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Categorical Type Theory Problems

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1 Preorders

Exercises 1.1

(1) Check that $\mathcal{P}(X)$ is a preorder–this is trivial.

(2) Check that X^{op} is a preorder if X is–this too is easy.

(3) If X and Y are preorders (that is, sets equipped with preorders $\leq_X$ and $\leq_Y$) check that so too is $X \times Y$.

(4) Check that $Sub(X)$ is a preorder.

Exercises 1.2

(1) If $f : X \to Y$ is any set function, then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by

$$f^{-1}(B) \stackrel{\text{def}}{=} \{\, x \in X \mid f(x) \in B \,\}$$

is monotone. Verify the details.

(2) If $f : X \to Y$ and $g : Y \to Z$ are both monotone functions, then so too is the **composition** $f \circ g : X \to Z$ defined by $(g \circ f)(x) \stackrel{\text{def}}{=} g(f(x))$ for any $x \in X$. Verify this fact. *Do the collection of preorders and monotone functions form a category? Why?*

(3) Let X and Y be preorders and $X \times Y$ their cartesian product. Check that there are monotone functions $\pi_X : X \times Y \to X,\ (x,y) \mapsto x$ and $\pi_Y : X \times Y \to Y,\ (x,y) \mapsto y$ where $(x,y) \in X \times Y$.

(4) Verify that given monotone functions $f : Z \to X$ and $g : Z \to Y$ where Z is any given preorder, there is a unique monotone function $m : Z \to X \times Y$ for which $f = \pi_X \circ m$ and $g = \pi_Y \circ m$. *Can you explain what this means using categorical language? (If not, you may just need to learn a little more about universal constructions . . .).*

(5) Find a counterexample to the following statement. A monotone function $f : X \to Y$ between posets X and Y which is a bijection is necessarily an isomorphism.

(6) * Let X be a poset and define a relation on the set X by saying that $x \prec y$ just in case $x < y$ and there is no $z \in X$ for which $x < z < y$. Now let X be any set and Y be a poset. Let $X \Rightarrow Y$ be the poset of functions $X \to Y$ ordered pointwise. Show that $f \prec g$ (where $f,g \in X \Rightarrow Y$) iff

(a) There is $\hat{x} \in X$ for which $f(\hat{x}) \prec g(\hat{x})$ in Y, and
(b) $f(x) = g(x)$ for each $x \in X \setminus \{\hat{x}\}$.

Now let X be a finite poset, and $X \Rightarrow Y$ the poset of **monotone** functions $X \to Y$. Show that $f \prec g$ iff (a) and (b) remains true, with this new definition of $X \Rightarrow Y$.

Exercises 1.3

(1) Write down some simple examples of isomorphic *elements* in any preorder $Sub(X)$ of injective functions with target X. *This is often referred to as the category of subobjects of* X.

(2) Verify that the relation $\cong$ of isomorphism *on preorders* is an equivalence relation.

Exercises 1.4

(1) In $(\mathcal{P}(X), \subseteq)$, binary meets and joins are given by the operations of *intersection* and *union*. Verify this. What are the top and bottom elements?

(2) Define the order $d \mid n$ to mean that $(\exists k \in \mathbb{N})(n = k * d)$. With this order, binary meets and joins are given simply by *highest common factor* and *lowest common multiple* respectively. Give some informal arguments to show that this is correct.

(3) Think of some simple finite preordered sets in which meets and joins do not exist.

(4) Suppose that X is a poset (a preorder satisfying anti-symmetry). Show that meets in a poset are unique if they exist. *Hint: Suppose that, in each case, there are at least two possibilities m and m' and prove that m and m' are equal.*

(5) * Try to work out how to compute meets (or joins) in $Sub(X)$. See if you can solve this problem without assitance; if you can't, then once you know what pullbacks are, have another attempt.

Exercises 1.5

(1) Verify that $\mathcal{P}(X)$ is a Heyting prelattice where $A \Rightarrow A' \stackrel{\text{def}}{=} (X \setminus A) \cup A'$.

(2) Verify that any finite predistributive lattice X is a Heyting prelattice in which

$$y \Rightarrow z \stackrel{\text{def}}{=} \bigvee \{\, l \in X \mid l \wedge y \leq z \,\}$$

(3) Consider the inverse image function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$. Verify that this is a homomorphism of Heyting prelattices.

(4) * Let X be a Heyting lattice, and for each $x \in X$ make the definition $\neg x \stackrel{\text{def}}{=} x \Rightarrow \bot$. Prove that for any $x, y \in X$, $\neg(x \vee y) = \neg x \wedge \neg y$.

2 Categories

Exercises 2.1

(1) Choose three examples of categories of your choice and check in detail that the axioms of a category hold.

(2) Given categories $\mathcal{C}$ and $\mathcal{D}$, the objects of the category $\mathcal{C} \times \mathcal{D}$ are pairs (A,B) of objects from $\mathcal{C}$ and $\mathcal{D}$ respectively. Convince yourself that there is such a category $\mathcal{C} \times \mathcal{D}$.

(3) The category $\mathcal{M}on$ has objects monoids, that is, sets M with an associative operation $\cdot : M \times M \to M$ and identity for the operation $e \in M$. (For example, if A is a set, the set $[A]$ of lists over A is a monoid with list concatenation and empty list.) Check the axioms.

(4) The category $\mathcal{P}art$ has objects sets and morphisms partial functions. Show that this is indeed a category by writing down the "most obvious" composition of morphisms and checking the axioms of identity and associativity.

(5) Make sure you understand the definition of $\mathcal{C}^{op}$.

Exercises 2.2

(1) Write down the definition you would expect of the product functor $F \times G : \mathcal{C} \times \mathcal{C}' \to \mathcal{D} \times \mathcal{D}'$ and check that $F \times G$ is a functor.

(2) Check that we can define a functor $\mathcal{P} : \mathcal{S}et^{op} \to \mathcal{S}et$ by setting

$$f : B \to A \quad \mapsto \quad f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A),$$

where $f : A \to B$ is a function in $\mathcal{S}et$, and the function f^{-1} is defined by

$$f^{-1}(B') \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in B'\}$$

where $B' \in \mathcal{P}(B)$.

(3) Define $G : \mathcal{S}et \to \mathcal{M}on$ by $GA \stackrel{\text{def}}{=} [A]$ and $Gf \stackrel{\text{def}}{=} mapsq(f)$, where $mapsq(f) : [A] \to [B]$ is defined by

$$mapsq(f)([a_1, \ldots, a_n]) = [f^2(a_1), \ldots, f^2(a_n)],$$

with $[a_1, \ldots, a_n]$ any element of $[A]$ and $f : A \to B$ a function. Show that G is a not a functor.

(4) Verify that the definition of a comma category does indeed give rise to a category.

(5) Let us say that a category $\mathcal{C}$ is **tiny** if the collection of objects forms a set and $\mathcal{C}$ is discrete, that is, the only morphisms are identities; prove that a category $\mathcal{C}$ is tiny iff given any category $\mathcal{D}$ with a set of objects $ob\,\mathcal{D}$ and any set function $f : ob\,\mathcal{C} \to ob\,\mathcal{D}$, then f extends uniquely to a functor $F : \mathcal{C} \to \mathcal{D}$. (Extends means that if A is an object of $\mathcal{C}$, then $FA = f(A) \in ob\,\mathcal{D}$.)

Exercises 2.3

(1) Given categories $\mathcal{C}$ and $\mathcal{D}$, verify that the functor category $[\mathcal{C},\mathcal{D}]$ is indeed a category.

(2) Given a diagram of categories and functors

$$\mathcal{C} \xrightarrow{I} \mathcal{D} \xrightarrow{F,G,H} \mathcal{E} \xrightarrow{J} \mathcal{F}$$

and natural transformations $\alpha : F \to G$ and $\beta : G \to H$, we can define $J^* : [\mathcal{D},\mathcal{E}] \to [\mathcal{D},\mathcal{F}]$ by $J^*(F) \stackrel{\text{def}}{=} J \circ F$ on any object F and $(J^*(\alpha))_D \stackrel{\text{def}}{=} J(\alpha_D)$ where D is an object of $\mathcal{D}$. Show that $J^*(\beta \circ \alpha) = J^*(\beta) \circ J^*(\alpha)$. There is also a functor $I_* : [\mathcal{D},\mathcal{E}] \to [\mathcal{C},\mathcal{E}]$. Try to define I_* and show that $I_*(\beta \circ \alpha) = I_*(\beta) \circ I_*(\alpha)$.

Note: make sure you understand in which categories the compositions are defined.

(3) Verify that $F_X : \mathcal{S}et \to \mathcal{S}et$ is a functor and that $ev : F_X \to id_{\mathcal{S}et}$ is a natural transformation (see slides).

(4) Let $\mathcal{S}$ be the category of non-empty sets and set functions. Define a functor $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ by sending $f : X \to Y$ in $\mathcal{S}$ to the function

$$\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y) \qquad A \mapsto f(A) \stackrel{\text{def}}{=} \{f(a) \mid a \in A\}.$$

Show that there is no natural transformation $\alpha : \mathcal{P} \to id_{\mathcal{S}}$. ($\mathcal{P}(f)$ is sometimes written f_*.)

Exercises 2.4

(1) Let $\mathcal{C}$ be a category and let $f : A \to B$ and $g, h : B \to A$ be morphisms. If $f \circ h = id_B$ and $g \circ f = id_A$ show that $g = h$. Deduce that any morphism f has a **unique** inverse if such exists.

(2) Let $\mathcal{C}$ be a category and $f : A \to B$ and $g : B \to C$ be morphisms. If f and g are isomorphisms, show that $g \circ f$ is too. What is its inverse?

Exercises 2.5

(1) Two categories are said to be equivalent, if, roughly speaking, we can write down a one to one correspondence between isomorphism classes of objects obtained from the categories. More precisely, two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\varepsilon : F \circ G \cong id_{\mathcal{D}}$ and $\eta : id_{\mathcal{C}} \cong G \circ F$. We say that F is an *equivalence* with an *inverse equivalence* G and denote the equivalence by $F : \mathcal{C} \simeq \mathcal{D} : G$.

Let $\mathcal{P}art$ be the category of sets and partial functions. Write 1 for a singleton set. An object of the category $1/\mathcal{S}et$ is a function $f : 1 \to A$ where A is a set (and hence in particular A is non-empty). A morphism $m : f \to f'$ (where $f' : 1 \to A'$) is a function $m : A \to A'$ for which $m \circ f = f'$. Prove that $\mathcal{P}art \simeq 1/\mathcal{S}et$. *Hint: Note that an object $f : 1 \to A$ amounts to specifying an element $a \in A$.*

(2) The slice category $\mathcal{S}et/B$ is often referred to as the category of B-indexed families of sets with functions preserving the indexing. First try to work out the definition of this category, by referring to the previous exercise. Then to understand the description of the category, note that a function $f : X \to B$ gives rise to the family of sets $(f^{-1}(b) \mid b \in B)$, and the family of sets $(X_b \mid b \in B)$ gives rise to the function

$$f : \{(x,b) \mid x \in X_b, b \in B\} \to B$$

where $f(x,b) \stackrel{\text{def}}{=} b$. Note that we can regard the set B as a discrete category; then there is an equivalence between the functor category $[B, \mathcal{S}et]$ and the slice $\mathcal{S}et/B$. Formulate this equivalence carefully and prove that your definitions really do give an equivalence.

Exercises 2.6

(1) Show that a category $\mathcal{C}$ has finite products just in case it has binary products and a terminal object.

(2) Let $\mathcal{C}$ be a category with finite products and let

$$\begin{array}{lll} l : X \to A & f : A \to B & g : A \to C \\ h : B \to D & k : C \to E & \end{array}$$

be morphisms of $\mathcal{C}$. Show that $(h \times k) \circ \langle f, g\rangle = \langle h \circ f, k \circ g\rangle$ and $\langle f, g\rangle \circ l = \langle f \circ l, g \circ l\rangle$.

(3) Investigate the notion of a binary product in a category $\mathcal{C}^{op}$.

(4) Prove the coproduct of any set-indexed family of objects is unique up to isomorphism if it exists.

(5) * Find an example of a functor $F : \mathcal{C} \to \mathcal{D}$ for which

$$F(A \times B) \cong FA \times FB$$

in $\mathcal{D}$ for all pairs of objects A and B in $\mathcal{C}$, but such that F does not preserve binary products. **Hint: think about countably infinite sets.**

Exercises 2.7

(1) Show that the category $\mathcal{S}et$ is bicartesian closed.

(2) Show that any category $[\mathcal{C},\mathcal{S}et]$ has finite products and coproducts.

(3) Let A be an object of a cartesian closed category $\mathcal{C}$. If $\mathcal{C}$ is locally small, show that $\mathcal{C}(A,-)$ preserves finite products.

Exercises 2.8

(1) If $\Delta: X \to X \times X$ is given by $\Delta(x) \stackrel{\text{def}}{=} (x,x)$, verify that there are adjoints $(\vee \dashv \Delta \dashv \wedge)$.

(2) Verify that there is a natural bijection

$$\overline{(-)} : \mathcal{M}on([A],M) \cong \mathcal{S}et(A,UM) : \widehat{(-)}$$

(3) Verify that the diagonal functor $\Delta: \mathcal{S}et \to \mathcal{S}et \times \mathcal{S}et$ taking a function $f: A \to B$ to $(f,f): (A,A) \to (B,B)$ has right adjoint Π taking any morphism $(f,g): (A,A') \to (B,B')$ of $\mathcal{S}et \times \mathcal{S}et$ to $f \times g \stackrel{\text{def}}{=} \langle f \circ \pi_A, g \circ \pi_B \rangle : A \times A' \to B \times B'$.

(4) Let $\mathcal{C}$ be a cartesian closed category and $f: A \to B$ and $g: B \to (C \Rightarrow D)$ be morphisms of $\mathcal{C}$. Show that $(g \circ f)^\star = g^\star \circ (f \times id_C)$, where if $h: X \to (Y \Rightarrow Z)$ then $h^\star \stackrel{\text{def}}{=} ev \circ (h \times id_Y)$.

(5) Let $f: A \times B \to C$ and $g: C \to D$ be morphisms of a cartesian closed category. Show that $\lambda(g \circ f) = \lambda(g \circ ev) \circ \lambda(f)$.

(6) Formulate precisely the definitions of the functors $A \Rightarrow (-) : \mathcal{C} \to \mathcal{C}$ and $(-) \Rightarrow A : \mathcal{C}^{op} \to \mathcal{C}$, where A is an object of a cartesian closed category $\mathcal{C}$.

(7) Let A be an object of a cartesian closed category $\mathcal{C}$. Show that $A \Rightarrow (-)$ preserves finite products.

(8) Formulate the notion of a finite coproduct preserving functor and show that $A \times (-) : \mathcal{C} \to \mathcal{C}$ is such a functor *provided* that $\mathcal{C}$ is cartesian closed.

(9) Make sure you can formulate the definition of a functor that preserves exponentials.

Exercises 2.9

(1) Prove the Yoneda Lemma.

(2) Let X be a preorder and let $F: X \to \mathcal{S}et$ be a functor where we will write $x \mapsto Fx$ for the operation on objects and $x \leq y \quad \mapsto \quad f_{x,y}: Fx \to Fy$ for the operation on morphisms.

(a) If Fx is the empty set $\varnothing$, what can we say about $x \in X$?

(b) Let $a \in X$. Show that to give a natural transformation $\alpha: H^a \to F$ is to give an element $e_x \in Fx$ for each $x \in X$ satisfying $a \leq x$, such that $f_{x,y}(e_x) = e_y$ whenever $y \in X$ and $x \leq y$.

(c) Investigate the Yoneda lemma in this situation.

(3) * Can you show that $[\mathcal{C},\mathcal{S}et]$ is cartesian closed? *Suppose that exponentials exist, and apply the Yoneda Lemma.*

3 Categorical Type Theory

Exercises 3.1

(1) If $Sg \rhd \Gamma \vdash M : \sigma$, then we have $Sg \rhd \pi\Gamma \vdash M : \sigma$. Work through the details of the proof.

(2) Prove that if $Sg \rhd \Gamma \vdash M : \sigma$, then the free variables of M appear in Γ.

(3) Fix a signature. Let $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash N : \tau$ be a proved term, and $\Gamma \vdash M_i : \sigma_i$ proved terms for each i. Prove that $\Gamma \vdash N[\vec{M}/\vec{x}] : \tau$ is a proved term.

Exercises 3.2

(1) Work through the details of the derivation of the semantics of binary product types. Be careful to understand the crucial fact that because the procedures of deriving proved terms and performing substitutions *commute*, the procedures of deriving a proved term can be modelled in an appropriate categorical structure by operations which are *natural* in their arguments.

(2) (a) Use the Yoneda lemma to deduce that to soundly interpret rules

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathsf{Inl}_\tau(M) : \sigma + \tau} \qquad \frac{\Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Inr}_\sigma(N) : \sigma + \tau}$$

it is necessary and sufficient to give morphisms $i : A \to A + B$ and $j : B \to A + B$ of $\mathcal{C}$ for all objects A and B, where we may define

$$\begin{aligned} [\![\Gamma \vdash \mathsf{Inl}_\tau(M) : \sigma + \tau]\!] &\stackrel{\text{def}}{=} i \circ [\![\Gamma \vdash M : \sigma]\!] \\ [\![\Gamma \vdash \mathsf{Inr}_\sigma(N) : \sigma + \tau]\!] &\stackrel{\text{def}}{=} j \circ [\![\Gamma \vdash N : \tau]\!]. \end{aligned}$$

(b) By writing down an appropriate family of functions on morphism sets which will give a sound interpretation to

$$\frac{\Gamma \vdash S : \sigma + \tau \quad \Gamma, x : \sigma \vdash E : \delta \quad \Gamma, y : \tau \vdash F : \delta}{\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F) : \delta}$$

and considering naturality conditions, prove that your functions may be specified in terms of a family of functions

$$\Phi_C : \mathcal{C}(C \times A, D) \times \mathcal{C}(C \times B, D) \longrightarrow \mathcal{C}(C \times (A + B), D)$$

which are natural in C. We can then define

$$\begin{aligned} &[\![\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F) : \delta]\!] \stackrel{\text{def}}{=} \\ &\qquad \Phi_{[\![\Gamma]\!]}([\![\Gamma, x : \sigma \vdash E : \delta]\!], [\![\Gamma, y : \tau \vdash F : \delta]\!]) \circ \langle id_{[\![\Gamma]\!]}, [\![\Gamma \vdash S : \sigma + \tau]\!]\rangle. \end{aligned}$$

(c) Using the semantics assigned to proved terms, write down the equations which must hold between morphisms of $\mathcal{C}$ in order that the equations-in-context involving coproduct types are always satisfied. Deduce that the function

$$\mathcal{C}(C \times (A+B), D) \longrightarrow \mathcal{C}(C \times A, D) \times \mathcal{C}(C \times B, D)$$

given by $f \mapsto (f \circ (id_C \times i), f \circ (id_C \times j))$ is a bijection. Hence show that the object $A+B$ is indeed the binary coproduct of A and B.

(d) By considering the bijection

$$\mathcal{C}(C \times A, D) \times \mathcal{C}(C \times B, D) \;\cong\; \mathcal{C}((C \times A) + (C \times B), D),$$

use the Yoneda lemma to prove that the binary products of $\mathcal{C}$ must **distribute** over binary coproducts, that is for all objects A, B and C of $\mathcal{C}$ we have

$$C \times (A+B) \cong (C \times A) + (C \times B).$$

Thus to interpret the case syntax soundly we require a category with finite products and binary coproducts, for which binary products distribute over binary coproducts.

Exercises 3.3

(1) Look at the details of the categorical semantics of $\lambda \times +$-theories and understand the ideas of such a model.

(2) Work through the details of the proof of the result that substitution is modelled by categorical composition.

(3) Work through the details of the soundness theorem.

Exercises 3.4

(1) Prove that there is a canonical isomorphism $[\![\sigma]\!]_{F_*\mathbf{M}} \cong F[\![\sigma]\!]_{\mathbf{M}}$.

(2) Given a proved term $\Gamma \vdash M : \sigma$ show by induction that the morphism $[\![\Gamma \vdash M : \sigma]\!]_{F_*\mathbf{M}}$ is given by the composition

$$[\![\sigma_1]\!]_{F_*\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{F_*\mathbf{M}} \cong F([\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{\mathbf{M}}) \stackrel{F[\![\Gamma \vdash M:\sigma]\!]_{\mathbf{M}}}{\longrightarrow} F[\![\sigma]\!]_{\mathbf{M}}.$$

Hence deduce that $F_*\mathbf{M}$ is a model of Th.

(3) Try to verify that $Cl(Th)$ is a bicartesian closed category.

(4) Define $\mu : Cl(Th) \to \mathcal{D}$ by

$$(x : \sigma \mid M) : \sigma \longrightarrow \tau \quad \longmapsto \quad [\![x : \sigma \vdash M : \tau]\!]_{\mathbf{M}} : [\![\sigma]\!]_{\mathbf{M}} \longrightarrow [\![\tau]\!]_{\mathbf{M}}$$

Write down some of the details required to see that μ is a bicartesian closed functor.

(5) Prove that $\mu \cong \mu'$ is indeed a *natural* isomorphism.

4 Applications

Exercises 4.1

(1) In the case of an algebraic theory, check that $Cl(Th)$ is a category.

(2) Verify that the Yoneda embedding $\mathcal{C}(-,+) : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{S}et]$ is a full and faithful functor for any locally small category $\mathcal{C}$.

(3) Using the fact that $I : Cl(Th) \to Cl(Th')$ is full and faithful:

$$Cl(Th)(\gamma, \gamma') \xrightarrow{\cong} Cl(Th')(I\gamma, I\gamma')$$

prove the “existence” part of the Conservative Extension result.

(4) Write down some of the details that $\mathcal{G}l$ is a bicartesian closed category and the obvious functor $\pi_2 : \mathcal{G}l \to \mathcal{D}$ is a morphism of BCCCs.

(5) Convince yourself of the fact that $I : Cl(Th) \to Cl(Th')$ is indeed full and faithful.