# Categorical Type Theory

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# Overview

- We aim to study some of the connections between type theory and category theory.
- We shall assume knowledge of basic logic, and an appreciation of basic type theory from programming.
- We shall cover the category theory and type theory required for understanding the connections . . .
- . . . with certain topics being taught in more detail by José, so we include only notational summaries.

# High Level Topics

- Order Theory: We shall study properties of orders, and show how these give rise to basic examples of categories and categorical structures.
- Category Theory: We shall study some simple category theory, enough to model simple type theories.

# High Level Topics

- Type Theory: We will define a simple type system, derive a categorical semantics from first principles, and show how a theory in the type system "corresponds" to a special category.
- Applications: We apply the correspondence to obtain a result about type theory.
- If there is time we will also look at an application to logic.

# Order Theory

- A (mathematical) order makes precise our intuitions about relations such as less than or less than or equal to.
- What's the point? Order relations are used extensively in computing; and a particular kind of order gives rise to simple examples of categories.
- We review some formal definitions of order relations.
- We progress to basic mathematical structures that can be defined using order relations.

# Order Theory

- Why? These structures are simple examples of the common structures found in categories!
- We also define functions which preserve order structure.
- Why? Such functions will give us examples of functors, which are mappings between categories, and are fundamental to category theory.

# Basic Definitions

- A binary relation $R$ on a set $X$ is any subset $R \subseteq X \times X$. If $x, y \in X$, then we will write $x\,R\,y$ for $(x, y) \in R$.

- $R$ is reflexive if whenever $x \in X$ we have $x\,R\,x$;

- transitive if whenever $x, y, z \in X$, $(x\,R\,y$ and $y\,R\,z)$ implies $x\,R\,z$;

- symmetric if whenever $x, y \in X$ then $x\,R\,y$ implies $y\,R\,x$;
- anti-symmetric if whenever $x, y \in X$, $(x\,R\,y$ and $y\,R\,x)$ implies $x = y$.
- $R$ is an equivalence relation if it is reflexive, symmetric and transitive.
- We will not make much use of the definitions on this slide, but they are (of course) used throughout computer science.

# Preordered Sets

- A preorder on a set $X$ is a binary relation $\leq$ on $X$ which is reflexive and transitive.
- A preordered set $(X, \leq)$ is a set $X$, equipped with a preorder $\leq$ on the set $X$.
- NOTE: We often just refer to a "preorder $X$".
- Every preorder is an example of a category! Elements $x$ in $X$ are objects and each relationship $x \leq x'$ is a morphism $x \to x'$.
- The axioms that make $X$ a preorder are exactly those required to make $X$ a category.

## Examples of Preordered Sets

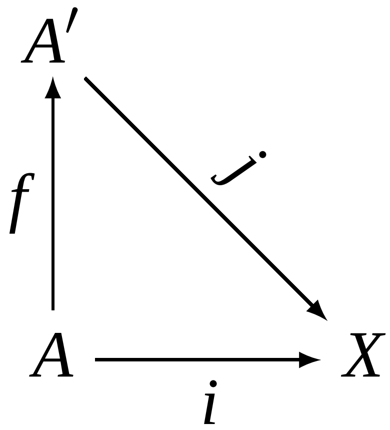
- The set $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ is called the powerset of $X$. The powerset is a preorder with order given by inclusion of subsets, $A \subseteq A'$.

- Given preorders $X$ and $Y$, their cartesian product has underlying set

$$X \times Y \stackrel{\text{def}}{=} \{(x,y) \mid x \in X, y \in Y\}$$

with order given pointwise, that is $(x,y) \leq (x',y')$ iff $x \leq_X x'$ and $y \leq_Y y'$ (using the obvious notation).

- Fix any set $X$. Consider the set $Sub(X)$ of all injective functions $i : A \to X$, and order $i : A \to X \le j : A' \to X$ provided there is $f : A \to A'$ such that




- If $X$ is a preorder, then $X^{op}$ is a preorder given by changing the order to $\le^{op}$ where for $x, x' \in X^{op} \stackrel{\text{def}}{=} X$, we define $x \le^{op} x'$ if and only if $x' \le x$.

# Monotone Functions between Preordered Sets

- Let $f : X \to Y$ be a function, with $X$ and $Y$ equipped with preorders. $f$ is monotone if for all $x, x' \in X$ we have $x \leq x'$ implies $f(x) \leq f(x')$.
- $f$ is also called a homomorphism of preorders, or sometimes simply a morphism.
- If $X$ is regarded as a category then such a function is an example of a functor between categories.

# Examples of Monotone Functions

- The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) \stackrel{\text{def}}{=} n+3$ is monotone; there are many such functions on $\mathbb{N}$ (and of course $\mathbb{R}$).
- The projection functions $\pi : X \times Y \to X$ and $\pi' : X \times Y \to Y$ are monotone if $X \times Y$ is cartesian product.
- Given a set $U$, there is a function $S : \mathcal{P}(U) \to \mathcal{P}(\mathcal{P}(U))$ given by $S(X) \stackrel{\text{def}}{=} \mathcal{P}(X)$. This is monotone.
- If $f : X \to Y$ is any set function, then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $f^{-1}(B) \stackrel{\text{def}}{=} \{\, x \in X \mid f(x) \in B \,\}$ is monotone.

# Notions of Isomorphism

- The preorders $X$ and $Y$ are isomorphic if there are monotone functions $f : X \to Y$ and $g : Y \to X$ for which $g \circ f = id_X$ and $f \circ g = id_Y$.
- The monotone function $g$ is an inverse for $f$; and vice versa.
- The elements $x$ and $x'$ in $X$ are isomorphic if $x \leq x'$ and $x' \leq x$. We write $x \cong x'$.
- We can regard $\cong$ as a relation on $X$, which is in fact an equivalence relation.

# Partially Preordered Sets

- A partial order on a set $X$ is a binary relation $\leq$ which is reflexive, transitive and anti-symmetric.
- A partially ordered set (poset) $(X, \leq)$ is a set $X$ equipped with a partial order $\leq$ on the set $X$.
- Trivially, any poset is of course a preorder in which isomorphic elements are always equal. Many of our examples of preorders will in fact be posets.

# Properties in Ordered Sets

- Suppose that $X$ is a preorder and $A$ is a subset of $X$. An element $x \in X$ is an upper bound for $A$ if for every $a \in A$ we have $a \leq x$ (sometimes written $A \leq x$).
- An element $x \in X$ is a greatest element of $A$ if it is an upper bound of $A$ which belongs to $A$;
- Lower bounds and least elements are defined analogously.
- Greatest and least elements are unique up to isomorphism; so too for lower and upper bounds.

# Binary Meets and Joins

- Meets and joins are very simple examples of universal constructions from category theory.

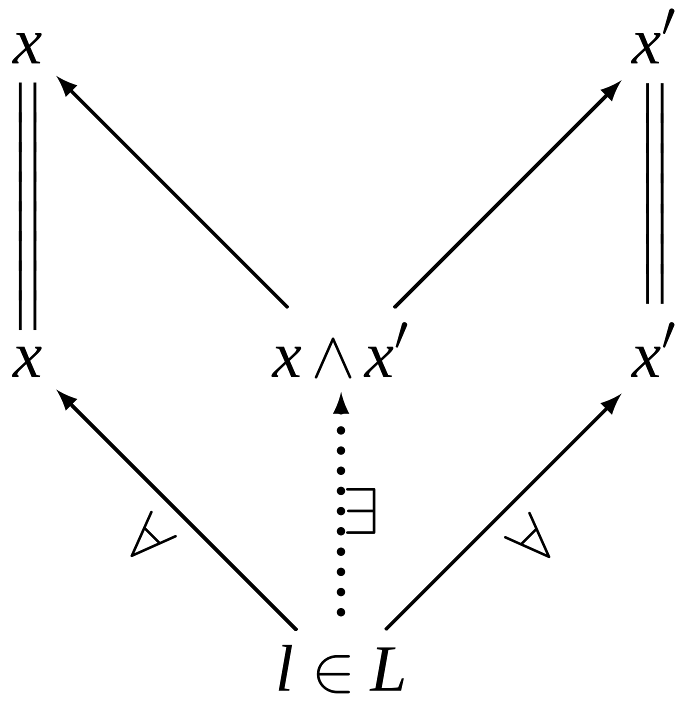
- Given a pair of elements $x, x' \in X$ in a preorder $X$, the set $L$ of lower bounds for $\{x, x'\}$ is

$$L \stackrel{\text{def}}{=} \{\, l \in X \mid l \leq x \text{ and } l \leq x' \,\}$$

- A meet of a pair of elements $x, x' \in X$, if it exists, is a greatest element in the set $L$ of lower bounds for $\{x, x'\}$.

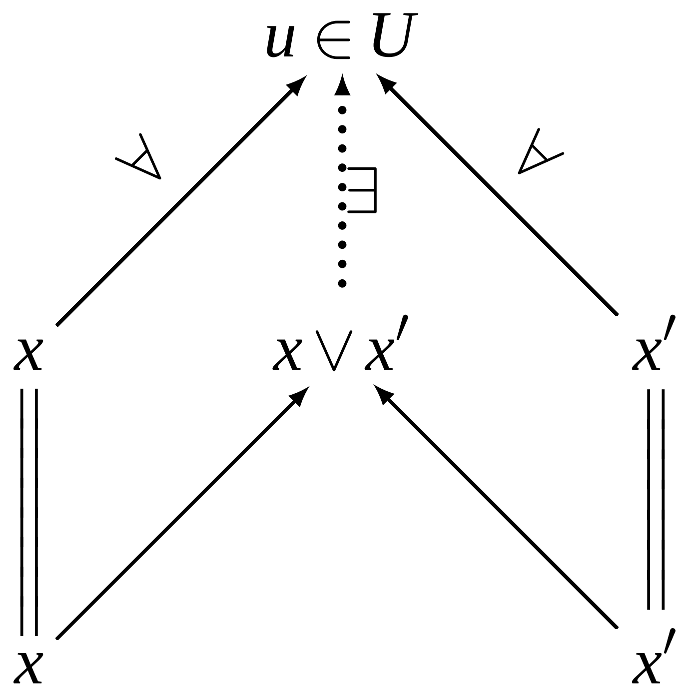
- This is a simple example of a universal construction in category theory. A binary meet is an example of a binary product.

- Writing $\rightarrow$ instead of $\leq$ we have



- A join of a pair of elements $x, x' \in X$, if it exists, is a least element in the set $U$ of upper bounds for $\{x, x'\}$.

- Writing $\rightarrow$ instead of $\leq$ we have



## Top and Bottom Elements; Uniqueness Properties

- For a preorder $X$, a top element $\top \in X$ satisfies $x \leq \top$ for all $x \in X$.
- For a preorder $X$, a bottom element $\bot \in X$ satisfies $\bot \leq x$ for all $x \in X$.
- In a preorder, if a meet, join, top or bottom element exists, then it is unique up to isomorphism.

## Some Examples

- With the usual order on the natural numbers $\mathbb{N}$, binary meets and joins are given by simply taking the least and greatest elements (of the pair).

- With the inclusion order on $\mathcal{P}(X)$, binary meets and joins are given by the operations of intersection and union. What are the top and bottom elements?

- Define the order $d \mid n$ to mean that $(\exists k \in \mathbb{N})(n = k * d)$. With this order, binary meets and joins are given simply by highest common factor and lowest common multiple respectively. Are there top and bottom elements?

# Arbitrary Meets and Joins

- Let $X$ be a preordered set and $A \subseteq X$. A join of $A$ is a least element in the set of upper bounds for $A$. All joins are isomorphic.
- A meet of $A$ is a greatest element in the set of lower bounds for $A$. All meets are isomorphic.
- If $A$ has at least one join (it might not!) we write $\bigvee A$ for a choice of one of the joins of $A$. Write also $x \vee x'$ for $\bigvee\{x, x'\}$.
- $\bigwedge A$ is a choice of meet $A$. Write also $x \wedge x'$ for $\bigwedge\{x, x'\}$.
- If a preorder has all meets we say it is complete.
- If a preorder has all joins we say it is cocomplete.

# Prelattices

- A prelattice is a preordered set which has binary meets and joins, plus top and bottom elements.
- In fact the examples $(\mathbb{N}, |)$ and $(\mathcal{P}(X), \subseteq)$ are prelattices.
- So too is $(Sub(X), \leq)$ but it requires a little more work to verify than the other three examples.
- $(\mathbb{N}, \leq)$ is not a prelattice.

# Heyting Prelattices

- A Heyting prelattice $X$ is a prelattice in which for each pair of elements $y, z \in X$ there is an element $y \Rightarrow z \in X$ such that

$$x \leq y \Rightarrow z \qquad \text{iff} \qquad x \wedge y \leq z.$$

We call $y \Rightarrow z$ the Heyting implication of $y$ and $z$.

- In a Heyting prelattice $X$, the Heyting implication of $y$ and $z$ is unique up to isomorphism.

Suppose that $a$ and $a'$ are two candidates for the element $y \Rightarrow z \in X$. Then $a \leq a$ implies $a \wedge y \leq z$ implies $a \leq a'$; the converse is similar.

# Distributive Prelattices and Examples

Let $X$ be a prelattice. Then $X$ is distributive if it satisfies $x \wedge (y \vee z) \cong (x \wedge y) \vee (x \wedge z)$ for all $x, y, z$ in $X$.

- $\mathcal{P}(X)$ is a Heyting prelattice where $A \Rightarrow A' \stackrel{\text{def}}{=} (X \setminus A) \cup A'$.
- Any finite distributive prelattice $X$ is a Heyting prelattice. One may define

$$y \Rightarrow z \stackrel{\text{def}}{=} \bigvee \{\, l \in X \mid l \wedge y \leq z \,\}$$

## Homomorphisms of Prelattices and Heyting Prelattices

- A homomorphism of prelattices is a function $f : X \to Y$ (with $X$ and $Y$ prelattices) which preserves finite meets and joins, that is

$$f(\bigwedge\{x_1, \ldots, x_n\}) \cong \bigwedge\{f(x_1), \ldots, f(x_n)\}$$

and

$$f(\bigvee\{x_1, \ldots, x_n\}) \cong \bigvee\{f(x_1), \ldots, f(x_n)\}$$

and also $f(\top) \cong \top$ and $f(\bot) \cong \bot$.

- A homomorphism of Heyting prelattices is as above but also preserves Heyting implications.

# Examples of Homomorphisms of Prelattices

- Consider the inverse image function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$. This is a homomorphism of Heyting prelattices.

- The function that multiplies by a natural number $k$ preserves lowest common multiples, that is, preserves binary joins in $(\mathbb{N}, |)$ where recall $d \mid m$ means that $d$ divides exactly into $m$.

# Definition of a Category

A category $\mathcal{C}$ is specified by the following data:

- A collection $ob\,\mathcal{C}$ of entities called objects, written $A$, $B$, $C\ldots$
- A collection $mor\,\mathcal{C}$ of entities called morphisms written $f$, $g$, $h\ldots$
- For each morphism $f$ a source $src(f)$ which is an object of $\mathcal{C}$ and a target $tar(f)$ also an object of $\mathcal{C}$. We shall write $f : src(f) \longrightarrow tar(f)$ or perhaps $f : A \to B$.

- Morphisms $f$ and $g$ are composable if $tar(f) = src(g)$. If $f : A \to B$ and $g : B \to C$, then there is a morphism $g \circ f : A \to C$.

- For each object $A$ of $\mathcal{C}$ there is an identity morphism $id_A : A \to A$, where

$$
\begin{aligned}
id_{tar(f)} \circ f &= f \\
f \circ id_{src(f)} &= f
\end{aligned}
$$

- Composition is associative, that is given morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$ then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

- If the collection of morphisms from $A$ to $B$ forms a set, then we write $\mathcal{C}(A, B)$ for this set. Such categories are called locally small.

- Sets and total functions, $\mathcal{S}et$. The objects are sets and morphisms are $(A, f, B)$ where $f \subseteq A \times B$ is a function. Composition is given by

$$(B,g,C) \circ (A,f,B) = (A, g \circ f, C)$$

Finally, if $A$ is any set, the identity is $(A, id_A, A)$.

- Any preordered set $(X, \leq)$ is a category. The objects are elements of $X$. The collection of morphisms is the set of pairs $(x,y)$ where $x \leq y$. Composition is $(y,z) \circ (x,y) \stackrel{\text{def}}{=} (x,z)$ (because $\leq$ is transitive). The identities are the pairs $(x,x)$ (because $\leq$ is reflexive).

# Definition of a Functor

A functor $F : \mathcal{C} \to \mathcal{D}$ is specified by

- an operation taking objects $A$ in $\mathcal{C}$ to objects $FA$ in $\mathcal{D}$, and
- an operation sending morphisms $f : A \to B$ in $\mathcal{C}$ to morphisms $Ff : FA \to FB$ in $\mathcal{D}$,

for which $F(id_A) = id_{FA}$, and whenever the composition of morphisms $g \circ f$ is defined in $\mathcal{C}$ we have $F(g \circ f) = Fg \circ Ff$.

# Examples of Functors

- The functors between two preorders $A$ and $B$ are precisely the monotone functions from $A$ to $B$.

- The set $[A]$ of finite lists over a set $A$ gives a monoid via list concatenation.

  Hence we may define $F : \mathcal{S}et \longrightarrow \mathcal{M}on$ by $FA \stackrel{\text{def}}{=} [A]$ and $Ff \stackrel{\text{def}}{=} map(f)$, where $map(f) : [A] \longrightarrow [B]$ is defined by

$$map(f)([a_1,\ldots,a_n]) = [f(a_1),\ldots,f(a_n)],$$

  with $[a_1,\ldots,a_n]$ any element of $[A]$.

To see that $F(g \circ f) = Fg \circ Ff$ where $A \xrightarrow{f} B \xrightarrow{g} C$ note that

$$
\begin{aligned}
F(g \circ f)([a_1, \ldots, a_n]) &\stackrel{\text{def}}{=} map(g \circ f)([a_1, \ldots, a_n]) \\
&= [(g \circ f)(a_1), \ldots, (g \circ f)(a_n)] \\
&= [g(f(a_1)), \ldots, g(f(a_n))] \\
&= map(g)([f(a_1), \ldots, f(a_n)]) \\
&= map(g)(map(f)([a_1, \ldots, a_n])) \\
&= (Fg \circ Ff)([a_1, \ldots, a_n]).
\end{aligned}
$$

- Let $\mathcal{C}$ be a category. The identity functor $id_{\mathcal{C}}$ is defined by $id_{\mathcal{C}}(A) \stackrel{\text{def}}{=} A$ where $A$ is an object of $\mathcal{C}$ and $id_{\mathcal{C}}(f) \stackrel{\text{def}}{=} f$ where $f$ is a morphism of $\mathcal{C}$.

- Given a set $A$, recall that the powerset $\mathcal{P}(A)$ is the set of subsets of $A$. We can define a functor $\mathcal{P} : \mathcal{S}et \longrightarrow \mathcal{S}et$ which is given by

$$f : A \longrightarrow B \quad \mapsto \quad f_* : \mathcal{P}(A) \longrightarrow \mathcal{P}(B),$$

where $f : A \longrightarrow B$ is a function and $f_*$ is defined by $f_*(A') \stackrel{\text{def}}{=} \{f(a') \mid a' \in A'\}$ where $A' \in \mathcal{P}(A)$. We call $P : \mathcal{S}et \longrightarrow \mathcal{S}et$ the covariant powerset functor.

- Given a category $\mathcal{C}$, the category $\mathcal{C}^{op}$ has objects those of $\mathcal{C}$, and morphisms $f^{op} : A \to B$ in $\mathcal{C}^{op}$ are morphisms $f : B \to A$ in $\mathcal{C}$. Composition is $g^{op} \circ f^{op} \stackrel{\text{def}}{=} (f \circ g)^{op}$.

- We can define a functor $\mathcal{P} : \mathcal{S}et^{op} \to \mathcal{S}et$ by setting

$$f : B \to A \quad \mapsto \quad f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A),$$

where $f : A \to B$ is a function in $\mathcal{S}et$, and the function $f^{-1}$ is defined by $f^{-1}(B') \stackrel{\text{def}}{=} \{a \in A \mid f(a) \in B'\}$ where $B' \in \mathcal{P}(B)$.

- Note that the source of the functor is an opposite category. We refer to $\mathcal{P}$ as the contravariant powerset functor.

■ Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C}' \to \mathcal{D}$ be functors. The comma category $(F \downarrow G)$ has objects triples $(A, f, A')$ where $A$ and $A'$ are objects of $\mathcal{C}$ and $\mathcal{C}'$ respectively and $f : FA \to GA'$ is a morphism of $\mathcal{D}$. A morphism $(A, f, A') \to (B, f', B')$ is a pair $(g, h)$ where $g : A \to B$ in $\mathcal{C}$ and $h : A' \to B'$ in $\mathcal{C}'$ for which the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Fg} & FB \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ GA' & \xrightarrow[Gh]{} & GB' \end{array}$$

# Definition of a Natural Transformation

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then a natural transformation $\alpha$ from $F$ to $G$, written $\alpha : F \to G$, is specified by giving a morphism $\alpha_A : FA \to GA$ in $\mathcal{D}$ for each object $A$ in $\mathcal{C}$, such that for any $f : A \to B$ in $\mathcal{C}$, we have

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\alpha_B]{} & GB \end{array}$$

# Examples of Natural Transformations

- Recall $F : \mathcal{S}et \rightarrow \mathcal{M}on$ where $FA \stackrel{\text{def}}{=} [A]$ and $F\,f \stackrel{\text{def}}{=} map(f)$ and $map(f) : [A] \rightarrow [B]$. We can define a natural transformation $rev : F \rightarrow F$ by

$$rev_A([a_1, \ldots, a_n]) \stackrel{\text{def}}{=} [a_n, \ldots, a_1]$$

We check

$$(F\,f \circ rev_A)([a_1, \ldots, a_n]) = [f(a_n), \ldots, f(a_1)] = (rev_B \circ F\,f)([a_1, \ldots, a_n]).$$

- Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F$, $G$, $H$ be functors from $\mathcal{C}$ to $\mathcal{D}$. Also let $\alpha : F \longrightarrow G$ and $\beta : G \longrightarrow H$ be natural transformations. We can define a natural transformation $\beta \circ \alpha : F \longrightarrow H$ by setting the components to be

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A.$$

This yields a category $[\mathcal{C}, \mathcal{D}]$ with objects functors from $\mathcal{C}$ to $\mathcal{D}$, morphisms natural transformations between such functors, and composition as given above.

- Define a functor $F_X : \mathcal{S}et \longrightarrow \mathcal{S}et$ by
  - $F_X(A) \stackrel{\text{def}}{=} (X \Rightarrow A) \times X$ on objects
  - $F_X(f) \stackrel{\text{def}}{=} (f \circ -) \times id_X$ on morphisms

Then define a natural transformation $ev : F_X \longrightarrow id_{\mathcal{S}et}$ by $ev_A(g,x) \stackrel{\text{def}}{=} g(x)$ where $(g,x) \in (X \Rightarrow A) \times X$. To see that we have defined a natural transformation $ev_A : (X \Rightarrow A) \times X \longrightarrow A$ let $f : A \longrightarrow B$ and $(g,x) \in (X \Rightarrow A) \times X$ and note that

$$
\begin{aligned}
(id_{\mathcal{S}et}(f) \circ ev_A)(g,x) &= f(ev_A(g,x)) \\
&= \ldots (ev_B \circ F_X(f))(g,x).
\end{aligned}
$$

## Isomorphisms and Equivalences

- A morphism $f : A \rightarrow B$ is an isomorphism if there is some $g : B \rightarrow A$ for which $f \circ g = id_B$ and $g \circ f = id_A$.
- We shall say $g$ is an inverse for $f$ and vise versa.
- We say that $A$ is isomorphic to $B$, $A \cong B$, if such a mutually inverse pair of morphisms exists.

- An isomorphism $\alpha : F \cong G$ in a functor category is referred to as a natural isomorphism. This is the same as having a collection of isomorphisms $FA \cong GA$ for each object $A$ which are "natural in $A\ldots$"

- Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\varepsilon : FG \cong id_{\mathcal{D}}$ and $\eta : id_{\mathcal{C}} \cong GF$. We say that $F$ is an equivalence with an inverse equivalence $G$ and denote the equivalence by $F : \mathcal{C} \simeq \mathcal{D} : G$.

# Examples

- Bijections in $\mathcal{S}et$ are isomorphisms.

- In the category determined by a partially ordered set, the only isomorphisms are the identities, and in a preorder $X$ with $x, y \in X$ we have $x \cong y$ iff $x \leq y$ and $y \leq x$. Note that in this case there can be only one pair of mutually inverse morphisms witnessing the fact that $x \cong y$.

- Let an object of $1/\mathcal{S}et$ be a pair $(A, a)$ where $a \in A$ and a morphism $g : (A, a) \rightarrow (B, b)$ be a function $g : A \rightarrow B$ for which $b = g(a)$. Let $\mathcal{P}art$ be the category of sets and partial functions. Then $\mathcal{P}art \simeq 1/\mathcal{S}et$.

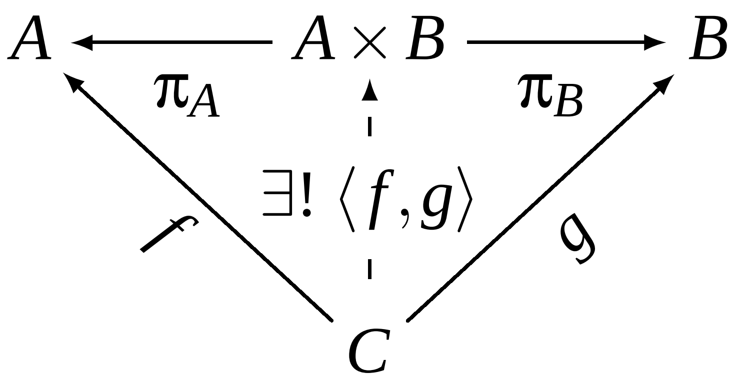
# Definition of Binary Products

A binary product of objects $A$ and $B$ in a category $\mathcal{C}$ is specified by

- an object $A \times B$ of $\mathcal{C}$, together with
- two projection morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$,

for which given any object $C$ and morphisms $f : C \to A$, $g : C \to B$, there is a unique morphism $\langle f, g \rangle : C \to A \times B$ for which $\pi_A \circ \langle f, g \rangle = f$ and $\pi_B \circ \langle f, g \rangle = g$.

- The data for a binary product is more readily understood as a commutative diagram,




The unique morphism $\langle f, g \rangle : C \to A \times B$ is called the mediating morphism for $f$ and $g$.

- The definition can be extended to families of objects $(A_i \mid i \in I)$.

# Definition of Finite Products

Given a family of objects in $\mathcal{C}$, a product is specified by

- an object $\Pi_{i \in I} A_i$ in $\mathcal{C}$, and
- for every $j \in I$, a morphism $\pi_j : \Pi_{i \in I} A_i \to A_j$ in $\mathcal{C}$ called the $j$th product projection,

such that for any $C$ and $(f_i : C \to A_i \mid i \in I)$ there is a unique morphism

$$\langle f_i \mid i \in I \rangle : C \to \Pi_{i \in I} A_i$$

for which given any $j \in I$, we have $\pi_j \circ \langle f_i \mid i \in I \rangle = f_j$.

# Examples

- A binary product of $x$ and $y$ in a preordered set $X$ is given by $x \wedge y$ with projections $x \wedge y \leq x$ and $x \wedge y \leq y$.

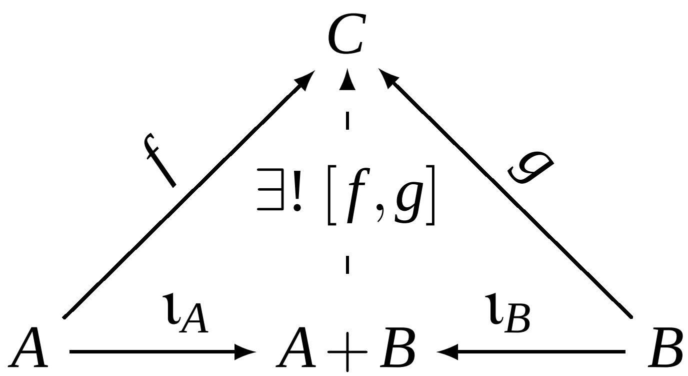
- A (non-empty) finite product of $(A_i \mid i \in I)$ in $\mathcal{S}et$ is given by the cartesian product $\Pi A_{i \in I}$. The product of the empty family is a terminal object 1, with the property that there is a unique morphism $!_A : A \longrightarrow 1$ for every $A$.

# Definition of Binary Coproducts

A binary coproduct of $A$ and $B$ is specified by

- an object $A+B$, together with
- two insertion morphisms $\iota_A : A \to A+B$ and $\iota_B : B \to A+B$,

such that there is a unique $[f,g]$ for which



## Definition of Cartesian Closed Categories

- $\mathcal{C}$ is cartesian closed if it has finite products, and for any $B$ and $C$ there is $B \Rightarrow C$ and morphism

$$ev : (B \Rightarrow C) \times B \longrightarrow C$$

such that for any $f : A \times B \longrightarrow C$ there is a unique morphism $\lambda(f) : A \longrightarrow (B \Rightarrow C)$ such that $f = ev \circ (\lambda(f) \times id_B)$.

- $B \Rightarrow C$ is called the exponential of $B$ and $C$

- $\lambda(f)$ is the exponential mate of $f$.

# Examples

- The category $\mathcal{S}et$.
  - The terminal object is $\{\varnothing\}$ and binary products are given by cartesian product.
  - $B \Rightarrow C$ is the set of functions from $B$ to $C$.
  - The function $ev : (B \Rightarrow C) \times B \to C$ is given by $ev(h, b) = h(b)$, where $b \in B$ and $h : B \to C$ is a function.
  - Given $f : A \times B \to C$ we define $\lambda(f) : A \to (B \Rightarrow C)$ by $\lambda(f)(a)(b) = f(a, b)$.
- A Heyting prelattice viewed as a category is indeed cartesian closed, with Heyting implications as exponentials. In fact such a prelattice also has finite coproducts.

## Definition of Distributive and Bicartesian Closed Categories

- A category with finite products and coproducts is said to be distributive if the mediating morphisms

$$[id_A \times i, id_A \times j] : (A \times B) + (A \times C) \xrightarrow{\cong} A \times (B + C)$$

and $!_{A\times 0} : 0 \xrightarrow{\cong} A \times 0$ are isomorphisms.

- A category $\mathcal{C}$ is a bicartesian closed category if it is a cartesian closed category which has finite coproducts.

# Examples

- The category $\mathcal{S}et$.
- Any category $[\mathcal{C}, \mathcal{S}et]$. Categorical structure is defined pointwise meaning, for example, that $(F \times G)(A) \stackrel{\text{def}}{=} FA \times GA$ and so on.
- Any Heyting prelattice which is regarded as a category.
- In fact any bicartesian closed category is automatically distributive–we will see why this is so later on.

# Functors Preserving Products

- The functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite products if for any finite family of objects $(A_1, \ldots, A_n)$ in $\mathcal{C}$ the morphism

$$m \stackrel{\text{def}}{=} \langle F\pi_i \mid i \in I \rangle : F(A_1 \times \ldots \times A_n) \to FA_1 \times \ldots \times FA_n$$

is an isomorphism.

- We refer to $m$ as the canonical isomorphism.
- $F$ is strict if the above isomorphisms are identities.
- The functor $\mathcal{C}(C, -)$ preserves finite products.

## Functors Preserving Coproducts and Exponentials

- A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be bicartesian closed if it preserves finite products, coproducts and exponentials.
- We shall also call such a functor a morphism of bicartesian closed categories.

# Adjunctions between Preorders

- A pair of monotone functions

$$X \underset{r}{\overset{l}{\rightleftarrows}} Y$$

is said to be an adjunction if for all $x \in X$ and $y \in Y$,

$$l(x) \leq y \iff x \leq r(y)$$

- We say that $l$ is left adjoint to $r$ and that $r$ is right adjoint to $l$. We write $l \dashv r$.

# Examples

- Let $1 \stackrel{\text{def}}{=} \{*\}$ be the one element preorder. Then there are adjunctions $(\bot \dashv\, ! \dashv \top)$

$$X \underset{\bot}{\overset{!}{\rightleftarrows}} 1 \qquad X \underset{\top}{\overset{!}{\rightleftarrows}} 1$$

provided that $X$ has both top and bottom elements. For example, for any $x \in X$,

$$!(x) \stackrel{\text{def}}{=} * \leq * \iff x \leq \top(*) \stackrel{\text{def}}{=} \top$$

# Examples

- Define $\Delta : X \to X \times X$ by $\Delta(x) \overset{\text{def}}{=} (x,x)$. Then there are adjoints $(\vee \dashv \Delta \dashv \wedge)$

$$X \underset{\vee}{\overset{\Delta}{\rightleftarrows}} X \times X \qquad X \underset{\wedge}{\overset{\Delta}{\rightleftarrows}} X \times X$$

provided that $X$ has both binary meets and joins. For example, for any $l \in X$,

$$\Delta(l) \overset{\text{def}}{=} (l,l) \leq (x,x') \iff l \leq \wedge(x,x') \overset{\text{def}}{=} x \wedge x'$$

# Adjunctions between Categories

■ Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. $L$ is left adjoint to $R$, written $L \dashv R$, if given any objects $A$ of $\mathcal{C}$ and $B$ of $\mathcal{D}$ we have

• a bijection between morphisms $LA \to B$ in $\mathcal{D}$ and $A \to RB$ in $\mathcal{C}$,

$$\frac{f : LA \to B}{\overline{f} : A \to RB} \qquad \frac{g : A \to RB}{\widehat{g} : LA \to B}$$

• this bijection is *natural in* $A$ *and* $B$: given morphisms $a : A' \to A$ in $\mathcal{C}$ and $b : B \to B'$ in $\mathcal{D}$ we have

$$\overline{b \circ f \circ La} = Rb \circ \overline{f} \circ a \quad \text{and} \quad (Rb \circ g \circ a)^{\wedge} = b \circ \widehat{g} \circ La.$$

# Notation for Adjunctions

Let $A$ and $B$ be objects of a locally small category $\mathcal{C}$. We define a functor

$$\mathcal{C}(-,+) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}et$$

by taking any morphism $(f,g) : (A,B) \to (A',B')$ in $\mathcal{C}^{op} \times \mathcal{C}$ to the set-theoretic function

$$\mathcal{C}(f,g) : \mathcal{C}(A,B) \to \mathcal{C}(A',B')$$

where $\mathcal{C}(f,g)(h) = g \circ h \circ f$ for each morphism $h : A \to B$. (Note that $f$ is a morphism $A' \to A$ in $\mathcal{C}$).

We can also define

$$\mathcal{C}(A,+) : \mathcal{C} \longrightarrow \mathcal{S}et.$$

This takes $B$ of $\mathcal{C}$ to the set $\mathcal{C}(A,B)$, and if $g : B \longrightarrow B'$ is a morphism of $\mathcal{C}$ then the functor $\mathcal{C}(A,+)$ takes $g : B \longrightarrow B'$ to the function

$$\mathcal{C}(A,g) : \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,B')$$

defined by setting

$$\mathcal{C}(A,g)(h) \stackrel{\text{def}}{=} g \circ h,$$

where $h : A \longrightarrow B$.

Similarly, we can define a functor $\mathcal{C}(-,B) : \mathcal{C}^{op} \longrightarrow \mathcal{S}et$.

If a categories $\mathcal{C}$ and $\mathcal{D}$ are locally small, then $L \dashv R$ provided that there is an isomorphism

$$\boxed{\mathcal{D}(-,+)\circ(L^{op}\times id) \stackrel{\text{def}}{=}} \quad \mathcal{D}(L-,+) \cong \mathcal{C}(-,R+) \quad \boxed{\stackrel{\text{def}}{=} \mathcal{C}(-,+)\circ(id\times R)}$$

in the functor category $[\mathcal{C}^{op}\times\mathcal{D},\mathcal{S}et]$ where $L^{op}:\mathcal{C}^{op}\longrightarrow\mathcal{D}^{op}$ is defined by

$$L^{op}(f:A'\longrightarrow A) \stackrel{\text{def}}{=} Lf:LA'\longrightarrow LA$$

We also say that $R$ is right adjoint to $L$.

# Examples

- The *forgetful* functor $U : \mathcal{M}on \to \mathcal{S}et$ taking a monoid to its underlying set, and the functor $[-] : \mathcal{S}et \to \mathcal{M}on$ taking a set to lists over the set, are adjoints: $[-] \dashv U$.

Given a monoid $M$ and a set $A$ any function $g : A \to UM$ corresponds to a unique monoid morphism $\widehat{g} : [A] \to M$.

Indeed, there is a bijection for each $A$ and $M$

$$\overline{(-)} : \mathcal{M}on([A], M) \cong \mathcal{S}et(A, UM) : \widehat{(-)}$$

given by

$$g : A \longrightarrow UM \quad \longmapsto \quad \widehat{g} : [A] \xrightarrow{[a_1, \ldots, a_n] \mapsto g(a_1) \ldots g(a_n)} M,$$

$$f : [A] \longrightarrow M \quad \longmapsto \quad \overline{f} : A \xrightarrow{a \mapsto f([a])} UM.$$

and a natural isomorphism

$$\Psi : \mathcal{M}on([-],+) \cong \mathcal{S}et(-,U(+)) : \mathcal{S}et^{op} \times \mathcal{M}on \to \mathcal{S}et$$

where at any object $(A,M)$ of $\mathcal{S}et^{op} \times \mathcal{M}on$ we have

$$(\Psi)_{(A,M)} \stackrel{\text{def}}{=} \overline{(-)}$$

and (say)

$$\Psi^{-1}_{(A,M)} \stackrel{\text{def}}{=} \widehat{(-)}.$$

- There are other examples of forgetful functors. The functor $U : \mathcal{C} \to \mathit{Graph}$ taking a category to its underlying graph has a left adjoint taking a graph to the free category over the graph.
- The functor $U : \mathit{Group} \to \mathcal{S}\mathit{et}$ taking a group to its underlying set has a left adjoint taking a set to the free group over the set.

- $F : \mathcal{S}et \to \mathcal{V}ec_K$ is the functor taking set $X$ to the vector space $FX$ with vectors linear combinations $\Sigma_{i\in n}k_i x_i$ where $x_i \in X$ and $k_i \in K$. Given function $f : X \to Y$, the linear map $Ff : FX \to FY$ is

$$Ff(\Sigma_{i\in n}k_i x_i) \stackrel{\text{def}}{=} \Sigma_{i\in n}k_i f(x_i).$$

The functor $U : \mathcal{V}ec_K \to \mathcal{S}et$ is forgetful. Any function $g : X \to UV$ has a *unique extension* to a linear map $\widehat{g} : FX \to V$. The assignment $g \mapsto \widehat{g}$ has an inverse: any linear $f : FX \to V$ restricts to function $\overline{f} : X \to UV$. Thus we have a natural bijection

$$\overline{(-)} : \mathcal{S}et(X, UV) \rightleftarrows \mathcal{V}ec_K(FX, U) : \widehat{(-)}$$

- The diagonal functor $\Delta : \mathcal{S}et \to \mathcal{S}et \times \mathcal{S}et$ taking a function $f : A \to B$ to $(f, f) : (A, A) \to (B, B)$ has right and left adjoints $\Pi$ and $\Sigma$ taking any morphism $(f, g) : (A, A') \to (B, B')$ of $\mathcal{S}et \times \mathcal{S}et$ to $f \times g \stackrel{\text{def}}{=} \langle f \circ \pi_A, g \circ \pi_B \rangle : A \times A' \to B \times B'$ and $f + g \stackrel{\text{def}}{=} [\iota_B f, \iota_{B'} g] : A + A' \to B + B'$ respectively, where

$$A \xleftarrow{\pi_A} A \times A' \xrightarrow{\pi_{A'}} A'$$

$$B \xrightarrow{\iota_B} B + B' \xleftarrow{\iota_{B'}} B'$$

This example remains valid if we replace $\mathcal{S}et$ by any category $\mathcal{C}$, where we leave the reader to define the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

- Let $\mathcal{C}$ be a category with finite products. Existence of a right adjoint $R$ to the functor $(-) \times B : \mathcal{C} \to \mathcal{C}$ for each object $B$ of $\mathcal{C}$, is equivalent to $\mathcal{C}$ being cartesian closed.

(⇒) Given an object $B$ of $\mathcal{C}$ set $B \Rightarrow C \stackrel{\text{def}}{=} R(C)$ for any object $C$ of $\mathcal{C}$. Given a morphism $f : A \times B \to C$ we define $\lambda(f) : A \to (B \Rightarrow C)$ to be the mate of $f$ across the given adjunction. The morphism

$$ev : (B \Rightarrow C) \times B \to C$$

is the mate $(id_{B \Rightarrow C})^{\star}$ of the identity $id_{B \Rightarrow C} : (B \Rightarrow C) \to (B \Rightarrow C)$.

($\Leftarrow$) Conversely, let $B$ be an object of $\mathcal{C}$. We define a right adjoint to $(-) \times B$ denoted by $B \Rightarrow (-)$, by setting

$$c : C \longrightarrow C' \quad \mapsto \quad B \Rightarrow c \stackrel{\text{def}}{=} \lambda(c \circ ev) : (B \Rightarrow C) \to (B \Rightarrow C')$$

for each morphism $c : C \to C'$ of $\mathcal{C}$. We define a bijection by declaring the mate of $f : A \times B \to C$ to be $\lambda(f) : A \to (B \Rightarrow C)$ and the mate of $g : A \to (B \Rightarrow C)$ to be

$$g^{\star} \stackrel{\text{def}}{=} ev \circ (g \times id_B) : A \times B \to C.$$

# The Yoneda Lemma — Preliminaries

Let $\mathcal{C}$ be a locally small category, $F : \mathcal{C} \to \mathcal{S}et$ a functor and $A$ an object of $\mathcal{C}$. Then the collection $Nat(\mathcal{C}(A,+),F)$ of natural transformations $\mathcal{C}(A,+) \to F$ is a set and so we can define a functor

$$Nat(\mathcal{C}(-,+),+) : \mathcal{C} \times [\mathcal{C},\mathcal{S}et] \longrightarrow \mathcal{S}et$$

– The morphism $(g,\mu) : (A,F) \to (A',F')$ in $\mathcal{C} \times [\mathcal{C},\mathcal{S}et]$ is taken to the function

$$Nat(\mathcal{C}(g,+),\mu) : Nat(\mathcal{C}(A,+),F) \to Nat(\mathcal{C}(A',+),F')$$

– $Nat(\mathcal{C}(g,+),\mu)(\alpha) \stackrel{\text{def}}{=} \mu \circ \alpha \circ \mathcal{C}(g,+)$ where $\alpha : \mathcal{C}(A,+) \to F$ is a natural transformation.

# The Yoneda Lemma

There is an “evaluation” functor

$$Ev : \mathcal{C} \times [\mathcal{C}, \mathcal{S}et] \longrightarrow \mathcal{S}et.$$

Then there is a natural isomorphism

$$\Phi : Nat(\mathcal{C}(-,+),+) \cong Ev : \Psi$$

If $A$ is an object of $\mathcal{C}$, this amounts to saying that there is an isomorphism (set-theoretic bijection)

$$\Phi_{(A,F)} : Nat(\mathcal{C}(A,+),F) \cong FA : \Psi_{(A,F)}$$

and this isomorphism is natural in $(A,F)$.

# Categorical Type Theory

- We shall define an equational type theory with products, sums, and functions.
- Working from first principles, we shall derive a semantics.
  - First we examine the rules for deriving type assignments, and show that basic properties lead naturally to categorical models.
  - Second, we examine each of the rules for deriving equations, and extract constraints on our models which guarantee soundness.

# Categorical Type Theory

- We show how structure preserving functors can transform one model into another . . .
- and use this to show how theories correspond to categories with a universal property.

# Signatures

A $\lambda\times+$-signature, $Sg$, is given by :

- A collection of ground types. The collection of *types* is inductively defined:

$$\frac{}{\gamma} \qquad \frac{}{\mathsf{unit}} \qquad \frac{}{\mathsf{null}} \qquad \frac{\sigma \quad \tau}{\sigma\times\tau} \qquad \frac{\sigma \quad \tau}{\sigma+\tau} \qquad \frac{\sigma \quad \tau}{\sigma\Rightarrow\tau}$$

- A collection of function symbols $f : \sigma_1 \ldots \sigma_a \to \sigma$ which may be constants $k : \sigma$ when $a = 0$.

# Raw Terms

We define the raw terms generated by a $\lambda\times+$-signature:

$$\frac{}{x} \qquad \frac{}{k} \qquad \frac{M_1 \quad \ldots \quad M_a}{f(M_1,\ldots,M_a)} \qquad \frac{}{\langle\rangle} \qquad \frac{M \quad N}{\langle M,N\rangle} \qquad \frac{P}{\mathsf{Fst}(P)} \qquad \frac{P}{\mathsf{Snd}(P)}$$

$$\frac{S}{\mathsf{Emp}_\sigma(S)} \qquad \frac{M}{\mathsf{Inr}_\tau(M)} \qquad \frac{M}{\mathsf{Inl}_\tau(M)} \qquad \frac{S \quad E \quad F}{\mathsf{Case}(S, x.E \mid y.F)}$$

$$\frac{M}{\lambda x:\sigma.M} \qquad \frac{F \quad A}{F\,A}$$

- We will use simultaneous substitution of raw terms for free variables, $T[\vec{U}/\vec{v}]$. For example, $\langle x,y\rangle[\mathsf{Inl}(y),x/x,y] = \langle \mathsf{Inl}(y),x\rangle$.

# Proved Terms

- A context is a finite list of (variable, type) pairs, usually written as $\Gamma = [x_1 : \sigma_1, \ldots, x_n : \sigma_n]$, where the variables are required to be distinct.
- A term-in-context is a judgement of the form $\Gamma \vdash M : \sigma$
- Given a signature $Sg$, the proved terms are those terms-in-context which are inductively generated by the following rules.

$$\frac{}{\Gamma, x : \sigma, \Gamma' \vdash x : \sigma} \qquad \frac{}{\Gamma \vdash k : \sigma} \qquad \frac{\Gamma \vdash M_1 : \sigma_1 \quad \ldots \quad \Gamma \vdash M_a : \sigma_a}{\Gamma \vdash f(M_1, \ldots, M_a) : \tau}$$

$$\frac{}{\Gamma \vdash \langle\rangle : \mathsf{unit}} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau}$$

$$\frac{\Gamma \vdash P : \sigma \times \tau}{\Gamma \vdash \mathsf{Fst}(P) : \sigma} \qquad \frac{\Gamma \vdash P : \sigma \times \tau}{\Gamma \vdash \mathsf{Snd}(P) : \tau}$$

$$\frac{\Gamma \vdash S : \mathsf{null}}{\Gamma \vdash \mathsf{Emp}_{\sigma}(S) : \sigma} \qquad \frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathsf{Inl}_{\tau}(M) : \sigma + \tau} \qquad \frac{\Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Inr}_{\sigma}(N) : \sigma + \tau}$$

$$\frac{\Gamma \vdash S : \sigma + \tau \quad \Gamma, x : \sigma \vdash E : \delta \quad \Gamma, y : \tau \vdash F : \delta}{\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F) : \delta}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \Rightarrow \tau} \qquad \frac{\Gamma \vdash F : \sigma \Rightarrow \tau \quad \Gamma \vdash A : \sigma}{\Gamma \vdash F A : \tau}$$

# Admissible Properties

Whenever $Sg \;\triangleright\; \Gamma \vdash M : \sigma$, we have $Sg \;\triangleright\; \pi\Gamma \vdash M : \sigma$.

We use rule induction. More precisely we prove

$$\forall\; Sg \;\triangleright\; \Gamma \vdash M : \sigma. \qquad \boxed{Sg \;\triangleright\; \pi\Gamma \vdash M : \sigma}$$

We give some examples of property closure.

$$\frac{\Gamma \vdash M_1 : \sigma_1 \quad \ldots \quad \Gamma \vdash M_a : \sigma_a}{\Gamma \vdash f(M_1, \ldots, M_a) : \sigma} \quad (f : \sigma_1, \ldots, \sigma_a \to \sigma)$$

(Property Closure for the inductive rule for function symbols): The inductive hypotheses are $Sg \ \triangleright \ \pi\Gamma \vdash M_i : \sigma_i$ for each $i$, that is, there is a derivation for each term-in-context. But now we can just apply an instance of the rule to these derivations to deduce that $Sg \ \triangleright \ \pi\Gamma \vdash f(M_1, \ldots, M_a) : \sigma$, as required.

# Theories

- A $\lambda\times+$-theory, $Th$, is a pair $(Sg, Ax)$ where $Ax$ is a collection of equations-in-context for $Sg$.

- An equation-in-context is a judgement $\Gamma \vdash M = M' : \sigma$ where $\Gamma \vdash M : \sigma$ and $\Gamma \vdash M' : \sigma$ are proved terms.

- The theorems of $Th$ consist of the judgements of the form $\Gamma \vdash M = M' : \sigma$ inductively generated by the rules on the following slides—it is a consequence of the rules that $Sg \ \triangleright \ \Gamma \vdash M : \sigma$ and $Sg \ \triangleright \ \Gamma \vdash M' : \sigma$.

$$\frac{Ax \ \triangleright \ \Gamma \vdash M = M' : \sigma}{\Gamma \vdash M = M' : \sigma}$$

$$\frac{\Gamma \vdash M = M' : \sigma}{\pi\Gamma \vdash M = M' : \sigma} \quad \text{(where } \pi \text{ is a permutation)}$$

$$\frac{\Gamma \vdash M = M' : \sigma}{\Gamma' \vdash M = M' : \sigma} \quad \text{(where } \Gamma \subseteq \Gamma'\text{)}$$

$$\frac{\Gamma, x : \sigma \vdash N = N' : \tau \quad \Gamma \vdash M = M' : \sigma}{\Gamma \vdash N[M/x] = N'[M'/x] : \tau}$$

plus rules to ensure that $=$ is an equivalence relation.

$$\frac{Sg \ \triangleright \ \Gamma \vdash M : \mathsf{unit}}{\Gamma \vdash M = \langle\rangle : \mathsf{unit}}$$

$$\frac{Sg \ \triangleright \ \Gamma \vdash M : \sigma \quad Sg \ \triangleright \ \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Fst}(\langle M, N\rangle) = M : \sigma} \qquad \frac{Sg \ \triangleright \ \Gamma \vdash M : \sigma \quad Sg \ \triangleright \ \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Snd}(\langle M, N\rangle) = N : \tau}$$

$$\frac{Sg \ \triangleright \ \Gamma \vdash P : \sigma \times \tau}{\Gamma \vdash \langle \mathsf{Fst}(P), \mathsf{Snd}(P)\rangle = P : \sigma \times \tau}$$

$$\frac{Sg \ \triangleright \ \Gamma \vdash S : \mathsf{null} \qquad Sg \ \triangleright \ \Gamma, x : \mathsf{null} \vdash M : \sigma}{\Gamma \vdash \mathsf{Emp}_{\sigma}(S) = M[S/x] : \sigma}$$

$$\frac{Sg \ \triangleright \ \Gamma \vdash M : \sigma \qquad Sg \ \triangleright \ \Gamma, x : \sigma \vdash E : \delta \qquad Sg \ \triangleright \ \Gamma, y : \tau \vdash F : \delta}{\Gamma \vdash \mathsf{Case}(\mathsf{Inl}_{\tau}(M), x.E \mid y.F) = E[M/x] : \delta}$$

$$\frac{Sg \ \triangleright \ \Gamma \vdash N : \tau \qquad Sg \ \triangleright \ \Gamma, x : \sigma \vdash E : \delta \qquad Sg \ \triangleright \ \Gamma, y : \tau \vdash F : \delta}{\Gamma \vdash \mathsf{Case}(\mathsf{Inr}_{\sigma}(N), x.E \mid y.F) = F[N/x] : \delta}$$

$$\frac{Sg \;\triangleright\; \Gamma \vdash S : \sigma + \tau \quad Sg \;\triangleright\; \Gamma, z : \sigma + \tau \vdash L : \delta}{\Gamma \vdash \mathsf{Case}(S, x.L[\mathsf{Inl}_\tau(x)/z] \mid y.L[\mathsf{Inr}_\sigma(y)/z]) = L[S/z] : \delta} \quad \text{(provided } x, y \notin fv(L))$$

$$\frac{\Gamma \vdash S = S' : \sigma + \tau \quad \Gamma, x : \sigma \vdash E = E' : \delta \quad \Gamma, y : \tau \vdash F = F' : \delta}{\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F) = \Gamma \vdash \mathsf{Case}(S', x.E' \mid y.F') : \delta}$$

$$\frac{Sg \ \triangleright \ \Gamma, x:\sigma \vdash M:\tau \quad Sg \ \triangleright \ \Gamma \vdash A:\sigma}{\Gamma \vdash (\lambda x:\sigma.M)A = M[A/x]:\tau}$$

$$\frac{Sg \ \triangleright \ \Gamma \vdash F:\sigma \Rightarrow \tau}{\Gamma \vdash \lambda x:\sigma.(Fx) = F:\sigma \Rightarrow \tau} \quad (\text{provided } x \notin fv(F))$$

$$\frac{\Gamma, x:\sigma \vdash M = M':\tau}{\Gamma \vdash \lambda x:\sigma.M = \lambda x:\sigma.M':\sigma \Rightarrow \tau}$$

# Deriving a Semantics For Proved Terms

- Suppose we model (or interpret) $\sigma$ and $\tau$ by "objects" $A$ and $B$. Let us model $x:\sigma \vdash M:\tau$ as a "relationship" $A \xrightarrow{m} B$.

- We first think about the process of substitution. Let

$$[\![x:\sigma \vdash M:\tau]\!] = A \xrightarrow{m} B \qquad [\![y:\tau \vdash N:\gamma]\!] = B \xrightarrow{n} C$$

Then

$$[\![x:\sigma \vdash N[M/y]:\gamma]\!] = A \xrightarrow{\square(n,m)} C$$

- Let $z:\gamma \vdash L:\delta$ be a further proved term. Note that we shall identify the semantics of the proved terms

$$x:\sigma \vdash (L[N/z])[M/y]:\delta \quad \text{and} \quad x:\sigma \vdash L[N[M/y]/z]:\delta$$

Thus

$$\Box(\Box(l,n),m) = \Box(l,\Box(n,m))$$

- We will have to model $x:\sigma \vdash x:\sigma$ as a relationship $A \xrightarrow{\star_A} A$. We can deduce that if $E \xrightarrow{e} A$, then $\Box(\star_A, e) = e$ because $x[E/x] = E$.

We summarise our deductions, writing $n \circ m$ for $\Box(n,m)$ and $id_A$ for $\star_A$, which amount to the definition of a category:

– Types are interpreted by “objects,” say $A$, $B\ldots$ and proved terms are interpreted by “relationships,” say $A \stackrel{m}{\longrightarrow} B\ldots$

– For each object $A$ there is a relationship $id_A$.

– Given relationships $A \stackrel{m}{\longrightarrow} B$ and $B \stackrel{n}{\longrightarrow} C$, there is a relationship $A \stackrel{n \circ m}{\longrightarrow} C$.

– Given relationships $E \stackrel{e}{\longrightarrow} A$ and $A \stackrel{m}{\longrightarrow} B$, then we have $id_A \circ e = e$ and $m \circ id_A = m$.

– For any $A \stackrel{m}{\longrightarrow} B$, $B \stackrel{n}{\longrightarrow} C$ and $C \stackrel{l}{\longrightarrow} D$, we have $l \circ (n \circ m) = (l \circ n) \circ m$.

# Summary

- We will model a proved term $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau$ in a category with finite products as a morphism of the form

$$[\![\Gamma \vdash M : \tau]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$$

where $\Gamma \stackrel{\text{def}}{=} x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and $[\![\Gamma]\!]$ stands for $[\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!]$.

- Substitution of terms will be modelled by categorical composition . . .

# Deriving a Semantics for Theories

- First we consider the types of $Sg$. We have to give an object $[\![\gamma]\!]$ of $\mathcal{C}$ to interpret each of the ground types $\gamma$, $[\![\mathsf{unit}]\!]$ to interpret $\mathsf{unit}$, and $[\![\mathsf{null}]\!]$ to interpret $\mathsf{null}$.
- We define $[\![\sigma \times \tau]\!] \stackrel{\text{def}}{=} [\![\sigma]\!] \Box [\![\tau]\!]$, etc
- We choose a morphism $[\![f]\!] : [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!] \to [\![\sigma]\!]$ in $\mathcal{C}$ for each function symbol.
- Recall that the interpretation of $\Gamma \vdash M : \sigma$ is given by $[\![\Gamma \vdash M : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$. By looking at how to soundly interpret the theorems of $Th$ we will deduce what the interpretation must be.

A typical rule looks like

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathsf{R}(M) : \tau} \quad (\mathsf{R})$$

Now suppose that $m \stackrel{\text{def}}{=} [\![\Gamma \vdash M : \sigma]\!]$ which is an element of $\mathcal{C}([\![\Gamma]\!], [\![\sigma]\!])$. How do we model $[\![\Gamma \vdash \mathsf{R}(M) : \tau]\!] \in \mathcal{C}([\![\Gamma]\!], [\![\tau]\!])$? All we can say at the moment is that this will depend on $m$, and we can model this idea by having a function

$$\Phi : \mathcal{C}([\![\Gamma]\!], [\![\sigma]\!]) \longrightarrow \mathcal{C}([\![\Gamma]\!], [\![\tau]\!])$$

and setting $[\![\Gamma \vdash \mathsf{R}(M) : \tau]\!] \stackrel{\text{def}}{=} \Phi(m)$.

Suppose that $x:\gamma \vdash M:\sigma$ and $y:\gamma' \vdash N:\gamma$ are any two given proved terms. If $m \stackrel{\text{def}}{=} [\![x:\gamma \vdash M:\sigma]\!]$ and $n \stackrel{\text{def}}{=} [\![y:\gamma' \vdash N:\gamma]\!]$ then $[\![y:\gamma' \vdash M[N/x]:\sigma]\!] = m \circ n$. Note that there are (definitionally) equal proved terms

$$y:\gamma' \vdash \mathsf{R}(M)[N/x]:\tau \qquad \text{and} \qquad y:\gamma' \vdash \mathsf{R}(M[N/x]):\tau.$$

and so

$$\Phi(m) \circ n = \Phi(m \circ n). \qquad (*)$$

$(*)$ will hold if there are natural transformations

$$\Phi : \mathcal{C}(-,A) \longrightarrow \mathcal{C}(-,B) : \mathcal{C}^{op} \longrightarrow \mathcal{S}et.$$

Recall that the rule for introducing product terms is

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash \langle M, N \rangle : \sigma \times \tau}$$

In order to soundly interpret this rule we shall need a natural transformation

$$\Phi : \mathcal{C}(-, A) \times \mathcal{C}(-, B) \longrightarrow \mathcal{C}(-, A \Box B)$$

for all objects $A$ and $B$ of $\mathcal{C}$.

Now let $m : C \to A$ and $n : C \to B$ be morphisms of $\mathcal{C}$. Applying naturality in $C$ at the morphism $\langle m, n \rangle : C \to A \times B$ we deduce

$$\Phi_C(\pi_A \circ \langle m, n \rangle, \pi_B \circ \langle m, n \rangle) = \Phi_{A \times B}(\pi_A, \pi_B) \circ \langle m, n \rangle,$$

that is $\Phi_C(m, n) = \Phi_{A \times B}(\pi_A, \pi_B) \circ \langle m, n \rangle$. Now let us define the morphism $q_{A,B} : A \times B \to A \Box B$ to be $\Phi_{A \times B}(\pi_A, \pi_B)$. Then we can make the definition

$$[\![\Gamma \vdash \langle M, N \rangle : A \times B]\!] \stackrel{\text{def}}{=}$$

$$[\![\Gamma]\!] \xrightarrow{\langle [\![\Gamma \vdash M : \sigma]\!], [\![\Gamma \vdash N : \sigma]\!] \rangle} [\![\sigma]\!] \times [\![\tau]\!] \xrightarrow{q_{[\![\sigma]\!],[\![\tau]\!]}} [\![\sigma]\!] \Box [\![\tau]\!].$$

$$\frac{\Gamma \vdash H : \sigma \times \tau}{\Gamma \vdash \mathsf{Fst}(H) : \sigma}$$

To model this rule we shall need a natural transformation $\Phi : \mathcal{C}(-, A \Box B) \longrightarrow \mathcal{C}(-, A)$. Using the Yoneda lemma (see notes), the components of $\Phi$ are given by $\theta \mapsto p \circ \theta$ for some $p : A \Box B \to A$. So now we can define

$$[\![\Gamma \vdash \mathsf{Fst}(H) : \sigma]\!] \stackrel{\text{def}}{=} [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash H : \sigma \times \tau]\!]} [\![\sigma]\!] \Box [\![\tau]\!] \xrightarrow{p_{[\![\sigma]\!],[\![\tau]\!]}} [\![\sigma]\!].$$

Now we think about the equations

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Fst}(\langle M, N \rangle) = M : \sigma} \quad (1)$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{Snd}(\langle M, N \rangle) = N : \sigma} \quad (2)$$

$$\frac{\Gamma \vdash H : \sigma \times \tau}{\Gamma \vdash \langle \mathsf{Fst}(H), \mathsf{Snd}(H) \rangle = H : \sigma \times \tau} \quad (3)$$

If we put $h \stackrel{\text{def}}{=} [\![\Gamma \vdash H : \sigma \times \tau]\!] : C \to A \Box B$, $m \stackrel{\text{def}}{=} [\![\Gamma \vdash M : \sigma]\!] : C \to A$ and $n \stackrel{\text{def}}{=} [\![\Gamma \vdash N : \tau]\!] : C \to B$, and our categorical interpretation satisfies the equations-in-context, this forces

$$p_{A,B} \circ q_{A,B} \circ \langle m, n \rangle = m \quad (1)$$

$$p'_{A,B} \circ q_{A,B} \circ \langle m, n \rangle = n \quad (2)$$

$$q_{A,B} \circ \langle p_{A,B} \circ h, p'_{A,B} \circ h \rangle = h \quad (3)$$

These equations imply that, up to isomorphism, $A \Box B$ and $A \times B$ are the same. Thus we may soundly interpret binary product types by binary categorical product.

To soundly interpret the rule

$$\frac{\Gamma \vdash S : \mathsf{null}}{\Gamma \vdash \mathsf{Emp}_\sigma(S) : \sigma}$$

we shall need a natural transformation $\Phi : \mathcal{C}(-, N) \longrightarrow \mathcal{C}(-, A)$, where $N = [\![\mathsf{null}]\!]$. The Yoneda Lemma tells us that the components of $\Phi$ are given by $\theta \mapsto n_A \circ \theta$ where $n_A : N \to A$ is a morphism, one for each $A$. So now we can define

$$[\![\Gamma \vdash \mathsf{Emp}_\sigma(S) : \sigma]\!] \stackrel{\text{def}}{=} [\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash S : \mathsf{null}]\!]} N \xrightarrow{n_{[\![\sigma]\!]}} [\![\sigma]\!].$$

If we write $s \stackrel{\text{def}}{=} [\![\Gamma \vdash S : \mathsf{null}]\!] : C \longrightarrow N$, and $m \stackrel{\text{def}}{=} [\![\Gamma, x : \mathsf{null} \vdash M : \sigma]\!] : C \times N \longrightarrow A$ then

$$\Gamma \vdash \mathsf{Emp}_\sigma(S) = M[S/x] : \sigma$$

will be soundly modelled providing that

$$n_A \circ s = m \circ \langle id_C, s \rangle \qquad (\dagger)$$

holds for any such morphisms. Suppose that $t : N \longrightarrow A$. Taking $s$ to be $id_N$ and $m$ to be $t \circ \pi_N$, then

$$n_A = t \circ \pi_N \circ \langle id_N, id_N \rangle = t$$

Thus $N$ is an initial object in the category $\mathcal{C}$. (In fact $(\dagger)$ forces $N$ to be distributive, that is $\pi_N : C \times N \longrightarrow N$ is an isomorphism for every $C$.)

# Formal Semantics of Proved Terms

Let $\mathcal{C}$ be a BCC. Then a structure, **M**, for some $Sg$ in $\mathcal{C}$ is specified by:

- For every ground type $\gamma$ an object $[\![\gamma]\!]$ of $\mathcal{C}$,
- for every function symbol $f : \sigma_1 \ldots \sigma_n \to \tau$ a morphism $[\![f]\!] : [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!] \to [\![\tau]\!]$, where we define $[\![\sigma]\!]$ by recursion, setting $[\![\mathsf{unit}]\!] \stackrel{\text{def}}{=} 1$, $[\![\sigma \times \tau]\!] \stackrel{\text{def}}{=} [\![\sigma]\!] \times [\![\tau]\!]$ etc.

Then for every proved term $\Gamma \vdash M : \sigma$ we specify a morphism

$$[\![\Gamma \vdash M : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$$

by recursion.

$$\frac{}{[\![\Gamma, x : \sigma, \Gamma' \vdash x : \sigma]\!] \stackrel{\text{def}}{=} \pi : [\![\Gamma]\!] \times [\![\sigma]\!] \times [\![\Gamma']\!] \longrightarrow [\![\sigma]\!]}$$

$$\frac{}{[\![\Gamma \vdash k : \sigma]\!] \stackrel{\text{def}}{=} [\![k]\!] \circ ! : [\![\Gamma]\!] \longrightarrow 1 \longrightarrow [\![\sigma]\!]} \quad (k : \sigma)$$

$$\frac{[\![\Gamma \vdash M_1 : \sigma_1]\!] = m_1 : [\![\Gamma]\!] \longrightarrow [\![\sigma_1]\!] \quad \ldots}{[\![\Gamma \vdash f(\vec{M}) : \tau]\!] = [\![f]\!] \circ \langle m_1, \ldots, m_n \rangle : [\![\Gamma]\!] \longrightarrow ([\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!]) \longrightarrow [\![\tau]\!]}$$

$$\frac{}{[\![\Gamma \vdash \langle\rangle : \mathsf{unit}]\!] \stackrel{\text{def}}{=} ! : [\![\Gamma]\!] \to 1}$$ (where 1 is the terminal object of $\mathcal{C}$)

$$\frac{[\![\Gamma \vdash P : \sigma \times \tau]\!] = p : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!])}{[\![\Gamma \vdash \mathsf{Fst}(P) : \sigma]\!] = \pi_1 \circ p : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!]) \to [\![\sigma]\!]}$$

$$\frac{[\![\Gamma \vdash P : \sigma \times \tau]\!] = p : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!])}{[\![\Gamma \vdash \mathsf{Snd}(P) : \tau]\!] = \pi_2 \circ p : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!]) \to [\![\tau]\!]}$$

$$\frac{[\![\Gamma \vdash M : \sigma]\!] = m : [\![\Gamma]\!] \to [\![\sigma]\!] \quad [\![\Gamma \vdash N : \tau]\!] = n : [\![\Gamma]\!] \to [\![\tau]\!]}{[\![\Gamma \vdash \langle M, N\rangle : \sigma \times \tau]\!] = \langle m, n\rangle : [\![\Gamma]\!] \to ([\![\sigma]\!] \times [\![\tau]\!])}$$

$$\frac{[\![\Gamma \vdash S : \mathsf{null}]\!] = s : [\![\Gamma]\!] \longrightarrow 0}{[\![\Gamma \vdash \mathsf{Emp}_\sigma(S) : \sigma]\!] = \,!\circ \cong \circ\langle id_{[\![\Gamma]\!]}, s\rangle : [\![\Gamma]\!] \to [\![\Gamma]\!] \times 0 \cong 0 \to [\![\sigma]\!]}$$

(where 0 is the initial object of $\mathcal{C}$)

$$\frac{[\![\Gamma \vdash M : \sigma]\!] = m : [\![\Gamma]\!] \to [\![\sigma]\!]}{[\![\Gamma \vdash \mathsf{Inl}_\tau(M) : \sigma + \tau]\!] = i \circ m : [\![\Gamma]\!] \to [\![\sigma]\!] \to [\![\sigma]\!] + [\![\tau]\!]}$$

$$\frac{[\![\Gamma \vdash N : \tau]\!] = n : [\![\Gamma]\!] \to [\![\tau]\!]}{[\![\Gamma \vdash \mathsf{Inr}_\sigma(N) : \sigma + \tau]\!] = j \circ n : [\![\Gamma]\!] \to [\![\tau]\!] \to [\![\sigma]\!] + [\![\tau]\!]}$$

$$\frac{\left\{\begin{array}{l} [\![\Gamma \vdash S : \sigma + \tau]\!] = s : [\![\Gamma]\!] \to [\![\sigma]\!] + [\![\tau]\!] \\ [\![\Gamma, x : \sigma \vdash E : \delta]\!] = e : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\delta]\!] \\ [\![\Gamma, y : \sigma \vdash F : \delta]\!] = f : [\![\Gamma]\!] \times [\![\tau]\!] \to [\![\delta]\!] \end{array}\right.}{\begin{array}{l} [\![\Gamma \vdash \mathsf{Case}(S, x.E \mid y.F) : \delta]\!] = \\ [e, f] \circ \cong \circ \langle id_{[\![\Gamma]\!]}, s\rangle : [\![\Gamma]\!] \to [\![\Gamma]\!] \times ([\![\sigma]\!] + [\![\tau]\!]) \\ \qquad \cong ([\![\Gamma]\!] \times [\![\sigma]\!]) + ([\![\Gamma]\!] \times [\![\tau]\!]) \to [\![\delta]\!] \end{array}}$$

$$\frac{[\![\Gamma, x:\sigma \vdash M:\tau]\!] = m : [\![\Gamma]\!] \times [\![\sigma]\!] \to [\![\tau]\!]}{[\![\Gamma \vdash \lambda x:\sigma.M : \sigma \Rightarrow \tau]\!] = \lambda(m) : [\![\Gamma]\!] \to [\![\sigma]\!] \Rightarrow [\![\tau]\!]}$$

$$\frac{[\![\Gamma \vdash F : \sigma \Rightarrow \tau]\!] = f : [\![\Gamma]\!] \to ([\![\sigma]\!] \Rightarrow [\![\tau]\!]) \quad [\![\Gamma \vdash A : \sigma]\!] = a : [\![\Gamma]\!] \to [\![\sigma]\!]}{[\![\Gamma \vdash F\, A : \tau]\!] \stackrel{\text{def}}{=} ev \circ \langle f, a \rangle : [\![\Gamma]\!] \to ([\![\sigma]\!] \Rightarrow [\![\tau]\!]) \times [\![\sigma]\!] \to [\![\tau]\!]}$$

# Modelling Composition

Let $\Gamma' \vdash N : \tau$ be a proved term where $\Gamma' = [x_1 : \sigma_1, x_2 : \sigma_2]$ and let $\Gamma \vdash M_i : \sigma_i$ be proved terms for $i = 1,2$. Then one can show that $\Gamma \vdash N[M_1, M_2/x_1, x_2] : \tau$ and

$$[\![\Gamma \vdash N[M_1, M_2/x_1, x_2] : \tau]\!] = [\![\Gamma' \vdash N : \tau]\!] \circ \langle [\![\Gamma \vdash M_1 : \sigma_1]\!], [\![\Gamma \vdash M_2 : \sigma_2]\!] \rangle$$

**Proof:** By rule induction on the derivation of the judgement $\Gamma' \vdash N : \tau$.

# Soundness

Let **M** be a structure for a $\lambda\times+$-signature in a bicartesian closed category $\mathcal{C}$. **M** satisfies the equation-in-context $\Gamma\vdash M=M':\sigma$ if $[\![\Gamma\vdash M:\sigma]\!]$ and $[\![\Gamma\vdash M':\sigma]\!]$ are equal. We say that **M** is a model of a $\lambda\times+$-theory $Th=(Sg,Ax)$ if **M** satisfies the axioms.

Then **M** satisfies any equation-in-context which is a theorem of $Th$.

**Proof:** This can be shown by rule induction using the rules for deriving theorems.

Let

$$m \stackrel{\text{def}}{=} [\![\Gamma, x:\sigma \vdash M:\tau]\!] : [\![\Gamma]\!] \times [\![\sigma]\!] \rightarrow [\![\tau]\!]$$

and $a \stackrel{\text{def}}{=} [\![\Gamma \vdash A:\sigma]\!] : [\![\Gamma]\!] \rightarrow [\![\sigma]\!]$. Then we have

(Property Closure for the (base) rule):

$$\frac{Sg \;\rhd\; \Gamma, x:\sigma \vdash M:\tau \quad Sg \;\rhd\; \Gamma \vdash A:\sigma}{\Gamma \vdash (\lambda x:\sigma.M)A = M[A/x]:\tau}$$

$$\begin{aligned}
[\![\Gamma \vdash (\lambda x:\sigma.M)A:\tau]\!] &= ev \circ \langle [\![\Gamma \vdash \lambda x:\sigma.M:\tau]\!], [\![\Gamma \vdash A:\sigma]\!] \rangle \\
&= ev \circ \langle \lambda(m), a \rangle \\
&= ev \circ (\lambda(m) \times id) \circ \langle id, a \rangle \\
&= m \circ \langle id, a \rangle \\
&= [\![\Gamma \vdash M[A/x]:\tau]\!]
\end{aligned}$$

# Transporting Models

Suppose that we are given a morphism of bicartesian closed categories $F : \mathcal{C} \to \mathcal{D}$. Let $\mathbf{M}$ be a model of $Th$ in $\mathcal{C}$. We shall show how to define a new model, of $Th$ in $\mathcal{D}$, denoted by $F_*\mathbf{M}$. We shall need a lemma, that may be proved by induction over types:

If we set $[\![\gamma]\!]_{F_*\mathbf{M}} \stackrel{\text{def}}{=} F[\![\gamma]\!]_{\mathbf{M}}$ where $\gamma$ is a ground type of $Th$, then it follows from this that there is a canonical isomorphism $[\![\sigma]\!]_{F_*\mathbf{M}} \cong F[\![\sigma]\!]_{\mathbf{M}}$ where $\sigma$ is any type of $Th$.

A structure $F_*\mathbf{M}$ is given by $[\![\gamma]\!]_{F_*\mathbf{M}} \stackrel{\text{def}}{=} F[\![\gamma]\!]_{\mathbf{M}}$ on ground types and $[\![f]\!]_{F_*\mathbf{M}}$ is given by the composition

$$[\![\sigma_1]\!]_{F_*\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{F_*\mathbf{M}} \cong F[\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times F[\![\sigma_n]\!]_{\mathbf{M}} \cong' \\ F([\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{\mathbf{M}}) \xrightarrow{F[\![f]\!]_{\mathbf{M}}} F[\![\tau]\!]_{\mathbf{M}} \cong [\![\tau]\!]_{F_*\mathbf{M}}$$

where $f : \sigma_1, \ldots, \sigma_n \to \tau$ is a function symbol of *Th*, the isomorphims $\cong$ exist because of the lemma, and $\cong'$ arises from $F$ preserving finite products.

In fact $F_*\mathbf{M}$ is a model of $Th$.

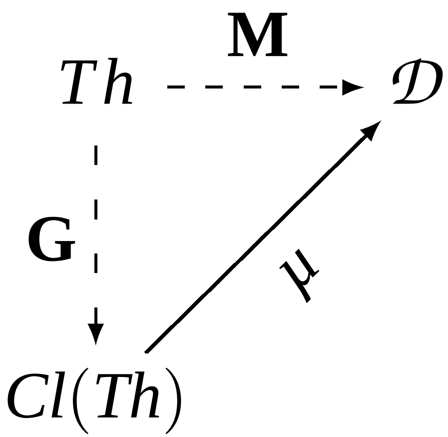
Given a proved term $\Gamma \vdash M : \sigma$ one can show by induction that the morphism $[\![\Gamma \vdash M : \sigma]\!]_{F_*\mathbf{M}}$ is given by the composition

$$[\![\sigma_1]\!]_{F_*\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{F_*\mathbf{M}} \cong F([\![\sigma_1]\!]_{\mathbf{M}} \times \ldots \times [\![\sigma_n]\!]_{\mathbf{M}}) \stackrel{F[\![\Gamma \vdash M:\sigma]\!]_{\mathbf{M}}}{\longrightarrow} F[\![\sigma]\!]_{\mathbf{M}}.$$

If we are given proved terms $\Gamma \vdash M : \sigma$ and $\Gamma \vdash N : \sigma$ for which $[\![\Gamma \vdash M : \sigma]\!]_{\mathbf{M}} = [\![\Gamma \vdash N : \sigma]\!]_{\mathbf{M}}$ then certainly $[\![\Gamma \vdash M : \sigma]\!]_{F_*\mathbf{M}} = [\![\Gamma \vdash N : \sigma]\!]_{F_*\mathbf{M}}$. Thus if $\mathbf{M}$ is a model of $Th$ in $\mathcal{C}$ then $F_*\mathbf{M}$ is a model of $Th$ in $\mathcal{D}$.

# Classifying Categories

Let $Th$ be a $\lambda \times +$-theory. A bicartesian closed category $Cl(Th)$ is called the classifying category of $Th$ if there is a model $\mathbf{G}$ of $Th$ in $Cl(Th)$ for which given any category $\mathcal{D}$ with finite products, and a model $\mathbf{M}$ of $Th$ in $\mathcal{D}$, then there is a functor $\mu : Cl(Th) \to \mathcal{D}$ such that




where $\mu_* \mathbf{G} = \mathbf{M}$.

# Constructing Classifiers

Every $\lambda\times+$-theory *Th* has a classifying category $Cl(Th)$. We can construct a canonical classifying category using the syntax of *Th*.

**Proof:**

- The objects of $Cl(Th)$ are the types of *Th*.
- A morphism $\sigma\to\tau$ is an equivalence class $(x:\sigma\mid M)$ of pairs $(x:\sigma, M)$ where $Sg\ \triangleright\ x:\sigma\vdash M:\tau$, with equivalence relation

$$(x:\sigma, M)\sim(x':\sigma, M') \qquad \text{iff} \qquad Th\ \triangleright\ x:\sigma\vdash M=M'[x/x']:\tau.$$

- Given $\sigma$ and $\tau$, the binary product is $\sigma \times \tau$ with projection $\pi_\sigma : \sigma \times \tau \to \sigma$ given by $(z : \sigma \times \tau \mid \mathsf{Fst}(z))$. If $(x : \gamma \mid M) : \gamma \to \sigma$ and $(y : \gamma \mid N) : \gamma \to \tau$, then the mediating morphism is

$$(z : \gamma \mid \langle M[z/x], N[z/y] \rangle) : \gamma \to \sigma \times \tau.$$

- $(x : \sigma \mid \langle\rangle)$ is the unique morphism $\sigma \to \mathsf{unit}$ so that $\mathsf{unit}$ is a terminal object for $Cl(Th)$.

- $(x : \mathsf{null} \mid \mathsf{Emp}_\sigma(x))$ is the unique morphism $\mathsf{null} \to \sigma$ so that $\mathsf{null}$ is a terminal object for $Cl(Th)$.

- We define a structure $\mathbf{G}$ for $Sg$ in $Cl(Th)$. $[\![\gamma]\!]_{\mathbf{G}} \stackrel{\text{def}}{=} \gamma$ (and hence it follows that $[\![\sigma]\!]_{\mathbf{G}} = \sigma$ for any type $\sigma$).

- Also define for $f : \sigma_1, \sigma_2 \to \tau$

$$[\![f]\!]_{\mathbf{G}} \stackrel{\text{def}}{=} (z : \sigma_1 \times \sigma_2 \mid f(\mathsf{Fst}(z), \mathsf{Snd}(z)))$$

Certainly we have

$$Sg \;\rhd\; z : \sigma_1 \times \sigma_2 \vdash f(\mathsf{Fst}(z), \mathsf{Snd}(z)) : \tau$$

- If $k : \sigma$ then $[\![k]\!]_{\mathbf{G}} \stackrel{\text{def}}{=} (x : \mathsf{unit} \mid k)$.

We check that **G** is indeed a model of $Th = (Sg, Ax)$. Suppose that $Sg \rhd x:\sigma, y:\tau \vdash M:\rho$. Then we can prove by induction that

$$[\![x:\sigma, y:\tau \vdash M:\rho]\!]_{\mathbf{G}} = (z:\sigma\times\tau \mid M[\mathsf{Fst}(z)/x, \mathsf{Snd}(z)/y])$$

Now, if we have $Th \rhd x:\sigma, y:\tau \vdash M = M':\rho$, then

$$Th \rhd z:\sigma\times\tau \vdash M[\mathsf{Fst}(z)/x, \mathsf{Snd}(z)/y] = M'[\mathsf{Fst}(z)/x, \mathsf{Snd}(z)/y]:\rho$$

and hence that $[\![x:\sigma, y:\tau \vdash M:\rho]\!]_{\mathbf{G}} = [\![x:\sigma, y:\tau \vdash M':\rho]\!]_{\mathbf{G}}$.

- Now let **M** be a model of *Th* in $\mathcal{D}$. We define $\mu : Cl(Th) \to \mathcal{D}$ by

$$(x : \sigma \mid M) : \sigma \longrightarrow \tau \quad \longmapsto \quad [\![x : \sigma \vdash M : \tau]\!]_{\mathbf{M}} : [\![\sigma]\!]_{\mathbf{M}} \longrightarrow [\![\tau]\!]_{\mathbf{M}}$$

The soundness theorem says that the definition makes sense. It is easy to see that $\mu$ is a bicartesian closed functor.

It is routine to verify that $\mu_*\mathbf{G} = \mathbf{M}$. For example, consider a function symbol $f : \sigma_1, \sigma_2 \to \tau$. Then

$$
\begin{aligned}
[\![f]\!]_{\mu_*\mathbf{G}} &= \mu(z : \sigma_1 \times \sigma_2 \mid f(\mathsf{Proj}_1(z), \mathsf{Proj}_2(z))) \\
&= [\![z : \sigma_1 \times \sigma_2 \vdash f(\mathsf{Proj}_1(z), \mathsf{Proj}_2(z)) : \tau]\!]_{\mathbf{M}} \\
&= [\![f]\!]_{\mathbf{M}} \circ \langle \pi, \pi' \rangle \\
&= [\![f]\!]_{\mathbf{M}}.
\end{aligned}
$$

Suppose that there is another bicartesian closed functor $\mu' : Cl(Th) \to \mathcal{D}$ for which $\mu'_* \mathbf{G} = \mathbf{M}$. If $\sigma$ is an object of $Cl(Th)$ then

$$\mu\sigma \stackrel{\text{def}}{=} [\![\sigma]\!]_{\mathbf{M}} = [\![\sigma]\!]_{\mu'_* \mathbf{G}} \cong \mu' [\![\sigma]\!]_{\mathbf{G}} = \mu'\sigma$$

using a previous lemma that establishes the isomorphism, and this gives rise to a natural isomorphism $\mu \cong \mu'$.

## Some Applications

- We show that by starting with a very simple type theory, the expressive power (in a sense to be made precise) is not increased by adding products, sums and functions. This is proved by establishing an equivalent categorical problem, and solving it using categorical methods.

# Algebraic Theories

An algebraic theory is a $\lambda \times +$-theory in which there are no product, sum, and function types. More precisely, an algebraic theory $Th = (Sg, Ax)$ consists of

- a collection of types and function symbols;
- raw terms generated from these data, using only the rules

$$\frac{}{x} \qquad \frac{}{k} \qquad \frac{M_1 \quad \dots \quad M_a}{f(M_1, \dots, M_a)}$$

- proved terms, generated as expected; and
- theorems, generated by the rules of equality.

# Classifiers for Algebraic Theories

Every algebraic theory *Th* has a classifying theory $Cl(Th)$.

- The objects of $Cl(Th)$ are finite lists of types from the algebraic signature *Sg* of *Th*, for example $\vec{\sigma} \stackrel{\text{def}}{=} [\sigma_1, \ldots, \sigma_n]$.
- The morphisms with source $\vec{\sigma}$ and target $\vec{\tau}$, where $\tau \stackrel{\text{def}}{=} [\tau_1, \ldots, \tau_m]$ and both $\vec{\sigma}$ and $\vec{\tau}$ are non-empty lists, are given by finite lists of the form

$$[(\Gamma \mid M_1), \ldots, (\Gamma \mid M_m)] : \vec{\sigma} \to \vec{\tau}$$

where the types $\vec{\sigma}$ appear in $\Gamma$ and we have $Sg \;\triangleright\; \Gamma \vdash M_j : \tau_j$ for $1 \leq j \leq m$.

# A Conservative Extension

Let $Th = (Sg, Ax)$ be an algebraic theory. Let $Th' = (Sg', Ax')$ be the $\lambda\times+$-theory with ground types and function symbols those of $Sg$, and $Ax' \stackrel{\text{def}}{=} Ax$. Let $\Gamma \stackrel{\text{def}}{=} [x_1 : \gamma_1, \ldots, x_n : \gamma_n]$. Suppose that

$$Sg' \;\rhd\; [x_1 : \gamma_1, \ldots, x_n : \gamma_n] \vdash E : \gamma$$

Then there exists $M$ for which

$$Sg \;\rhd\; \Gamma \vdash M : \gamma \qquad \text{and} \qquad Th' \;\rhd\; \Gamma \vdash E = M : \gamma.$$

Moreover, if there is $M'$ for which $Sg \;\rhd\; \Gamma \vdash M' : \gamma$ and also $Th' \;\rhd\; \Gamma \vdash E = M' : \gamma$ then we have $Th \;\rhd\; \Gamma \vdash M = M' : \gamma$.

# Free Bicartesian Closed Cateories

Let $\mathcal{C}$ be a category with finite products. Then $\mathcal{F}\mathcal{C}$ is the relatively free BCCC generated by $\mathcal{C}$ if there is a finite product preserving functor $I : \mathcal{C} \to \mathcal{F}\mathcal{C}$ such that if $F : \mathcal{C} \to \mathcal{D}$ is finite product preserving and $\mathcal{D}$ is a BCCC then there is a BCCC functor $\overline{F} : \mathcal{F}\mathcal{C} \to \mathcal{D}$ for which $\phi : \overline{F}I \cong F$ and $\overline{F}$ is unique up to isomorphism.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{I} & \mathcal{F}\mathcal{C} \\ & \searrow^{F} & \downarrow{\overline{F}} \\ & & \mathcal{D} \end{array}$$

## Relating $Th$ and $Th'$ Categorically

We define a functor $I : Cl(Th) \to Cl(Th')$. Very roughly, if

$$(x : \gamma \mid M)_{Th} : \gamma \to \gamma'$$

then we set

$$I(x : \gamma \mid M)_{Th} \stackrel{\text{def}}{=} (x : \gamma \mid M)_{Th'}$$

Warning: the objects of $Cl(Th)$ are in fact lists of types (in the example above the source $\gamma$ and target $\gamma'$ are lists of length one) and the precise definition of $I$ is rather messy . . . .

On an object $\vec{\gamma}$ of $Cl(Th)$ set

$$I(\vec{\gamma}) \stackrel{\text{def}}{=} (\ldots(\gamma_1 \times \gamma_2) \times \ldots) \times \gamma_n$$

and given a morphism $(\Gamma \mid \vec{M})_{Th} : \vec{\gamma} \to \vec{\gamma}'$ (where the subscript $Th$ denotes equivalence up to provable equality in $Th$), then we set

$$I(\Gamma \mid \vec{M})_{Th} \stackrel{\text{def}}{=} (z : \Pi\gamma_i \mid \langle \ldots \langle \widehat{M_1}, \widehat{M_2} \rangle, \ldots, \widehat{M_m} \rangle)_{Th'}$$

in which we have written $\Pi\gamma_i$ for $(\ldots(\gamma_1 \times \gamma_2) \times \ldots) \times \gamma_n$ and also

$$\widehat{M_j} \stackrel{\text{def}}{=} M_j[\mathsf{Proj}_1(z)/x_1, \ldots, \mathsf{Proj}_j(z)/x_j, \ldots, \mathsf{Proj}_n(z)/x_n]$$

where $\mathsf{Proj}_j(z)$ is a term for $j$-th projection.

# Full and Faithful Functors

- $F : \mathcal{C} \to \mathcal{D}$ is faithful if given a parallel pair of morphisms $f, g : A \to B$ in $\mathcal{C}$ for which $Ff = Fg$, then $f = g$. Thus

$$\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$$

is 1-1.

- $F$ is full if given objects $A$ and $B$ in $\mathcal{C}$ and a morphism $g : FA \to FB$ in $\mathcal{D}$, then there is some $f : A \to B$ in $\mathcal{C}$ for which $Ff = g$. Thus

$$\mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$$

is onto.

# Outlining a Proof of the Con. Extension

1. Show that $I : Cl(Th) \to Cl(Th')$ yields a free BCCC.
2. Prove a purely categorical result called the "logical relations" gluing lemma.
3. Apply the gluing lemma and the free BCCC property, to show that $I$ is full and faithful ...

$$Cl(Th)(\gamma,\gamma') \xrightarrow{\cong} Cl(Th')(I\gamma,I\gamma')$$

**Existence:** Suppose that $Sg' \ \triangleright \ x:\gamma \vdash E:\gamma'$. Then we certainly have

$$e \stackrel{\text{def}}{=} (x:\gamma \mid E)_{Th'} : I\gamma \longrightarrow I\gamma'$$

in $Cl(Th')$. Using the fullness of $I$, there is a morphism $(x:\gamma \mid M)_{Th} : \gamma \longrightarrow \gamma'$ which is taken to $e$ by $I$. But this implies

$$Th' \ \triangleright \ x:\gamma \vdash M = E:\gamma'$$

as required.

# A Free BCCC

The functor $I : Cl(Th) \to Cl(Th')$ presents $Cl(Th')$ as the relatively free BCC generated by $Cl(Th)$.

**Proof:** Let $F : Cl(Th) \to \mathcal{C}$ preserve finite products where $\mathcal{C}$ is a BCCC. We shall define a functor $\overline{F} : Cl(Th') \to \mathcal{C}$ by recursion over the syntactic structure of $Cl(Th')$. For example

- $\overline{F}\gamma \stackrel{\text{def}}{=} F[\gamma]$ where $\gamma$ is a ground type of $Sg'$,
- $\overline{F}(\sigma \times \tau) \stackrel{\text{def}}{=} \overline{F}\sigma \times \overline{F}\tau$,
- $\overline{F}(z : \delta \mid \langle\rangle) \stackrel{\text{def}}{=} ! : \overline{F}\delta \to 1_{\mathcal{C}}$,
- $\overline{F}(z : \delta \mid \mathsf{Fst}(P)) \stackrel{\text{def}}{=} \pi_1 \overline{F}(z : \delta \mid P)$ where $\pi_1 : \overline{F}\sigma \times \overline{F}\tau \to \overline{F}\sigma$,

# Gluing Lemma by Logical Relations

Let $\mathcal{D}$ be a BCC and let $I : \mathcal{C} \to \mathcal{D}$ preserve finite products. We define a category $\mathcal{G}l$ as follows:

- Objects of $\mathcal{G}l$ are $(F, \lhd, D)$ where $F : \mathcal{C}^{op} \to \mathcal{S}et$ is a functor, $D$ is an object of $\mathcal{D}$, and for each object $C$ of $\mathcal{C}$, $\lhd_C \subseteq FC \times \mathcal{D}(IC, D)$.
- A morphism $(\alpha, d) : (F, \lhd, D) \to (F', \lhd', D')$ is given by a natural transformation $\alpha : F \to F'$ and a morphism $d : D \to D'$ in $\mathcal{D}$ for which if $x \lhd_C u$ then $\alpha_C(x) \lhd'_C d \circ u$, where of course $x \in FC$ and $u \in \mathcal{D}(IC, D)$.

Then $\mathcal{G}l$ is a bicartesian closed category and the obvious functor $\pi_2 : \mathcal{G}l \to \mathcal{D}$ is a morphism of BCCCs.

**Proof** The structure of $\mathcal{G}l$ is specified by a “logical relations” procedure on the subset $\lhd_C$.

(*Binary Products*): We set

$$(F, \lhd, D) \times (F', \lhd', D') \stackrel{\text{def}}{=} (F \times F', \lhd \times \lhd', D \times D')$$

where $(x, x')(\lhd \times \lhd')_C u$ just in case $x \lhd_C \pi u$ and $x' \lhd'_C \pi' u$ where of course $\pi : D \times D' \to D$ and $\pi' : D \times D' \to D'$ in $\mathcal{D}$. The projections in $\mathcal{G}l$ are given by pairing of projections in $[\mathcal{C}^{op}, \mathcal{S}et]$ and $\mathcal{D}$, such as:

$$\pi_{(F,\lhd,D)} \stackrel{\text{def}}{=} (\pi_F, \pi_D) : (F \times F', \lhd \times \lhd', D \times D') \longrightarrow (F, \lhd, D).$$

# Freeness Implies Full and Faithful

Let $\mathcal{C}$ be a locally small category, and $\mathcal{F}\mathcal{C}$ the freely generated bicartesian closed category. Then the canonical functor $I : \mathcal{C} \to \mathcal{F}\mathcal{C}$ is full and faithful.

**Proof** We apply the gluing lemma to $I$ (so $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{F}\mathcal{C}$). We define a functor $J : \mathcal{C} \to \mathcal{G}l$: on objects $C$ of $\mathcal{C}$ define $JC$ by $(\mathcal{C}(-,C), \lhd^{C}, IC)$ where the subset

$$\lhd^{C}_{C'} \subseteq \mathcal{C}(C',C) \times \mathcal{F}\mathcal{C}(IC',IC)$$

is defined by just requiring $c \lhd^{C}_{C'} Ic$ for each morphism $c : C' \to C$ in $\mathcal{C}$. On morphisms $c$ of $\mathcal{C}$ we set $Jc \stackrel{\text{def}}{=} (\mathcal{C}(-,c), Ic)$.

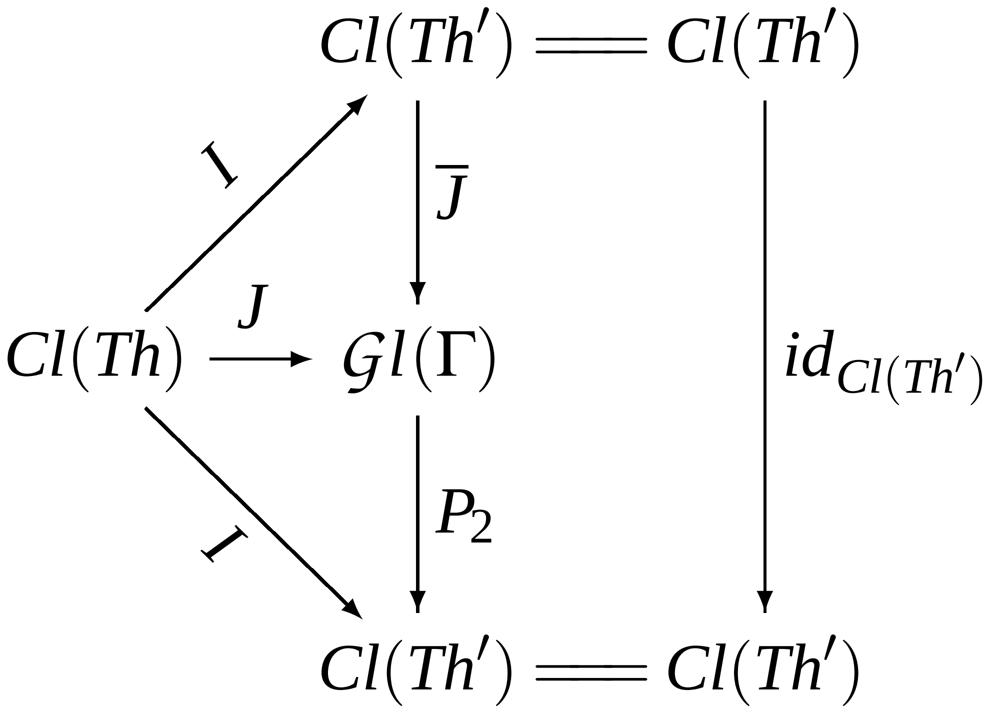
The Yoneda functor $\mathcal{C}(-,+) : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \mathcal{S}et]$ is full and faithful, where $c : C \to C' \mapsto \mathcal{C}(-,c) : \mathcal{C}(-,C) \to \mathcal{C}(-,C')$.

$J$ is faithful for $\mathcal{C}(-,+)$ is faithful. For fullness, let $(\alpha, d) : JC \to JC'$. Hence $\alpha : \mathcal{C}(-,C) \to \mathcal{C}(-,C')$ and so $\alpha = \mathcal{C}(-,c)$ for some $c : C \to C'$ in $\mathcal{C}$. Now certainly $id_C \lhd_C^C id_{IC}$ and so

$$\alpha_C(id_C) = \mathcal{C}(C,c)(id_C) = c \lhd_C^{C'} d \circ id_{IC} = d$$

implying $d = Ic$; therefore $Jc = (\alpha, d)$, that is $J$ is full.

Consider the following diagram



By freeness, the functor $\overline{J}$ exists and $\overline{J} \circ I \cong J$ naturally. By definition, $P_2 \circ J = I$. It follows that $P_2 \circ \overline{J} \circ I \cong I$ naturally, that is $(P_2 \circ \overline{J}) \circ I \cong I$, and as $id_{Cl(Th')} \circ I \cong I$ (trivially!) it follows from the universal property of relatively free bicartesian closed categories that $id_{Cl(Th')} \cong P_2 \circ \overline{J}$ naturally. This latter isomorphism implies that $\overline{J}$ is faithful. This fact, together with $J$ full and faithful proved above, and $\overline{J} \circ I \cong J$ implies that $I$ is full and faithful.