Introducing open games

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Joint work with

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February 24, 2017
Motivation: powerful machinery vs. hard problems

- Functional programming
- Type systems
- Higher order logic
- Monads
- Algebraic effects
- Problems of economics
- Monoidal categories
- String diagrams
- Compositionality

Source: Land of Lisp
What game theory is

- Mathematical theory of interacting “rational” agents
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- **Players** make *observations* and then make *choices*
- Group choices determine *payoffs*
- “Local view” of rationality: players act to maximise payoff
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- Mathematical theory of interacting “rational” agents
- Players make observations and then make choices
- Group choices determine payoffs
- “Local view” of rationality: players act to maximise payoff
- “Global view”: equilibrium strategies
Example: penalty shootout

\[ a, b \in \{L, R\} \]
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\[ \pi(a, b) = \begin{cases} 
(+1, -1) & \text{if } a \neq b \\
(-1, +1) & \text{if } a = b 
\end{cases} \]

Unique (probabilistic) equilibrium:

\[ a = b = \frac{1}{2} |L\rangle + \frac{1}{2} |R\rangle \]

Nash's theorem generalises this situation.
Example: penalty shootout

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Example: the $$$ auction
Game theory has some issues

- Well known: equilibrium as behavioural prediction is experimentally falsified (e.g. ultimatum game)
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- Ordinary games do not compose/scale
Beliefs have causal effects
What is compositionality?

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A theory is **compositional** if:

- Objects of interest are **black boxes with interfaces**
- Objects can be composed **without knowing** how they are defined
- Consequence: can easily change one component in a large structure
- All reasoning is by structural induction on composition
Examples of compositional systems

- Any serious programming language
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Non-examples:
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- Economic systems
If a theory is compositional, then reasoning with it is scalable.
The compositionality hypothesis

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Hypothesis

Compositionality is the only way to be scalable
Compositionality is delicate

Compositionality is not all-or-nothing
Compositionality is delicate

Compositionality is not all-or-nothing

In programming languages it is harmed by:

- Goto
- Mutable global state
- Inheritance
- Type classes *a la* Haskell

More generally:

- Leaky abstractions
- Emergent behaviour
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Compositionality via symmetric monoidal categories (1)

Objects (aka. interfaces, types, systems) $X$

\[ X \rightarrow X \]
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Objects (aka. interfaces, types, systems) $X$

\[ X \]

Morphisms (aka. black boxes, processes) $f : X \to Y$

\[ f \]

\[ Y \]

\[ X \]
Compositionality via symmetric monoidal categories (1)

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```
X
```

Morphisms (aka. black boxes, processes) $f : X \to Y$

```
\[
f \quad X \to Y
\]

Compound object $X_1 \otimes X_2$

```
\[
\begin{tikzcd}
X_1 \ar{u}
\ar{dr}

\bigotimes

\
\ar{u}

\begin{tikzcd}
X_2
\end{tikzcd}
\end{tikzcd}
```

$X_1$ $X_2$
Compositionality via symmetric monoidal categories (1)

Objects (aka. interfaces, types, systems) $X$

$X$

Morphisms (aka. black boxes, processes) $f : X \rightarrow Y$

$Y$

Compound object $X_1 \otimes X_2$

$X_1$

Monoidal product (aka. tensor product, simultaneous/spatial composition)

$f_1 \otimes f_2 : X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2$

$Y_1$

$f_1$

$X_1$

$f_2$

$X_2$
Categorical composition (aka. sequential/temporal composition) $g \circ f : X \rightarrow Z$
Categorical composition (aka. sequential/temporal composition) $g \circ f : X \to Z$

\[ f : I \to X \otimes Y \]
\[ g : X \to Z \]
\[ h : X \otimes Z \to I \]
\[ h \circ \sigma_{Z,X} \circ (g \otimes Y) \circ f : I \to I \]
Open games

A closed game consists of:
- A set $\Sigma$ of strategy profiles
- A best response function $B : \Sigma \to \mathcal{P}(\Sigma)$

An open game

$$G : (X, S) \to (Y, R)$$

consists of:
- A set $\Sigma$ of strategy profiles
- A play function $P_G : \Sigma \times X \to Y$
- A coplay function $C_G : \Sigma \times X \times R \to S$
- A best response function

$$B_G : X \times (Y \to R) \times \Sigma \to \mathcal{P}(\Sigma)$$
Bimatrix game

\[ \begin{array}{c}
X \\
A_1
\end{array} \quad U \quad \begin{array}{c}
Y \\
A_2
\end{array} \]
Perfect information game
Imperfect information