Operational Semantics Abstract Machines and Correctness

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Introduction

By the end of this introduction, you should be able to

- briefly explain the meaning of syntax and semantics;
- give a snap-shot overview of the course;
- explain what inductively defined sets are; and
- do simple rule inductions.

What's Next? Background

- What is a Programming Language?
- What is Syntax?
- What is Semantics?

Some Answers

- Programming Languages are formal languages used to "communicate" with a "computer".
- Programming languages may be "low level". They give direct instructions to the processor (instruction set architecture).
- Or "high level". The instructions are indirect—being (eg) compiled for the processor—but much closer to concepts understood by the user (Java, C++, ...).

- Syntax refers to particular arrangements of "words and letters" eg David hit the ball or

  if $t > 2$ then $H = \mathrm{Off}$.

- A grammar is a set of rules which can be used to specify how syntax is created.
- Examples can be seen in automata theory, or programming manuals.
- Theories of syntax and grammars can be developed—ideas are used in compiler construction.

- Semantics is the study of "meaning".
- In particular, syntax can be given meaning. The word run can mean
  - execution of a computer program,
  - spread of ink on paper, ...
- Programming language syntax can be given a semantics—at least in theory!. We need this to write meaningful programs ...

Semantic descriptions are often informal. Consider

while (*expression*) *command* ;

adapted from Kernighan and Ritchie 1978/1988, p 224:

The command is executed repeatedly so long as the value of the expression remains unequal to 0; the expression must have arithmetic or pointer type. The execution of the (test) expression, including all side effects, occurs before each execution of the command.

We want to be more precise, more succinct.

Top Level view of Course

- Define syntax for programs $\boxed{P}$ and types $\boxed{\sigma}$;
- (define type assignments $\boxed{P :: \sigma}$);
- define operational semantics looking like

  $$\boxed{(P\,,s) \Downarrow (V\,,s')} \qquad \boxed{P \Downarrow V};$$

- and compile P and V to abstract machine instructions $\boxed{P \mapsto [\![P]\!]}$ and $\boxed{V \mapsto (\![V]\!)}$
- Then prove correctness: $P \Downarrow V$ *iff* $[\![P]\!] \longmapsto^{t} (\![V]\!)$

What's Next? Inductively Defined Sets

- Specify inductively defined sets; programs, types etc will be defined this way. BNF grammars are a form of inductive definition; abstract syntax trees are also defined inductively.
- Define Rule Induction; properties of programs will be proved using this. It is important.

Example Inductive Definition

Let *Var* be a set of propositional variables. Then the set *Prpn* of propositions of propositional logic is inductively defined by the rules

$$\frac{}{P}\,[P \in Var]\ (A) \qquad \frac{\phi \quad \psi}{\phi \wedge \psi}\,(\wedge)$$

$$\frac{\phi \quad \psi}{\phi \vee \psi}\,(\vee) \qquad \frac{\phi \quad \psi}{\phi \to \psi}\,(\to) \qquad \frac{\phi}{\neg\phi}\,(\neg)$$

Each proposition is created by a deduction ...

Inductively Defined Sets in General

- Given a set of rules, a deduction is a finite tree such that
  - each leaf node label c occurs as a base rule $(\varnothing, c) \in \mathcal{R}$
  - for any non-leaf node label c, if H is the set of children of c then $(H, c) \in \mathcal{R}$ is an inductive rule.
- The set I inductively defined by $\mathcal{R}$ consists of those elements e which have a deduction with root node e. One may prove $\forall e \in I. \boxed{\phi(e)}$ for a property $\phi(e)$ by rule induction. See the notes ...

Example of Rule Induction

Consider the set of trees $\mathcal{T}$ defined inductively by

$$\frac{}{n}\,[n \in \mathbb{Z}] \qquad \frac{T_1 \quad T_2}{+(T_1, T_2)}$$

Let $L(T)$ be the number of leaves in T, and $N(T)$ be the number of $+$-nodes of T. We prove (see board)

$$\forall T \in \mathcal{T}. \quad \boxed{L(T) = N(T) + 1}$$

where the functions $L, N \colon \mathcal{T} \to \mathbb{N}$ are defined recursively by

- $L(n) = 1$ and $L(+(T_1, T_2)) = L(T_1) + L(T_2)$
- $N(n) = 0$ and $N(+(T_1, T_2)) = N(T_1) + N(T_2) + 1$

Chapter 1

By the end of this chapter, you should be able to

- describe the programs (syntax) of a simple imperative language called IMP;
- give a type system to IMP and derive types;
- explain the idea of evaluation relations;
- derive example evaluations.

What's Next? Types and Expressions

- We define the types and expressions of $\mathbb{IMP}$.
- We give an inductive definition of a formal type system.

Program Expressions and Types for IMP

The program expressions are given (inductively) by

P ::=	$\underline{c}$	constant
|	l	memory location
|	$P\ iop\ P'$	integer operator
|	$P\ bop\ P'$	boolean operator
|	$l := P'$	assignment
|	$P\ ;\ P'$	sequencing
|	if P then P' else P''	conditional
|	while P do P'	while loop

- The types of the language $\mathbb{IMP}$ are given by the grammar

$$\sigma \quad ::= \quad \mathsf{int} \mid \mathsf{bool} \mid \mathsf{cmd}$$

- A location environment $\mathcal{L}$ is a finite set of (location, type) pairs, with type being just int or bool:

$$\mathcal{L} = l_1 :: \mathsf{int}, \ldots, l_n :: \mathsf{int},\ l_{n+1} :: \mathsf{bool}, \ldots, l_m :: \mathsf{bool}$$

- Given $\mathcal{L}$, then any P whose locations all appear in $\mathcal{L}$ can (sometimes) be assigned a type; we write $P :: \sigma$ to indicate this, and define such type assignments inductively.

$$\frac{}{\underline{n} :: \mathsf{int}}\ [\text{any } n \in \mathbb{Z}] \qquad \frac{}{\underline{T} :: \mathsf{bool}} \qquad \frac{}{\underline{F} :: \mathsf{bool}}$$

$$\frac{}{l :: \mathsf{int}}\ [l :: \mathsf{int} \in \mathcal{L}] \qquad \frac{P_1 :: \mathsf{int} \quad P_2 :: \mathsf{int}}{P_1\ bop\ P_2 :: \mathsf{bool}}\ [\ bop\ \in BOpr]$$

$$\frac{}{\mathsf{skip} :: \mathsf{cmd}} \qquad \frac{l :: \sigma \quad P :: \sigma}{l := P :: \mathsf{cmd}}$$

$$\frac{P_1 :: \mathsf{bool} \quad P_2 :: \mathsf{cmd} \quad P_3 :: \mathsf{cmd}}{\mathsf{if}\ P_1\ \mathsf{then}\ P_2\ \mathsf{else}\ P_3 :: \mathsf{cmd}} \qquad \frac{P_1 :: \mathsf{bool} \quad P_2 :: \mathsf{cmd}}{\mathsf{while}\ P_1\ \mathsf{do}\ P_2 :: \mathsf{cmd}}$$

Example: Deduction of a Type Assignment

$$\dfrac{\dfrac{\dfrac{}{l :: \mathsf{int}} \quad \dfrac{}{\underline{5} :: \mathsf{int}}}{l \geq \underline{5} :: \mathsf{bool}} \qquad \mathcal{D}2 \qquad \dfrac{\mathcal{D}3 \quad \mathcal{D}4}{l := l - 1\,;\, l' := l' * l :: \mathsf{cmd}}}{\mathsf{if}\ l \geq 5\ \mathsf{then}\ l' := \underline{1}\ \mathsf{else}\ (l := l + 1\,;\, l' := l' * l) :: \mathsf{cmd}}$$

What's Next? An Evaluation Relation

- We define a notion of state.
- We define an evaluation relation for $\mathbb{IMP}$.
- We look at an example.

States

- A state s is a finite partial function $Loc \to \mathbb{Z} \cup \mathbb{B}$.
- For example $s = \langle l_1 \mapsto 4, l_2 \mapsto T, l_3 \mapsto 21 \rangle$
- There is a state denoted by $s\{l \mapsto c\} : Loc \to \mathbb{Z} \cup \mathbb{B}$ which is the partial function

$$(s\{l \mapsto c\})(l') \stackrel{\text{def}}{=} \begin{cases} c & \text{if } l' = l \\ s(l') & \text{otherwise} \end{cases}$$

- We say that state s is updated at l by c.

An Evaluation Relation

Consider the following evaluation relationship

$$(\ l' := \underline{T}\,;\, l := \underline{4} + \underline{1}\ ,\ \langle\rangle\ ) \Downarrow (\ \mathsf{skip}\ ,\ \langle l' \mapsto T, l \mapsto 5 \rangle\ )$$

The idea is

Starting program $\Downarrow$ *final result*

We describe an operational semantics which has assertions which look like

$$(P, s) \Downarrow (\underline{c}, s) \qquad \text{and} \qquad (P, s_1) \Downarrow (\mathsf{skip}, s_2)$$

$$\frac{}{(l, s) \Downarrow (\underline{s(l)}, s)}\ [\text{ provided } l \in \text{ domain of } s] \Downarrow\text{LOC}$$

$$\frac{(P_1, s) \Downarrow (\underline{n_1}, s) \quad (P_2, s) \Downarrow (\underline{n_2}, s)}{(P_1\ op\ P_2, s) \Downarrow (\underline{n_1\ op\ n_2}, s)} \Downarrow\text{OP}$$

$$\frac{(P, s) \Downarrow (\underline{c}, s)}{(l := P, s) \Downarrow (\mathsf{skip}, s\{l \mapsto c\})} \Downarrow\text{ASS}$$

$$\frac{(P_1, s_1) \Downarrow (\mathsf{skip}, s_2) \quad (P_2, s_2) \Downarrow (\mathsf{skip}, s_3)}{(P_1\,;\, P_2, s_1) \Downarrow (\mathsf{skip}, s_3)} \Downarrow\text{SEQ}$$

$$\frac{(P, s_1) \Downarrow (\underline{F}, s_1) \quad (P_2, s_1) \Downarrow (\mathsf{skip}, s_2)}{(\mathsf{if}\ P\ \mathsf{then}\ P_1\ \mathsf{else}\ P_2, s_1) \Downarrow (\mathsf{skip}, s_2)} \Downarrow\text{COND}_2$$

$$\frac{(P_1, s_1) \Downarrow (\underline{T}, s_1) \quad (P_2, s_1) \Downarrow (\mathsf{skip}, s_2) \quad (\mathsf{while}\ P_1\ \mathsf{do}\ P_2, s_2) \Downarrow (\mathsf{skip}, s_3)}{(\mathsf{while}\ P_1\ \mathsf{do}\ P_2, s_1) \Downarrow (\mathsf{skip}, s_3)}$$

$$\frac{(P_1, s) \Downarrow (\underline{F}, s)}{(\mathsf{while}\ P_1\ \mathsf{do}\ P_2, s) \Downarrow (\mathsf{skip}, s)} \Downarrow\text{LOOP}_2$$

Example Evaluations

We derive deductions for

$$((\underline{3} + \underline{2}) * \underline{6}, s) \Downarrow (\underline{30}, s)$$

and

$$(\mathsf{while}\ l = \underline{1}\ \mathsf{do}\ l := l - \underline{1}, \langle l \mapsto 1 \rangle) \Downarrow (\mathsf{skip}, \langle l \mapsto 0 \rangle)$$

Chapter 2

By the end of this chapter you should be able to

- describe the "compiled" CSS machine, which executes compiled IMP programs;
- show how to compile to CSS instruction sequences;
- give some example executions.

Motivating the CSS Machine

An operational semantics gives a useful model of $\mathbb{IMP}$—we seek a more direct, "computational" method for evaluating configurations. If $P \Downarrow^e V$, how do we "mechanically produce" V from P?

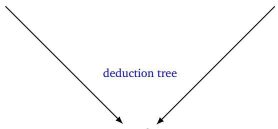
$$P \equiv P_0 \mapsto P_1 \mapsto P_2 \mapsto \ldots \mapsto P_n \equiv V$$

"Mechanically produce" can be made precise using a relation $P \longmapsto P'$ defined by rules with no hypotheses.

$$\frac{}{\underline{n}+\underline{m} \longmapsto \underline{m+n}}$$

$$P_0 \mapsto P_1 \mapsto P_2 \mapsto P_3 \mapsto P_4 \ldots \mapsto V$$

Re-Write Rules (Abstract Machine)



$$P \quad \Downarrow^e \quad V$$

Evaluation Semantics

An Example

Let $s(l) = 6$. Execute $\underline{10} - l$ on the CSS machine.

First, compile the program.

$$\llbracket \underline{10} - l \rrbracket \quad = \quad \mathsf{FETCH}(l) : \mathsf{PUSH}(\underline{10}) : \mathsf{OP}(-)$$

Then

$$\begin{array}{ll} & \boxed{\mathsf{FETCH}(l) : \mathsf{PUSH}(\underline{10}) : \mathsf{OP}(-) \parallel - \parallel s} \\ \longmapsto & \boxed{\mathsf{PUSH}(\underline{10}) : \mathsf{OP}(-) \parallel \underline{6} \parallel s} \\ \longmapsto & \boxed{\mathsf{OP}(-) \parallel \underline{10} : \underline{6} \parallel s} \\ \longmapsto & \boxed{- \parallel \underline{4} \parallel s} \end{array}$$

Defining the CSS Machine

- A CSS code C is a list:

$$C \quad ::= \quad - \mid ins : C$$

$$\begin{array}{lll} ins & ::= & \mathsf{PUSH}(\underline{c}) \mid \mathsf{FETCH}(l) \mid \mathsf{OP}(op) \mid \mathsf{SKIP} \\ & & \mid \mathsf{STO}(l) \mid \mathsf{BR}(C,C) \mid \mathsf{LOOP}(C,C) \end{array}$$

The objects ins are CSS instructions. We will overload $:$ to denote append; and write ξ for $\xi : -$ (ditto below).

- A stack S is produced by the grammar

$$S ::= - \mid \underline{c} : S$$

- A CSS configuration is a triple (C, S, s).
- A CSS re-write takes the form

$$(C_1\,,S_1\,,s_1) \longmapsto (C_2\,,S_2\,,s_2)$$

and re-writes are specified inductively by rules with no hypotheses (such rules are often called axioms)

$$\frac{}{(C_1\,,S_1\,,s_1) \longmapsto (C_2\,,S_2\,,s_2)}\,R$$

- Note that the CSS re-writes are deterministic.

$$\begin{array}{rcl} \boxed{\mathsf{PUSH}(\underline{c}) : C \parallel S \parallel s} & \longmapsto & \boxed{C \parallel \underline{c} : \sigma \parallel s} \\ \boxed{\mathsf{FETCH}(l) : C \parallel S \parallel s} & \longmapsto & \boxed{C \parallel s(l) : S \parallel s} \\ \boxed{\mathsf{OP}(\,op\,) : C \parallel \underline{n_1} : \underline{n_2} : S \parallel s} & \longmapsto & \boxed{C \parallel \underline{n_1 \; op \; n_2} : S \parallel s} \\ \boxed{\mathsf{STO}(l) : C \parallel \underline{c} : S \parallel s} & \longmapsto & \boxed{C \parallel S \parallel s\{l \mapsto c\}} \\ \boxed{\mathsf{BR}(C_1,C_2) : C \parallel \underline{F} : S \parallel s} & \longmapsto & \boxed{C_2 : C \parallel S \parallel s} \\ \boxed{\mathsf{LOOP}(C_1,C_2) : C \parallel S \parallel s} & \longmapsto & \end{array}$$

$$\boxed{C_1 : \mathsf{BR}(C_2 : \mathsf{LOOP}(C_1,C_2), \mathsf{SKIP}) : C \parallel S \parallel s}$$

$$\begin{array}{rcl} \llbracket \underline{c} \rrbracket & \stackrel{\text{def}}{=} & \mathsf{PUSH}(\underline{c}) \\ \llbracket l \rrbracket & \stackrel{\text{def}}{=} & \mathsf{FETCH}(l) \\ \llbracket P_1 \; op \; P_2 \rrbracket & \stackrel{\text{def}}{=} & \llbracket P_2 \rrbracket : \llbracket P_1 \rrbracket : \mathsf{OP}(op) \\ \llbracket l := P \rrbracket & \stackrel{\text{def}}{=} & \llbracket P \rrbracket : \mathsf{STO}(l) \\ \llbracket \mathsf{skip} \rrbracket & \stackrel{\text{def}}{=} & \mathsf{SKIP} \\ \llbracket P_1 \,;\, P_2 \rrbracket & \stackrel{\text{def}}{=} & \llbracket P_1 \rrbracket : \llbracket P_2 \rrbracket \\ \llbracket \mathsf{if}\ P\ \mathsf{then}\ P_1\ \mathsf{else}\ P_2 \rrbracket & \stackrel{\text{def}}{=} & \llbracket P \rrbracket : \mathsf{BR}(\llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket) \\ \llbracket \mathsf{while}\ P_1\ \mathsf{do}\ P_2 \rrbracket & \stackrel{\text{def}}{=} & \mathsf{LOOP}(\llbracket P_1 \rrbracket, \llbracket P_2 \rrbracket) \end{array}$$

Chapter 3

By the end of this chapter you should be able to

- describe the "interpreted" CSS machine, which executes IMP programs;
- explain the outline of a proof of correctness;
- explain some of the results required for establishing correctness, and the proofs of these results.

Architecture of the Machine

- A CSS code C is a list of instructions which is produced by the following grammars:

$$C ::= - \mid ins : C \qquad ins ::= P \mid\ op\ \mid \mathsf{STO}(l) \mid \mathsf{BR}(P_1, P_2)$$

We will overload : to denote append; and write ξ for $\xi : -$ (ditto below).

- A stack S is produced by the grammar

$$S ::= - \mid \underline{c} : S$$

$$\boxed{\underline{n} : C}\boxed{S}\boxed{s} \longmapsto \boxed{C}\boxed{\underline{n} : \sigma}\boxed{s}$$

$$\boxed{P_1\ op\ P_2 : C}\boxed{S}\boxed{s} \longmapsto \boxed{P_2 : P_1 :\ op\ : C}\boxed{S}\boxed{s}$$

$$\boxed{op\ : C}\boxed{\underline{n_1} : \underline{n_2} : S}\boxed{s} \longmapsto \boxed{C}\boxed{\underline{n_1\ op\ n_2} : S}\boxed{s}$$

$$\boxed{l := P : C}\boxed{S}\boxed{s} \longmapsto \boxed{P : \mathsf{STO}(l) : C}\boxed{S}\boxed{s}$$

$$\boxed{\mathsf{STO}(l) : C}\boxed{\underline{n} : S}\boxed{s} \longmapsto \boxed{C}\boxed{S}\boxed{s\{l \mapsto n\}}$$

$$\boxed{\mathsf{while}\ P_1\ \mathsf{do}\ P_2 : C}\boxed{S}\boxed{s} \longmapsto$$

$$\boxed{P_1 : \mathsf{BR}((P_2\ ;\ \mathsf{while}\ P_1\ \mathsf{do}\ P_2), \mathsf{skip}) : C}\boxed{S}\boxed{s}$$

A Correctness Theorem

For all $n \in \mathbb{Z}$, $b \in \mathbb{B}$, $P_1 :: \mathsf{int}$, $P_2 :: \mathsf{bool}$, $P_3 :: \mathsf{cmd}$ and $s, s_1, s_2 \in States$ we have

$$(P_1\,,s) \Downarrow (\underline{n}\,, s) \quad \textit{iff} \quad \boxed{P_1}\boxed{-}\boxed{s} \longmapsto^t \boxed{-}\boxed{\underline{n}}\boxed{s}$$

$$(P_2\,,s) \Downarrow (\underline{b}\,, s) \quad \textit{iff} \quad \boxed{P_2}\boxed{-}\boxed{s} \longmapsto^t \boxed{-}\boxed{\underline{b}}\boxed{s}$$

$$(P_3\,,s_1) \Downarrow (\mathsf{skip}\,, s_2) \quad \textit{iff} \quad \boxed{P_3}\boxed{-}\boxed{s_1} \longmapsto^t \boxed{-}\boxed{-}\boxed{s_2}$$

where $\longmapsto^t$ denotes the transitive closure of $\longmapsto$.

Proof Method

- $\Longrightarrow_{onlyif}$ by Rule Induction for $\Downarrow$.
- $\Longleftarrow_{if}$ by Mathematical Induction on k. Recall $\kappa \longmapsto^t \kappa'$ iff $(\exists\, k \geq 1)(\kappa \longmapsto^k \kappa')$, where for $k \geq 1$, $\kappa \longmapsto^k \kappa'$ means that

$$(\forall 1 \leq i \leq k)(\exists \kappa_i)(\kappa \longmapsto \kappa_1 \longmapsto \ldots \longmapsto \kappa_k = \kappa')$$

Then note if the $\square$ are configurations with ξ parameters

$$(\forall \xi)(\ (\exists k)(\square \longmapsto^k \square)\ \textit{implies}\ \ \square \Downarrow \square\ \ )$$

$\equiv$

$$(\forall k)\underbrace{(\forall \xi)\boxed{(\square \longmapsto^k \square\ \textit{implies}\ \ \square \Downarrow \square)}}_{\phi(k)}$$

Code and Stack Extension

For all $k \in \mathbb{N}$, and for all appropriate codes, stacks and states,

$$\boxed{C_1}\boxed{S_1}\boxed{s_1} \longmapsto^k \boxed{C_2}\boxed{S_2}\boxed{s_2}$$

implies

$$\boxed{C_1 : C_3}\boxed{S_1 : S_3}\boxed{s_1} \longmapsto^k \boxed{C_2 : C_3}\boxed{S_2 : S_3}\boxed{s_2}$$

where $\longmapsto^0$ is reflexive closure of $\longmapsto$.

Code Splitting

For all $k \in \mathbb{N}$, and for all appropriate codes, stacks and states, if

$$\boxed{C_1 : C_2}\boxed{S}\boxed{s} \longmapsto^k \boxed{-}\boxed{S''}\boxed{s''}$$

then there is a stack and state S' and s', and $k_1, k_2 \in \mathbb{N}$ for which

$$\boxed{C_1}\boxed{S}\boxed{s} \longmapsto^{k_1} \boxed{-}\boxed{S'}\boxed{s'}$$

$$\boxed{C_2}\boxed{S'}\boxed{s'} \longmapsto^{k_2} \boxed{-}\boxed{S''}\boxed{s''}$$

where $k_1 + k_2 = k$.

Typing and Termination Yields Values

For all $k \in \mathbb{N}$, and for all appropriate codes, stacks, states,

$$P :: \mathsf{int} \quad \textit{and} \quad \boxed{P}\boxed{S}\boxed{s} \longmapsto^k \boxed{-}\boxed{S'}\boxed{s'} \quad \textit{implies}$$

$$s = s' \quad \textit{and} \quad S' = \underline{n} : S \text{ some } n \in \mathbb{Z}$$

$$\textit{and} \quad \boxed{P}\boxed{-}\boxed{s} \longmapsto^k \boxed{-}\boxed{\underline{n}}\boxed{s}$$

and similarly for Booleans.

Proving the Theorem

($\Longrightarrow_{onlyif}$): Rule Induction for $\Downarrow$

(*Case* $\Downarrow$ OP$_1$): The inductive hypotheses are

$$\boxed{P_1 \mid - \mid s} \longmapsto^t \boxed{- \mid \underline{n_1} \mid s} \qquad \boxed{P_2 \mid - \mid s} \longmapsto^t \boxed{- \mid \underline{n_2} \mid s}$$

Then

$$\begin{aligned} \boxed{P_1\ op\ P_2 \mid - \mid s} &\longmapsto \boxed{P_2 : P_1 :\ op \mid - \mid s} \\ &\longmapsto^t \boxed{P_1 :\ op \mid \underline{n_2} \mid s} \equiv \boxed{P_1 :\ op \mid \underline{n_2} \mid s} \\ &\longmapsto^t \boxed{op \mid \underline{n_1} : \underline{n_2} \mid s} \\ &\longmapsto \boxed{- \mid \underline{n_1\ op\ n_2} \mid s} \end{aligned}$$

($\Longleftarrow_{if}$): We prove by induction for all k, for all $P :: \mathsf{int}, n, s$,

$$\underbrace{\boxed{P \mid - \mid s} \longmapsto^k \boxed{- \mid \underline{n} \mid s} \quad \textit{implies} \quad (P\,,s) \Downarrow (\underline{n}\,,s)}_{\phi(k)}$$

(*Proof of* $\forall k_0 \in \mathbb{N},\ \phi(k)_{k \le k_0}$ *implies* $\phi(k_0+1)$): Suppose that for some arbitrary k_0, $P :: \mathsf{int}$, n and s

$$\boxed{P \mid - \mid s} \longmapsto^{k_0+1} \boxed{- \mid \underline{n} \mid s} \qquad (*)$$

and then we prove $(P\,,s) \Downarrow (\underline{n}\,,s)$ by considering cases on P.

(*Case* P is $P_1\ op\ P_2$): Suppose that

$$\boxed{P_1\ op\ P_2 \mid - \mid s} \longmapsto^{k_0+1} \boxed{- \mid \underline{n} \mid s}$$

and so

$$\boxed{P_2 : P_1 :\ op \mid - \mid s} \longmapsto^{k_0} \boxed{- \mid \underline{n} \mid s}.$$

Using splitting and termination we have, noting $P_2 :: \mathsf{int}$, that

$$\begin{aligned} \boxed{P_2 \mid - \mid s} &\longmapsto^{k_1} \boxed{- \mid \underline{n_2} \mid s} \\ \boxed{P_1 :\ op \mid \underline{n_2} \mid s} &\longmapsto^{k_2} \boxed{- \mid \underline{n} \mid s} \end{aligned}$$

where $k_1 + k_2 = k_0$,

and repeating for the latter re-write we get

$$\begin{aligned} \boxed{P_1 \mid \underline{n_2} \mid s} &\longmapsto^{k_{21}} \boxed{- \mid \underline{n_1} : \underline{n_2} \mid s} \\ \boxed{op \mid \underline{n_1} : \underline{n_2} \mid s} &\longmapsto^{k_{22}} \boxed{- \mid \underline{n} \mid s} \end{aligned} \qquad (1)$$

where $k_{21} + k_{22} = k_2$. So as $k_1 \le k_0$, by induction we deduce that $(P_2\,,s) \Downarrow (\underline{n_2}\,,s)$, and from termination that

$$\boxed{P_1 \mid - \mid s} \longmapsto^{k_{21}} \boxed{- \mid \underline{n_1} \mid s}.$$

Also, as $k_{21} \le k_0$, we have inductively that $(P_1\,,s) \Downarrow (\underline{n_1}\,,s)$ and hence

$$(P_1\ op\ P_2\,,s) \Downarrow (\underline{n_1\ op\ n_2}\,,s).$$

But from determinism and (1) we see that $\underline{n_1\ op\ n_2} = \underline{n}$ and we are done.

Chapter 4

By the end of this chapter you should be able to

- describe the expressions and type system of a language with higher order functions;
- explain how to write simple programs;
- specify an eager evaluation relation;
- prove properties such as determinism.

What's Next? Expressions and Types for FUN

- Define the expression syntax and type system.

Examples of FUN Declarations

```
g :: Int -> Int -> Int
g x y = x+y

l1 :: [Int]
l1 = 5:(6:(8:(4:(nil))))

h :: Int
h  = hd (5:6:8:4:nil)

length :: [Bool] -> Int
length l  = if elist(l) then 0 else (1 + length t)
```


FUN Types

- The types of $\mathbb{FUN}^e$ are

$$\sigma \quad ::= \quad \mathsf{int} \mid \mathsf{bool} \mid \sigma \to \sigma \mid [\sigma]$$

- We shall write

$$\sigma_1 \to \sigma_2 \to \sigma_3 \to \ldots \to \sigma_n \to \sigma$$

for

$$\sigma_1 \to (\sigma_2 \to (\sigma_3 \to (\ldots \to (\sigma_n \to \sigma)\ldots))).$$

Thus for example $\sigma_1 \to \sigma_2 \to \sigma_3$ means $\sigma_1 \to (\sigma_2 \to \sigma_3)$.

FUN Expressions

The expressions are

E	$::=$	x	variables
	$\mid$	$\underline{c}$	constants
	$\mid$	K	constant identifier
	$\mid$	F	function identifier
	$\mid$	$E_1\ E_2$	function application
	$\mid$	$\mathsf{tl}(E)$	tail of list
	$\mid$	$E_1 : E_2$	cons for lists
	$\mid$	$\mathsf{elist}(E)$	Boolean test for empty list

Bracketing conventions apply ...

What's Next? A Formal FUN Type System

- Show how to declare the types of variables and identifiers.
- Give some examples.
- Define a type assignment system.

Contexts (Variable Environments)

- When we write a FUN program, we shall declare the types of variables, for example

$$x :: \mathsf{int}, y :: \mathsf{bool}, z :: \mathsf{bool}$$

- A context, variables assumed distinct, takes the form

$$\Gamma = x_1 :: \sigma_1, \ldots, x_n :: \sigma_n.$$

Identifier Environments

- When we write a FUN program, we want to declare the types of constants and functions.
- A simple example of an identifier environment is

$$\mathsf{K} :: \mathsf{bool},\ \ \mathsf{map} :: (\mathsf{int} \to \mathsf{int}) \to [\mathsf{int}] \to [\mathsf{int}],\ \ \mathsf{suc} :: \mathsf{int} \to \mathsf{int}$$

- An identifier type looks like $\sigma_1 \to \sigma_2 \to \sigma_3 \to \ldots \to \sigma_a \to \sigma$ where $a \geq 0$ and σ is **NOT a function type**.
- An identifier environment looks like

$$I = I_1 :: \iota_1, \ldots, I_m :: \iota_m.$$

Example Type Assignments

- With the previous identifier environment

$$x :: \mathsf{int}, y :: \mathsf{int}, z :: \mathsf{int} \vdash \mathsf{map}\,\mathsf{suc}\,(x : y : z : \mathsf{nil}_{\mathsf{int}}) :: [\mathsf{int}]$$

- We have

$$\varnothing \vdash \mathsf{if}\ \underline{T}\ \mathsf{then}\ \mathsf{hd}(\underline{2} : \mathsf{nil}_{\mathsf{int}})\ \mathsf{else}\ \mathsf{hd}(\underline{4} : \underline{6} : \mathsf{nil}_{\mathsf{int}}) :: \mathsf{int}$$

Inductively Defining Type Assignments

Start with an identifier environment I and a context Γ. Then

$$\frac{}{\Gamma \vdash x :: \sigma}\ (\text{ where } x :: \sigma \in \Gamma) \quad ::\ \textsc{var} \qquad \frac{}{\Gamma \vdash \underline{n} :: \mathsf{int}}\ ::\ \textsc{int}$$

$$\frac{\Gamma \vdash E_1 :: \mathsf{int} \quad \Gamma \vdash E_2 :: \mathsf{int}}{\Gamma \vdash E_1\ iop\ E_2 :: \mathsf{int}}\ ::\ \textsc{op}_1$$

$$\frac{\Gamma \vdash E_1 :: \sigma_2 \to \sigma_1 \quad \Gamma \vdash E_2 :: \sigma_2}{\Gamma \vdash E_1\ E_2 :: \sigma_1}\ ::\ \textsc{ap}$$

$$\frac{}{\Gamma \vdash I :: \iota}\ (\text{ where } I :: \iota \in I) \quad ::\ \textsc{idr}$$

$$\frac{}{\Gamma \vdash \mathsf{nil}_\sigma :: [\sigma]}\ ::\ \textsc{nil} \qquad \frac{\Gamma \vdash E_1 :: \sigma \quad \Gamma \vdash E_2 :: [\sigma]}{\Gamma \vdash E_1 : E_2 :: [\sigma]}\ ::\ \textsc{cons}$$

What's Next? Function Declarations and Programs

- Show how to code up functions.
- Define what makes up a FUN program.
- Give some examples.

Introducing Function Declarations

- To declare plus can write $\mathsf{plus}\, x\, y = x + y$.
- To declare fac

$$\mathsf{fac}\, x = \mathsf{if}\ x == \underline{1}\ \mathsf{then}\ \underline{1}\ \mathsf{else}\ x * \mathsf{fac}(x - \underline{1})$$

- And to declare that true denotes $\underline{T}$ we write $\mathsf{true} = \underline{T}$.
- In $\mathbb{FUN}^e$, can specify (recursive) declarations

$$K = E \qquad Fx = E' \qquad G\, x\, y = E'' \ldots$$

An Example Declaration

Let $I = I_1 :: [\mathsf{int}] \to \mathsf{int} \to \mathsf{int}, I_2 :: \mathsf{int} \to \mathsf{int}, I_3 :: \mathsf{bool}$. Then an example of an identifier declaration dec_I is

$$\begin{array}{rcl} I_1\, l\, y & = & \mathsf{hd}(\mathsf{tl}(\mathsf{tl}(l))) + I_2\, y \quad \stackrel{\text{def}}{=} \quad E_{I_1} \\ I_2 x & = & x * x \quad \stackrel{\text{def}}{=} \quad E_{I_2} \\ I_3 & = & \underline{T} \quad \stackrel{\text{def}}{=} \quad E_{I_3} \\ I_4\, u\, v\, w & = & u + v + w \quad \stackrel{\text{def}}{=} \quad E_{I_4} \end{array}$$

An Example Program

Let $I = F :: \mathsf{int} \to \mathsf{int} \to \mathsf{int}, K :: \mathsf{int}$. Then an identifier declaration dec_I is

$$\begin{array}{rcl} F\, x\, y & = & x + \underline{7} - y \quad \stackrel{\text{def}}{=} \quad E_F \\ K & = & \underline{10} \end{array}$$

An example of a program is $\boxed{dec_I \quad in\ F\ \underline{8}\ \underline{1} \leq K}$. Note that

$$\varnothing \vdash F\ \underline{8}\ \underline{1} \leq K :: \mathsf{bool}$$

and

$$\underbrace{x :: \mathsf{int}, y :: \mathsf{int}}_{\Gamma_F} \vdash x + \underline{7} - y :: \underbrace{\mathsf{int}}_{\sigma_F} \qquad \text{and} \qquad \varnothing \vdash K :: \mathsf{int}$$

Defining Programs

A program in $\mathbb{FUN}^e$ is a judgement of the form

$$dec_I \quad in\ P$$

where dec_I is a given identifier declaration and the program expression P satisfies a type assignment of the form

$$\varnothing \vdash P :: \sigma \qquad (\text{written} \quad P :: \sigma)$$

and $\quad \forall\ F\vec{x} = E_F \ \in dec_I$

$$\Gamma_F \vdash E_F :: \sigma_F$$

What's Next? Values and the Evaluation Relation

- Look at the notion of evaluation order.
- Define values, which are the results of eager program executions.
- Define an eager evaluation semantics: $P \Downarrow^e V$.
- Give some examples.

Evaluation Orders

- The operational semantics of $\mathbb{FUN}^e$ says when a program P evaluates to a value V. It is like the IMP evaluation semantics.
- Write this in general as $P \Downarrow^e V$, and examples are

$$\underline{3} + \underline{4} + \underline{10} \Downarrow^e \underline{17} \qquad \mathsf{hd}(\underline{2} : \mathsf{nil}_{\mathsf{int}}) \Downarrow^e \underline{2}$$

- Let $F\, x\, y = x + y$. We would expect $F\ (\underline{2} * \underline{3})\ (\underline{4} * \underline{5}) \Downarrow^e \underline{26}$.
- We could
  - evaluate $\underline{2} * \underline{3}$ to get value $\underline{6}$ yielding $F\ \underline{6}\ (\underline{4} * \underline{5})$,
  - then evaluate $\underline{4} * \underline{5}$ to get value $\underline{20}$ yielding $F\ \underline{6}\ \underline{20}$.
- We then call the function to get $\underline{6} + \underline{20}$, which evaluates to $\underline{26}$. This is call-by-value or eager evaluation.
- Or the function could be called first yielding $(\underline{2} * \underline{3}) + (\underline{4} * \underline{5})$ and then we continue to get $\underline{6} + (\underline{4} * \underline{5})$ and $\underline{6} + \underline{20}$ and $\underline{26}$. This is called call-by-name or lazy evaluation.

Defining and Explaining (Eager) Values

- Let dec_I be an identifier declaration, with typical typing

$$F :: \sigma_1 \to \sigma_2 \to \sigma_3 \to \ldots \to \sigma_a \to \sigma$$

Informally a is the maximum number of inputs taken by F. A value expression is any expression V produced by

$$V ::= \underline{c} \mid \mathsf{nil}_\sigma \mid F\, \vec{V} \mid V : V$$

where $\vec{V}$ abbreviates $V_1\ V_2\ \ldots\ V_{k-1}\ V_k$ and $0 \leq k < a$.

- Note also that k is strictly less than a, and that if $a = 1$ then $F\, \vec{V}$ denotes F.

- A value is any value expression for which dec_I in V is a valid $\mathbb{FUN}^e$ program.
- Suppose that $F :: \mathsf{int} \to \mathsf{int} \to \mathsf{int} \to \mathsf{int}$ and that $P_1 \Downarrow^e \underline{2}$ and $P_2 \Downarrow^e \underline{5}$ and $P_3 \Downarrow^e \underline{7}$ with P_i not values. Then

P	V
	F
$F\ P_1$	$F\ \underline{2}$
$F\ \underline{2}\ P_2$	$F\ \underline{2}\ \underline{5}$

P	V
$F\ \underline{2}\ \underline{5}\ P_3$	
$F\ \underline{2}\ \underline{5}\ \underline{7}$	$\underline{14}$
$F\ P_1\ P_2\ P_3$	$\underline{14}$

The Evaluation Relation

$$\frac{}{V \Downarrow^e V}\ \Downarrow^e\text{VAL} \qquad \frac{P_1 \Downarrow^e \underline{m} \quad P_2 \Downarrow^e \underline{n}}{P_1\ op\ P_2 \Downarrow^e \underline{m\ op\ n}}\ \Downarrow^e\text{OP}$$

$$\frac{P_1 \Downarrow^e \underline{T} \quad P_2 \Downarrow^e V}{\mathsf{if}\ P_1\ \mathsf{then}\ P_2\ \mathsf{else}\ P_3 \Downarrow^e V}\ \Downarrow^e\text{COND}_1$$

$$\frac{\left\{\begin{array}{l} P_1 \Downarrow^e F\vec{V} \quad P_2 \Downarrow^e V_2 \quad F\vec{V}\,V_2 \Downarrow^e V \\ \text{where either } P_1 \text{ or } P_2 \text{ is not a value}\end{array}\right.}{P_1\,P_2 \Downarrow^e V}\ \Downarrow^e\text{AP}$$

$$\frac{E_F[V_1,\dots,V_a/x_1,\dots,x_a] \Downarrow^e V}{F V_1 \dots V_a \Downarrow^e V}\ [F\vec{x} = E_F \text{ declared in } dec_I]\ \Downarrow^e\text{FID}$$

$$\frac{E_K \Downarrow^e V}{K \Downarrow^e V}\ [K = E_K \text{ declared in } dec_I]\ \Downarrow^e\text{CID}$$

$$\frac{P \Downarrow^e V : V'}{\mathsf{hd}(P) \Downarrow^e V}\ \Downarrow^e\text{HD} \qquad \frac{P \Downarrow^e V : V'}{\mathsf{tl}(P) \Downarrow^e V'}\ \Downarrow^e\text{TL}$$

$$\frac{P_1 \Downarrow^e V \quad P_2 \Downarrow^e V'}{P_1 : P_2 \Downarrow^e V : V'}\ \Downarrow^e\text{CONS}$$

$$\frac{P \Downarrow^e \mathsf{nil}_\sigma}{\mathsf{elist}(P) \Downarrow^e \underline{T}}\ \Downarrow^e\text{ELIST}_1 \qquad \frac{P \Downarrow^e V : V'}{\mathsf{elist}(P) \Downarrow^e \underline{F}}\ \Downarrow^e\text{ELIST}_2$$

Examples of Evaluations

Suppose that dec_I is

$$\begin{array}{rcl} G\,x &=& x * \underline{2} \\ K &=& \underline{3} \end{array}$$

$$\frac{\frac{}{G \Downarrow^e G}\ \text{VAL} \qquad \frac{\frac{}{\underline{3} \Downarrow^e \underline{3}}\ \text{VAL}}{K \Downarrow^e \underline{3}}\ \text{CID} \qquad \frac{\frac{\frac{}{\underline{3} \Downarrow^e \underline{3}}\ \text{VAL} \quad \frac{}{\underline{2} \Downarrow^e \underline{2}}\ \text{VAL}}{(x*\underline{2})[\underline{3}/x] = \underline{3}*\underline{2} \Downarrow^e \underline{6}}\ \text{OP}}{G\,\underline{3} \Downarrow^e \underline{6}}\ \text{FID}}{G\,K \Downarrow^e \underline{6}}\ \text{AP}$$

We can prove that

$$F\,\underline{2}\,\underline{3}\,(\underline{4}+\underline{1}) \Downarrow^e \underline{10}$$

where $F\,x\,y\,z = x+y+z$ as follows:

$$\frac{\frac{}{F\,\underline{2}\,\underline{3} \Downarrow^e F\,\underline{2}\,\underline{3}}\ \Downarrow^e\text{VAL} \qquad \frac{\frac{}{\underline{4} \Downarrow^e \underline{4}} \quad \frac{}{\underline{1} \Downarrow^e \underline{1}}}{\underline{4}+\underline{1} \Downarrow^e \underline{5}} \qquad T}{F\,\underline{2}\,\underline{3}\,(\underline{4}+\underline{1}) \Downarrow^e \underline{10}}\ \Downarrow^e\text{AP}$$

where T is the tree

$$\frac{\frac{\frac{\frac{\frac{}{\underline{2} \Downarrow^e \underline{2}} \quad \frac{}{\underline{3} \Downarrow^e \underline{3}}}{\underline{2}+\underline{3} \Downarrow^e \underline{5}} \qquad \frac{}{\underline{5} \Downarrow^e \underline{5}}}{\underline{2}+\underline{3}+\underline{5} \Downarrow^e \underline{10}}}{(x+y+z)[\underline{2},\underline{3},\underline{5}/x,y,z] \Downarrow^e \underline{10}}}{F\,\underline{2}\,\underline{3}\,\underline{5} \Downarrow^e \underline{10}}\ \Downarrow^e\text{FID}$$

What's Next? FUN Properties of Eager Evaluation

- Explain and define determinism.
- Explain and define subject reduction, that is, preservation of types during program execution.

Properties of FUN

- The evaluation relation for $\mathbb{FUN}^e$ is deterministic. More precisely, for all P, V_1 and V_2, if

$$P \Downarrow^e V_1 \qquad \textit{and} \qquad P \Downarrow^e V_2$$

then $V_1 = V_2$. (Thus $\Downarrow^e$ is a partial function.)

- Evaluating a program dec_I in P does not alter its type. More precisely,

$$(\varnothing \vdash P :: \sigma \ \textit{and}\ P \Downarrow^e V) \quad \textit{implies} \quad \varnothing \vdash V :: \sigma$$

for any P, V, σ and dec_I. The conservation of type during program evaluation is called subject reduction.

Chapter 5

By the end of this chapter you should be able to

- describe the SECD machine, which executes compiled $\mathbb{FUN}^e$ programs; here the expressions *Exp* are defined by $E ::= x \mid \underline{n} \mid F \mid E\ E$;
- show how to compile to SECD instruction sequences;
- write down example executions.

Architecture of the Machine

- The SECD machine consists of rules for transforming SECD configurations (S,E,C,D).
- The non-empty stack S is generated by

$$S ::= \begin{array}{c} \underline{n} \\ \uparrow \end{array} \quad \Big| \quad \begin{array}{c} S_l \ldots S_1 \\ clo_F \\ \uparrow \end{array}$$

- Each node occurs at a level ≥ 1.
- A stack S has a height the maximum level of any clo_F, or 0 otherwise.

- If the (unique) left-most closure node clo_F at level α exists, call it the α-prescribed node, and write $\alpha \quad S$.
- For any stack $\alpha \quad S$ of height ≥ 1 there is a sub-stack S' of shape

$$\begin{array}{c} S_l \ \ldots \ S_1 \\ {}_{\bullet} clo_F \\ \uparrow \end{array}$$

Given any other stack S_{l+1} there is a stack S''

$$\begin{array}{c} S_{l+1}\ S_l \ \ldots \ S_1 \\ {}_{\bullet} clo_F \\ \uparrow \end{array}$$

- Write $S_{l+1} \oplus S$ for S with S' replaced by S''.

- The environment E takes the form $x_1 = ?S_1 : \ldots : x_n = ?S_n$.
- The value of each ? is determined by the form of an S_i.
- If S_i is $\begin{array}{c} \underline{n} \\ \uparrow \end{array}$ then ? is 0; if S_i is $\begin{array}{c} clo_F \\ \uparrow \end{array}$ then ? is 1; in any other case, ? is $_{\mathsf{Av}}1$.

- A SECD code C is a list which is produced by the following grammars:

$$\begin{array}{rcl} ins & ::= & x \mid \underline{n} \mid F \mid \mathsf{APP} \\ C & ::= & - \mid ins : C \end{array}$$

- A typical dump looks like

$$(S_1, E_1, C_1, (S_2, E_2, C_2, \ldots (S_n, E_n, C_n, -) \ldots))$$

- We will overload : to denote append; and write ξ for $\xi : -$.

We define a compilation function $[\![-]\!] : Exp \to SECDcodes$ which takes an SECD expression and turns it into code.

- $[\![x]\!] \stackrel{\text{def}}{=} x$
- $[\![\underline{n}]\!] \stackrel{\text{def}}{=} \underline{n}$
- $[\![F]\!] \stackrel{\text{def}}{=} F$
- $[\![E_1\ E_2]\!] \stackrel{\text{def}}{=} [\![E_1]\!] : [\![E_2]\!] : \mathsf{APP}$

There is a representation of program values as stacks, given by

- $(\!|\underline{n}|\!) \stackrel{\text{def}}{=} \begin{array}{c}\underline{n}\\ \uparrow\end{array}$
- $$(\!|F\ V_1\ldots V_k|\!) \stackrel{\text{def}}{=} \begin{array}{c}(\!|V_k|\!)\ \ldots\ (\!|V_1|\!)\\ clo_F\\ \uparrow\end{array} = (\!|V_k|\!)\oplus\ldots\oplus(\!|V_1|\!)\oplus \begin{array}{c}clo_F\\ \uparrow\end{array}$$
- Recall $k < a$ with a the arity of F.

The Re-writes

A number is pushed onto the stack (the initial stack can be of any status):

$$\begin{array}{|c||c|}\hline \mathbf{S} & [_{\mathsf{Av}}]\alpha \quad S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & \underline{n}:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{num}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & \alpha \quad \begin{array}{c}\underline{n}\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & C\\\hline \mathbf{D} & D\\\hline\end{array}$$

A function is pushed onto the stack (the initial stack can be of any status):

$$\begin{array}{|c||c|}\hline \mathbf{S} & [_{\mathsf{Av}}]\alpha \quad S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & F:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{fn}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & \alpha+1 \quad \begin{array}{c}\bullet clo_F\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & C\\\hline \mathbf{D} & D\\\hline\end{array}$$

A variable's value is pushed onto the stack, provided that the environment E contains $x = ?T \equiv [_{\mathsf{Av}}]\delta \quad T$ (where δ is 0 or 1).

Note that by definition, the status of T determines the status of the re-written stack:

$$\begin{array}{|c||c|}\hline \mathbf{S} & [_{\mathsf{Av}}]\alpha \quad S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & x:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{var}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & [_{\mathsf{Av}}]\delta+\alpha \quad T\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & C\\\hline \mathbf{D} & D\\\hline\end{array}$$

An APP command creates an application value, type 0:

$$\begin{array}{|c||c|}\hline \mathbf{S} & \alpha \quad \begin{array}{c}S_k\ldots S_1\\ \bullet clo_F\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & \mathsf{APP}:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{cav0}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & \mathsf{Av}\,\alpha \quad \begin{array}{c}S_k\ldots S_1\\ \bullet clo_F\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & C\\\hline \mathbf{D} & D\\\hline\end{array}$$

An APP command creates an application value, type 1:

$$\begin{array}{|c||c|}\hline \mathbf{S} & \alpha \quad \begin{array}{c}\bullet clo_H \quad S_{k-1}\ldots S_1\\ \uparrow\\ clo_F\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & \mathsf{APP}:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{cav1}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & \mathsf{Av}\,\alpha-1 \quad \begin{array}{c}clo_H \quad S_{k-1}\ldots S_1\\ \uparrow\\ \bullet clo_F\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & C\\\hline \mathbf{D} & D\\\hline\end{array}$$

An APP command produces an application value from an application value:

$$\begin{array}{|c||c|}\hline \mathbf{S} & \mathsf{Av}\,\alpha \quad \begin{array}{c}S_k\ldots S_1\\ \bullet clo_F \quad S'_{k'-1}\ldots S'_1\\ \uparrow\\ clo_G\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & \mathsf{APP}:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{avtav}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & \mathsf{Av}\,\alpha-1 \quad \begin{array}{c}S_k\ldots S_1\\ clo_F \quad S'_{k'-1}\ldots S'_1\\ \uparrow\\ \bullet clo_G\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & C\\\hline \mathbf{D} & D\\\hline\end{array}$$

An APP command calls a function, type 0:

$$\begin{array}{|c||c|}\hline \mathbf{S} & \alpha \quad \begin{array}{c}S_a\ldots S_1\\ \bullet clo_F\\ \uparrow\end{array}\oplus S\\\hline \mathbf{E} & E\\\hline \mathbf{C} & \mathsf{APP}:C\\\hline \mathbf{D} & D\\\hline\end{array} \overset{\text{call0}}{\longmapsto} \begin{array}{|c||c|}\hline \mathbf{S} & -\\\hline \mathbf{E} & x_a = ?S_a : \ldots : x_1 = ?S_1 : E\\\hline \mathbf{C} & [\![E_F]\!]\\\hline \mathbf{D} & (\alpha-1 \quad S, E, C, D)\\\hline\end{array}$$

An APP command calls a function, type 1:

S	α	$.clo_H$ ↑ clo_F ↑ $\quad S_{a-1}\ldots S_1 \quad \oplus S$
E		E
C		$\mathsf{APP} : C$
D		D

$\xmapsto{\text{call1}}$

S	$-$
E	$x_a = ?S_a : \ldots : x_1 = ?S_1 : E$
C	$[\![E_F]\!]$
D	$(\alpha - 2 \quad S, E, C, D)$

An APP command calls a function, type 2:

S	$_{\mathsf{Av}}\,\alpha$	$S_k \ldots S_1$ $.clo_F$ ↑ clo_G ↑ $\quad S'_{a-1}\ldots S'_1 \quad \oplus S$
E		E
C		$\mathsf{APP} : C$
D		D

$\xmapsto{\text{call2}}$

S	$-$
E	$x_a = ?S'_a : \ldots : x_1 = ?S'_1 : E$
C	$[\![E_G]\!]$
D	$(\alpha - 2 \quad S, E, C, D)$

Restore, where the final status is determined by the initial status:

S	$[_{\mathsf{Av}}]\beta \quad T$
E	E'
C	$-$
D	$(\alpha \quad S, E, C, D)$

$\xmapsto{\text{res}}$

S	$[_{\mathsf{Av}}]\alpha + \beta \quad T \oplus S$
E	E
C	C
D	D

Suppose that K, N and MN are functions which are also values, and that

$$F\,x\,y = x \qquad I\,a\,b = b$$
$$L\,u\,v = u \qquad H\,z = L\,(M\,N)\,z$$

Then $(F\ (H\ \underline{4}))\ (I\ \underline{2}\ K) \Downarrow^e M\,N$. Note that

$[\![(F\ (H\ \underline{4}))\ (I\ \underline{2}\ K)]\!] =$

$(11. \stackrel{\text{def}}{=} F) : H : \underline{4} : \mathsf{APP} : \mathsf{APP} : I : \underline{2} : \mathsf{APP} : K : \mathsf{APP} : (\mathsf{APP} \stackrel{\text{def}}{=} 1.)$

and

$[\![L\ (M\ N)\ z]\!] \stackrel{\text{def}}{=} 7. \stackrel{\text{def}}{=} L : M : N : \mathsf{APP} : \mathsf{APP} : z : \mathsf{APP} \stackrel{\text{def}}{=} 1.$

S	$0 \quad -$
E	$-$
C	$11.$
D	$-$

$\xmapsto{\text{num/fn}}{}^3$

S	2	$\underline{4}$ ↑ $.clo_H$ ↑ clo_F ↑
E		$-$
C		$8. \equiv \mathsf{APP} : 7.$
D		$-$

$\xmapsto{\text{call0}}$

S	$0 \quad -$
E	$E' \stackrel{\text{def}}{=} z = 0$ $\quad \underline{4}$ ↑
C	$[\![L\ (M\ N)\ z]\!]$
D	$\xi \stackrel{\text{def}}{=} (1 \quad clo_F$ ↑ $, -, 7., -)$

$\xmapsto{\text{fn}}{}^3$

S	3	$.clo_N$ ↑ clo_M ↑ clo_L ↑
E		E'
C		$4.$
D		ξ

$\xmapsto{\text{cav1}}$

S	$_{\mathsf{Av}}\,2$	clo_N ↑ $.clo_M$ ↑ clo_L ↑
E		E'
C		$3.$
D		ξ

$\xmapsto{\text{avtav}}$

S	$_{\mathsf{Av}}\,1$	clo_N ↑ clo_M ↑ $.clo_L$ ↑
E		E'
C		$2.$
D		ξ

Chapter 6

By the end of this chapter you should be able to

- explain the outline of a proof of correctness;
- explain some of the results required for establishing correctness, and the proofs of these results.

A Correctness Theorem

For all programs $dec_I \quad in\ P$ for which $\varnothing \vdash P :: \sigma$ we have

$$P \Downarrow^e V \qquad \textit{iff} \qquad \begin{array}{|c|c|}\hline \mathbf{S} & - \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![P]\!] \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^t \begin{array}{|c|c|}\hline \mathbf{S} & (\![V]\!) \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

Code and Stack Extension

For any stacks, environments, codes, and dumps, if C_1 is non-empty

$$M \stackrel{\text{def}}{=} \begin{array}{|c|c|}\hline \mathbf{S} & S_1 \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_1 \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^k \begin{array}{|c|c|}\hline \mathbf{S} & S_2 \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_2 \\ \hline \mathbf{D} & D \\ \hline \end{array} \stackrel{\text{def}}{=} M'$$

implies

$$\overline{M} \stackrel{\text{def}}{=} \begin{array}{|c|c|}\hline \mathbf{S} & S_1 \oplus S_3 \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_1 : C_3 \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^k \begin{array}{|c|c|}\hline \mathbf{S} & S_2 \oplus S_3 \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_2 : C_3 \\ \hline \mathbf{D} & D \\ \hline \end{array} \stackrel{\text{def}}{=} \overline{M'}$$

- Need to prove "lemma plus": if $D \equiv (S', E', C', D')$ we can also similarly arbitrarily extend any of the stacks and codes in D (say to $\overline{D}$).
- We use induction on k. Suppose lemma plus is true $\forall k \leq k_0$. Must prove we can extend any re-write $M \longmapsto^{k_0+1} M'$ to $\overline{M} \longmapsto^{k_0+1} \overline{M'}$. By determinism, we have $M \longmapsto^1 M'' \longmapsto^{k_0} M'$.
- If no function call during $M \longmapsto^1 M''$, trivial to extend to get $\overline{M} \longmapsto^1 \overline{M''}$. And by induction, $\overline{M''} \longmapsto^{k_0} \overline{M'}$.

If there is a function call, there are k_1 and k_2 such that

$$M \stackrel{\text{def}}{=} \begin{array}{|c|c|}\hline \mathbf{S} & T \oplus S \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & \mathrm{APP} : C \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^1 \begin{array}{|c|c|}\hline \mathbf{S} & - \\ \hline \mathbf{E} & E' \\ \hline \mathbf{C} & [\![E_F]\!] \\ \hline \mathbf{D} & (S,E,C,D) \\ \hline \end{array} \longmapsto^{k_1} \begin{array}{|c|c|}\hline \mathbf{S} & S'' \\ \hline \mathbf{E} & E'' \\ \hline \mathbf{C} & C'' \\ \hline \mathbf{D} & (S,E,C,D) \\ \hline \end{array}$$

$$\stackrel{\text{res}}{\longmapsto}{}^1 \begin{array}{|c|c|}\hline \mathbf{S} & S'' \oplus S \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^{k_2} M'$$

where there are no function calls in the k_2 re-writes.

By induction, we have

$$\overline{M''} \stackrel{\text{def}}{=} \begin{array}{|c|c|}\hline \mathbf{S} & - \\ \hline \mathbf{E} & E' \\ \hline \mathbf{C} & [\![E_F]\!] \\ \hline \mathbf{D} & (S \oplus S_3, E, C : C_3, \overline{D}) \\ \hline \end{array} \longmapsto^{k_1} \begin{array}{|c|c|}\hline \mathbf{S} & S'' \\ \hline \mathbf{E} & E'' \\ \hline \mathbf{C} & C'' \\ \hline \mathbf{D} & (S \oplus S_3, E, C : C_3, \overline{D}) \\ \hline \end{array}$$

It is easy to see that $\overline{M} \longmapsto \overline{M''}$, and obviously

$$\begin{array}{|c|c|}\hline \mathbf{S} & S'' \\ \hline \mathbf{E} & E'' \\ \hline \mathbf{C} & C'' \\ \hline \mathbf{D} & (S \oplus S_3, E, C : C_3, \overline{D}) \\ \hline \end{array} \longmapsto^1 \begin{array}{|c|c|}\hline \mathbf{S} & S'' \oplus S \oplus S_3 \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C : C_3 \\ \hline \mathbf{D} & \overline{D} \\ \hline \end{array}$$

If $k_2 = 0$ then we are done.

If $k_2 \geq 1$ then we can similarly extend the stack and code of the final $k_2 \geq 1$ transitions by induction

$$\begin{array}{|c|c|}\hline \mathbf{S} & S'' \oplus S \oplus S_3 \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C : C_3 \\ \hline \mathbf{D} & \overline{D} \\ \hline \end{array} \longmapsto^1 \overline{M'}.$$

and we are also done.

Code Splitting

For any stacks, environments, codes, and dumps, if C_1 and C_2 are non-empty then

$$\begin{array}{|c|c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_1 : C_2 \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^k \begin{array}{|c|c|}\hline \mathbf{S} & S'' \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & D \\ \hline \end{array}$$

implies that

$$\begin{array}{|c|c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_1 \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^{k_1} \begin{array}{|c|c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & D \\ \hline \end{array} \quad \textit{and} \quad \begin{array}{|c|c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & C_2 \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^{k_2} \begin{array}{|c|c|}\hline \mathbf{S} & S'' \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & D \\ \hline \end{array}$$

where $k = k_1 + k_2$.

Program Code Factors Through Value Code

For any well typed $\mathbb{FUN}^e$ program $dec_I \quad in \ P$ where $P :: \sigma$ and $P \Downarrow^e V$,

$$\begin{array}{|c||c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & [\![P]\!] \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^{k} \begin{array}{|c||c|}\hline \mathbf{S} & \hat{S} \\ \hline \mathbf{E} & \hat{E} \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & \hat{D} \\ \hline \end{array} \quad implies \quad (\exists \overline{k} \leq k) \quad \begin{array}{|c||c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & E \\ \hline \mathbf{C} & [\![V]\!] \\ \hline \mathbf{D} & D \\ \hline \end{array} \longmapsto^{\overline{k}} \begin{array}{|c||c|}\hline \mathbf{S} & \hat{S} \\ \hline \mathbf{E} & \hat{E} \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & \hat{D} \\ \hline \end{array}$$

with equality only if P is a value (and hence equal to V).

Proving the Theorem

($\Longleftarrow_{if}$): We shall prove that if $P :: \sigma$ then

$$\begin{array}{|c||c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![P]\!] \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k} \begin{array}{|c||c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array} \quad implies \quad (\exists V) \quad S' = (\!|V|\!) \oplus S \quad and \quad P \Downarrow^e V$$

from which the required result follows. Induction on k. If P is a number or a function the result is trivial. Else P has the form $P_1 P_2$.

Suppose that

$$\begin{array}{|c||c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![P_1]\!] : [\![P_2]\!] : \mathsf{APP} \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k_0+1} \begin{array}{|c||c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

Then appealing to splitting and the induction hypothesis, we get

$$\begin{array}{|c||c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![P_1]\!] \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k_1} \begin{array}{|c||c|}\hline \mathbf{S} & (\!|F\ \vec{V}|\!) \oplus S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

and

$$\begin{array}{|c||c|}\hline \mathbf{S} & (\!|F\ \vec{V}|\!) \oplus S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![P_2]\!] : \mathsf{APP} \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k_2} \begin{array}{|c||c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

where $P_1 \Downarrow^e F\ \vec{V}$.

Appealing to splitting again, and by induction,

$$\begin{array}{|c||c|}\hline \mathbf{S} & (\!|F\ \vec{V}|\!) \oplus S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![P_2]\!] \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k_{21}} \begin{array}{|c||c|}\hline \mathbf{S} & (\!|V_2|\!) \oplus (\!|F\ \vec{V}|\!) \oplus S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

and

$$\begin{array}{|c||c|}\hline \mathbf{S} & (\!|V_2|\!) \oplus (\!|F\ \vec{V}|\!) \oplus S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & \mathsf{APP} \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k_{22}} \begin{array}{|c||c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

where $P_2 \Downarrow^e V_2$.

By factorization on P_1 and P_2, and extension we have (check!)

$$\begin{array}{|c||c|}\hline \mathbf{S} & S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & [\![F\ \vec{V}\ V_2]\!] \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{\overline{k_1}+\overline{k_{21}}} \begin{array}{|c||c|}\hline \mathbf{S} & (\!|V_2|\!) \oplus (\!|F\ \vec{V}|\!) \oplus S \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & \mathsf{APP} \\ \hline \mathbf{D} & - \\ \hline \end{array} \longmapsto^{k_{22}} \begin{array}{|c||c|}\hline \mathbf{S} & S' \\ \hline \mathbf{E} & - \\ \hline \mathbf{C} & - \\ \hline \mathbf{D} & - \\ \hline \end{array}$$

and so if $P_1\ P_2$ is not a value then

$$\overline{k_1} + \overline{k_{21}} + k_{22} < k_0 + 1$$

and by induction $S' = (\!|V|\!) \oplus S$ for some V where $F\ \vec{V}\ V_2 \Downarrow^e V$.
Hence $P_1\ P_2 \Downarrow^e V$ as required.

If $P_1\ P_2$ is a value, refer to part ($\Longrightarrow_{onlyif}$) of the proof, case $\Downarrow^e$ VAL