) \\
& \cong & col_k(DFk \times DGk)
\end{array}
$$

The steps follow by: induction on F and G; $(col_j FDj) \times (-)$ has a right adjoint so preserves colimits; $(-) \times DGi$ also has a right adjoint; the earlier fact that a colimit for $\Delta\colon \omega \times \omega \to \mathcal{C}$ and $\Delta'\colon \omega \to \mathcal{C}$ where $\Delta'(k) \stackrel{\text{def}}{=} \Delta(k, k)$ are isomorphic.

Datatype Initial Algebra, Categorically

Suppose that a functor $F\colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F ::= P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D\colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$
\begin{array}{rcl}
(F \times G)(col_i Di) & \stackrel{\text{def}}{=} & (F col_i Di) \times (G col_i Di) \\
& \cong & (col_j FDj) \times (col_i GDi) \\
& \cong & col_i((col_j DFj) \times DGi) \\
& \cong & col_i(col_j(DFi \times DGj)) \\
& \cong & col_k(DFk \times DGk)
\end{array}
$$

The steps follow by: induction on F and G; $(col_j FDj) \times (-)$ has a right adjoint so preserves colimits; $(-) \times DGi$ also has a right adjoint; the earlier fact that a colimit for $\Delta\colon \omega \times \omega \to \mathcal{C}$ and $\Delta'\colon \omega \to \mathcal{C}$ where $\Delta'(k) \stackrel{\text{def}}{=} \Delta(k,k)$ are isomorphic.

Datatype Initial Algebra, Categorically

Suppose that a functor $F\colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F ::= P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D\colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$
\begin{array}{rcl}
(F \times G)(col_i Di) & \stackrel{\text{def}}{=} & (F col_i Di) \times (G col_i Di) \\
& \cong & (col_j FDj) \times (col_i GDi) \\
& \cong & col_i((col_j DFj) \times DGi) \\
& \cong & col_i(col_j(DFi \times DGj)) \\
& \cong & col_k(DFk \times DGk)
\end{array}
$$

The steps follow by: induction on F and G; $(col_j FDj) \times (-)$ has a right adjoint so preserves colimits; $(-) \times DGi$ also has a right adjoint; the earlier fact that a colimit for $\Delta\colon \omega \times \omega \to \mathcal{C}$ and $\Delta'\colon \omega \to \mathcal{C}$ where $\Delta'(k) \stackrel{\text{def}}{=} \Delta(k,k)$ are isomorphic.

Datatype Initial Algebra, Categorically

Suppose that a functor $F\colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F ::= P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D\colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$
\begin{array}{rcl}
(F \times G)(col_i Di) & \stackrel{\mathrm{def}}{=} & (F col_i Di) \times (G col_i Di) \\
& \cong & (col_j FDj) \times (col_i GDi) \\
& \cong & col_i((col_j DFj) \times DGi) \\
& \cong & col_i(col_j(DFi \times DGj)) \\
& \cong & col_k(DFk \times DGk)
\end{array}
$$

The steps follow by: induction on F and G; $(col_j FDj) \times (-)$ has a right adjoint so preserves colimits; $(-) \times DGi$ also has a right adjoint; the earlier fact that a colimit for $\Delta\colon \omega \times \omega \to \mathcal{C}$ and $\Delta'\colon \omega \to \mathcal{C}$ where $\Delta'(k) \stackrel{\mathrm{def}}{=} \Delta(k, k)$ are isomorphic.

Datatype Initial Algebra, Categorically

Suppose that a functor $F\colon \mathcal{S}et \to \mathcal{S}et$ is defined by a grammar $F ::= P \mid F \times F \mid F + F$ where P preserves colimits of diagrams $D\colon \omega \to \mathcal{S}et$. Then so too does F. This follows by induction. Suppose that F, G preserve such colimits.

$$\begin{array}{rcl} (F+G)(col_i Di) & \stackrel{\mathrm{def}}{=} & (F col_i Di) + (G col_i Di) \\ & \cong & (col_i FDi) + (col_i GDi) \\ & \cong & col_i(DFi + DGi) \end{array}$$

The first step follows by induction on F and G; the second step can be proven directly from the definition of a colimit (coproduct).

Hence any such F preserves $D\colon \omega \to \mathcal{S}et$ colimits.

Datatype Initial Algebra, Categorically

It follows from this, plus the fact that identity functors and constant functors preserve colimits of diagrams $D\colon \omega \to \mathcal{C}$ for any $\mathcal{C}$, that the datatype functor

$$F \stackrel{\mathbf{def}}{=} F_{T_1} + \ldots + F_{T_m} \colon \mathcal{S}et \longrightarrow \mathcal{S}et$$

preserves colimits of shape $D\colon \omega \longrightarrow \mathcal{S}et$. Since in fact $\mathcal{S}et$ has *all* colimits, by purely categorical reasoning it has an initial algebra $\sigma\colon F\mathsf{I} \longrightarrow \mathsf{I}$.

Mini Project

Find out what nominal sets are, and learn the basic properties of the category $\mathcal{N}om$ (of nominal sets and finitely supported functions) such as finite products and coproducts. Follow this up by learning what a nominal algebraic datatype is. Then see if you can construct an initial algebra model of expressions for such a datatype, proving the relevant properties, and further show that initial algebras exist for purely categorical reasons, much as we did in these slides for (ordinary) algebraic datatypes.

References

- Steve Awodey. Category Theory, Oxford Logic Guides, Oxford University Press, 2006. Second edition 2010.
- Roy L Crole. Categories for Types, Cambridge University Press, 1994.
- P.J. Freyd and A. Scedrov. Categories, Allegories, Elsevier Science Publishers, 1990.
- S. Mac Lane. Categories for the Working Mathematician, Springer Verlag, 1971, vol. 5 of Graduate Texts in Mathematics.
- Benjamin C. Pierce. Basic Category Theory for Computer Scientists. MIT Press, 1991.
- Andrew M. Pitts. Nominal Sets, Cambridge University Press, 2013.
- Harold Simmons. An Introduction to Category Theory, Cambridge University Press, 2011.