MODAL LOGIC

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Note

These notes originated when the module MC 212 was taught during the second semester of the academic year 1994–1995. This particular set of notes is essentially a copy, with minor changes and corrections.

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Preface

It is assumed that readers have attended the course "First Order Predicate Logic". We shall refer to this as *Logic A*.

These notes are to accompany the first half of the course MC 212. They contain all of the core material for this course. For more motivation and background, as well as further comments about some of the details of proofs, please attend the lectures.

Please do let me know about any typos or other errors which you find in the notes. If you have any other (constructive) comments, please tell me about them.

Books recommended for the first eighteen lectures of MC 212 are

- **Modal Logic** by Brian F. Chellas. Cambridge University Press 1980.
- **A Companion to Modal Logic** by G. E. Hughes and M. J. Cresswell. Methuen 1984.

Thanks to Dr S. Ambler, Dr N. Measor and Dr R. Thomas for their help in the preparation of these notes. Any errors are the responsibility of the author.

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Mathematical Preliminaries

1.1 Introduction

Discussion 1.1.1 We shall begin by reviewing some mathematics which will be used throughout this course on Modal Logic. Some of the material you have seen before. For the material that is new, you may need to flesh out the definitions and concepts using books or other sets of notes. However, most of the *basic* ideas you have met in Logic A.

We shall adopt a few conventions:

• If we give a definition, the entity being defined will be written in an **italic** typeface; and when we emphasise something it appears in a *slanted* typeface.

• If we wish to define a set A whose elements are known as widgets, then we shall simply say "let A be the set of **widgets**."

• $\iff$ means "if and only if". Some authors write $\Leftrightarrow$ instead of iff. In proofs of theorems which are of the form

$$\textit{statement 1} \iff \textit{statement 2}$$

we shall write

($\Rightarrow$) ... proof that *statement 1* implies *statement 2* ...

($\Leftarrow$) ... proof that *statement 2* implies *statement 1* ...

• Suppose we wish to speak of a set A, and indicate that the set A happens to be a subset of a set X. We will write "consider the set $A \subseteq X$ ..." for this. For example, we might say "let $O \subseteq \mathbb{N}$ be the set of odd numbers" to emphasise that we are considering the set of odd numbers denoted by O, which happen to be a subset of the natural numbers (denoted by $\mathbb{N}$).

1.2 A Review of Sets

Discussion 1.2.1 We assume that the idea of a *set* is understood, being an unordered collection of objects. A capital letter such as A or B or X or Y will be used to denote an arbitrary set. If a is any object in a set A, we say that a is an **element** of A, and write $a \in A$ for this. If a is not an element of A, we write $a \notin A$. The idea of *union* $A \cup B$, *intersection* $A \cap B$, and *difference* $A \setminus B$, of sets should already be known. We collect the definitions here:

$$
\begin{array}{rrcl}
\textit{Union} & A \cup B & \stackrel{\text{def}}{=} & \{\, x \mid x \in A \text{ or } x \in B \,\} \\
\textit{Intersection} & A \cap B & \stackrel{\text{def}}{=} & \{\, x \mid x \in A \text{ and } x \in B \,\} \\
\textit{Difference} & A \setminus B & \stackrel{\text{def}}{=} & \{\, x \mid x \in A \text{ and } x \notin B \,\} \\
\textit{Powerset} & \mathcal{P}(A) & \stackrel{\text{def}}{=} & \{\, S \mid S \subseteq A \,\} \\
\textit{FinitePowerset} & \mathcal{P}_{fin}(A) & \stackrel{\text{def}}{=} & \{\, S \mid S \subseteq A \text{ and } S \text{ is finite } \}
\end{array}
$$

Recall that the **empty** set, $\varnothing$, is the set with no elements, and that we say a set S is a **subset** of a set A, written $S \subseteq A$, if any element of S is an element of A. Thus given sets A and S we could write this definition of subset as

$$S \subseteq A \qquad \Longleftrightarrow \qquad (x \in S \Longrightarrow x \in A). \qquad\qquad (*)$$

Note that $\Longleftrightarrow$ stands for "if and only if" and is used to give definitional equivalences. We could read $(*)$ as $S \subseteq A$ "is by definition the same as" $x \in S \Longrightarrow x \in A$. Note that $\varnothing \subseteq A$ for any set A, because $x \in \varnothing$ is always false. So $\varnothing \in \mathcal{P}(A)$. We regard $\varnothing$ as a *finite* set.

Two sets A and B are **equal**, written $A = B$, if they have the same elements. So, for example, $\{\,1,2\,\} = \{\,2,1\,\}$. Here, the critical point is whether an object is an element of a set or not: if we write down the elements of a set, it is irrelevant what order they are written down in. But we shall need a way of writing down "a set of objects" in which the order is important.

To see this, think about the map references "1 along and 2 up" and "2 along and 1 up." These two references are certainly different, both involve the numbers 1 and 2, but we cannot use the sets $\{\,1,2\,\}$ and $\{\,2,1\,\}$ as a mathematical notation for the map references because the sets are equal. Thus we introduce the idea of a pair to model this. If A and B are sets, with $a \in A$ and $b \in B$, we shall write (a,b) for the **pair** of a and b. The crucial property of pairs is that (a,b) and (a',b') are said to be **equal** iff $a = a'$ and $b = b'$. We write

$$(a,b) = (a',b')$$

to indicate that the two pairs are indeed equal. We could write $(1,2)$ and $(2,1)$ for our map references. Note that the definition of equality of pairs captures the exact property required of map references. We can also consider n-**tuples** $(a_1,\ldots,a_n)$ and regard such an n-tuple as equal to another n-tuple $(a'_1,\ldots,a'_n)$ iff $a_i = a'_i$ for each $1 \le i \le n$. Note that a pair is a 2-tuple.

The **cartesian product** of A and B, written $A \times B$, is a set given by

$$A \times B \stackrel{\text{def}}{=} \{\, (a,b) \mid a \in A \text{ and } b \in B \,\}.$$

For example,

$$\{\,1,2\,\} \times \{\,a,b,c\,\} = \{\,(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\,\}.$$

Examples 1.2.2

(1) $\{1,2,3\} \cup \{x,y\} = \{1,2,3,x,y\} = \{x,1,y,3,2\} = \ldots$ The written order of the elements is irrelevant.

(2) $\{a,b\} \setminus \{b\} = \{a\}$.

(3) $A \setminus A = \varnothing$.

(4) $\mathcal{P}(\{1,2\}) = \{\{1,2\},\{1\},\{2\},\varnothing\}$.

(5) $\{a\} \times \{b\} = \{(a,b)\}$.

(6) $(x,y) = (2,100) \iff x = 2$ and $y = 100$.

1.3 Relations

Discussion 1.3.1 The idea here is to see how we can formalise the the notion of relationships. Some examples of relationships are

• Ron *is_the_father_of* Roy;

• $0 \le 5$;

• London *is_south_of* Leicester;

In each case, we have a pair of objects (for example 0 and 5) which are related in some way. Note that the order in which the objects are written down is important: $0 \le 5$, but not $5 \le 0$. Let us look for a general framework into which all of our examples fit.

Given sets A and B, a **relation** R between A and B is a subset $R \subseteq A \times B$. Informally, R is the set whose elements are pairs (a,b) for which a is related to b. Given a set A, a **binary relation** on A is a subset $R \subseteq A \times A$.

Remark 1.3.2 Note that a *relation* is a set: it is the set of all pairs for which the first element of the pair is in a relationship to the second element of the pair. If $R \subseteq A \times B$ is a relation, and $(a,b) \in R$, we may also write $a\ R\ b$ for this. So *is_the_father_of* is a relation on the set *Humans* of humans, and if we have $(\mathrm{Ron},\mathrm{Roy}) \in \mathit{Humans}$ then we can write instead Ron *is_the_father_of* Roy. Note that if $(a,b) \notin R$ then we write $a \not R\ b$ for this.

Example 1.3.3 Being strictly less than is a binary relation, written $<$, on the natural numbers $\mathbb{N}$. So $< \subseteq \mathbb{N} \times \mathbb{N}$, and (by definition)

$$< \stackrel{\text{def}}{=} \{(0,1),(0,2),(0,3),(0,4)\ldots,(1,2),(1,3)\ldots,(2,3),\ldots\}.$$

Being less than or equal to is also a binary relation on $\mathbb{N}$, written $\le$. By definition,

$$\le \stackrel{\text{def}}{=} \{(0,0),(0,1),(0,2),(0,3),\ldots,(1,1),(1,2),\ldots\}.$$

Discussion 1.3.4 We will be interested in binary relations which satisfy certain important properties. Let A be any set and R any binary relation on A. Then

(i) R is **reflexive** iff for all $a \in A$ we have $a\ R\ a$;

(ii) R is **symmetric** iff for all $a, b \in A$, $a\ R\ b$ implies $b\ R\ a$;

(iii) R is **transitive** iff for all $a, b, c \in A$, $a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$; and

(iv) R is **anti-symmetric** iff for all $a, b \in A$, $a\ R\ b$ and $b\ R\ a$ implies $a = b$.

(v) R is **euclidean** iff for all $a, b, c \in A$, $a\ R\ b$ and $a\ R\ c$ implies $b\ R\ c$.

Examples 1.3.5 Let $A \stackrel{\text{def}}{=} \{\alpha, \beta, \gamma\}$ be a three element set, and recall the binary relations $<$ and $\leq$ on $\mathbb{N}$ from Example 1.3.3.

(1) $R \stackrel{\text{def}}{=} \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma)\}$ is reflexive, but $<$ is not reflexive.

(2) $R \stackrel{\text{def}}{=} \{(\alpha, \beta), (\beta, \alpha), (\gamma, \gamma)\}$ is symmetric, but $\leq$ is not.

(3) $R \stackrel{\text{def}}{=} \{(\alpha, \beta), (\beta, \gamma), (\alpha, \gamma)\}$ is transitive, as are $<$ and $\leq$.

(4) $R \stackrel{\text{def}}{=} \{(\alpha, \beta), (\beta, \gamma), (\alpha, \gamma)\}$ is anti-symmetric. Both $<$ and $\leq$ are anti-symmetric.

(5) $R \stackrel{\text{def}}{=} \{(\alpha, \beta), (\beta, \gamma)\}$ is euclidean.

(6) Note that R in (1) is also transitive—what other properties hold of the other examples?

Discussion 1.3.6 A **partial order** $\preceq$ on a set A is a binary relation which is reflexive, transitive and anti-symmetric. An example is the relation $\leq$ on the natural numbers. A **partially ordered set**, or **poset**, is a pair $(A, \preceq)$ where A is any set and $\preceq$ is any partial order on A. So $(\mathbb{N}, \leq)$ is a poset, but $(\mathbb{N}, <)$ is not.

1.4 Inductively Defined Sets

Discussion 1.4.1 Let us first introduce some notation. Consider

$$\textit{statement 1} \Longrightarrow \textit{statement 2}.$$

It is sometimes convenient to write this as

$$\frac{\textit{statement 1}}{\textit{statement 2}}$$

Consider

$$\textit{statement 1} \Longleftrightarrow \textit{statement 2}.$$

It is sometimes convenient to write this as

$$\frac{\textit{statement 1}}{\overline{\overline{\textit{statement 2}}}}$$

For example, we can write "$x \leq 4 \Longrightarrow x \leq 6$" as

$$\frac{x \leq 4}{x \leq 6}$$

The reason for using this notation will become clear very soon—it can make implications involving many hypotheses much easier to read.

Discussion 1.4.2 As motivation for this section, consider the following:

The set $E \subseteq \mathbb{N}$ of even natural numbers is the least subset of the natural numbers satisfying

(a) $0 \in E$, *and*

(b) if $n \in E$ *then* $n + 2 \in E$.

Note that "*least*" means that if another subset $S \subseteq \mathbb{N}$ satisfies (a) and (b) (by which we mean $0 \in S$, and if $n \in S$ then $n + 2 \in S$) then $E \subseteq S$. The above definition of E amounts to saying that the elements of E are created by the rules (a) and (b), and that there can be no other elements in E. We say that E is *inductively defined* by the *rules* (a) and (b). So $E = \{ 0, 2, 4, 6, 8, \dots \}$, another set satisfying (a) and (b) is (for example) $S \stackrel{\text{def}}{=} \{ 0, 2, 4, 5, 6, 7, 8, 9, \dots \}$, and indeed $E \subseteq S$.

We shall now give a general formal recipe for defining a set S inductively. First we need some machinery. A **rule** R is a pair (F, x) where F is any finite set, and x is an element. Note that F might be $\varnothing$, in which case the rule R is of **Form 1** . If F is non-empty we say R is of **Form 2** . In the case that F is non-empty we might write $F = \{ x_1, \dots, x_n \}$ where $1 \leq n$. We can write down a rule $R = (\varnothing, x)$ of Form 1 using the following notation

Form 1

$$\frac{}{x}\ (R)$$

and a rule $R = (\{ x_1, \dots, x_n \}, x)$ of Form 2 as

Form 2

$$\frac{x_1 \quad x_2 \quad \dots \quad x_n}{x}\ (R)$$

A set I is **closed under a rule** R of Form 1 if $x \in I$; and is **closed under a rule** R of Form 2 if whenever $x_1 \in I, x_2 \in I, \dots, x_n \in I$, then $x \in I$. The set I is **closed under** $\mathcal{R}$ if I is closed under each rule in $\mathcal{R}$. We can now say that:

Inductively Defined Sets

A set I is **inductively defined** by a set of rules $\mathcal{R}$ if

IC I is closed under $\mathcal{R}$; and

IL for *every* set S which is closed under $\mathcal{R}$, we have $I \subseteq S$.

Remark 1.4.3 Rules for defining the set E of even numbers are

$$\frac{}{0} \qquad \frac{n}{n+2}$$

IC means that the elements of the inductively defined set are built up by applying the rules: thus the elements of E are 0, $0+2=2$, $2+2=4$ and so on. **IL** amounts to saying that there can be no elements of E other than those arising by application of the rules: any other set S closed under the rules must contain E as a subset. An example of such an S is $\{0,2,4,6,7,8,9,10,\dots\}$.

This is of course true in general: **IC** means that certain elements of the **I**nductively defined set I are **C**onstructed by applying the rules in $\mathcal{R}$, and **IL** captures precisely the idea that I is the **L**east set satisfying the rules, that is, there can be no elements of I other than those constructed by the rules.

Discussion 1.4.4 If I is inductively defined by a set of rules $\mathcal{R}$, and $x \in I$, a **deduction** that x is an element of I is given by a list

$$y_1 \in I, y_2 \in I, \dots, y_m \in I$$

where

(i) y_1 is a conclusion to a rule of Form 1;

(ii) for any $1 \le i \le m$, y_i is the conclusion of some rule R for which the hypothesis of R is a subset of $\{y_1, \dots, y_i\}$; and

(iii) $y_m = x$.

A **labelled** deduction that $x \in I$ looks like

$$\begin{array}{ll} y_1 \in I & (R1) \\ y_2 \in I & (R2) \\ \dots & \\ y_m \in I & (Rm) \end{array}$$

in which the sequence of $y_i \in I$'s is a deduction that $x \in I$, and each Ri is the rule from $\mathcal{R}$ which has been used to deduce that $y_i \in I$.

Examples 1.4.5

(1) The set I of integer multiples of 3 can be inductively defined by a set of rules

$$\mathcal{R} \stackrel{\text{def}}{=} \{\ (\varnothing, 0), (\{n\}, n+3), (\{n\}, n-3) \mid n \in \mathbb{Z}\ \}.$$

We can write these rules more expressively as

$$\frac{}{0}\,(a) \qquad \frac{n}{n+3}\,(b) \qquad \frac{n}{n-3}\,(c)$$

Note that rules (b) and (c) are used when n is *any* element of $\mathbb{Z}$. We call each of (b) and (c) a **rule schema**, meaning that each "written rule" is a shorthand for a collection of rules. For example I being closed under (b) means that if n is any element of I, so too is $n+3$. A deduction that $9 \in I$ is given by $0 \in I, 3 \in I, 6 \in I, 9 \in I$, and a labelled version of this deduction would be

$$\begin{array}{ll} 0 \in I & (a) \\ 3 \in I & (b) \\ 6 \in I & (b) \\ 9 \in I & (b) \end{array}$$

(2) Suppose that Σ is any set, which we think of as an **alphabet**. Each element l of Σ is called a **letter**. We inductively define the set Σ^* of **words** over the alphabet Σ by the set of rules $\mathcal{R} \stackrel{\text{def}}{=} \{1, 2, 3\}$ (so 1, 2 and 3 are just labels for rules!) given by

$$\frac{-}{\epsilon}\,(1) \qquad \frac{-}{l}\,(2) \qquad \frac{w \quad w'}{ww'}\,(3)$$

A word is just a list of letters. The symbol ϵ denotes the **empty** word, which is the word with no letters. **IC** says that Σ^* is closed under the rules 1,2,3— so Rule 1 says that ϵ is an element of Σ^*, that is, ϵ is a word. Rule 2 says that any letter is a word. Rule 3 says that if w and w' are any two words, the list of letters ww' obtained by writing down the list of letters w followed immediately by the list of letters w' is a word. Note also that "rules" 2 and 3 are in fact rule schemas: the l in rule 2 ranges over *any* letter; w and w' range over any words.

(3) We can use sets of rules to define the language of propositional logic. Let *Var* be a set of **propositional variables** with typical elements written p, q or r. Then the set *Prop* of **propositions** of propositional logic is inductively defined by the rules (more precisely, rule schemas)

$$\frac{-}{p}\,(V) \qquad \frac{\phi \quad \psi}{\phi \wedge \psi}\,(\wedge) \qquad \frac{\phi \quad \psi}{\phi \vee \psi}\,(\vee) \qquad \frac{\phi \quad \psi}{\phi \to \psi}\,(\to) \qquad \frac{\phi}{\neg\psi}\,(\neg)$$

Clause **IC** of the definition of an inductively defined set says that *Prop* is closed under each of these rules. Thus the first rule (schema) says that any propositional variable p is a proposition. Also (for example) if $\phi \in \mathit{Prop}$ and $\psi \in \mathit{Prop}$ then $\phi \wedge \psi \in \mathit{Prop}$. Clause **IL** captures formally the requirement that propositions can only arise through applications of the above rules. Finally, a labelled deduction that $(p \to q) \vee (q \to p) \wedge r$ is a proposition might be

$$\begin{array}{ll} p & (V) \\ q & (V) \\ p \to q & (\to) \\ q \to p & (\to) \\ r & (V) \\ (p \to q) \vee (q \to p) & (\vee) \\ (p \to q) \vee (q \to p) \wedge r & (\wedge) \end{array}$$

Remark 1.4.6 We can use a **BNF grammar** to write sets of rules informally but concisely. The set *Prop* of propositions is specified by the BNF grammar

$$\phi \ ::= \ p \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid \neg\phi$$

where p is any element of the set Var. One reads the grammar as "any element of *Prop* is either a propositional variable, or of the form $\phi \wedge \psi$ where ϕ and ψ are any propositions, or ..." As an exercise, write a BNF grammar for example (2).

Remark 1.4.7 We have said that a convenient way of writing a rule

$$R = (\{ x_1, \ldots, x_n \}, x)$$

is

$$\frac{x_1 \quad \ldots \quad x_n}{x}\,(R)$$

and that a set X is closed under the rule if whenever $x_i \in X$ for each $1 \leq i \leq n$ then $x \in X$. It is therefore useful to write

$$\frac{x_1 \in X \quad \ldots \quad x_n \in X}{x \in X}\,(R)$$

to mean that X is closed under the rule R. We might also say (for example) that *Prop* is inductively defined by rules such as

$$\frac{\phi \in Prop \quad \psi \in Prop}{\phi \wedge \psi \in Prop}\,(\wedge)$$

If we then say that a set S is closed under rule $\wedge$, what we mean is that

$$\frac{\phi \in S \quad \psi \in S}{\phi \wedge \psi \in S}\,(\wedge)$$

1.5 Principles of Induction

Discussion 1.5.1 In this section we see how inductive techniques of proof which the reader has met before fit into the framework of inductively defined sets.

First we shall see how the Principle of Structural Induction for the propositions of first order logic fits into our general framework of inductively defined sets. Recall that this says in order to prove that a property $P(\phi)$ holds for all propositions ϕ it is enough to show that

- $P(r)$ holds for each propositional variable r;
- if $P(\phi)$ and $P(\psi)$ hold for any ϕ and ψ, then so do $P(\phi \wedge \psi)$, $P(\phi \vee \psi)$, $P(\phi \rightarrow \psi)$ and $P(\neg\phi)$.

Now, we have specified the collection *Prop* of propositions as an inductively defined set. If we put

$$S \stackrel{\text{def}}{=} \{\ \phi \in Prop \ \mid\ P(\phi)\ \}$$

then $S \subseteq Prop$ by definition. If also S is closed under the rules defining *Prop*, then $Prop \subseteq S$ by property **IL**, and so $Prop = S$. But then for any proposition ϕ we must have $\phi \in S$, and so $P(\phi)$. Thus: *showing S is closed under the rules defining Prop will prove that $P(\phi)$ holds for all ϕ.* Let us examine one (typical) part of proving that S is closed under the rules for defining *Prop*. Take the rule

$$\frac{\phi \quad \psi}{\phi \wedge \psi}$$

Showing S is closed under this rule amounts to showing that if $\phi \in S$ and $\psi \in S$, then $\phi \wedge \psi \in S$. But this is exactly proving that if $P(\phi)$ and $P(\psi)$ hold, then so does $P(\phi \wedge \psi)$. We conclude by remarking that the Principle of Structural Induction arises as a special case of the clause **IL** of a general inductively defined set.

The Principle of Mathematical Induction also arises as a special case of a property of an inductively defined set. We can regard the set $\mathbb{N}$ as inductively by the rules

$$\frac{-}{0}\ (zero) \qquad\qquad \frac{n}{n+1}\ (add1)$$

Suppose we wish to show that $P(n)$ holds for all $n \in \mathbb{N}$. Let $S \stackrel{\text{def}}{=} \{\ n \in \mathbb{N} \ \mid\ P(n)\ \}$. If $S = \mathbb{N}$ we are done. But (by **IL**) we can prove $S = \mathbb{N}$ by showing that S is closed under the rules *zero* and *add1* and this amounts to precisely what one needs to verify for Mathematical Induction:

- S is closed under *zero* iff $0 \in S$ iff $P(0)$; and
- S is closed under *add1*, iff $n \in S$ implies $n+1 \in S$, iff $P(n)$ implies $P(n+1)$.

We finish this chapter by noting a useful inductive principle which subsumes the two principles given above—we call it **rule induction**.

Rule Induction

Let I be inductively defined by a set of rules $\mathcal{R}$. Suppose we wish to show that a
property $P(i)$ holds for all elements $i \in I$. Then all we need to do is *show that the set*

$$S \stackrel{\text{def}}{=} \{\ i \ \mid\ i \in I \text{ and } P(i)\ \}$$

is closed under $\mathcal{R}$.
For $S \subseteq I$ by definition, and from **IL** we get $I \subseteq S$, that is $S = I$. So if i is *any*
element of I, then $i \in S$, and so $P(i)$.

2

Logics

2.1 The Notion of A Logic

Discussion 2.1.1 It is assumed that you have met the idea of classical first order logic. We begin by describing a very general notion of a *logic*. Classical first order logic will then be one example of a logic within our general formulation. In this course we will study a number of new logics. These new logics, while having different technical details, possess a common link. This link is that the new logics all have a so-called *modal* language, and are all examples of *modal* logics.

Discussion 2.1.2 A **logic** $\mathbb{L}$ is a pair $(\mathcal{L},\vdash)$ where $\mathcal{L}$ is a set and $\vdash$ is a subset of $\mathcal{P}_{fin}(\mathcal{L}) \times \mathcal{L}$, that is, a relation between $\mathcal{P}_{fin}(\mathcal{L})$[1] and $\mathcal{L}$. We call the set $\mathcal{L}$ a **language** and an element of $\mathcal{L}$ a **sentence**. The elements of $\vdash$ are called **sequents**. Note that $\mathcal{L}$ can be *any* set, but that it will usually be a set whose elements conform to a common-sense interpretation of our idea of linguistic sentences. An example of a language $\mathcal{L}$ might be the set *Prop* of sentences of classical first order logic—see page 7. Note that $\vdash$ specifies relationships between finite sets of sentences and individual sentences—an example of $\vdash$ would be the collection of natural deduction sequents of classical first order logic. Given a logic $(\mathcal{L},\vdash)$ with $\Gamma \vdash \phi$ (recall the notation for relations on page 3) we think of the elements of Γ as hypotheses, and ϕ as a conclusion. If $\Gamma \vdash \phi$, then (informally) we might say "ϕ can be deduced from (the elements of) Γ." Note that while the *element* (Γ, ϕ) of $\vdash$ is (by definition) called a sequent, we shall also refer to $\Gamma \vdash \phi$ as a **sequent**. Finally, we write $\vdash \phi$ for a sequent $\varnothing \vdash \phi$ (which is, of course, a notation for $(\varnothing, \phi) \in \vdash$).

Discussion 2.1.3 We need some rather general ways in which to compare different logics.

• $\mathbb{L} = (\mathcal{L}',\vdash')$ is an **extension** of $\mathbb{L} = (\mathcal{L},\vdash)$ iff $\mathcal{L} \subseteq \mathcal{L}'$ and $\vdash \subseteq \vdash'$. The idea is simply that we extend a logic by giving a richer language, and by giving a greater number of sequents.

• $(\mathcal{L}',\vdash')$ is a **conservative extension** of $(\mathcal{L},\vdash)$ iff $(\mathcal{L}',\vdash')$ is an extension of $(\mathcal{L},\vdash)$, and whenever $\Gamma \vdash' \phi$ with $\Gamma \in \mathcal{P}_{fin}(\mathcal{L})$ and $\phi \in \mathcal{L}$ we have $\Gamma \vdash \phi$. Informally, we can see that this means "no new sequents can be derived" in the extension logic if we use the language of the original logic.

Discussion 2.1.4 We shall also require some simple ways of combining logics. Suppose that we are given logics $\mathbb{L}$ and $\mathbb{L}'$ with the same languages, that the set $\vdash$ of sequents of

[1] See Discussion 1.2.1

$\mathbb{L}$ is defined inductively by a set of rules $\mathcal{R}$, and that $\vdash'$ is defined by $\mathcal{R}'$. Then we define the sum **sum** $\mathbb{L} + \mathbb{L}'$ of logics to be $(\mathcal{L}, \vdash'')$ where $\vdash''$ is the set of sequents defined by the rule set $\mathcal{R}'' \stackrel{\text{def}}{=} \mathcal{R} \cup \mathcal{R}'$. Informally, then, we add two logics by combining the given rules for deducing sequents (provided that the sets of sequents of each of the two logics is specified inductively by a set of rules).

If $\mathbb{L} = (\mathcal{L}, \vdash)$ is a logic with $\vdash$ specified inductively, and R is a rule of the form (F, x) where F is a finite subset of $\mathcal{L}$ and $x \in \mathcal{L}$, then we write $\mathbb{L} + R$ for the logic whose language is again $\mathcal{L}$ and whose set of sequents is inductively defined by the set of rules $\mathcal{R} \cup \{ R \}$, where $\mathcal{R}$ is the set of rules specifying $\vdash$.

2.2 Formalising Natural Deduction

Discussion 2.2.1 Recall the set *Prop* of propositions from page 7. In the natural deduction style presentation of classical logic, rules are given for deriving sequents of the form $\Gamma \vdash \phi$ where Γ is a finite subset of *Prop* and $\phi \in \mathit{Prop}$. Now, as we remarked above, a sequent is really a pair (Γ, ϕ), we write $\vdash$ for the set of sequents, and writing $\Gamma \vdash \phi$ is a convenient notation for $(\Gamma, \phi) \in \vdash$. In fact $\vdash$ is an inductively defined set. To see this we need to show that there is a set of rules $\mathcal{R}$ for defining $\vdash$. Recalling what this means, a typical rule should look like (F, x) where F is a finite set of elements of $\vdash$, say $\{ (\Gamma_1, \phi_1), \ldots, (\Gamma_n, \phi_n) \}$, and x is an element of $\vdash$, say (Γ, ϕ). Thus a rule

$$\frac{F}{x}$$

should take the form

$$\frac{(\Gamma_1, \phi_1) \quad \ldots \quad (\Gamma_n, \phi_n)}{(\Gamma, \phi)}$$

and mean that if $(\Gamma_i, \phi_i) \in \vdash$ for $1 \leq i \leq n$ then $(\Gamma, \phi) \in \vdash$. And indeed (as an example) the rule for conjunction introduction does take this form:

$$\frac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2} \qquad \text{which is really shorthand for} \qquad \frac{(\Gamma, \phi_1) \in \vdash \qquad (\Gamma, \phi_2) \in \vdash}{(\Gamma, \phi_1 \wedge \phi_2) \in \vdash}$$

So we have explained how we can make precise the definition of natural deduction sequents using set theory—you should note that a sequent $\Gamma \vdash \phi$ as described in your course on classical logic was simply given as some abstract notation with an intended meaning, without being specified set-theoretically.

2.3 Hilbert Style Classical Logic

Discussion 2.3.1 Here we recall the Hilbert style presentation of classical propositional logic. We suppose that we are given a countable set *Var* of propositional variables, with typical elements written p, q or r. We define the classical logic $\mathbb{H} = (\mathcal{L}_C, \vdash_{\mathbb{H}})$ as follows: The language $\mathcal{L}_C$ is specified inductively by the BNF grammar

$$\phi \quad ::= \quad p \mid \neg\phi \mid \phi \rightarrow \psi$$

where p is any element of Var. We call $\mathcal{L}_C$ the set of **classical propositions** , or sometimes just the set of **propositions**. The set $\vdash_{\mathbb{H}}$ of sequents of $\mathbb{H}$ is inductively defined[2] by the set of rules $\mathcal{R}_{\mathbb{H}} \stackrel{\text{def}}{=} \{ C1, C2, C3, MP \}$ where

$$\frac{}{\vdash_{\mathbb{H}} \phi \rightarrow (\psi \rightarrow \phi)}\,(C1) \qquad \frac{}{\vdash_{\mathbb{H}} (\phi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \theta))}\,(C2)$$

$$\frac{}{\vdash_{\mathbb{H}} (\neg\psi \rightarrow \neg\phi) \rightarrow (\phi \rightarrow \psi)}\,(C3) \qquad \frac{\vdash_{\mathbb{H}} \phi \rightarrow \psi \quad \vdash_{\mathbb{H}} \phi}{\vdash_{\mathbb{H}} \psi}\,(MP)$$

Note that each "rule" is in fact a "rule schema:" when you read a rule, ϕ, ψ and θ can be *any* classical propositions. When a rule is used to deduce that a particular sequent holds, we say that the sequent arises as an **instance** of the rule. Thus both $\vdash_{\mathbb{H}} p \rightarrow (q \rightarrow p)$ and $\vdash_{\mathbb{H}} p \rightarrow (p \rightarrow p)$ both arise through instances of $C1$. Also,

$$\vdash_{\mathbb{H}} (\neg p \rightarrow q) \rightarrow (\neg r \rightarrow (\neg p \rightarrow q))$$

is an instance of $C1$, where ϕ is $\neg p \rightarrow q$ and ψ is $\neg r$.

We shall specify conjunction, disjunction and equivalence of propositions as definitions:

- $\phi \wedge \psi \stackrel{\text{def}}{=} \neg(\phi \rightarrow \neg\psi)$;
- $\phi \vee \psi \stackrel{\text{def}}{=} \neg\phi \rightarrow \psi$; and
- $\phi \leftrightarrow \psi \stackrel{\text{def}}{=} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We shall use brackets "(" and ")" informally to show how a proposition should be read (parsed), and you should recall the bracketing conditions introduced in Logic A.

Remark 2.3.2 Recall Remark 1.4.7 and Discussion 2.1.2. Note that in this presentation of classical logic, every sequent is of the form $(\varnothing, \theta)$ where $\theta \in \mathcal{L}_C$, $(\varnothing, \theta) \in \vdash_{\mathbb{H}}$ and $\vdash_{\mathbb{H}}$ is a subset of $\mathcal{P}_{fin}(\mathcal{L}_C) \times \mathcal{L}_C$. For example, $C1$ is an abbreviation of

$$\frac{}{(\varnothing, \phi \rightarrow (\psi \rightarrow \phi)) \in \vdash_{\mathbb{H}}}\,(C1)$$

and formally the rule $C1$ is just the pair $(\varnothing, (\varnothing, \phi \rightarrow (\psi \rightarrow \phi)))$. *Care—the first element of the pair (the hypothesis of the rule $C1$) is $\varnothing$, and the second (the conclusion of $C1$) is $(\varnothing, \phi \rightarrow (\psi \rightarrow \phi))$.*

Note that in this presentation of classical logic, *all sequents have empty hypotheses*, and so the subset $\vdash_{\mathbb{H}}$ of $\mathcal{P}_{fin}(\mathcal{L}) \times \mathcal{L}$ amounts to a subset $\vdash_{\mathbb{H}}$ of $\mathcal{L}_C$, by identifying each $(\varnothing, \theta)$ with θ. We adopt this perspective in the rest of this course.

[2] See Remark 1.4.7

Remark 2.3.3 Recall from Logic A that the set $\{ \phi \mid \vdash_{\mathbb{H}} \phi \}$ comprises the collection of **theorems** of $\mathbb{H}$. If $\vdash_{\mathbb{H}} \phi$, ϕ is a theorem of $\mathbb{H}$ (as just defined), but we sometimes also refer to the **theorem** $\vdash_{\mathbb{H}} \phi$ when no confusion is likely to result. Recall from Logic A that a **tautology** is a classical proposition for which $[\![\phi]\!]_v = 1$ for all valuations v. We shall assume that $\mathbb{H}$ is **complete**, that is:

Theorem 2.3.4 If $\phi \in \mathcal{L}_C$ and $[\![\phi]\!]_v = 1$ for all valuations v, then ϕ is indeed a theorem of $\mathbb{H}$, that is $\vdash_{\mathbb{H}} \phi$.

Proof See Logic A. □

Discussion 2.3.5 We finish this chapter by listing a few theorems of $\mathbb{H}$. It's easy to verify that they *are* theorems by completeness. As an exercise, try to verify some by giving derivations (labelled deductions), but do not worry if you get stuck! Here, ϕ, ψ, and θ are any classical propositions.

- $$\vdash_{\mathbb{H}} (\phi \wedge \psi) \to \phi \qquad\qquad \vdash_{\mathbb{H}} (\phi \wedge \psi) \to \psi$$

- $$\vdash_{\mathbb{H}} \phi \to (\phi \vee \psi) \qquad\qquad \vdash_{\mathbb{H}} \psi \to (\phi \vee \psi)$$

- $$\frac{\vdash_{\mathbb{H}} \phi \to \psi \quad \vdash_{\mathbb{H}} \psi \to \theta}{\vdash_{\mathbb{H}} \phi \to \theta}$$

- $$\frac{\vdash_{\mathbb{H}} (\phi \wedge \psi) \to \theta}{\vdash_{\mathbb{H}} \phi \to (\psi \to \theta)} \text{ (double line)}$$

- $$\frac{\vdash_{\mathbb{H}} \phi \to \theta \quad \vdash_{\mathbb{H}} \psi \to \theta}{\vdash_{\mathbb{H}} (\phi \vee \psi) \to \theta} \text{ (double line)}$$

- $$\frac{\vdash_{\mathbb{H}} \theta \to \phi \quad \vdash_{\mathbb{H}} \theta \to \psi}{\vdash_{\mathbb{H}} \theta \to (\phi \wedge \psi)} \text{ (double line)}$$

- $$\frac{\vdash_{\mathbb{H}} \phi \to \psi}{\vdash_{\mathbb{H}} \neg\psi \to \neg\phi} \text{ (double line)}$$

3
Introducing Modal Logic

3.1 Introduction

Discussion 3.1.1 We begin by indicating the scope of this course. You have met first order logic as a way of formalising everyday statements and arguments which are all either true or false. We have seen in Chapter 2 how to extend logics, by first giving a very general definition of a logic (of which classical first order logic is but one example). What about statements that are not absolutely true or false? We can accommodate this problem by either extending classical logic, or giving alternatives to it.

Some things we consider in everyday life may be true on one day, but false on later on. At the beginning of this term, it is true that Dr Crole is giving a course of lectures on modal logic. But if he wins the lottery, this might become false, because he decides he'd rather be sun-bathing in Barbados. So we have notions of truth which depend on the passage of time—we call these notions **temporal**.

The statement that "2+4 = 6" is true, and in fact is necessarily true. But the statement that "Major is Prime Minister" is not necessarily true. However "Major is Prime Minister" is possibly true. This course deals with the logic of *necessity* and *possibility*. You should note that these "connectives" are not truth-functional, in the sense that if they "act" on true sentences, the resulting sentence might be true or false:

$2 + 4 = 6$	is	true
necessarily$(2 + 4 = 6)$	is	true
Major is Prime Minister	is	true
necessarily(Major is Prime Minister)	is	false
possibly(Major is Prime Minister)	is	true

So we cannot formalise these properties within classical logic, because classically all propositions are true or false—period. In first order logic, a language involving the connectives "and" ($\wedge$), "or" ($\vee$) and so on was built up. In this course we extend this language to include connectives $\Box$—meaning "necessarily," and $\Diamond$—meaning "possibly". These meanings are informal semantics for $\Box$ and $\Diamond$. One should note that "possibly ϕ" can be regarded as being the same as "it is not the case that ϕ is necessarily false." We shall define $\Diamond$ as an abbreviation for $\neg\Box\neg$, just as (for example) we can define $\wedge$ in terms of $\neg$ and $\rightarrow$.

To summarise, we shall study logics which extend classical logic with connectives for necessity and possibility, and the things we study about such logics will be very similar to those found in Logic A.

3.2 A Modal Language

Discussion 3.2.1 The modal logics which we consider in this course are all extensions of Hilbert style classical propositional logic. In order to define our modal logics, we first need to give a language. Note that each modal logic we consider has the same language. We shall write $\mathcal{L}_M$ for it, and define it inductively by the BNF grammar

$$\phi \quad ::= \quad p \mid \neg\phi \mid \phi \to \psi \mid \Box\phi$$

where p is any propositional variable drawn from a fixed set *Var* of **propositional variables**. We call an element $\phi \in \mathcal{L}_M$ a **modal proposition**. We shall sometimes simply refer to ϕ as a proposition, when no confusion can result.

We shall define a macro for our language $\mathcal{L}_M$ by setting $\Diamond\phi \stackrel{\text{def}}{=} \neg\Box\neg\phi$. Each of the symbols $\Box$ and $\Diamond$ are called **modalities**. We call $\Box$ the **necessity** modality, and $\Diamond$ the **possibility** modality. Note that $\mathcal{L}_C \subseteq \mathcal{L}_M$, which is part of the requirements for each of our modal logics to be extensions of the classical logic $\mathbb{H}$. We shall write $\Box^n\phi$, where $n \geq 1$ is a natural number, to mean the modal proposition with n occurrences of $\Box$ before ϕ; $\Diamond^n\phi$ is similar.

We shall now introduce a number of modal logics, each new logic being a non-conservative extension of its predecessor.

3.3 The Modal Logic $\mathbb{K}$

Discussion 3.3.1 The logic $\mathbb{K}$ has language $\mathcal{L}_M$, and its set of sequents $\vdash_{\mathbb{K}}$ (in fact set of theorems) is defined by the rule set $\mathcal{R}_{\mathbb{K}} \stackrel{\text{def}}{=} \{ M1, M2, M3, MP, D, N \}$ where

• The rules $M1$, $M2$, $M3$ and MP are "the same as" the rules $C1$, $C2$, $C3$ and MP but the propositions ϕ, ψ and θ appearing in the rules are now modal propositions, that is, elements of $\mathcal{L}_M$ rather than $\mathcal{L}_C$.

• The rule D is given by

$$\frac{}{\vdash_{\mathbb{K}} \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)}\,(D)$$

• The rule N is given by

$$\frac{\vdash_{\mathbb{K}} \phi}{\vdash_{\mathbb{K}} \Box\phi}\,(N)$$

We say that a modal logic $(\mathcal{L}_M, \vdash)$ is **normal** if the set of rules for defining $\vdash$ contains $\mathcal{R}_{\mathbb{K}}$—so in particular $\mathbb{K}$ is normal. In fact all of the modal logics we shall consider are normal. One reason for this is that any normal modal logic has a semantics which takes on a rather pleasant form, and non-normal modal logics do not have such nice semantics. Another reason is that it is quite easy to prove that "equivalent propositions can replace each other in a theorem", provided that the modal logic is normal. We have a result

(Theorem 3.3.8) which formalises this idea, and we prove this in due course. First, we make further comments about normal modal logics extending classical logic.

Remark 3.3.2 Let $\mathbb{L} = (\mathcal{L}_M, \vdash)$ be any normal modal logic. $\mathbb{L}$ *extends* $\mathbb{K}$, and $\mathbb{K}$ *extends* $\mathbb{H}$, because $\mathcal{L}_C \subseteq \mathcal{L}_M$ (by definition) and also $\vdash_{\mathbb{H}} \subseteq \vdash_{\mathbb{K}} \subseteq \vdash$ (because $\mathbb{L}$ contains all of the rules for deriving theorems of $\mathbb{K}$, and $\mathbb{K}$ contains all of the rules for deriving theorems of $\mathbb{H}$). In particular, we might read $\vdash_{\mathbb{H}} \subseteq \vdash$ as saying that any classical theorem is a theorem of any normal modal logic.

Remark 3.3.3 However, we can also see that the set of theorems of $\mathbb{L}$ contains all of the theorems of $\mathbb{H}$, that is, the classical theorems, *in which propositions "appearing in" a classical theorem range over* $\mathcal{L}_M$. To see this, recall the definitions of the rules $M1$, $M2$, $M3$ and MP which $\mathbb{L}$ contains because it is normal. For example, $\vdash_{\mathbb{K}} p \rightarrow (p \vee \Box r)$ is a theorem, as is $\vdash_{\mathbb{K}} \phi \rightarrow (\psi \vee \phi)$ where ϕ and ψ are any *modal propositions*, because $\vdash_{\mathbb{H}} p \rightarrow (p \vee q)$ is a classical theorem. We make this more precise in a (very useful) lemma, after we have defined substitution:

Discussion 3.3.4 Let $\phi, \psi \in \mathcal{L}_M$, and let p and q be any distinct propositional variables. We shall define a new modal proposition, written $\phi[\psi/p]$, by induction over the structure of ϕ:

- $p[\psi/p] \stackrel{\text{def}}{=} \psi$ and $q[\psi/p] \stackrel{\text{def}}{=} q$ (remember $p \neq q$ by assumption);
- $(\neg\theta)[\psi/p] \stackrel{\text{def}}{=} \neg(\theta[\psi/p])$;
- $(\theta \rightarrow \theta')[\psi/p] \stackrel{\text{def}}{=} \theta[\psi/p] \rightarrow \theta'[\psi/p]$; and
- $(\Box\theta)[\psi/p] \stackrel{\text{def}}{=} \Box(\theta[\psi/p])$.

We say that the proposition $\phi[\psi/p]$ has resulted from the **substitution** of ψ for all **occurrences** of p in ϕ. Note that one should also check that this is a good definition, that is, $\phi[\psi/p]$ *really is* an element of $\mathcal{L}_M$. This follows from a simple (structural) induction—do it as an exercise.

Example 3.3.5 Suppose that $\phi \stackrel{\text{def}}{=} \neg(p \rightarrow (q \rightarrow p))$ and $\psi \stackrel{\text{def}}{=} \Box r$, where p, q and r are propositional variables. Then using the above definition we have

$$\begin{array}{ccl} \phi[\psi/p] & = & \neg(p \rightarrow (q \rightarrow p))[\Box r/p] \\ (1) & = & \neg(p[\Box r/p] \rightarrow (q \rightarrow p)[\Box r/p]) \\ (2) & = & \neg(p[\Box r/p] \rightarrow (q[\Box r/p] \rightarrow p[\Box r/p]) \\ & = & \neg(\Box r \rightarrow (q \rightarrow \Box r)). \end{array}$$

Lemma 3.3.6 Let $\mathbb{L} = (\mathcal{L}_M, \vdash)$ be any normal modal logic. Let ϕ be any classical proposition, that is $\phi \in \mathcal{L}_C$, where $\vdash_{\mathbb{H}} \phi$. If ψ is a modal proposition, then $\vdash \phi[\psi/p]$.

Proof We use Rule Induction—see page 9. Fix any modal proposition ψ. Set

$$S \stackrel{\text{def}}{=} \{ \phi \mid \vdash_{\mathbb{H}} \phi \text{ and} \vdash \phi[\psi/p] \} \subseteq \vdash_{\mathbb{H}}.$$

We can show (exercise) that S is closed under the rules defining $\vdash_{\mathbb{H}}$, hence $\vdash_{\mathbb{H}} \subseteq S$, and so $S = \vdash_{\mathbb{H}}$. The result follows—if $\vdash_{\mathbb{H}} \phi$, that is[1] $\phi \in \vdash_{\mathbb{H}}$, then $\phi \in S$ and so $\vdash \phi[\psi/p]$. □

Remark 3.3.7 We can capture the essence of Lemma 3.3.6 by saying that any normal modal logic "*contains classical reasoning*." These ideas are important, and will be used throughout the course. For example, let $\mathbb{L} = (\mathcal{L}_M, \vdash)$ be a normal modal logic and $\vdash \phi$ and $\vdash \psi$. By "classical reasoning" we would have $\vdash \phi \to \psi$ and thus we claim this is indeed a theorem of $\mathbb{L}$. We can prove this precisely: $\mathbb{L}$ is normal, so contains $\vdash p \to q \to (p \to q)$ (because this is a classical theorem) so using Lemma 3.3.6 twice we have $\vdash \phi \to \psi \to (\phi \to \psi)$. The conclusion follows from *MP* (applied twice) to the assumptions $\vdash \phi$ and $\vdash \psi$.

Theorem 3.3.8 Let $\mathbb{L} = (\mathcal{L}_M, \vdash)$ be a normal modal logic, and let $\vdash \psi \leftrightarrow \psi'$. Then for any $\phi \in \mathcal{L}_M$, $\vdash \phi[\psi/p] \Longleftrightarrow \vdash \phi[\psi'/p]$.

Proof We induct on the structure of ϕ; we give one case:

(*Case ϕ is $\Box\phi$*): Suppose that $\vdash (\Box\phi)[\psi/p]$. By induction we have that $\vdash \phi[\psi/p]$ implies $\vdash \phi[\psi'/p]$, and so $\vdash \phi[\psi/p] \to \phi[\psi'/p]$ is a theorem of $\mathbb{L}$ via classical reasoning. Hence from N we have $\vdash \Box(\phi[\psi/p] \to \phi[\psi'/p])$, and using the rules D and *MP* we have $\vdash \Box(\phi[\psi/p]) \to \Box(\phi[\psi'/p])$. Using this, the supposition, and *MP* we deduce $\vdash (\Box\phi)[\psi'/p]$. □

Remark 3.3.9 We shall refer to applications of Lemma 3.3.8 as "*substitution of equivalent propositions*." For example, let $\mathbb{L} = (\mathcal{L}_M, \vdash)$ be a normal modal logic. Then

$$\vdash (\neg\neg\neg\psi \to \neg\phi) \to (\phi \to \neg\neg\psi)$$

follows (from $M3$) by substitution of equivalent propositions (for $\vdash \psi \leftrightarrow \neg\neg\psi$ by classical reasoning).

Examples 3.3.10 In these examples we look at deriving theorems in the modal logic $\mathbb{K}$. We present the derivations as labelled deductions of elements of the inductively defined set $\vdash_{\mathbb{K}}$.

(i) If p and q are propositional variables, then $\vdash_{\mathbb{H}} p \vee \neg p$. If we take ϕ to be $p \vee \neg p$ and ψ to be $\Box\Box\Diamond q$ in Lemma 3.3.6 then we see that $\vdash_{\mathbb{K}} (\Box\Box\Diamond q) \vee (\neg\Box\Box\Diamond q)$.

[1] See end of Remark 2.3.2

(ii)

$$\begin{array}{lll}
\vdash_{\mathbb{K}} \phi \to (\phi \vee \psi) & & [1] \\
\vdash_{\mathbb{K}} \Box(\phi \to (\phi \vee \psi)) & (N) & \\
\vdash_{\mathbb{K}} \Box(\phi \to (\phi \vee \psi)) \to (\Box\phi \to \Box(\phi \vee \psi)) & (D) & \\
\vdash_{\mathbb{K}} \Box\phi \to \Box(\phi \vee \psi) & (MP) &
\end{array}$$

where [1] follows by classical reasoning.

(iii)

$$\begin{array}{ll}
\vdash_{\mathbb{K}} \phi \to (\psi \to \phi) & (M1) \\
\vdash_{\mathbb{K}} \Box(\phi \to (\psi \to \phi)) \to (\Box\phi \to \Box(\psi \to \phi)) & (D) \\
\vdash_{\mathbb{K}} \Box(\phi \to (\psi \to \phi)) & (N) \\
\vdash_{\mathbb{K}} \Box\phi \to \Box(\psi \to \phi) & (MP)
\end{array}$$

3.4 Three More Modal Logics

Discussion 3.4.1 Throughout this course we shall concentrate on four particular (normal) modal logics. These logics are some of the most fundamental of all modal logics. They have simple and interesting models, and will allow us to study the basic ideas behind semantics of modal logic, without having to become too involved with complex mathematical machinery. We have introduced $\mathbb{K}$; now we specify three more modal systems:

The Modal Logic $\mathbb{T}$

The modal logic $\mathbb{T}$ is a non-conservative extension of $\mathbb{K}$. The set of rules for defining the set $\vdash_{\mathbb{T}}$ of sequents of $\mathbb{T}$ is given by $\mathcal{R}_{\mathbb{T}} \stackrel{\text{def}}{=} \mathcal{R}_{\mathbb{K}} \cup \{\, T \,\}$, where the rule T is

$$\frac{}{\vdash_{\mathbb{T}} \Box\phi \to \phi}$$

Thus we have $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{K} + T$. As an example of a $\mathbb{T}$-theorem, we have:

Lemma 3.4.2 We have $\vdash_{\mathbb{T}} \phi \to \Diamond\phi$ in $\mathbb{T}$ for any $\phi \in \mathcal{L}_M$.

Proof

$$\begin{array}{lll}
\vdash_{\mathbb{T}} \Box\neg\phi \to \neg\phi & (T) & \\
\vdash_{\mathbb{T}} (\Box\neg\phi \to \neg\phi) \to (\neg\neg\phi \to \neg\Box\neg\phi) & & [1] \\
\vdash_{\mathbb{T}} \neg\neg\phi \to \Diamond\phi & (MP) & [2]
\end{array}$$

[1] follows from $M3$ and a careful substitution of equivalent propositions. The result follows from [2] by another substitution of equivalent propositions. □

The Modal Logic $\mathbb{S}4$

We define $\mathbb{S}4$ to be $\mathbb{T} + 4$ where the rule 4 is given by

$$\frac{}{\vdash_{\mathbb{S}4} \Box\phi \to \Box\Box\phi}\,(4)$$

The Modal Logic $\mathbb{S}5$

The modal logic $\mathbb{S}5$ is a non-conservative extension of $\mathbb{S}4$. In fact $\mathbb{S}5$ is defined to be $\mathbb{S}4 + E$ where the rule E is given by

$$\frac{}{\vdash_{\mathbb{S}5} \Diamond\phi \to \Box\Diamond\phi}\,(E)$$

3.5 Some Results about the Modal Logics

Discussion 3.5.1 In this section we build up some facts about our modal logics, and develop ways of manipulating theorems. A characteristic feature of $\mathbb{S}4$ is that any modal proposition of the form $\Box^n\phi$ with $n \geq 2$ which appears in a theorem ψ, can be replaced by the proposition $\Box\Box\phi$. A similar statement holds for $\Diamond$. We shall make this precise, but first require a lemma:

Lemma 3.5.2 Let $\mathbb{L} = (\mathcal{L}_M, \vdash)$ be any normal modal logic. Then a derived rule is

$$\frac{\vdash \phi \to \psi}{\vdash \Box\phi \to \Box\psi}$$

Proof Starting from the hypotheses of the rule, apply N, take the corresponding instance of D, and use MP. $\Box$

Proposition 3.5.3 We have $\vdash_{\mathbb{S}4} \Box\phi \leftrightarrow \Box\Box\phi$ for all $\phi \in \mathcal{L}_M$.

Proof Immediate from the rules 4 and T of $\mathbb{S}4$. $\Box$

Proposition 3.5.4 We have $\vdash_{\mathbb{S}4} \Diamond\phi \leftrightarrow \Diamond\Diamond\phi$ for all $\phi \in \mathcal{L}_M$.

Proof

(*Show* $\vdash_{\mathbb{S}4} \Diamond\Diamond\phi \to \Diamond\phi$):

$$\begin{array}{lll}
\vdash_{\mathbb{S}4} \Box\neg\phi \to \neg\neg\Box\neg\phi & & [1] \\
\vdash_{\mathbb{S}4} \Box\Box\neg\phi \to \Box\neg\neg\Box\neg\phi & & [2] \\
\vdash_{\mathbb{S}4} \Box\neg\phi \to \Box\Box\neg\phi & (4) & \\
\vdash_{\mathbb{S}4} \Box\neg\phi \to \Box\neg\neg\Box\neg\phi & & [3] \\
\vdash_{\mathbb{S}4} \neg\Box\neg\neg\Box\neg\phi \to \neg\Box\neg\phi & & [4]
\end{array}$$

[1] follows by classical reasoning and substitution of equivalent propositions. [2] follows from [1] by Lemma 3.5.2. [3] follows from its previous two deductions by classical reasoning. [4] follows from [3] by using $M3$ and MP.

(*Show* $\vdash_{\mathbb{S}4} \Diamond\phi \to \Diamond\Diamond\phi$): This follows from Lemma 3.4.2.

The proposition follows by classical reasoning. $\Box$

Discussion 3.5.5 We defined $\mathbb{S}5$ as an extension to $\mathbb{S}4$ by adding in just one rule (E) for defining sequents. In fact, $\mathbb{S}5$ can be obtained by adding E to $\mathbb{T}$. We make this precise in the next theorem:

Theorem 3.5.6 $\mathbb{S}4 + E = \mathbb{T} + E$.

Proof This is claiming that $(\mathcal{L}_M, \vdash') = (\mathcal{L}_M, \vdash)$, where $\vdash'$ is defined by $\mathcal{R}_\mathbb{T} \cup \{ E \} \cup \{ 4 \}$ and $\vdash$ is defined by $\mathcal{R}_\mathbb{T} \cup \{ E \}$. To show that $\vdash \subseteq \vdash'$ we need to show that $\vdash'$ is closed under $\mathcal{R}_\mathbb{T} \cup \{ E \}$, which is trivially so. To show that $\vdash' \subseteq \vdash$, we need to show that $\vdash$ is closed under $\mathcal{R}_\mathbb{T} \cup \{ E \} \cup \{ 4 \}$. Thus all we have to do is show that $\vdash$ is closed under 4, that is

$$\text{for all } \phi \in \mathcal{L}_M.\ \vdash \Box\phi \to \Box\Box\phi.$$

We have

$$
\begin{array}{lll}
\vdash \Diamond\neg\phi \to \Box\Diamond\neg\phi & (E) & \\
\vdash \neg\Box\Diamond\neg\phi \to \neg\Diamond\neg\phi & & [1] \\
\vdash \Box\phi \leftrightarrow \neg\Diamond\neg\phi & & [2] \\
\vdash \neg\Box\Diamond\neg\phi \to \Box\phi & & [3] \\
\vdash \Diamond\Box\neg\neg\phi \to \Box\phi & & [4] \\
\vdash \neg\neg\phi \leftrightarrow \phi & & [5] \\
\vdash \Diamond\Box\phi \to \Box\phi & & [6] \\
\vdash \Box\Diamond\Box\phi \to \Box\Box\phi & & [*]
\end{array}
$$

[1] follows from the first theorem using $M3$ and MP. [2] arises from classical reasoning and substitution. [3] follows from [1] and [2] using Theorem 3.3.8, [4] follows from [3] by definition of $\Diamond$, [6] follows from [4] and [5] by using Theorem 3.3.8, and [*] follows from [6] using Lemma 3.5.2.

We also have

$$
\begin{array}{llll}
\vdash \Box\phi \to \Diamond\Box\phi & & & [7] \\
\vdash \Diamond\Box\phi \to \Box\Diamond\Box\phi & (E) & & \\
\vdash \Box\phi \to \Box\Diamond\Box\phi & & [8] & [**]
\end{array}
$$

[7] follows from Lemma 3.4.2. [8] follows by classical reasoning.

Using classical reasoning on [*] and [**] yields the desired conclusion. □

3.6 Theorems of Modal Logic

Discussion 3.6.1 We list some more theorems of modal logic, some of which will be used later on in the course.

- $$\vdash_\mathbb{K} \Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi).$$

- $$\vdash_\mathbb{K} \Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi).$$

- $$\vdash_{\mathbb{K}} (\Box\phi \vee \Box\psi) \rightarrow \Box(\phi \vee \psi).$$

- $$\vdash_{\mathbb{K}} \Diamond(\phi \wedge \psi) \rightarrow (\Diamond\phi \wedge \Diamond\psi).$$

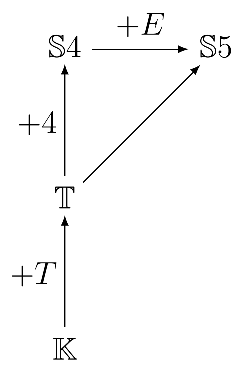
- $$\frac{\vdash_{\mathbb{K}} \phi \rightarrow \psi}{\vdash_{\mathbb{K}} \Diamond\phi \rightarrow \Diamond\psi}$$

3.7 A Summary

We can summarise the results of this Chapter as follows:

$$\begin{array}{rcl} \mathbb{S}5 & = & \mathbb{S}4 + E \;\; = \;\; \mathbb{T} + 4 + E \;=\; \mathbb{T} + E \\ \mathbb{S}4 & = & \mathbb{T} + 4 \\ \mathbb{T} & = & \mathbb{K} + T \end{array}$$

We illustrate the above pictorially:



The diagram shows a table of modal logic extensions. Recall also that each is normal, so extends the classical logic $\mathbb{H}$.

4
Kripke Semantics

4.1 Introduction

Discussion 4.1.1 So far we have explained the language of modal logic, given an intuitive meaning (or semantics) to the language, and developed a proof theory for deriving theorems. Everything thus far has dealt with the syntax of modal logic, and the manipulation of the syntax. We shall now look at the semantics of modal logic.

Recall the semantics of classical logic. A meaning is given to a classical proposition by asserting whether the proposition is true or false—and precise rules are given for computing the semantic truth or falsity of a proposition, given knowledge of the truth or falsity of the propositional variables appearing in the proposition. One very good reason for defining such a semantics is that one can prove the completeness of classical logic—thus we can use the model theory to verify whether propositions are theorems or not.

We have already seen that modal propositions cannot be regarded as simply true or false. We have also seen that modal propositions can be regarded as making statements about the future, that is, a notion of time is involved. In fact we shall describe a semantics in which there is a concept of changing worlds (think of the passage of time on the earth as giving rise to a changing world) and in which the modal propositions can then be regarded as true or false with respect to a particular world. In our semantics, a binary relation between worlds is specified, and if a pair of worlds appear in the relation, we can think of the second world as the first world seen at a later time—a future world. Very roughly, a modal proposition of the form $\Box\phi$ can be thought of as true in a given world iff ϕ is true in all future worlds, and $\Diamond\phi$ can be thought of as true in a given world iff ϕ is true in some future world.

One word of warning about this "real life" visualisation of the semantics which we give. It suggests that the relation between worlds is anti-symmetric—exercise: why? But in the formal definition of the semantics it can in fact be any relation! Perhaps a more accurate intuition is that if a world, say W, is related to a different world, say W', then W' is "accessible" from W—but W' might also be "accessible" from W, which is fine provided the relation does not have to be anti-symmetric. Bearing all this in mind, let's read on:

4.2 Kripke Frames and Models

Discussion 4.2.1 A **Kripke frame** is a pair $(\mathcal{W}, R)$ where $\mathcal{W}$ is any non-empty set of **worlds**, and R is any binary relation on $\mathcal{W}$. Though $\mathcal{W}$ can be any non-empty set, one should think of the elements of $\mathcal{W}$ as worlds or universes W in which a proposition may or may not hold. If W and W' are worlds, that is $W, W' \in \mathcal{W}$, and if $W\ R\ W'$, then you should think of the world W' as "reachable" or "accessible" from W.

A **Kripke model** is a triple $(\mathcal{W}, R, \Vdash)$ where $(\mathcal{W}, R)$ is a Kripke frame and $\Vdash$ is a relation between $\mathcal{W}$ and $\mathcal{L}_M$, that is, $\Vdash$ is a subset of $\mathcal{W} \times \mathcal{L}_M$, for which the following properties hold:

$$\frac{W \Vdash \neg\phi}{W \nVdash \phi} \qquad \frac{W \Vdash \phi \to \psi}{W \Vdash \phi \Longrightarrow W \Vdash \psi} \qquad \frac{W \Vdash \Box\phi}{W' \Vdash \phi \quad \text{for all } W' \text{ such that } W\ R\ W'}$$

If $\mathbf{F} = (\mathcal{W}, R)$ we sometimes write $(\mathbf{F}, \Vdash)$ for a model of the form $(\mathcal{W}, R, \Vdash)$. We shall say that a Kripke model $\mathbf{M}$ is **based** on a Kripke frame $\mathbf{F}$ if $\mathbf{M}$ is of the form $(\mathbf{F}, \Vdash)$. If $W \Vdash \phi$ then we say that the world W **forces** the proposition ϕ. We call the relation $\Vdash$ a **forcing** relation. We say that $\mathbf{F}$ (or $\mathbf{M}$) is **reflexive** if R is reflexive—with analogous definitions for other properties of relations. Note that a consequence of the definition of Kripke model is:

Proposition 4.2.2 Any Kripke model $(\mathcal{W}, R, \Vdash)$ satisfies

$$\frac{W \Vdash \phi \wedge \psi}{W \Vdash \phi \text{ and } W \Vdash \psi} \qquad \frac{W \Vdash \phi \vee \psi}{W \Vdash \phi \text{ or } W \Vdash \psi}$$

$$\frac{W \Vdash \Diamond\phi}{W \Vdash \phi \quad \text{for some world } W' \text{ such that } W\ R\ W'}$$

Proof A simple exercise, using the definitions of the connectives. □

Discussion 4.2.3 We shall say that a modal proposition ϕ is **valid in a Kripke model** $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ iff $W \Vdash \phi$ for all worlds $W \in \mathcal{W}$. We shall write $\models_{\mathbf{M}} \phi$ for ϕ is valid in the Kripke model $\mathbf{M}$. We say that ϕ is **valid in a Kripke frame** $\mathbf{F} = (\mathcal{W}, R)$ iff ϕ is valid in every Kripke model based on $\mathbf{F}$. We write $\models_{\mathbf{F}} \phi$ to indicate this. We say that ϕ is **valid** if $\models_{\mathbf{F}} \phi$ for *all* Kripke frames $\mathbf{F}$, and we write $\models \phi$ to mean that ϕ is valid. We have

Proposition 4.2.4 Let ϕ be a modal proposition. Then following are equivalent:

(i) $\models \phi$;

(ii) $\models_{\mathbf{F}} \phi$ for all Kripke frames $\mathbf{F}$; and

(iii) $\models_{\mathbf{M}} \phi$ for all Kripke models $\mathbf{M}$.

Proof (i) and (ii) are equivalent by definition. (ii) follows by a *careful* examination of the definitions. □

4.3 Soundness Results

Discussion 4.3.1 Recall the notion of *soundness* from Logic A. A model is sound for a logic if whenever a sequent is derived in the logic, the model satisfies the sequent. We present such ideas for modal logics:

If $\mathbb{L} = (\mathcal{L}_M, \vdash_{\mathbb{L}})$ is a modal logic for which $\vdash_{\mathbb{L}}$ is specified inductively by $\mathcal{R}_{\mathbb{L}}$, we say that a *frame* $\mathbf{F}$ is **sound** for $\mathbb{L}$ iff

$$\forall \phi \in \mathcal{L}_M. \quad \vdash_{\mathbb{L}} \phi \Longrightarrow \models_{\mathbf{F}} \phi.$$

We say that a *model* $\mathbf{M}$ is **sound** for $\mathbb{L}$ iff

$$\forall \phi \in \mathcal{L}_M. \quad \vdash_{\mathbb{L}} \phi \Longrightarrow \models_{\mathbf{M}} \phi.$$

We say that a *model* $\mathbf{M}$ is **sound** for a *rule* $R \in \mathcal{R}_{\mathbb{L}}$ if the set $\{ \phi \mid \models_{\mathbf{M}} \phi \}$ is closed under R. We say that a *frame* $\mathbf{F}$ is **sound** for a *rule* $R \in \mathcal{R}_{\mathbb{L}}$ if the set $\{ \phi \mid \models_{\mathbf{F}} \phi \}$ is closed under R.

Theorem 4.3.2 Let $\mathbf{F}$ be any Kripke frame. Then $\mathbf{F}$ is sound for $\mathbb{K}$.

Proof We have to show that for any $\phi \in \mathcal{L}_M$, $\vdash_{\mathbb{K}} \phi \Longrightarrow \models_{\mathbf{M}} \phi$ for any $\mathbf{M}$ based on $\mathbf{F}$. Thus [1] we show that if $\mathbf{M}$ is any such Kripke model, $\vdash_{\mathbb{K}} \phi \Longrightarrow \models_{\mathbf{M}} \phi$.

Let $P \stackrel{\text{def}}{=} \{ \phi \mid \vdash_{\mathbb{K}} \phi \text{ and} \models_{\mathbf{M}} \phi \}$ where $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ is any Kripke model based on $\mathbf{F}$. We shall show that P is closed under the rules defining $\vdash_{\mathbb{K}}$. Thus as $P \subseteq \vdash_{\mathbb{K}}$ by definition, we have $P = \vdash_{\mathbb{K}}$ and the theorem follows—see rule induction.

(*Closure of P under $M1$*): We have to show that $\models_{\mathbf{M}} \phi \to (\psi \to \phi)$ for any modal propositions ϕ and ψ, that is $W \Vdash \phi \to (\psi \to \phi)$ for any $W \in \mathcal{W}$. So pick any such W, and we have to verify that

$$W \Vdash \phi \Longrightarrow (W \Vdash \psi \Longrightarrow W \Vdash \phi)$$

which is of course true.

(*Closure of P under $M2$, $M3$, MP, N*): Easy exercises.

(*Closure of P under D*): We have to prove that $\models_{\mathbf{M}} \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. So we pick any $W \in \mathcal{W}$ for which $W \Vdash \Box(\phi \to \psi)$ and $W \Vdash \Box\phi$ and we shall prove that $W \Vdash \Box\psi$. This means we have to show that for all worlds W' with $W \; R \; W'$, one has $W' \Vdash \psi$. But for any such W' we have $W' \Vdash \phi \to \psi$ and $W' \Vdash \phi$ from the assumptions, and the result follows from (one of) the properties of $\Vdash$.

$\Box$

Discussion 4.3.3 We shall need to construct examples of Kripke models based on Kripke frames. Let $\mathbf{F} = (\mathcal{W}, R)$ be a frame, and $S \subseteq \mathcal{W} \times \mathit{Var}$ any subset. Then there is a model $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ based on $\mathbf{F}$ where $\Vdash$ is specified through the following clauses:

[1] Remember that "$\forall x.P \Longrightarrow Q(x)$" is equivalent to "$P \Longrightarrow \forall x.Q(x)$" when P does not depend on x. Above, x is $\mathbf{M}$, P is $\vdash_{\mathbb{K}} \phi$, and $Q(x)$ is $\models_{\mathbf{M}} \phi$.

(i) $W \Vdash p$ iff $(W, p) \in S$;

(ii) $W \Vdash \neg\phi$ iff $W \nVdash \phi$[2];

(iii) $W \Vdash \phi \to \psi$ iff $W \Vdash \phi$ implies $W \Vdash \psi$; and

(iv) $W \Vdash \Box\phi$ iff $W' \Vdash \phi$ for all worlds W' for which $W\ R\ W'$.

Note that $\Vdash$ will indeed give rise to a Kripke model! We say that such a model has been **generated** from S.

Theorem 4.3.4 Let $\mathbf{F}$ be any Kripke frame. Then

(a) $\mathbf{F}$ is reflexive iff $\mathbf{F}$ is sound for T.

(b) $\mathbf{F}$ is transitive iff $\mathbf{F}$ is sound for 4.

(c) $\mathbf{F}$ is euclidean iff $\mathbf{F}$ is sound for E.

Proof Note that parts (b) and (c) use the same proof *method* as part (a), but omit preliminary details which are analogous to those in part (a).

(a) ($\Rightarrow$) Let $\mathbf{F}$ be reflexive. We have to show that $A = \{\ \psi\ \mid \models_{\mathbf{F}} \psi\ \}$ is closed under T, which simply amounts to showing that $\Box\phi \to \phi \in A$ for any ϕ, that is $\models_{\mathbf{F}} \Box\phi \to \phi$. So choose any (arbitrary) model based on $\mathbf{F}$, say $\mathbf{M} = (\mathbf{F}, \Vdash)$, and we shall show that $\models_{\mathbf{M}} \Box\phi \to \phi$. To show the latter, we pick an arbitrary world W and show that $W \Vdash \Box\phi \to \phi$. Suppose that $W \Vdash \Box\phi$. Then $W' \Vdash \phi$ for all worlds W' where $W\ R\ W'$, and from reflexivity $W \Vdash \phi$, so we are done—W was arbitrary, so $W \Vdash \Box\phi \to \phi$ for all $W \in \mathcal{W}$, that is $\models_{\mathbf{M}} \Box\phi \to \phi$.

($\Leftarrow$) Let $\mathbf{F} = (\mathcal{W}, R)$ be a frame which is not reflexive. We shall show that $\mathbf{F}$ is not sound for T—which amounts to finding a model $\mathbf{M}$ based on $\mathbf{F}$, and a modal proposition ϕ, for which $\models_{\mathbf{M}} \phi$ does not hold. Define $S \subseteq \mathcal{W} \times \mathit{Var}$ by

$$S \stackrel{\text{def}}{=} \{\ (W', p)\ \mid\ U\ R\ W' \text{ and } W' \in \mathcal{W}\ \}$$

where p is a fixed propositional variable and the world U satisfies $U\ \not R\ U$. Note that $(U, p) \notin S$. Suppose that $\models_{\mathbf{M}} \Box p \to p$ where $\mathbf{M}$ is the model generated from S. Then in particular $U \Vdash \Box p \to p$, $U \Vdash \Box p$ holds because $\Vdash$ is generated from S, and so $U \Vdash p$. Hence $(U, p) \in S$ which is a contradiction. Therefore $\models_{\mathbf{M}} \Box p \to p$ does not hold, and so $\mathbf{F}$ is not sound for T.

(b) ($\Rightarrow$) Let $\mathbf{F}$ be transitive, and $\mathbf{M} = (\mathbf{F}, \Vdash)$ any model based on $\mathbf{F}$. It is enough to show that $\models_{\mathbf{M}} \Box\phi \to \Box\Box\phi$ for any proposition ϕ, so pick any world W and we show that $W \Vdash \Box\phi \to \Box\Box\phi$. Suppose that $W \Vdash \Box\phi$. Let W' and W'' be any worlds for which $W\ R\ W'\ R\ W''$. Then $W\ R\ W''$ and hence $W'' \Vdash \phi$ by the supposition, implying that $W' \Vdash \Box\phi$ (because W'' was arbitrary) and thus $W \Vdash \Box\Box\phi$ (because W' was arbitrary).

[2] See Remark 1.3.2

($\Leftarrow$) Let $\mathbf{F} = (\mathcal{W}, R)$ be a frame which is not transitive. Then there are worlds U, U', U'' with $U\ R\,U'\ R\,U''$ but $U \not R\, U''$. Put

$$S \stackrel{\text{def}}{=} \{\ (W', p)\ \mid\ U\ R\,W' \text{ and } W' \in \mathcal{W}\ \}$$

where p is a fixed propositional variable. Let $\mathbf{M} = (\mathbf{F}, \Vdash)$ be the model generated by S. Suppose that $\models_{\mathbf{M}} \Box p \to \Box\Box p$. Then in particular $U \Vdash \Box p$ implies $U \Vdash \Box\Box p$: but the former holds by definition of $\Vdash$, hence the latter holds too. But $U \Vdash \Box\Box p$ implies $U'' \Vdash p$ and so $U\ R\,U''$, a contradiction. So the supposition is false, that is, $\models_{\mathbf{M}} \Box p \to \Box\Box p$ does not hold.

(c) ($\Rightarrow$) Let $\mathbf{F}$ be euclidean, and $\mathbf{M} = (\mathbf{F}, \Vdash)$ any model based on $\mathbf{F}$. We shall show that $\models_{\mathbf{M}} \Diamond\phi \to \Box\Diamond\phi$ for any ϕ. Pick any arbitrary world W and suppose that $W \Vdash \Diamond\phi$. We have to show that $W' \Vdash \Diamond\phi$ for any world W' where $W\ R\,W'$. So choose any such W'. By the supposition, $W'' \Vdash \phi$ for some world W'' where $W\ R\,W''$, and hence $W'\ R\,W''$. Thus $W' \Vdash \Diamond\phi$ as required.

($\Leftarrow$) Let $\mathbf{F}$ be non-euclidean. Hence there are worlds U, U' and U'' with $U\ R\ U'$, $U\ R\ U''$ but $U' \not R\ U''$. Fix a propositional variable p and set $S \stackrel{\text{def}}{=} \{(U'', p)\}$. Let $\mathbf{M} = (\mathbf{F}, \Vdash)$ be generated by S. Suppose that $\models_{\mathbf{M}} \Diamond p \to \Box\Diamond p$. Note that we have $U \Vdash \Diamond p$ (because $U\ R\,U''$ and $U'' \Vdash p$) and so $U \Vdash \Box\Diamond p$. Hence $U' \Vdash \Diamond p$ and therefore $W \Vdash p$ for some world W where $U'\ R\,W$. But by the definition of $\Vdash$, W must be U'' and thus $U'\ R\,U''$, a contradiction. The supposition is false, so we are done. $\Box$

Corollary 4.3.5 Let $\mathbf{F}$ be any Kripke frame. Then

(a) $\mathbf{F}$ is sound for $\mathbb{K}$.

(b) $\mathbf{F}$ is reflexive iff $\mathbf{F}$ is sound for $\mathbb{T}$.

(c) $\mathbf{F}$ is reflexive and transitive iff $\mathbf{F}$ is sound for $\mathbb{S}4$.

(d) $\mathbf{F}$ is reflexive and euclidean iff $\mathbf{F}$ is sound for $\mathbb{S}5$.

Proof This follows immediately from Theorem 4.3.2 and Theorem 4.3.4. $\Box$

4.4 Constructing Counter Models

Discussion 4.4.1 Using the soundness results, in order to prove that a proposition is not a theorem of a logic, all we need do is construct a model in which the proposition is not valid. For if a proposition is not valid in a model, it is not valid in the frame based on the model, and is thus not a theorem by Corollary 4.3.5. We give an example:

Example 4.4.2 We want to prove that $\nvdash_{\mathbb{K}} \Box p \to p$. So we need to show that $\models_{\mathbf{F}} \Box p \to p$ does not hold, that is, $\models_{\mathbf{M}} \Box p \to p$ does not hold for some model $\mathbf{M}$. But the latter happens just in case there is a world W in $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ for which $W \nVdash \Box p \to p$.

The latter will happen if $W \Vdash \Box p$ but $W \nVdash p$—(1). So, can we find a model with these properties?

Well, $W \Vdash \Box p$ iff for all worlds W' where $W \; R \, W'$, we have $W' \Vdash p$—(2). So we need to satisfy both (1) and (2).

Bearing all this in mind, set $\mathcal{W} \stackrel{\text{def}}{=} \{ U, U' \}$ (just any two element set), $R \stackrel{\text{def}}{=} \{ (U, U') \}$. This defines a Kripke frame. Looking at (1) and (2) we specify $S \stackrel{\text{def}}{=} \{ (U', p) \}$ where p is any fixed propositional variable. Setting $\mathbf{M}$ to be the model generated by S, it is an exercise to check that indeed $U \nVdash \Box p \rightarrow p$, as required.

5
Completeness Results

5.1 Introduction

We have seen that any frame **F** is sound for $\mathbb{K}$. With a moment's thought (exercise!) we see that this tells us that

$$\vdash_{\mathbb{K}} \phi \Longrightarrow \Vdash \phi.$$

In this chapter we shall show that the converse holds, which is an example of a completeness result—if ϕ is valid in all frames, then ϕ is a theorem of $\mathbb{K}$. This will give us a useful method for checking whether a modal proposition is a theorem.

5.2 Background Definitions

Here we recall some facts from Logic A which will be used to prove our completeness theorems. Let $\mathbb{L}$ be any normal modal logic. A finite set $\Lambda = \{ \phi_1, \ldots \phi_n \}$ of modal propositions is **$\mathbb{L}$-inconsistent** if $\vdash_{\mathbb{L}} \neg(\phi_1 \wedge \ldots \wedge \phi_n)$. We say that Λ is **$\mathbb{L}$-consistent** if Λ is not inconsistent. An infinite set Λ of modal propositions is **$\mathbb{L}$-consistent** if every finite subset of Λ is $\mathbb{L}$-consistent. Any set Λ of modal propositions is **maximal** if for every modal proposition ϕ, either $\phi \in \Lambda$ or $\neg\phi \in \Lambda$. Finally, we say that Λ is maximal $\mathbb{L}$-consistent if it is both maximal and $\mathbb{L}$-consistent. Note that we shall often simply write "consistent" for "$\mathbb{L}$-consistent" when it is clear that we are refering to the logic $\mathbb{L}$.

Lemma 5.2.1 Let $\mathbb{L}$ be a normal modal logic and let Λ be a consistent set of modal propositions containing $\neg\Box\phi$ for some given proposition ϕ. Then the set

$$A = \{ \psi \in \mathcal{L}_M \mid \Box\psi \in \Lambda \} \cup \{ \neg\phi \}$$

is consistent.

Proof Let us suppose that A is inconsistent and derive a contradiction. Thus there is a finite subset $\{ \theta_1, \ldots, \theta_n \}$ of $\{ \psi \in \mathcal{L}_M \mid \Box\psi \in \Lambda \}$ such that $\vdash \neg(\theta_1 \wedge \ldots \wedge \theta_n \wedge \neg\phi)$ (*care!—why?*). Because $\mathbb{L}$ is a normal modal logic, we have

$$\begin{aligned}
& \vdash \neg(\theta_1 \wedge \ldots \wedge \theta_n \wedge \neg\phi) \\
\Longrightarrow\ & \vdash \theta_1 \wedge \ldots \wedge \theta_n \to \phi \\
\Longrightarrow\ & \vdash \Box(\theta_1 \wedge \ldots \wedge \theta_n \to \phi) \\
\Longrightarrow\ & \vdash \Box(\theta_1 \wedge \ldots \wedge \theta_n) \to \Box\phi \\
\Longrightarrow\ & \vdash \Box\theta_1 \wedge \ldots \wedge \Box\theta_n \to \Box\phi \\
\Longrightarrow\ & \vdash \neg(\Box\theta_1 \wedge \ldots \wedge \Box\theta_n \wedge \neg\Box\phi)
\end{aligned}$$

It is an exercise for you to check why these implications hold. So we have deduced that there is a finite subset of Λ which is inconsistent, which is a contradiction. □

Lemma 5.2.2 Let $\mathbb{L}$ be a normal modal logic and S a consistent set of modal propositions. Then there is a maximal consistent set Λ for which $S \subseteq \Lambda$.

Proof Omitted—the proof works in the same way as for classical logic—see Logic A. $\square$

Lemma 5.2.3 Let $\mathbb{L}$ be a normal modal logic, and let Λ be maximal $\mathbb{L}$-consistent. Then

(i) $\phi \in \mathcal{L}_M$ implies either $\phi \in \Lambda$ or $\neg\phi \in \Lambda$, but not both.

(ii) $\vdash_{\mathbb{L}} \phi \Longrightarrow \phi \in \Lambda$ and so $\{ \phi \}$ is consistent;

(iii) If $\phi \in \Lambda$ and $\phi \to \psi \in \Lambda$, then $\psi \in \Lambda$.

Proof

(i) By maximality, one of ϕ or $\neg\phi$ is in Λ. If both are, then $\{ \phi, \neg\phi \} \subseteq \Lambda$, contradicting the consistency of Λ ($\vdash_{\mathbb{L}} \neg(\phi \wedge \neg\phi)$ by classical reasoning).

(ii) If $\phi \notin \Lambda$, then $\neg\phi \in \Lambda$ and so $\{ \neg\phi \}$ is consistent. But this contradicts $\vdash_{\mathbb{L}} \neg\neg\phi$ which follows from the hypothesis. So $\phi \in \Lambda$.

(iii) Suppose that $\psi \notin \Lambda$, thus $\neg\psi \in \Lambda$. It is easy to see that $\{ \phi, \phi \to \psi, \neg\psi \}$ is an inconsistent subset of Λ, a contradiction. Hence $\psi \in \Lambda$.

$\square$

5.3 Canonical Models

If $\mathbb{L}$ is a normal modal logic, the **canonical** model $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ of $\mathbb{L}$ is specified by the following data:

• $\mathcal{W}$ is the set of all maximal $\mathbb{L}$-consistent sets, that is

$$\mathcal{W} \stackrel{\text{def}}{=} \{ W \subseteq \mathcal{L}_M \mid W \text{is maximal } \mathbb{L}\text{-consistent} \};$$

Note that $\mathcal{W}$ is indeed non-empty: There is at least one theorem ϕ of $\mathbb{L}$ (!) and so by Lemma 5.2.3 $\{ \phi \}$ is consistent. We then apply Lemma 5.2.2. We call the elements of $\mathcal{W}$ **canonical** worlds.

• $W \; R \; W'$ iff $\Box\phi \in W$ implies $\phi \in W'$. We call R the **canonical** relation; and

• the forcing relation $\Vdash$ is generated by the set $\{ (W, p) \mid p \in W, W \in \mathcal{W}, p \in \mathit{Var} \}$. We call this $\Vdash$ the **canonical** forcing relation. The **canonical** frame is the frame on which the canonical model is based.

Theorem 5.3.1 Let $\mathbb{L}$ be a normal modal logic and $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ its canonical model. Then for every world W and modal proposition ϕ, $W \Vdash \phi$ iff $\phi \in W$.

Proof Throughout the proof bear in mind Lemma 5.2.3. We prove the result by structural induction on modal propositions: recall that $\mathcal{L}_M$ is specified by the grammar

$$\phi \quad ::= \quad p \mid \neg\phi \mid \phi \to \psi \mid \Box\phi$$

(*Induction step for* p): Trivial.

(*Induction step for* $\neg\phi$): Easy exercise.

(*Induction step for* $\Box\phi$): Suppose the result holds for ϕ.

($\Rightarrow$) Let $W \Vdash \Box\phi$. We suppose that $\Box\phi \notin W$ for a contradiction. Hence $\neg\Box\phi \in W$ by maximal consistency, and from Lemma 5.2.1 we have that $A \stackrel{\text{def}}{=} \{ \psi \mid \Box\psi \in W \} \cup \{ \neg\phi \}$ is consistent. From Lemma 5.2.2, there is a world U containing A. Now, $\Box\psi \in W$ implies $\psi \in A \subseteq U$, so $W \; R \; U$ (property of the canonical model). Therefore $U \Vdash \phi$ and induction gives $\phi \in U$, contradicting the consistency of U.

($\Leftarrow$) Let $\Box\phi \in W$. Let W' be any world where $W \; R \; W'$. Then $\phi \in W'$, and by induction $W' \Vdash \phi$. So $W \Vdash \Box\phi$.

(*Induction step for* $\phi \to \psi$):

($\Rightarrow$) Let $W \Vdash \phi \to \psi$, so that $W \Vdash \phi$ implies $W \Vdash \psi$. By induction, this says exactly that $\phi \in W$ implies $\psi \in W$—($*$). Suppose, for a contradiction, that $\phi \to \psi \notin W$. Then $\neg(\phi \to \psi) \in W$. It follows that $\phi \in W$ *and* $\neg\psi \in W$—for if $\phi \notin W$ then $\neg\phi \in W$ and so $\{ \neg\phi, \neg(\phi \to \psi) \} \subseteq W$ which is an inconsistent subset. Hence $\phi \in W$ and we can show $\neg\psi \in W$ similarly. From ($*$) we deduce $\psi \in W$, implying $\{ \psi, \neg\psi \} \subseteq W$, a contradiction. Therefore $\phi \to \psi \in W$.

($\Leftarrow$) Trivial exercise. $\Box$

Theorem 5.3.2 Let $\mathbb{L}$ be a normal modal logic and write $\mathbf{M} = (\mathcal{W}, R, \Vdash)$ for its canonical model. Then for any modal proposition $\phi \in \mathcal{L}_M$, $\vdash_{\mathbb{L}} \phi$ iff $\models_{\mathbf{M}} \phi$.

Proof

($\Rightarrow$): Let $\vdash_{\mathbb{L}} \phi$. Note that by Lemma 5.2.3 ϕ must be a member of any maximal consistent set. Thus ϕ belongs to all canonical worlds, so by Theorem 5.3.1, every canonical world forces ϕ, that is $\models_{\mathbf{M}} \phi$.

($\Leftarrow$): Let $\models_{\mathbf{M}} \phi$. Suppose (for a contradiction) that $\not\vdash_{\mathbb{L}} \phi$. Clearly $\not\vdash_{\mathbb{L}} \neg\neg\phi$ and so $\{ \neg\phi \}$ is consistent, and so from Lemma 5.2.2 there is a canonical world W such that $\neg\phi \in W$. Hence from Theorem 5.3.1 we have $W \Vdash \neg\phi$, that is $W \nVdash \phi$, implying that $\models_{\mathbf{M}} \phi$ does not hold, a contradiction. $\Box$

5.4 Characterisation and Completeness

Discussion 5.4.1 Let $\mathbb{L}$ be a normal modal logic, $\mathbf{F}$ a Kripke frame and $\mathcal{F}$ any *class* of Kripke frames. We shall write $\models_{\mathcal{F}} \phi$ if $\models_{\mathbf{F}} \phi$ for all frames $\mathbf{F} \in \mathcal{F}$. Then

• **F** is **sound** for $\mathbb{L}$ iff

$$\forall \phi \in \mathcal{L}_M. \quad \vdash_{\mathbb{L}} \phi \Longrightarrow \models_{\mathbf{F}} \phi.$$

• **F** is **complete** for $\mathbb{L}$ iff

$$\forall \phi \in \mathcal{L}_M. \quad \models_{\mathbf{F}} \phi \Longrightarrow \vdash_{\mathbb{L}} \phi.$$

• $\mathbb{L}$ is **characterised** by a frame **F** iff **F** is sound and complete for $\mathbb{L}$.

• $\mathcal{F}$ is **sound** for $\mathbb{L}$ iff

$$\forall \phi \in \mathcal{L}_M. \quad \vdash_{\mathbb{L}} \phi \Longrightarrow \models_{\mathcal{F}} \phi.$$

• $\mathcal{F}$ is **complete** for $\mathbb{L}$ iff

$$\forall \phi \in \mathcal{L}_M. \quad \models_{\mathcal{F}} \phi \Longrightarrow \vdash_{\mathbb{L}} \phi.$$

• $\mathbb{L}$ is **characterised** by $\mathcal{F}$ iff $\mathcal{F}$ is sound and complete for $\mathbb{L}$.

One should note that if a class of frames is complete for $\mathbb{L}$ then it does not follow that every frame in the class is complete for $\mathbb{L}$. However, $\{\mathbf{F}\}$ is complete for $\mathbb{L}$ just in case **F** is complete for $\mathbb{L}$.

Theorem 5.4.2 Let $\mathbb{L}$ be a normal modal logic and **F** its canonical frame. Then **F** is (sound and) complete for $\mathbb{L}$ if **F** is sound for $\mathbb{L}$.

Proof

($\Rightarrow$) Follows by definition.

($\Leftarrow$) Let **F** be sound for $\mathbb{L}$. It remains to show completeness. So let $\models_{\mathbf{F}} \phi$ and suppose that $\nvdash_{\mathbb{L}} \phi$. But then Theorem 5.3.2 tells us that $\models_{\mathbf{M}} \phi$ cannot hold when **M** is the canonical model, and so $\models_{\mathbf{F}} \phi$ cannot hold, a contradiction. Hence **F** is (sound and) complete for $\mathbb{L}$, as required. □

Discussion 5.4.3 We shall show that for each of the logics $\mathbb{K}$, $\mathbb{T}$, $\mathbb{S}4$ and $\mathbb{S}5$ that the canonical frames are sound and complete for the respective logics. If $\mathbb{L}$ is any normal modal logic, we shall write $\mathbf{F}_{\mathbb{L}}$ for its canonical frame, and $\mathbf{M}_{\mathbb{L}}$ for its canonical model. First we need a lemma:

Lemma 5.4.4

(i) $\mathbf{F}_{\mathbb{T}}$ is reflexive.

(ii) $\mathbf{F}_{\mathbb{S}4}$ is reflexive and transitive.

(iii) $\mathbf{F}_{\mathbb{S}5}$ is reflexive and euclidean.

Proof Throughout the proof bear in mind Lemma 5.2.3.

(i) Let R be the canonical relation of $\mathbf{F}_{\mathbb{T}}$, and choose an arbitrary canonical world W. Suppose that $\Box\phi \in W$. As W is maximal $\mathbb{T}$-consistent, $\Box\phi \to \phi \in W$, and so $\phi \in W$. Hence $W\ R\ W$ by the definition of R, and thus R is reflexive as required because W was arbitrary.

(ii) Write $\mathbf{F}_{\mathbb{S}4} = (\mathcal{W}, R)$. We know that $\mathbf{F}_{\mathbb{S}4}$ must be reflexive, because $\mathbb{S}4$ extends $\mathbb{T}$; we prove that R is transitive. So let us pick any three arbitrary canonical worlds for which $W\ R\ W'\ R\ W''$. Suppose that $\Box\phi \in W$ where ϕ is any modal proposition. As W is maximal $\mathbb{S}4$-consistent, $\Box\Box\phi \in W$ (because $\vdash_{\mathbb{S}4} \Box\phi \to \Box\Box\phi$). From the definition of R, it follows that $\phi \in W''$ and recalling the supposition we see that $W\ R\ W''$, and so we are done.

(iii) We know that $\mathbf{F}_{\mathbb{S}5}$ must be reflexive and transitive, because $\mathbb{S}5$ extends $\mathbb{S}4$. Thus we show that $\mathbf{F}_{\mathbb{S}5}$ is symmetric. Write $\mathbf{F}_{\mathbb{S}5} = (\mathcal{W}, R)$. Pick any two arbitrary worlds for which $W\ R\ W'$. We shall show that if $\Box\phi \in W'$ then $\phi \in W$, for then $W'\ R\ W$. So let $\Box\phi \in W'$ and suppose that $\phi \notin W$. By maximality of W, $\neg\phi \in W$. Now, $\vdash_{\mathbb{S}5} \neg\phi \to \Box\neg\Box\phi$ (exercise! use Lemma 3.4.2) and so by maximal $\mathbb{S}5$-consistency we have $\Box\neg\Box\phi \in W$. As $W\ R\ W'$ we have that $\neg\Box\phi \in W'$, which contradicts the maximality of W', and we are done.

$\Box$

Theorem 5.4.5

(i) $\mathbf{F}_{\mathbb{K}}$ is sound and complete for $\mathbb{K}$.

(ii) $\mathbf{F}_{\mathbb{T}}$ is sound and complete for $\mathbb{T}$.

(iii) $\mathbf{F}_{\mathbb{S}4}$ is sound and complete for $\mathbb{S}4$.

(iv) $\mathbf{F}_{\mathbb{S}5}$ is sound and complete for $\mathbb{S}5$.

Proof Note that each of the logics is normal modal. Using Lemma 5.4.4 together with Corollary 4.3.5, we see that each of the canonical frames is sound for the respective logics. The theorem follows from Theorem 5.4.2. $\Box$

Discussion 5.4.6 We shall show that for each of the logics $\mathbb{K}$, $\mathbb{T}$, $\mathbb{S}4$ and $\mathbb{S}5$ that there is a particular class of frames which is sound and complete for the logic in question.

Theorem 5.4.7

(i) The class of all frames is sound and complete for $\mathbb{K}$.

(ii) The class of all reflexive frames is sound and complete for $\mathbb{T}$.

(iii) The class of all reflexive and transitive frames is sound and complete for $\mathbb{S}4$.

(iv) The class of all reflexive and euclidean frames is sound and complete for $\mathbb{S}5$.

Proof Each class of frames is sound for the respective logic, using Corollary 4.3.5. We look at $\mathbb{T}$; the other logics have analogous proofs. Write $\mathcal{F}_{ref}$ for the class of reflexive frames. In the case of $\mathbb{T}$, the corollary says that

$$\forall \mathbf{F} \in \mathcal{F}_{ref}. \quad \vdash_{\mathbb{T}} \phi \Longrightarrow \models_{\mathbf{F}} \phi$$

which is the same statement as

$$\vdash_{\mathbb{T}} \phi \Longrightarrow \models_{\mathcal{F}_{ref}} \phi.$$

For completeness, we look at $\mathbb{T}$: the other logics have analogous proofs. If $\models_{\mathcal{F}_{ref}} \phi$, then in particular $\models_{\mathbf{F}_{\mathbb{T}}} \phi$ and so $\vdash_{\mathbb{T}} \phi$ because $\mathbf{F}_{\mathbb{T}}$ is complete for $\mathbb{T}$. $\square$

6
Modal Tableaux

6.1 Introduction

Recall the use of classical tableaux from Logic A. If we restrict attention to theorems, rather than general sequents, the basic idea is that we want to show that all valuations of a proposition ϕ are of value **true**. To do this, we suppose (for a contradiction) that there is a valuation making ϕ **false**, that is $\neg\phi$ is **true**. We then systematically divide $\neg\phi$ ($\mathrm{F}\phi$ when using signed propositions) into sets of sub-propositions which when assigned the value **true** ensure that $\neg\phi$ has value **true** too, that is, if each set is satisfiable, so too is $\neg\phi$. The sub-propositions are written down in the form of a tree (called a tableau), whose leaves collect up the sub-propositions. If we can show that each leaf is not satisfiable (the tableau is closed) then neither is $\neg\phi$, a contradiction as required.

A tableaux for modal logic is similar to that for classical logic, but we begin with a model in which a world W forces a modal proposition $\neg\phi$, and then systematically divide $\neg\phi$ ($\mathrm{F}\phi$ when using signed propositions) into sets of sub-propositions which when forced, ensure that $\neg\phi$ is indeed forced by W. The rules for producing the sub-propositions arising from the classical connectives are the same as in Logic A—we give additional rules for the modalities.

6.2 Basic Definitions

Discussion 6.2.1 Recall that the set $\mathcal{L}_M$ of (modal) propositions is given by the grammar

$$\phi \quad ::= \quad p \mid \neg\phi \mid \phi \to \psi \mid \Box\phi.$$

The set of **signed propositions** is specified by the grammar $\sigma ::= \mathrm{T}\phi \mid \mathrm{F}\phi$ where T and F are formal syntactic symbols. A **modal tableaux** $\mathcal{T}$ is a tree whose nodes are sets of signed propositions. For example, if $p, q \in \mathit{Var}$ and $\theta \in \mathcal{L}_M$, then $\{\, \mathrm{T}p, \mathrm{F}(p \to q), \mathrm{F}\theta \,\}$ is a typical set of signed propositions. We shall simply refer to a modal tableaux as a **tableaux**, because all of our logics are modal.

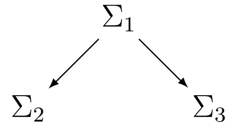
Lemma 6.2.2 Given a tableau $\mathcal{T}$, a leaf L of $\mathcal{T}$, a set of signed propositions L' and another modal tableau $\mathcal{T}'$ with root L, then

(i) there is a tableau $\mathcal{T}[L \rightsquigarrow L']$ which is $\mathcal{T}$ with L' replacing L; and

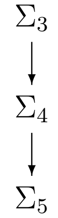
(ii) there is a tableau $\mathcal{T}[L \rightsquigarrow \mathcal{T}']$ which is formed by fusing the root L of $\mathcal{T}'$ to the leaf L of $\mathcal{T}$.

Proof A tedious (but very trivial) manipulation of the definition of a tree, which is omitted. $\Box$

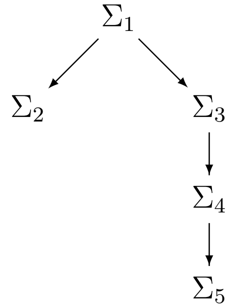
Example 6.2.3 Let $\mathcal{T}$ be



and $\mathcal{T}'$ be



then $\mathcal{T}[\Sigma_3 \rightsquigarrow \mathcal{T}']$ is the tableau



Discussion 6.2.4 A set Σ of signed propositions is **($\mathbb{K}/\mathbb{T}/\mathbb{S}4$)-satisfiable** if there exists (any / a reflexive /a reflexive and transitive) Kripke model $\mathbf{M}$ containing a world W, for which $W \Vdash \sigma$ for all signed propositions σ of Σ. A tableau $\mathcal{T}$ is **($\mathbb{K}$/ $\mathbb{T}$/ $\mathbb{S}4$)-satisfiable** if there exists a leaf L of $\mathcal{T}$ which is ($\mathbb{K}/\mathbb{T}/\mathbb{S}4$)-satisfiable—recall any such leaf is indeed a set of signed propositions. We say that a tableau $\mathcal{T}$ is **closed** if all leaves L of $\mathcal{T}$ contain both ϕ and $\neg\phi$ for some modal proposition ϕ. Otherwise $\mathcal{T}$ is said to be **open**.

Recall from Logic A that it is convenient to define rules for manipulating tableaux which make use of propositions in alpha and beta form. For modal logic we need two more kinds of propositional forms; here we list all of the forms we need:

- The set of **eta-propositions** is given by the grammar $\eta ::= \mathrm{T}p \mid \mathrm{F}p$ for all $p \in \mathit{Var}$;
- The set of **alpha propositions** is given by the grammar $\alpha ::= \mathrm{F}(\phi \to \psi) \mid \mathrm{F}(\neg\phi)$;
- the set of **beta propositions** is given by the grammar $\beta ::= \mathrm{T}(\phi \to \psi) \mid \mathrm{T}(\neg\phi)$;
- the set of **nu propositions** is given by the grammar $\nu := \mathrm{T}(\Box\phi) \mid \mathrm{F}(\Diamond\phi)$; and
- the set of **pi propositions** is given by the grammar $\pi := \mathrm{F}(\Box\phi) \mid \mathrm{T}(\Diamond\phi)$.

We shall refer to a proposition η to mean that η is an eta-proposition, and similarly for the other definitions.

α	α_1	α_2	β	β_1	β_2	ν	ν_0	π	π_0
$\mathrm{F}(\phi \to \psi)$	$\mathrm{T}\phi$	$\mathrm{F}\psi$	$\mathrm{T}(\phi \to \psi)$	$\mathrm{F}\phi$	$\mathrm{T}\psi$	$\mathrm{T}(\Box\phi)$	$\mathrm{T}\phi$	$\mathrm{F}(\Box\phi)$	$\mathrm{F}\phi$
$\mathrm{F}(\neg\phi)$	$\mathrm{T}\phi$	$\mathrm{T}\psi$	$\mathrm{T}(\neg\phi)$	$\mathrm{F}\phi$	$\mathrm{F}\phi$	$\mathrm{F}(\Diamond\phi)$	$\mathrm{F}\phi$	$\mathrm{T}(\Diamond\phi)$	$\mathrm{T}\phi$

We make the definitions that given a Kripke model $\mathbf{M}$ with a world W, $W \Vdash \mathrm{T}\phi$ iff $W \Vdash \phi$ and also $W \Vdash \mathrm{F}\phi$ iff $W \nVdash \phi$.

Lemma 6.2.5 Let $\mathbf{M}$ be any Kripke model in which there is a world W. For propositions α, β, ν and π we have

(i) $W \Vdash \alpha$ iff $W \Vdash \alpha_1$ and $W \Vdash \alpha_2$;

(ii) $W \Vdash \beta$ iff $W \Vdash \beta_1$ or $W \Vdash \beta_2$;

(iii) $W \Vdash \nu$ iff $W' \Vdash \nu_0$ for all worlds W' where $W\ R\,W'$; and

(iv) $W \Vdash \pi$ iff $W' \Vdash \pi_0$ for some world W where $W\ R\,W'$.

Proof We look at just two examples.

(i) Suppose that α is $\mathrm{F}(\phi \to \psi)$.

$$\begin{aligned} W \Vdash \mathrm{F}(\phi \to \psi) &\iff W \nVdash \phi \to \psi \\ &\iff W \Vdash \phi \text{ and } W \nVdash \psi \\ &\iff W \Vdash \alpha_1 \text{ and } W \Vdash \alpha_2 \end{aligned}$$

(ii) Suppose that ν is $\mathrm{T}(\Box\phi)$.

$$\begin{aligned} W \Vdash \mathrm{T}(\Box\phi) &\iff W \Vdash \Box\phi \\ &\iff W' \Vdash \phi \text{ for all worlds } W' \text{ where } W\ R\,W' \\ &\iff W' \Vdash \mathrm{T}\phi \text{ for all worlds } W' \text{ where } W\ R\,W' \end{aligned}$$

□

6.3 Modal Tableaux Rules

Discussion 6.3.1 If S is any set, and $s \notin S$, then we shall write $S\,;\,s$ for the set $S \cup \{\,s\,\}$. We shall write Σ to denote any set of signed propositions. We shall now specify rules for transforming tableaux—the rules make reference to Lemma 6.2.2.

• The $\mathbb{K}$ rules:

$$[\ \Sigma\ ;\ \eta \qquad \rightsquigarrow \qquad \Sigma\ ;\ \eta\ \ ]$$

$$[\ \Sigma\ ;\ \mathrm{T}(\phi \rightarrow \psi) \qquad \rightsquigarrow \qquad \left.\begin{array}{ccc} & \Sigma\ ;\ \mathrm{T}(\phi \rightarrow \psi) & \\ \swarrow & & \searrow \\ \Sigma\ ;\ \mathrm{F}\phi & & \Sigma\ ;\ \mathrm{T}\psi \end{array}\right]$$

$$[\ \Sigma\ ;\ \mathrm{F}(\phi \rightarrow \psi) \qquad \rightsquigarrow \qquad \Sigma\ ;\ \mathrm{T}\phi\ ;\ \mathrm{F}\psi\ \ ]$$

$$[\ \Sigma\ ;\ \mathrm{T}(\neg\phi) \qquad \rightsquigarrow \qquad \Sigma\ ;\ \mathrm{F}\phi\ \ ]$$

$$[\ \Sigma\ ;\ \mathrm{F}(\neg\phi) \qquad \rightsquigarrow \qquad \Sigma\ ;\ \mathrm{T}\phi\ \ ]$$

$$[\ \Sigma\ ;\ \pi \qquad \rightsquigarrow \qquad \{\ \nu_0\ \mid\ \nu \in \Sigma\ \}\ ;\ \pi_0\ \ ]$$

• The $\mathbb{T}$ rules: are those of $\mathbb{K}$ together with

$$[\Sigma\ ;\ \nu\ \rightsquigarrow\ \Sigma\ ;\ \nu_0]$$

• The $\mathbb{S}4$ rules: are those of $\mathbb{T}$ but with the rule for $\Sigma\ ;\ \pi$ replaced by

$$[\Sigma\ ;\ \pi\ \rightsquigarrow\ \{\ \nu\ \mid\ \nu \in \Sigma\ \}\ ;\ \pi_0]$$

We shall refer to a rule which applies to $\Sigma\ ;\ \sigma$ (where σ is any signed proposition) as a σ-**rule**.

6.4 Derived Tableaux Rules

Discussion 6.4.1 When using the tableau rules, it will be convenient not to have to bother to expand out the definitions of $\wedge$ and $\vee$ and $\Diamond$. It is easy to prove that the following rules all follow from the the rules given in the previous section:

Lemma 6.4.2 The following rules can all be derived from the basic $\mathbb{K}$-rules given in

Section 6.3 (and hence are trivially derivable from the $\mathbb{T}$-rules and the $\mathbb{S}4$-rules):

$$[\ \Sigma\,;\mathrm{T}(\phi\wedge\psi) \quad\rightsquigarrow\quad \Sigma\,;\mathrm{T}\phi\,;\mathrm{T}\psi\ ]$$

$$\left[\ \Sigma\,;\mathrm{F}(\phi\wedge\psi) \quad\rightsquigarrow\quad \begin{array}{ccc} & \Sigma\,;\mathrm{F}(\phi\wedge\psi) & \\ \swarrow & & \searrow \\ \Sigma\,;\mathrm{F}\phi & & \Sigma\,;\mathrm{F}\psi \end{array}\ \right]$$

$$\left[\ \Sigma\,;\mathrm{T}(\phi\vee\psi) \quad\rightsquigarrow\quad \begin{array}{ccc} & \Sigma\,;\mathrm{T}(\phi\vee\psi) & \\ \swarrow & & \searrow \\ \Sigma\,;\mathrm{T}\phi & & \Sigma\,;\mathrm{T}\psi \end{array}\ \right]$$

$$[\ \Sigma\,;\mathrm{F}(\phi\vee\psi) \quad\rightsquigarrow\quad \Sigma\,;\mathrm{F}\phi\,;\mathrm{F}\psi\ ]$$

Proof Easy exercise—do it! □

As a summary, we present all modal tableaux rules in Table 6.1.

6.5 Modal Tableau Theory

Let ϕ be a modal proposition. A **$\mathbb{K}$-tableau** for ϕ is a tableau $\mathcal{T}$ generated from the singleton tableau $\{\mathrm{F}\phi\}$ using the $\mathbb{K}$-rules. A **$\mathbb{T}$-tableau** for ϕ is a tableau $\mathcal{T}$ generated from the singleton tableau $\{\mathrm{F}\phi\}$ using the $\mathbb{T}$-rules. A **$\mathbb{S}4$-tableau** for ϕ is a tableau $\mathcal{T}$ generated from the singleton tableau $\{\mathrm{F}\phi\}$ using the $\mathbb{S}4$-rules.

Theorem 6.5.1

(i) Let a tableau $\mathcal{T}$ be $\mathbb{K}$-satisfiable. If $\mathcal{T}^*$ has been obtained from $\mathcal{T}$ by one of the $\mathbb{K}$-rules above then $\mathcal{T}^*$ is $\mathbb{K}$-satisfiable.

(ii) Let a tableau $\mathcal{T}$ be $\mathbb{T}$-satisfiable. If $\mathcal{T}^*$ has been obtained from $\mathcal{T}$ by one of the $\mathbb{T}$-rules above then $\mathcal{T}^*$ is $\mathbb{T}$-satisfiable.

(iii) Let a tableau $\mathcal{T}$ be $\mathbb{S}4$-satisfiable. If $\mathcal{T}^*$ has been obtained from $\mathcal{T}$ by one of the $\mathbb{S}4$-rules above then $\mathcal{T}^*$ is $\mathbb{S}4$-satisfiable.

Proof

(i) $\mathcal{T}$ is $\mathbb{K}$-satisfiable if there is a leaf L of $\mathcal{T}$ which is $\mathbb{K}$-satisfiable. So there is a model $\mathbf{M}$ and a world W of $\mathbf{M}$ for which $W \Vdash L$ (recall Discussion 6.2.4). We prove the result by considering (inductively) the different forms of L and the tableau rules which apply in each case. Either L is changed to a new leaf L', in which case we show that there is a

We write

- $\eta ::= \mathrm{T}p \mid \mathrm{F}p$ where $p \in \mathit{Var}$;
- $\nu := \mathrm{T}(\Box\phi) \mid \mathrm{F}(\Diamond\phi)$; and $\nu_0 = \mathrm{T}\phi$ if $\nu = \mathrm{T}(\Box\phi)$ and $\nu_0 = \mathrm{F}\phi$ if $\nu = \mathrm{F}(\Diamond\phi)$;
- $\pi := \mathrm{F}(\Box\phi) \mid \mathrm{T}(\Diamond\phi)$; and $\pi_0 = \mathrm{T}\phi$ if $\pi = \mathrm{T}(\Diamond\phi)$ and $\pi_0 = \mathrm{F}\phi$ if $\pi = \mathrm{F}(\Box\phi)$;
- Rules applicable to $\mathbb{K}$ are:

$$[\ \Sigma ; \eta \quad \rightsquigarrow \quad \Sigma ; \eta\ ]$$

$$\left[\ \Sigma ; \mathrm{T}(\phi \to \psi) \quad \rightsquigarrow \quad \begin{array}{ccc} & \Sigma ; \mathrm{T}(\phi \to \psi) & \\ \swarrow & & \searrow \\ \Sigma ; \mathrm{F}\phi & & \Sigma ; \mathrm{T}\psi \end{array}\ \right]$$

$$[\ \Sigma ; \mathrm{F}(\phi \to \psi) \quad \rightsquigarrow \quad \Sigma ; \mathrm{T}\phi ; \mathrm{F}\psi\ ]$$

$$[\ \Sigma ; \mathrm{T}(\neg\phi) \quad \rightsquigarrow \quad \Sigma ; \mathrm{F}\phi\ ]$$

$$[\ \Sigma ; \mathrm{F}(\neg\phi) \quad \rightsquigarrow \quad \Sigma ; \mathrm{T}\phi\ ]$$

$$[\ \Sigma ; \mathrm{T}(\phi \wedge \psi) \quad \rightsquigarrow \quad \Sigma ; \mathrm{T}\phi ; \mathrm{T}\psi\ ]$$

$$\left[\ \Sigma ; \mathrm{F}(\phi \wedge \psi) \quad \rightsquigarrow \quad \begin{array}{ccc} & \Sigma ; \mathrm{F}(\phi \wedge \psi) & \\ \swarrow & & \searrow \\ \Sigma ; \mathrm{F}\phi & & \Sigma ; \mathrm{F}\psi \end{array}\ \right]$$

$$\left[\ \Sigma ; \mathrm{T}(\phi \vee \psi) \quad \rightsquigarrow \quad \begin{array}{ccc} & \Sigma ; \mathrm{T}(\phi \vee \psi) & \\ \swarrow & & \searrow \\ \Sigma ; \mathrm{T}\phi & & \Sigma ; \mathrm{T}\psi \end{array}\ \right]$$

$$[\ \Sigma ; \mathrm{F}(\phi \vee \psi) \quad \rightsquigarrow \quad \Sigma ; \mathrm{F}\phi ; \mathrm{F}\psi\ ]$$

$$[\ \Sigma ; \pi \quad \rightsquigarrow \quad \{\ \nu_0 \mid \nu \in \Sigma\ \} ; \pi_0\ ]$$

- Rules applicable to $\mathbb{T}$ are those of $\mathbb{K}$ together with:

$$[\Sigma ; \nu \ \rightsquigarrow\ \Sigma ; \nu_0]$$

- Rules applicable to $\mathbb{S}4$ are those of $\mathbb{T}$ but with the rule for $\Sigma ; \pi$ replaced by:

$$[\Sigma ; \pi \ \rightsquigarrow\ \{\ \nu \mid \nu \in \Sigma\ \} ; \pi_0]$$

Table 6.1: The Modal Tableaux Rules

world W' satisfying $W' \Vdash L'$—or a tableau $\mathcal{T}'$ with root L is fused to the leaf L, in which case we show that there is W' and a leaf L' of $\mathcal{T}'$ where $W' \Vdash L'$.

(*Case L is* $\Sigma\ ;\ \eta$): $\mathcal{T}^*$ is $\mathcal{T}$, so is satisfiable.

(*Case L is* $\Sigma\ ;\ \alpha$): By examining the (two) forms that α can take we see that $\mathcal{T}^*$ will be $\mathbb{K}$-satisfiable if $W \Vdash \Sigma\ ;\ \alpha_1\ ;\ \alpha_2$. But by assumption, $W \Vdash \Sigma\ ;\ \alpha$, and so it remains to prove that $W \Vdash \alpha_1$ and $W \Vdash \alpha_2$. We are done by Lemma 6.2.5 part (i).

(*Case L is* $\Sigma\ ;\ \beta$): Use Lemma 6.2.5 part (ii).

(*Case L is* $\Sigma\ ;\ \pi$): We have $W \Vdash \Sigma\ ;\ \pi$. Hence $W \Vdash \nu$ for any $\nu \in \Sigma$ and thus from Lemma 6.2.5 part (iii) we have $W' \Vdash \nu_0$ for all W' where $W\ R\ W'$. Lemma 6.2.5 part (iv) tells us that $U \Vdash \pi_0$ for some U with $W\ R\ U$, and thus $U \Vdash \nu_0$ for any $\nu \in \Sigma$. Hence $U \Vdash \{\ \nu_0 \mid \nu \in \Sigma\ \}\ ;\ \pi_0$ and we are done.

(ii) $\mathcal{T}$ is $\mathbb{T}$-satisfiable if there is a leaf L of $\mathcal{T}$ which is $\mathbb{T}$-satisfiable. So there is a *reflexive* model $\mathbf{M}$ and a world W of $\mathbf{M}$ for which $W \Vdash L$. The proof is the same as for part (i), but we also have to consider

(*Case L is* $\Sigma\ ;\ \nu$): We have $W \Vdash \Sigma\ ;\ \nu$. So in particular $W \Vdash \nu$ and $W' \Vdash \nu_0$ for all W' where $W\ R\ W'$. But here, R is reflexive, so taking W' to be W we have $W \Vdash \nu_0$ and so $W \Vdash \Sigma\ ;\ \nu_0$.

(iii) $\mathcal{T}$ is $\mathbb{S}4$-satisfiable if there is a leaf L of $\mathcal{T}$ which is $\mathbb{S}4$-satisfiable. So there is a *reflexive and transitive* model $\mathbf{M}$ and a world W of $\mathbf{M}$ for which $W \Vdash L$. The proof is the same as for part (ii) except the case when L is $\Sigma\ ;\ \pi$:

(*Case L is* $\Sigma\ ;\ \pi$): We have $W \Vdash \Sigma\ ;\ \pi$. So $U \Vdash \pi_0$ for some U with $W\ R\ U$. Let W' be any world where $U\ R\ W'$. Then $W\ R\ W'$, for here R is (reflexive and) transitive. So if $\nu \in \Sigma$, $W \Vdash \nu$ by assumption and thus $W' \Vdash \nu_0$. Thus $U \Vdash \nu$, and altogether we have $U \Vdash \{\ \nu \mid \nu \in \Sigma\ \}\ ;\ \pi_0$.

□

Remark 6.5.2 [Comments on The Use of Modal Tableaux]

Let us consider tableaux for classical logic. It is a standard result that if a tableau $\mathcal{T}^*$ is obtained from $\mathcal{T}$ by applying a rule, then $\mathcal{T}$ is satisfiable iff $\mathcal{T}^*$ is satisfiable. Now let $\mathrm{F}\phi$ be a signed proposition, and $\mathcal{T}_C$ be its complete tableaux. Suppose that $\Sigma_1, \Sigma_2 \ldots \Sigma_k$ are the leaves of $\mathcal{T}_C$. It follows that

$$\mathrm{F}\phi \text{ is satis } \Leftrightarrow (\Sigma_1 \text{ is satis or } \Sigma_2 \text{ is satis or} \ldots\ \Sigma_k \text{ is satis }) \quad (*)$$

or equivalently

$$\mathrm{F}\phi \text{ not satis } \Leftrightarrow (\Sigma_1 \text{ not satis and } \Sigma_2 \text{ not satis and} \ldots\ \Sigma_k \text{ not satis }) \quad (**)$$

Hence (remembering that $\mathcal{T}_C$ is complete)

$$\begin{array}{rcl}
\mathcal{T}_C \text{ is } \textit{closed} & \Leftrightarrow & \text{each } \Sigma_i \text{ contains T}p \text{ and F}p \text{ for some variable } p \\
& \Rightarrow & (\Sigma_1 \text{ not satis and } \Sigma_2 \text{ not satis and} \ldots \ \Sigma_k \text{ not satis }) \\
(^{**}) & \Rightarrow & \text{F}\phi \text{ not satis} \\
& \Leftrightarrow & \Vdash \phi.
\end{array}$$

and

$$\begin{array}{rcl}
\mathcal{T}_C \text{ is } \textit{open} & \Leftrightarrow & \text{some } \Sigma_i \text{ does not contain T}p \text{ and F}p \text{ for any variable } p \\
& \Rightarrow & \text{some } \Sigma_i \text{ is satis} \\
& \Rightarrow & (\Sigma_1 \text{ is satis or } \Sigma_2 \text{ is satis or} \ldots \ \Sigma_k \text{ is satis }) \\
(^{*}) & \Rightarrow & \text{F}\phi \text{ is satis} \\
& \Leftrightarrow & \text{we do not have } \Vdash \phi.
\end{array}$$

Note that a complete classical tableaux is either open or closed.

In the case of modal logic, if tableaux $\mathcal{T}^*$ is obtained from $\mathcal{T}$ by applying a rule, then all we have (see Theorem 6.5.1) is that $\mathcal{T}$ is satisfiable implies $\mathcal{T}^*$ is satisfiable, *and NOT the converse.* Thus for modal logic, all we have is

$$\text{F}\phi \text{ is satis } \Rightarrow (\Sigma_1 \text{ is satis or } \Sigma_2 \text{ is satis or} \ldots \ \Sigma_k \text{ is satis }) \quad (*)$$

or equivalently

$$\text{F}\phi \text{ not satis } \Leftarrow (\Sigma_1 \text{ not satis and } \Sigma_2 \text{ not satis and} \ldots \ \Sigma_k \text{ not satis }) \quad (**)$$

and from (**) we can deduce (just as for classical logic) that if there is a closed tableau for Fϕ then $\Vdash \phi$.

To see that we cannot have the converse of (*), consider

$$\begin{array}{rcl}
\text{F}((p \wedge \neg p) \to \Box r) & \rightsquigarrow & \text{T}(p \wedge \neg p) \ ; \text{F}\Box r \\
& \rightsquigarrow & \text{F}r
\end{array}$$

where in the second tableau rewrite, the π proposition F$\Box r$ becomes the π_0 proposition Fr, and T$(p \wedge \neg p)$, which is not a ν, is deleted. Clearly Fr is satisfiable, but F$((p \wedge \neg p) \to \Box r)$ is not.

Note also that the tableaux rules are not deterministic—Fϕ may lead to both open and closed tableaux.

Theorem 6.5.3 Let $\phi \in \mathcal{L}_M$, and write $\mathcal{F}_{ref}$ for the class of reflexive frames, $\mathcal{F}_{ref\&tran}$ for the class of frames which are both reflexive and transitive.

(i) If there is a closed $\mathbb{K}$-tableau for ϕ, then $\Vdash \phi$, and so $\vdash_{\mathbb{K}} \phi$.

(ii) If there is a closed $\mathbb{T}$-tableau for ϕ, then $\models_{\mathcal{F}_{ref}} \phi$, and so $\vdash_{\mathbb{T}} \phi$.

(iii) If there is a closed $\mathbb{S}4$-tableau for ϕ, then $\models_{\mathcal{F}_{ref\&tran}} \phi$, and so $\vdash_{\mathbb{S}4} \phi$.

Proof

(i) Let $\mathcal{T}$ be such a $\mathbb{T}$-tableau for ϕ, and suppose (for a contradiction) that $\Vdash \phi$ does not hold. Then there is a Kripke model **M** and a world W in **M** for which $W \nVdash \phi$, that is $W \Vdash$ Fϕ. Therefore $\{$ F$\phi \}$ is $\mathbb{K}$-satisfiable and thus so is $\mathcal{T}$ by Theorem 6.5.1. But if $\mathcal{T}$ is $\mathbb{K}$-satisfiable then it cannot be closed, a contradiction. Hence $\Vdash \phi$ does hold. That $\vdash_{\mathbb{T}} \phi$ then follows by completeness—see Theorem 5.4.7.

(ii) Analogous to part (i).

(iii) Analogous to part (i).

□

Discussion 6.5.4 In fact we have (soundness and) completeness results for modal tableaux. We shall not prove completeness here, but quote a result which you are free to use.

Theorem 6.5.5 Let $\phi \in \mathcal{L}_M$.

(i) There is a closed $\mathbb{K}$-tableau for ϕ iff $\vdash_{\mathbb{K}} \phi$.

(ii) There is a closed $\mathbb{T}$-tableau for ϕ iff $\vdash_{\mathbb{T}} \phi$.

(iii) There is a closed $\mathbb{S}4$-tableau for ϕ iff $\vdash_{\mathbb{S}4} \phi$.

6.6 Using Modal Tableaux

Discussion 6.6.1 The use of modal tableaux is of course just the same as for classical tableaux which you used in Logic A. By Theorem 6.5.3, if we can show that a proposition has a closed tableau, then the proposition is in fact a theorem. We give a few examples:

Examples 6.6.2

(1) We show that $\vdash_{\mathbb{K}} (\Box p \wedge \Box q) \to \Box(p \wedge q)$. Let us produce the $\mathbb{K}$-tableau:

$$\mathrm{F}((\Box p \wedge \Box q) \to \Box(p \wedge q)) \quad \rightsquigarrow \quad \mathrm{T}(\Box p \wedge \Box q)\ ;\ \mathrm{F}\Box(p \wedge q)$$

$$\rightsquigarrow \quad \mathrm{T}\Box p\ ;\ \mathrm{T}\Box q\ ;\ \mathrm{F}\Box(p \wedge q)$$

$$\rightsquigarrow_{(1)} \quad \mathrm{T}p\ ;\ \mathrm{T}q\ ;\ \mathrm{F}(p \wedge q)$$

$$\rightsquigarrow \quad \begin{array}{c} \mathrm{T}p\ ;\ \mathrm{T}q\ ;\ \mathrm{F}(p \wedge q) \\ \swarrow \qquad \searrow \\ \mathrm{T}p\ ;\ \mathrm{T}q\ ;\ \mathrm{F}p \qquad \mathrm{T}p\ ;\ \mathrm{T}q\ ;\ \mathrm{F}q \end{array}$$

So the tableau is closed as required. Note that step (1) follows by the $\mathbb{K}$ π-rule, where π is of course $\mathrm{F}\Box(p \wedge q)$.

(2) We show that $\vdash_{\mathbb{T}} p \to \Diamond p$. We have

$$\mathrm{F}(p \to \Diamond p) \quad \rightsquigarrow \quad \mathrm{T}p\ ;\ \mathrm{F}\Diamond p$$

$$\rightsquigarrow \quad \mathrm{T}p\ ;\ \mathrm{F}p$$

where the last step follows from Lemma 6.4.2. The tableau is closed. Note that if you ever get stuck remembering a derived rule, simply expand out the definition of the connective (here $\Diamond$ is $\neg\Box\neg$) and use the basic rules.

(3) We show that $\vdash_{\mathbb{S}4} \Box p \to \Box\Box p$. Let us produce the $\mathbb{K}$-tableau:

$$\mathrm{F}(\Box p \to \Box\Box p) \quad \rightsquigarrow \quad \mathrm{T}\Box p \ ; \mathrm{F}\Box\Box p$$

$$\rightsquigarrow \quad \mathrm{T}\Box p \ ; \mathrm{F}\Box p$$

The tableau is closed. The last step follows by the $\mathbb{S}4$ π-rule.

Remark 6.6.3 In the formal presentation of the tableau rules, it suggests implicitly that when a rule is applied, the entire tableau is re-written. In practice, one would not do this, but simply expand out the tableau as you did in Logic A. For example, Example (1) is best written down as:

$$\mathrm{F}((\Box p \wedge \Box q) \to \Box(p \wedge q))$$

$$\mathrm{T}\Box p \ ; \mathrm{T}\Box q \ ; \mathrm{F}\Box(p \wedge q)$$

$$\mathrm{T}p \ ; \mathrm{T}q \ ; \mathrm{F}(p \wedge q)$$

$$\mathrm{T}p \ ; \mathrm{T}q \ ; \mathrm{F}p \qquad\qquad \mathrm{T}p \ ; \mathrm{T}q \ ; \mathrm{F}q$$