Category Theory

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MGS

Examples of Categories

any function

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[loop left, "id_A"] \arrow[rr, dashed, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \arrow[r, "h"] & D \end{tikzcd}$$

any set

Set

$$h \circ (g \circ f) = (h \circ g) \circ f$$

$$f \circ id_A = f \qquad f \circ id_B = f$$

$$\begin{tikzcd} 1 \\ 0 \arrow[u, "\leq"] \arrow[loop left, "\leq"] \\ -1 \arrow[u, "\leq"] \arrow[uu, bend left, "\leq \circ \leq"'] \end{tikzcd}$$

$$\begin{array}{ccc} -1 & \leq & -1 \\ 0 & \leq & 0 \\ 1 & \leq & 1 \end{array}$$

reflexive

$$\frac{-1 \leq 0 \qquad 0 \leq 1}{-1 \leq 1}$$

transitive

$$\leq \circ (\leq \circ \leq) = (\leq \circ \leq) \circ \leq$$

exc

$$
\begin{tikzcd}
 & & * \arrow[rr, bend left, "G(x_0,K)"] & & \\
x_0 \arrow[loop left] & * \arrow[ur, bend left, "F(x_0)"] & & & \\
 & * \arrow[uul, "K"] \arrow[uur, dashed] \arrow[uurr, dashed, "G(F(K),K)"] & & &
\end{tikzcd}
$$

e.g. $F(x_0) \circ K \overset{\text{def}}{=} F(x_0)[K/x_0]$

$$\boxed{t := x_0 \mid K \mid F(t) \mid G(t,t)}$$

WHAT IS A CATEGORY?

- A theory of "typed functions"
- "typed functions"
  * possess identities
  * are composable
  * compose associatively

- remark: objects are "recoverable" from the identities
- regard the above very informally!

General Definition of Category $\mathbb{C}$

- See slide for sets...
- Collection of objects $A, B, C \ldots$
- $\mathbb{C}(A,B) =$

$$\{ f \mid A \xrightarrow{f} B \}$$

collection of morphisms for each A, B

- satisfying axioms that define a category ....

More Examples

identities, transitivities omitted

- Opposite of $0 \to 1 \to 2 \to 3 \to \cdots$

  is $0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$

  [isomorphic to $\cdots \to {}^{-}2 \to {}^{-}1 \to 0$]

- $(X, \leq)$ "is" a category, say $\mathbb{C}$.

  * $\mathrm{ob}\,\mathbb{C} = X$
  * $\mathbb{C}(x, x') = \begin{cases} \{*\} & \Leftarrow x \leq x' \\ \emptyset & \Leftarrow \text{otherwise} \end{cases}$

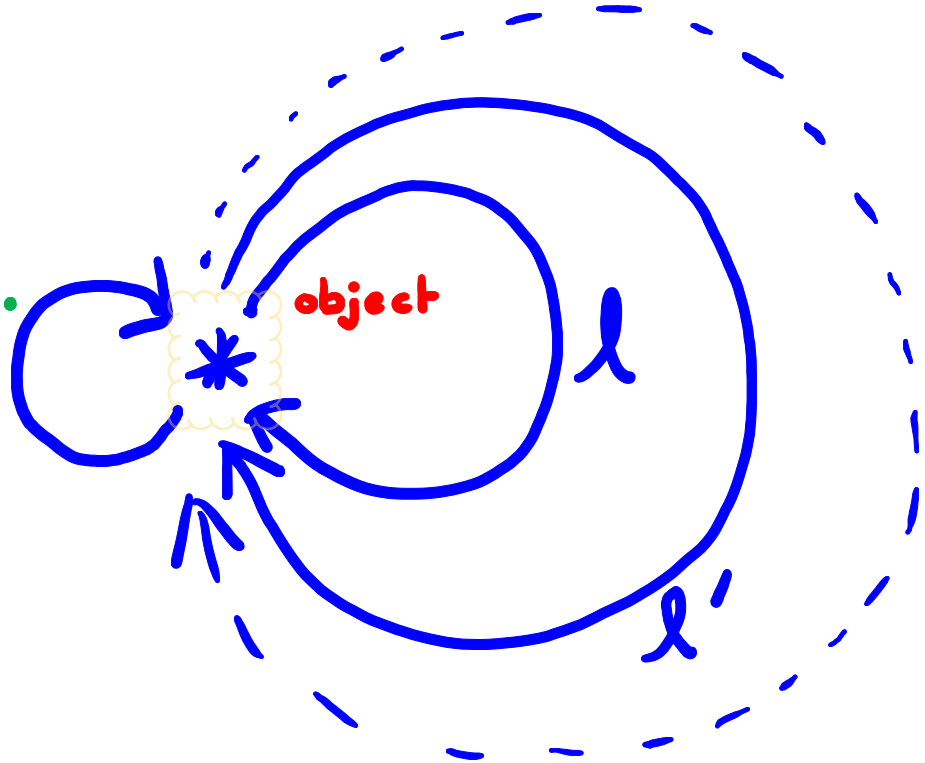
- $\mathbb{C}^{op} = (X, \leq^{op})$ where

  $\forall x, x' \in X.\ x \leq^{op} x' \Leftrightarrow x' \leq x$

A monoid is a category with one object $*$

for example $(list(A), ++, [])$

$id_* \dashrightarrow []$



$l ++ l'$

$l, l' \in list(A)$ — $++$ append associative

morphisms

— $[] ++ l = l = l ++ []$

... and Mon is a category (!)

$$m_1 \cdot m_2 \xmapsto{h} h(m_1) \cdot h(m_2) \xmapsto{h'} h'(h(m_1)) \cdot h'(h(m_2))$$

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[rr, bend right, dashed, "h' \circ h"'] & M' \arrow[loop right, "id"] \arrow[d, "h'"] \\ & M'' \end{tikzcd}$$

objects: M, M', M''

morphisms: h, h', $h' \circ h$, id

— homomorphisms compose

— identity function is a homomorphism

WHAT IS A FUNCTOR?

- A mapping between categories that preserves the (fundamental) structure.
- maps objects to objects
- maps morphisms to morphisms
- preserves identities & composition

Examples of Functors

- $F : \underline{\mathbf{Set}} \longrightarrow \underline{\mathbf{Mon}}$

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} \mathrm{list}(A) \arrow[d, "\mathrm{map}(f)"] \\ \mathrm{list}(B) \end{tikzcd} \qquad \boxed{\begin{tikzcd} FA \arrow[d, "Ff"] \\ FB \end{tikzcd}}$$

- $F(\mathrm{id}_A) = \mathrm{map}(\mathrm{id}_A) = \mathrm{id}_{\mathrm{list}(A)}$

$$\begin{tikzcd} & B \arrow[dr, bend left, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & C \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} & \mathrm{list}(B) \arrow[dr, bend left, "\mathrm{map}(g)"', "Fg"] & \\ \mathrm{list}(A) \arrow[ur, bend left, "\mathrm{map}(f)", "Ff"'] \arrow[rr, dashed, "\mathrm{map}(g \circ f)", "F(g \circ f)"'] \arrow[rr, mapsto, "\oplus\ ?"] & & \mathrm{list}(C) \end{tikzcd}$$

FB, FA, FC

$$[a_1, a_2] \longmapsto \mathrm{map}(g \circ f)\,[a_1, a_2]$$

$$= [(g \circ f)\,a_1,\ (g \circ f)\,a_2]$$

$\oplus$ ✓

$$= [g(f(a_1)),\ g(f(a_2))]$$

$$= \mathrm{map}(g) \circ \mathrm{map}(f)\,[a_1, a_2]$$

General Definition of Functor $F : \mathbb{C} \to \mathbb{D}$

- $A \longmapsto FA$

- $\begin{tikzcd} A \arrow[loop below, "id_A"] \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[loop below, "id_{FA}"] \end{tikzcd}$

- $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[dr, dashed, "g \circ f"'] & B \arrow[d, bend left, "g"] \\ & C \end{tikzcd} \longmapsto \begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[dr, dashed] & FB \arrow[d, bend left, "Fg"] \\ & FC \end{tikzcd}$

$$F(g \circ f) = (Fg) \circ (Ff)$$

Example of Natural Transformation

- $\mathrm{rev} : F \longrightarrow F$ ← Think: polymorphic function ☺
- specified by $\mathrm{rev}_A : FA \to FA$
- such that

$$\begin{tikzcd}
FA \arrow[r, "\mathrm{rev}_A"] \arrow[d, "Ff"] & FA \arrow[d, "Ff"] \\
FB \arrow[r, "\mathrm{rev}_B"'] & FB
\end{tikzcd}$$

$$[a_1, a_2] \longmapsto [a_2, a_1]$$

$$[fa_1, fa_2] \longmapsto [fa_2, fa_1]$$

$$Ff \overset{\text{def}}{=} \mathrm{map}(f)$$

$$FA \overset{\text{def}}{=} \mathrm{list}(A)$$

$(\mathrm{id}_F)_A \overset{\text{def}}{=} \mathrm{id}_{FA}$ $\qquad$ $(\beta \circ \alpha)_A \overset{\text{def}}{=}$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift right=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"'] & \mathbb{D}
\end{tikzcd}$$

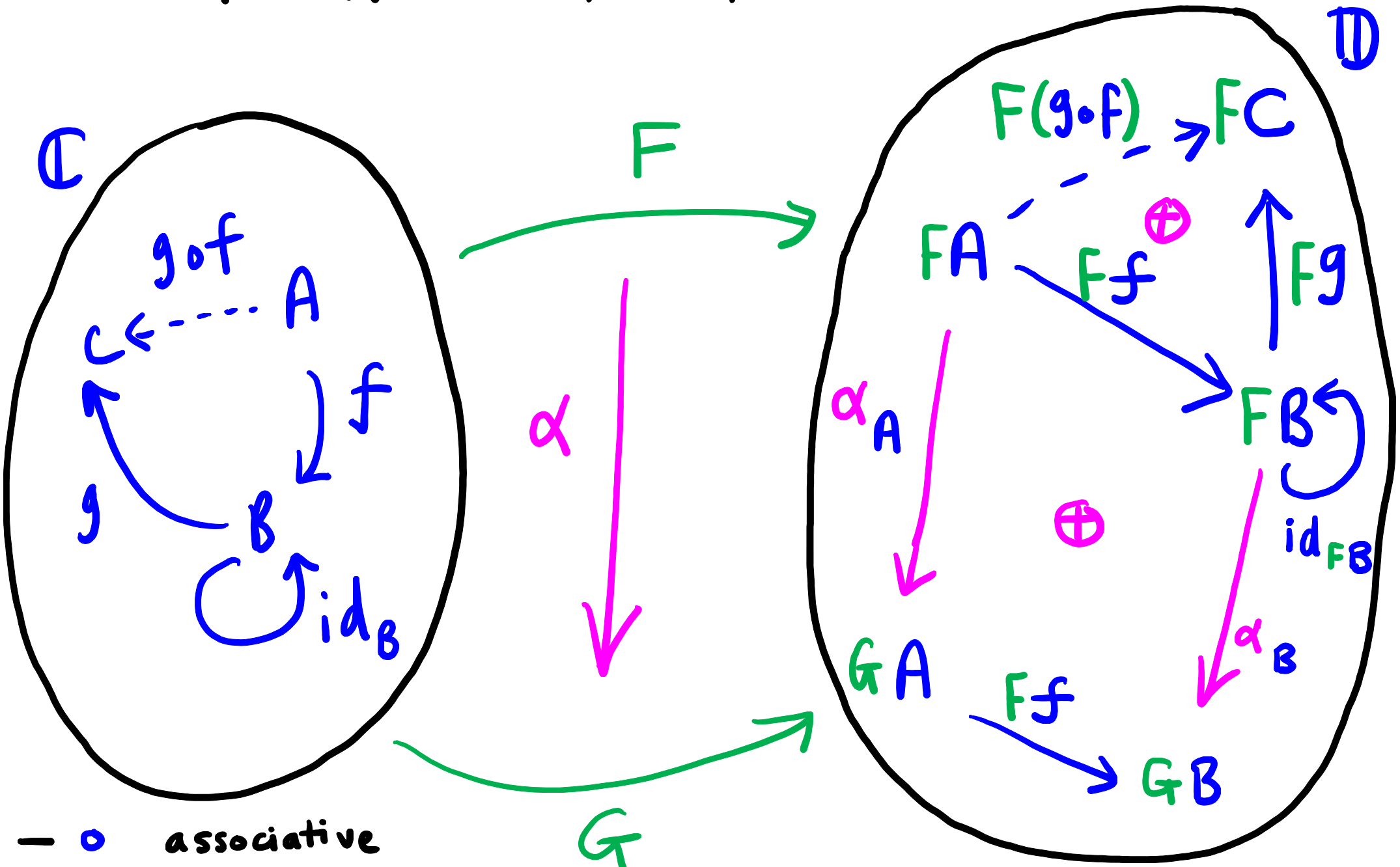
$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[rr, bend left, dashed] \arrow[d, "Ff"] \arrow[dr, phantom, "\oplus"] & GA \arrow[r, "\beta_A"] \arrow[d, "Gf"] \arrow[dr, phantom, "\oplus"] & HA \arrow[d, "Hf"] \\
FB \arrow[r, "\alpha_B"] & GB \arrow[r, "\beta_B"] & HB
\end{tikzcd}$$

$\mathbb{D}^{\mathbb{C}}$ — objects $F, G, H \ldots$

— morphisms $\alpha, \beta, \gamma, \delta \ldots$

SUMMARY

CATEGORIES, FUNCTORS, NATURAL TRANSFORMATIONS



- ∘ associative
- ∘ id unitary

Universal Properties: Structure up to Isomorphism

$\forall X.\ \exists!$ $\quad f_X : X \longrightarrow T$

$\qquad\qquad\qquad\quad g_X : X \longrightarrow T'$

Hence

$$\begin{tikzcd}
T' \arrow[r, "\exists\, f_{T'}"] \arrow[rr, bend right, "\mathrm{id}_{T'}"'] & T \arrow[r, "\exists\, g_T"] \arrow[d, phantom, "\Downarrow"] & T' \\
& g_T \circ f_{T'} \ \ \| \ !
\end{tikzcd}$$

& via a symmetrical argument

$$\boxed{T \cong T'}$$

Example of Binary Product of A, B

$A = \{2, 4\}$ $B = \{5, 6\}$ $C = \{\alpha, \beta\}$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
& C \arrow[ul, bend left, "\forall\ \hat{2}"] \arrow[u, dashed, "\langle \hat{2}, \hat{5} \rangle^{\exists!}"] \arrow[ur, bend right, "\forall\ \hat{5}"'] &
\end{tikzcd}$$

$\{(2,5), (2,6), (4,5), (4,6)\}$ $\oplus$

x $\forall$

$\oplus$

$\langle \hat{2}, \hat{5} \rangle$ is the unique morphism such that $\oplus$ $\oplus$

$$\langle \hat{2}, \hat{5} \rangle (x) = (\hat{2}(x), \hat{5}(x)) = (2, 5)$$

Example of Binary Coproduct of A, B

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[dr, bend right, "\hat{\alpha}"'] & A + B \arrow[d, dashed, "{[\hat{\alpha}, \hat{\beta}]}"] & B \arrow[l, "\iota_B"'] \arrow[dl, bend left, "\hat{\beta}"] \\
& C &
\end{tikzcd}$$

$\oplus$ $\oplus$

$\{2, 4, 5, 6\}$

α β

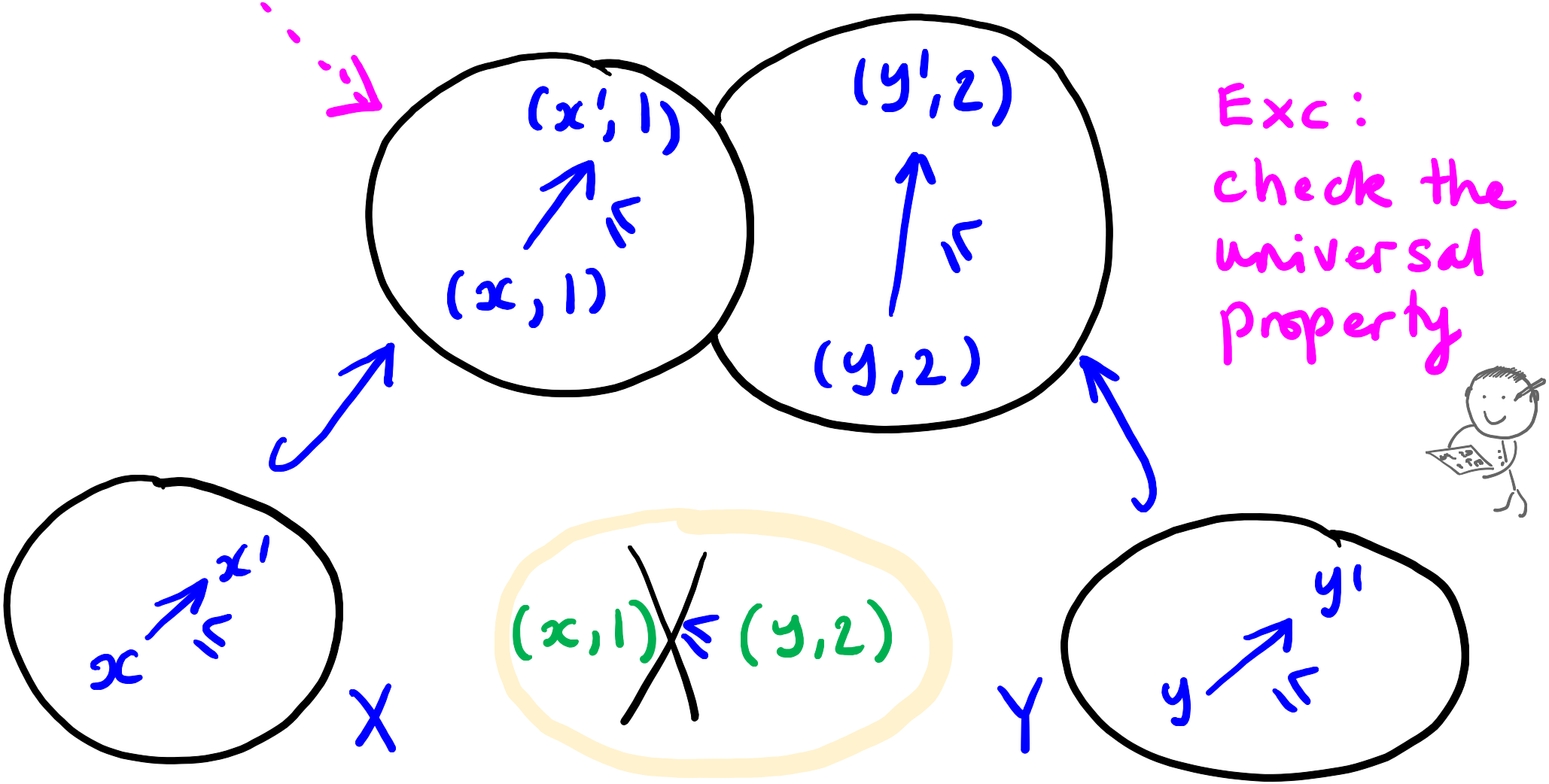
$[\hat{\alpha}, \hat{\beta}]$ is the unique morphism such that

$\oplus$ $\oplus$

$$[\hat{\alpha}, \hat{\beta}](\xi) = \text{case } \xi \text{ of } \begin{array}{l} 2, 4 \longmapsto \alpha \\ 5, 6 \longmapsto \beta \end{array}$$

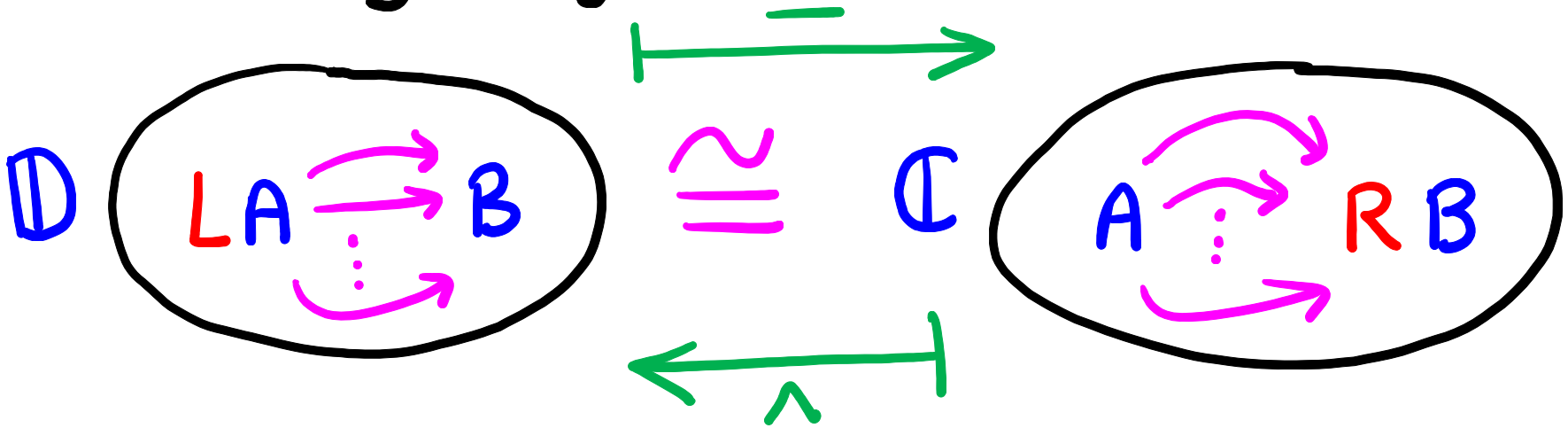
Example of Binary Coproduct in PreSet

$$X + Y = \{(x,1), (y,2) \mid x \in X, y \in Y\}$$



Exc: check the universal property

Illustrating Adjunctions $L \dashv R$



a natural bijection

$$\mathbb{C}(-, R+),\ \mathbb{D}(L^{op}-, +) : \mathbb{C}^{op} \times \mathbb{D} \to \underline{\mathbf{Set}}$$

are isomorphic in $\underline{\mathbf{Set}}^{\mathbb{C}^{op} \times \mathbb{D}}$:

$-$ there are natural transformations $\wedge$

$$\Phi_{(A,B)} : \mathbb{C}(A, RB) \rightleftarrows \mathbb{D}(L^{op}A, B) : \Psi_{(A,B)}$$

such that $\cong$

$$\begin{tikzcd}
LA' \arrow[r, "L\varphi"] & LA \arrow[r, "\bar{f}"] & B \arrow[r, "\psi"] & B' & \text{in } \mathbb{D} \\
A' \arrow[r, "\varphi"] & A \arrow[r, "f"] & RB \arrow[r, "R\psi"] & RB' & \text{in } \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C}(A, RB) \arrow[r, "\bar{-}"] \arrow[dd, "\mathbb{C}(\varphi^{op}, R\psi)"'] & \mathbb{D}(LA, B) \arrow[dd, "\mathbb{C}(L^{op}\varphi^{op}, \psi)"] \\
f \arrow[r, mapsto] \arrow[d, mapsto] & \bar{f} \arrow[d, mapsto] \\
 & \psi \circ \bar{f} \circ L\varphi \\
R\psi \circ f \circ \varphi \arrow[r, mapsto] & \overline{R\psi \circ f \circ \varphi} \\
\mathbb{C}(A', RB') \arrow[r, "\bar{-}"] & \mathbb{D}(LA', B')
\end{tikzcd}$$

Examples of Adjunctions

$\mathrm{list} \dashv \mathcal{U}$

— $\mathcal{U} : \underline{\mathrm{Mon}} \rightleftarrows \underline{\mathrm{Set}} : \mathrm{list}$

$$\begin{tikzcd}
{[a_1, a_2]} \arrow[d, mapsto, "\hat{g}"] & \mathrm{list}(A) \arrow[d, "f"] \arrow[r, phantom, "\cong_{bij}"] & A \arrow[d, "g"] & a \arrow[d, mapsto, "\bar{f}"'] \\
g(a_1) \circ g(a_2) & M & \mathcal{U}M & f([a])
\end{tikzcd}$$

$$f([a_1, a_2]) = f([a_1] ++ [a_2]) = f([a_1])\, f([a_2]) = \bar{f}(a_1)\, \bar{f}(a_2)$$

$\therefore\ f([a_1, a_2]) = \hat{\bar{f}}([a_1, a_2]) \Rightarrow \cong_{bij}$

$$\Delta : \underline{\mathrm{Set}} \rightleftarrows \underline{\mathrm{Set}} \times \underline{\mathrm{Set}} : \sqcap \qquad \Delta \dashv \sqcap$$

$$\begin{tikzcd}
(x,y) \arrow[d, mapsto, "\hat{m}"] \\
(\pi_A \circ m\, x,\ \pi_B \circ m\, y)
\end{tikzcd}
\qquad
\begin{tikzcd}
(c,c) \arrow[d, equal] \\
\Delta c \arrow[d, "(f,g)"] \\
(A,B)
\end{tikzcd}
\cong_{bij}
\begin{tikzcd}
C \arrow[d, "m"] \\
\sqcap(A,B) \arrow[d, equal] \\
A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \\
A \qquad\qquad B
\end{tikzcd}
\qquad
\begin{tikzcd}
z \arrow[d, mapsto, "\overline{(f,g)}"] \\
(fz, gz)
\end{tikzcd}$$

$$(f,g)(x,y) = (fx, gy) = (\pi_A \circ \langle f,g\rangle x,\ \pi_B \circ \langle f,g\rangle y)$$

$$= (\pi_A \circ \overline{(f,g)}\, x,\ \pi_B \circ \overline{(f,g)}\, y) \quad \therefore\ (f,g)(x,y) = \widehat{\overline{(f,g)}}\,(x,y) \Rightarrow \cong_{bij}$$

Example of Algebra and Homomorphism

— Take a monoid (M, b, e)

— $\hat{e} : 1 \longrightarrow M \qquad b : M \times M \longrightarrow M$

Define* $F(-) = 1 + (- \times -)$ and note

$$1 + (M \times M) \xrightarrow{[\hat{e}, b]} M$$

$$* \xmapsto{\hat{e}} e$$

$$(m, n) \xmapsto{b} m \cdot n$$

$$\boxed{FM \xrightarrow{[\hat{e}, b]} M \text{ is an algebra}}$$

* what is the effect on morphisms?

Example of Algebra and Homomorphism

- Similarly (S, β, ε). Then $\theta : M \to S$ is a homomorphism $\Leftrightarrow$ θ is a homomorphism of algebras

- homomorphism means

  * $\theta(e) = \varepsilon$

  * $\theta(m \cdot n) = \theta(m) \cdot \theta(n)$

$$\Longrightarrow$$

$$\begin{tikzcd}
\overbrace{1 + (M \times M)}^{FM} \arrow[r, "{[\hat{e}, b]}"] \arrow[dd, "F\theta = 1+(\theta\times\theta)"'] & M \arrow[dd, "\theta"] \\
* \arrow[r, mapsto] \arrow[d, mapsto] & e \arrow[d, mapsto] \\
(m,n) \arrow[r, mapsto] \arrow[d, mapsto] & m \circ n \arrow[d, mapsto] \\
(\theta m, \theta n) \arrow[r, mapsto] & \theta(m \circ n) = \theta m \circ \theta n \\
* \arrow[r, mapsto] & \varepsilon \\
\underbrace{1 + (S \times S)}_{FS} \arrow[r, "{[\hat{\varepsilon}, \beta]}"] & S
\end{tikzcd}$$

Natural Numbers as an Initial Algebra

$$1 + A \xrightarrow{id_1 + g} 1 + B$$

$$\begin{tikzcd}
1 + \mathbb{N} \arrow[rr, "{[z, s]}"] \arrow[dd, "1+\bar{f}"'] & & \mathbb{N} \arrow[dd, "\bar{f}"] \\
* \arrow[rr, mapsto] \arrow[d, mapsto] & & 0 \arrow[d, mapsto] \\
 & n \arrow[r, mapsto] \arrow[d, mapsto] & n+1 \arrow[d, mapsto] \\
 & \bar{f}(n) \arrow[r, mapsto] & \bar{f}(n+1) = \phi(\bar{f}(n)) \\
* \arrow[rr, mapsto] & & x \\
1 + X \arrow[rr, "f"] & & X
\end{tikzcd}$$

$$= [\hat{x}, \phi]$$

by properties of $1 + X$

CASE STUDY

The Mini Yoneda Lemma
for Type Theorists

– for details see slides...

plus...

$$
\begin{tikzcd}
\mathbb{C}(A,A) \arrow[rr, "\rho_A"] \arrow[dd, "\mathbb{C}(m,\mathrm{id}_A)"'] & & \mathbb{C}(A,B) \arrow[dd, "\mathbb{C}(m,\mathrm{id}_B)"] \\
\mathrm{id}_A \arrow[rr, mapsto] \arrow[d, mapsto] & & \rho_A(\mathrm{id}_A) \arrow[d, mapsto] \\
\mathrm{id}_A \circ m \arrow[rr, mapsto] & & \rho_A(\mathrm{id}_A)\circ m \;=\; \rho_G(m) \\
\mathbb{C}(G,A) \arrow[rr, "\rho_G"] \arrow[dd, "\mathbb{C}(n,\mathrm{id}_A)"'] & & \mathbb{C}(G,B) \arrow[dd, "\mathbb{C}(n,\mathrm{id}_B)"] \\
m \arrow[rr, mapsto] \arrow[d, mapsto] & & \rho_G(m) \arrow[d, mapsto] \\
m\circ n \arrow[rr, mapsto] & & \rho_G(m)\circ n \;=\; \rho_{G'}(m\circ n) \;(\natural) \\
\mathbb{C}(G',A) \arrow[rr, "\rho_{G'}"] & & \mathbb{C}(G',B)
\end{tikzcd}
$$

$\therefore$ $(\natural) \iff \rho_- : \mathbb{C}(-,A) \to \mathbb{C}(-,B)$ nat. trans.

$\iff \rho_{G_\forall}(m_\forall) = \rho_A(\mathrm{id}_A)\circ m_\forall$ (Why does $\Leftarrow$ hold?)

CASE STUDY

A ccc can be defined in terms of an adjunction

– for details see slides...

plus...

Cartesian Closed Categories

- For any B, C there is $B \Rightarrow C$ and
  $ev : (B \Rightarrow C) \times B \longrightarrow C$
- For any $f : A \times B \to C$ there is
  $\lambda f : A \longrightarrow (B \Rightarrow C)$ unique such that

$$\begin{tikzcd}
A \times B \arrow[r, "f"] \arrow[d, "\lambda f \times id_B"'] \arrow[dr, phantom, "\oplus"] & C \\
(B \Rightarrow C) \times B \arrow[ur, "ev"'] &
\end{tikzcd}$$

λf is the currying of f

Example of <u>Set</u>

— $ev : (B \Rightarrow C) \times B \longrightarrow C$

$$(f, b) \longmapsto f(b)$$

where $B \Rightarrow C = \{ f \mid f : B \to C \}$

— for $f : A \times B \longrightarrow C$ define

$$\lambda f : A \longrightarrow (B \Rightarrow C)$$

$$a \longmapsto \lambda b . f(a, b)$$

— EXERCISE check ⊕

CASE STUDY

Recursive types can be modelled by initial algebras

- for details see slides, plus...

Illustrative notes

silly lists

- Take $D = K_1(\mathrm{Int} \times \mathrm{Unit})$
  $\quad\quad\quad\quad\quad\quad\quad\quad | \; K_2(\mathrm{Int} \times D)$

eg
$(3, (4, (5, ())))$
minus constructors K_i

- $F : \underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Set}}$ defined from D

$$F\,X = (\mathbb{Z} \times 1) + (\mathbb{Z} \times X)$$

complex definition in the slides...

— Define $I \stackrel{\text{def}}{=} \{ E \mid E :: D \}$

— Then $F\,I \xrightarrow{\sigma} I$ is initial

definition in the slides

— Roughly speaking

$$(\mathbb{Z} \times 1) + (\mathbb{Z} \times I) \xrightarrow{\sigma} I$$

$$(z, *) \longmapsto K_1(\underline{z}, ())$$

$$(z, E) \longmapsto K_2(\underline{z}, E)$$

where $\underline{z} :: \mathrm{Int}$ for each $z \in \mathbb{Z}$

– initiality by example

$$
\begin{tikzcd}
(\mathbb{Z}\times 1)+(\mathbb{Z}\times I) \arrow[rr, "\sigma"] \arrow[dd, "F\bar{f}"'] & & I \arrow[dd, "\bar{f}"] \\
 & (z,E) \arrow[r, mapsto] \arrow[dd, mapsto] & K_2(z,E) \arrow[dd, mapsto] \\
(z,*) \arrow[r, mapsto] \arrow[d, mapsto] & K_1(z,()) \arrow[d, mapsto] & \\
(z,*) \arrow[r, mapsto] & \bar{f}K_1(z,()) = z & \bar{f}K_2(z,E) = (z+\bar{f}E) \\
 & (z,\bar{f}E) \arrow[r, mapsto] & (z+\bar{f}E)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mathbb{Z}\times 1)+(\mathbb{Z}\times X) \arrow[r] & X
\end{tikzcd}
$$

$$f = [\pi_{\mathbb{Z}}, +]$$

Take $X \overset{\text{def}}{=} \mathbb{Z}$

Hence if $\xi = K_2(\underline{3}, K_2(\underline{4}, K_1(\underline{8}, ()))$

then we have

$$\begin{aligned}
\bar{f}(\xi) &= 3 + \bar{f}\, K_2(\underline{4}, K_1(\underline{8}, ())) \\
&= 3 + 4 + \bar{f}\, K_1(\underline{8}, ()) \\
&= 3 + 4 + 8 \\
&= 3 + 12 \\
&= 15 \in \mathbb{Z} \quad (= X)
\end{aligned}$$

CASE STUDY

Colimits - Building Initial Algebras

- for details see slides....

The end....

$$\int_{C} R(LC, LC)$$