INFINITE MULTI-BASES

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Abstract. Infinite multi-bases can have infinite and multiple type declarations for the same variable. They can be used as a proof-technique to manipulate only one common basis along the proof. However, no proper definition and precise study of typed lambda calculus with infinite multi-bases appear in the literature. This paper introduces type assignment systems with infinite multi-bases and studies the basic meta-theoretic properties. As an application of our study of multi-bases, we prove that a function on λ-terms satisfies the type semantics property if and only if this function defines a λ structure which coincides with the usual filter structure.

Introduction

In typed lambda calculus, a basis is usually defined as a *finite* set (or sequence) of type declarations, i.e.

$$\Gamma = \{x_1 : A_1, \ldots, x_n : A_n\}$$

It is also required that the variables $x_1, \ldots, x_n$ be all different [2]. In contrast, infinite multi-bases can have infinite and multiple type declarations for the same variable. They can be used as a proof-technique to manipulate only one common basis along the proof. The need of multiple declarations for the same variable x seems natural in the presence of type assignment systems with intersection types [4, 3, 5, 7]. In order to prove that the interpretation of a term is the set of its type (Type Semantics Theorem for intersection types), we could use an infinite multi-basis:

$$\Gamma_\rho = \{x : B \mid B \in \rho(x)\}$$

where ρ is an assignment of variables in the filter model. Then we proceed to prove that:

$$[\![M]\!]_\rho = \{A \mid \Gamma_\rho \vdash M : A\}$$

However, no proper definition and precise study of typed lambda calculus with multi-bases appear in the literature. This paper introduces type assignment systems with multi-bases and studies basic meta-theoretic properties.

The main problem with multi-bases concerns rule ($\to$I).

$$(\to\text{I})\ \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$$

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This rule is not valid if Γ is a multi-basis. We cannot derive that $\Gamma \vdash \lambda x.M : A \to B$ from $\Gamma, x : A \vdash M : B$ because there may be some other declaration for x in Γ besides $x : A$. If the set $\Gamma(x) = \{x : A_1, \dots, x : A_n\}$ of declarations for x in Γ is finite then we could consider the following variant of ($\to$I).

$$(\to\text{I' })\frac{\Gamma \vdash M : B}{\Gamma \vdash \lambda x.M : A_1 \cap A_2 \cap \ldots \cap A_n \to B} \quad \text{where } \Gamma(x) = \{x : A_1, \dots, x : A_n\}$$

The above solution does not work in general since the set $\Gamma(x)$ of declarations for x in Γ can be infinite. In general, we have to consider the following variant of ($\to$I) that makes use of principal filters:

$$(\to\text{I" })\frac{\Gamma[x := \uparrow A] \vdash^{\mathcal{T}} M : B}{\Gamma \vdash^{\mathcal{T}} \lambda x.M : A \to B}.$$

Since it is not clear when the typing rules will be valid, we also have to be careful how we define typability for multi-bases. We define that a term is typable in a multi-basis Γ if it is typable in a finite basis that is contained in Γ. Having this definition in mind, we investigate whether the typing rules are derivable for multi-bases.
In Section 1 we simplify matters and study infinite bases (with no multiple declarations for the same variable) for the simply typed lambda calculus.
In Section 2 we study multi-bases for intersection type assignment systems. We first give examples that show that ($\to$I) and ($\cap$I) are not derivable. We, then, show that all rules, except ($\to$I), are derivable if we require that the bases are closed under $\cap$. Finally, we show that the rule ($\to$I") shown above is indeed derivable.
In Section 3, we present an application of our study of multi-bases. We prove that a function on λ-terms satisfies the type semantics property if and only if this function coincides with the usual filter interpretation on λ-terms. The "only if" is the well-known type semantics theorem which we will prove using multi-bases. The other direction is an interesting observation since it means that there is only one way of defining application and abstraction on the set of filters when the type semantics property holds.

1. Infinite Bases in Simply Typed Lambda Calculus

In this section we recall the definition of simply typed lambda calculus [2] where bases are assumed to be finite. For the extension to infinite bases, we do not redefine the typing system. Instead, we define that a term is typable in an infinite basis Δ if it is typable in a finite basis that is contained in Δ. Then, we show a variant of rule ($\to$I) that is derivable.

Definition 1.1. Let $\mathbb{A}$ be a set of atomic types. The set $\mathbb{T}$ of simply types is defined by:

$$\mathbb{T} = \mathbb{A} \mid \mathbb{T} \to \mathbb{T}$$

Definition 1.2. (1) A *declaration* is a statement of the form $x : A$ with $A \in \mathbb{T}$.
(2) A (singled) *basis* Γ is a set of declarations with all variables distinct, i.e. there is only one declaration per variable.

Definition 1.3. Let $M \in \Lambda$, $A \in \mathbb{T}$ and Γ a finite singled basis. The *simply typed lambda calculus* derives assertions $\Gamma \vdash M : A$ by the following axioms and rules.

$$\begin{array}{lll}
(\text{ax}) & \Gamma \vdash x{:}A & \text{if } (x{:}A \in \Gamma) \\[1ex]
(\to\text{I}\ ) & \dfrac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} & \\[2ex]
(\to\text{E}) & \dfrac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} &
\end{array}$$

Definition 1.4. Let Δ be a basis that is allowed to be infinite. We define $\Delta \vdash M : A$ iff there exists a finite basis $\Gamma \subseteq \Delta$ such that $\Gamma \vdash M : A$.

Lemma 1.5. *The axiom (ax) and rule (→E) are derivable when the basis* Γ *is infinite.*

Since the basis Γ can contain all variables, the rule (→I) is derivable for infinite bases only if it is reformulated in the following way.

Lemma 1.6. *Let* Γ *be a possible infinite basis. The following variant of rule (→I) is derivable:*

$$\frac{\Gamma_x, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M) : (A \to B)}$$

where Γ_x *is the result of removing the declaration of* x *from* Γ.

2. Multi-bases for Type Assignment with Intersection Types

In this section we first recall the notions of intersection type theory and type assignment systems [4, 3, 5, 7] where bases are assumed to be finite and singled (no multiple declarations of the same variable). Similarly to the simply typed lambda calculus, we define that a term is typable in a (possibly infinite) multi-basis Δ if that term is typable in a finite singled basis contained in Δ. We show conditions to ensure the derivability of rules (→∩) and (→I). We also prove Generation Lemma for infinite multi-bases.

Definition 2.1. Let $\mathbb{A}$ be a set of atomic types with a special symbol U called *universal top*. The set $\mathbb{T}^\cap$ of intersection types is defined by induction as follows:

$$\mathbb{T}^\cap = \mathbb{A} \mid \mathbb{T}^\cap \to \mathbb{T}^\cap \mid \mathbb{T}^\cap \cap \mathbb{T}^\cap$$

Definition 2.2. An *intersection type theory over a set* $\mathbb{A}$ *of type atoms* is a set $\mathcal{T}$ of sentences of the form $A \leq B$ (to be read: A is a subtype of B), with $A, B \in \mathbb{T}^\cap$, satisfying at least the following axioms and rules.

$$\begin{array}{llllll}
(\text{refl}) & A \leq A & (\text{incl}_L) & A \cap B \leq A & (\text{incl}_R) & A \cap B \leq B \\[1ex]
(\text{trans}) & \dfrac{A \leq B \quad B \leq C}{A \leq C} & (\text{glb}) & \dfrac{C \leq A \quad C \leq B}{C \leq A \cap B} & (\mathsf{U}) & A \leq \mathsf{U}
\end{array}$$

Definition 2.3.

(1) A *declaration* is a statement of the form $x : A$ with $A \in \mathbb{T}^\cap$.
(2) A (singled) *basis* Γ is a set of declarations with all variables distinct, i.e. there is only one declaration per variable.

Definition 2.4. Let $M \in \Lambda$, $A \in \mathbb{T}^{\cap}$ and Γ a finite singled basis. The *(intersection) type assignment system* over an intersection type theory $\mathcal{T}$ derives assertions $\Gamma \vdash^{\mathcal{T}} M : A$ by adding the following rules to the axioms and rules of Definition 1.3.

$$(\cap\mathrm{I}\ )\quad \frac{\Gamma \vdash^{\mathcal{T}} M : A \quad \Gamma \vdash^{\mathcal{T}} M : B}{\Gamma \vdash^{\mathcal{T}} M : A \cap B} \qquad (\mathsf{U}\mathrm{I})\quad \Gamma \vdash^{\mathcal{T}} M : \mathsf{U}$$

$$(\leq_{\mathcal{T}})\quad \frac{\Gamma \vdash^{\mathcal{T}} M : A \quad A \leq_{\mathcal{T}} B}{\Gamma \vdash^{\mathcal{T}} M : B}$$

Definition 2.5 (Multiplication of bases)**.**

$$\begin{array}{rcl} \Gamma \uplus \Gamma' & = & \{x\!: A \cap B \mid x\!: A \in \Gamma \text{ and } x\!: B \in \Gamma'\} \\ & \cup & \{x\!: A \mid x\!: A \in \Gamma \text{ and } x \notin \Gamma'\} \\ & \cup & \{x\!: B \mid x\!: B \in \Gamma' \text{ and } x \notin \Gamma\}. \end{array}$$

For example, $\{x\!:\!A, y\!:\!B\} \uplus \{x\!:\!C, z\!:\!D\} = \{x\!:\!A \cap C, y\!:\!B, z\!:\!D\}$.

Lemma 2.6. *The following rules are admissible in all intersection type assignment systems.*

$$(\leq \vdash^{\mathcal{T}} \mathrm{L})\qquad \frac{\Gamma, x\!:\!B \vdash^{\mathcal{T}} M : A \quad C \leq_{\mathcal{T}} B}{\Gamma, x\!:\!C \vdash^{\mathcal{T}} M : A}$$

$$(\textit{multiple weakening})\quad \frac{\Gamma_1 \vdash^{\mathcal{T}} M : A}{\Gamma_1 \uplus \Gamma_2 \vdash^{\mathcal{T}} M : A}$$

Definition 2.7. A *multi-basis* Γ is a set of declarations, in which the requirement that

$$x\!:\!A, y\!:\!B \in \Gamma \Rightarrow x \equiv y \Rightarrow A \equiv B$$

is dropped. Then, the set $\Gamma(x)$ is defined as $\{A \mid x : A \in \Gamma\}$.

For an infinite multi-basis Γ, the set $\Gamma(x)$ could be infinite.

Definition 2.8. Let Δ be a (possibly infinite) multi-basis. We define $\Delta \vdash^{\mathcal{T}} M : A$ iff there exists a singled (only one declaration per variable) basis $\Gamma \subseteq \Delta$ such that $\Gamma \vdash^{\mathcal{T}} M : A$.

Let CDS be the smallest intersection type theory over the set $\mathbb{A} = \{\alpha_1, \alpha_2, \ldots\} \cup \{\mathsf{U}\}$ of atoms originally defined in [4]. The following examples show that the rules $(\cap\mathrm{I})$, $(\to\mathrm{E})$ and $(\to\mathrm{I})$ are not derivable for multi-bases.

Example 2.9. Rule $(\cap\mathrm{I})$ is not derivable.
$x\!:\!\alpha_1, x\!:\!\alpha_2 \nvdash^{\mathrm{CDS}} x : \alpha_1 \cap \alpha_2$.

Example 2.10. Rule $(\to\mathrm{E})$ is not derivable:
$x\!:\!\alpha_1 \to \alpha_2, x\!:\!\alpha_1 \nvdash^{\mathrm{CDS}} xx : \alpha_2$.

Example 2.11. Rule $(\to\mathrm{I})$ is not derivable.
Consider $\Gamma = \{x\!:\!\alpha_1 \cap \alpha_2, x\!:\!\alpha_1\}$;
$A = \alpha_2$;
$B = (\alpha_1 \to \alpha_2 \to \alpha_3) \to \alpha_3$;
$M = \lambda y.yxx$.
We have that $\Gamma, x\!:\!A \vdash^{\mathrm{CDS}} M : B$, but $\Gamma \nvdash^{\mathrm{CDS}} (\lambda x.M) : (A \to B)$.

Definition 2.12. We say that a multi-basis Γ is closed under $\cap$ if for all $x \in \mathrm{dom}(\Gamma)$ the set $\Gamma(x)$ is closed under $\cap$, i.e. $A, B \in \Gamma(x) \Rightarrow A \cap B \in \Gamma(x)$, up to equality of types in the type theory $\mathcal{T}$ under consideration.

Theorem 2.13. *All the typability rules of the definition of type assignment, except for (→I), are derivable for (possibly infinite) multi-bases that are closed under* $\cap$*.*

Proof. We only prove that rule (→E) is derivable. The other cases are similar or simpler. Let Γ be a (possible infinite) multi-basis. Assume $\Gamma \vdash^{\mathcal{T}} M : A \to B$ and $\Gamma \vdash^{\mathcal{T}} N : A$. There exist finite singled bases $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that $\Gamma_1 \vdash^{\mathcal{T}} M : A \to B$ and $\Gamma_2 \vdash^{\mathcal{T}} N : A$. By Lemma 2.6, applying rule (multiple weakening), we get $\Gamma_1 \uplus \Gamma_2 \vdash^{\mathcal{T}} M : A \to B$ and $\Gamma_1 \uplus \Gamma_2 \vdash^{\mathcal{T}} N : A$. Then, applying rule (→E) on finite bases, we have that $\Gamma_1 \uplus \Gamma_2 \vdash^{\mathcal{T}} MN : B$. Since Γ is closed under $\cap$, $\Gamma_1 \uplus \Gamma_2 \subseteq \Gamma$. Hence, $\Gamma \vdash^{\mathcal{T}} MN : B$. □

Definition 2.14. Let Γ be a multi-basis. We define the replacement of all declarations of x in Γ by a set $X \subseteq \mathbb{T}^{\cap}$ as follows.

$$\Gamma[x := X] = \{y : A \mid A \in \Gamma(y) \& y \neq x\} \cup \{x : B \mid B \in X\}.$$

The condition of being closed under $\cap$ is not strong enough to ensure that rule (→I) is derivable for infinite multi-basis. Take, for instance, $X = \mathbb{T}^{\cap}$, the set of all types. Then the premisse of(→I) is $\Gamma[x := \mathbb{T}^{\cap}] \vdash^{\mathcal{T}} M : B$ does not give any information about the type A that has been used to type the free variables of M.

Theorem 2.15. *Let* Γ *be a (possible) infinite multi-basis. The following variant of (→I) using principal filters is derivable*

$$(\to I")\frac{\Gamma[x :=\uparrow A] \vdash^{\mathcal{T}} M : B}{\Gamma \vdash^{\mathcal{T}} \lambda x.M : A \to B}.$$

Proof. Let Γ be a (possible) infinite multi-basis. Assume $\Gamma[x :=\uparrow A] \vdash^{\mathcal{T}} M : B$. There exists $\Gamma_1, x : B \subseteq \Gamma[x :=\uparrow A]$ such that $\Gamma_1, x : B \vdash^{\mathcal{T}} M : A$. Then $A \leq_{\mathcal{T}} A$. By applying rule ($\leq\vdash^{\mathcal{T}}$L) we obtain $\Gamma_1, x : A \vdash^{\mathcal{T}} M : A$. □

The generation lemma for (infinite) multi-bases is similar to the one for finite bases.

Lemma 2.16. (Generation Lemma for multi-bases).
Let Γ *be a multi-basis and* $\Gamma \vdash^{\mathcal{T}} M : A$*.*

1. *If* $M = x$ *and* $A \neq \mathsf{U}$*, then there exists* $B \in \Gamma(x)$ *such that* $B \leq A$*.*
2. *If* $M = \lambda x.N$*, then there exist* B_i, C_i *such that* $\bigcap_{i=1}^{n}(B_i \to C_i) \leq A$ *and* $\Gamma[x :=\uparrow C_i] \vdash^{\mathcal{T}} N : C_i$*.*
3. *If* $M = NP$ *and* $A \neq \mathsf{U}$*, then there exist* B_i *and* C_i *such that* $\bigcap_{i=1}^{n} C_i \leq A$*,* $\Gamma \vdash^{\mathcal{T}} N : B_i \to C_i$ *and* $\Gamma \vdash P : B_i$*.*

Proof. Using Generation Lemma for finite bases. □

3. Application of multi-bases to filter models

In this section we exploit the fact that multi-bases on a certain intersection type assignment system correspond to environments of the filter model induced by that system.

Definition 3.1. Let $\mathcal{T}$ be an intersection type theory and $X \subseteq \mathbb{T}^{\cap}$. Then X is a *filter* over $\mathcal{T}$ if the following hold.

1. X is non-empty;
2. if $A \in X$ and $A \leq B$ then $B \in X$;
3. $A, B \in X$ implies $A \cap B \in X$.

We define $\mathcal{F}^{\mathcal{T}} = \{X \subseteq \mathbb{T}^{\cap} \mid X \text{ is a filter over } \mathcal{T}\}$.

Lemma 3.2. *If* $\Gamma(x)$ *is a filter then* $\Gamma(x) = \{A \mid \Gamma \vdash x : A\}$.

Definition 3.3. Let $D \subseteq \mathcal{P}(\mathbb{T}^{\cap})$, $\rho \in \mathcal{V} \to D$ and $f_\rho \in \Lambda \to D$.

(1) $\Gamma_\rho = \{x : A \mid A \in \rho(x)\}$.
(2) We say that f satisfies the type semantics property in $\vdash^{\mathcal{T}}$ if

$$f_\rho(M) = \{A \mid \Gamma_\rho \vdash^{\mathcal{T}} M : A\} \text{ for all } M \in \Lambda, \rho \in \mathcal{V} \to D$$

Next we define the notion of filter interpretation [4, 3, 5]. A filter interpretation may not be a λ-model [1] since it may not satisfy β-reduction.

Definition 3.4. Let $\mathcal{T}$ be an intersection type theory and $\rho \in \mathcal{V} \to \mathcal{F}^{\mathcal{T}}$. The filter interpretation $[\![\ ]\!]_\rho$ is a function from Λ to $\mathcal{F}^{\mathcal{T}}$ defined as follows.

$$\begin{array}{ll} [\![x]\!]_\rho & = \rho(x) \\ [\![\lambda x.N]\!]_\rho & = \uparrow \{B \to C \mid C \in [\![N]\!]_{\rho(x:=\uparrow B)}\} \\ [\![NP]\!]_\rho & = \uparrow \{B \mid \exists C \in [\![P]\!]_\rho.(B \to C) \in [\![N]\!]_\rho\} \end{array}$$

We now prove that a function that satisfies the type semantics property coincides with the filter interpretation. This means that the only way of interpreting the abstraction and the application on $\mathcal{F}^{\mathcal{T}}$ is the one given in Definition 3.4.

Theorem 3.5. *The function* f *satisfies the type semantics property in* $\vdash^{\mathcal{T}}$ *if and only if* $D \subseteq \mathcal{F}^{\mathcal{T}}$ *and* $f_\rho = [\![\ ]\!]_\rho$ *for all* $\rho \in \mathcal{V} \to D$.

Proof.
($\Rightarrow$) If f satisfies the type semantics property in $\vdash^{\mathcal{T}}$ then it is easy to prove that $f_\rho(M)$ is a filter using the definition of $\vdash^{\mathcal{T}}$.
Let $\rho \in \mathcal{V} \to D$. We prove that $f_\rho(M) = [\![M]\!]_\rho$ for all $M \in \Lambda$ by induction on M.

(1) Case $M = x$.

$$\begin{array}{ll} f_\rho(x) & = \{A \mid \Gamma_\rho \vdash^{\mathcal{T}} x : A\} \text{ by the type semantics property} \\ & = \Gamma_\rho(x) \text{ by Lemma 2.16} \\ & = \rho(x) \\ & = [\![x]\!]_\rho \end{array}$$

(2) Case $M = \lambda x.N$.

$$\begin{array}{ll} f_\rho(\lambda x.N) & = \{A \mid \Gamma_\rho \vdash^{\mathcal{T}} \lambda x.N : A\} \text{ by the type semantics property} \\ & = \{A \mid \bigcap_{i=1}^n (B_i \to C_i) \leq A \& \Gamma_\rho[x :=\uparrow B_i] \vdash^{\mathcal{T}} N : C_i\} \text{ by Lemma 2.16} \\ & =\uparrow \{B \to C \mid \Gamma_\rho[x :=\uparrow B] \vdash^{\mathcal{T}} N : C\} \\ & =\uparrow \{B \to C \mid C \in f_{\rho(x:=\uparrow B)}(N)\} \text{ by the type semantics property} \\ & =\uparrow \{B \to C \mid C \in [\![N]\!]_{\rho(x:=\uparrow B)}\} \text{ by induction hypothesis} \\ & = [\![\lambda x.N]\!]_\rho \end{array}$$

(3) The case of the application is similar to the previous one.

($\Leftarrow$) We prove that $[\![M]\!]_\rho = \{A \mid \Gamma_\rho \vdash^{\mathcal{T}} M : A\}$ by induction on M.

(1) The case $M = x$ follows from Lemma 3.2.

(2) If $M = \lambda x.N$, then

$$\begin{aligned} [\![\lambda x.N]\!]_\rho &= \uparrow \{B \to C \mid C \in [\![N]\!]_{\rho(x:=\uparrow B)}\} \\ &= \uparrow \{B \to C \mid \Gamma_\rho[x := \uparrow B] \vdash^{\mathcal{T}} N : C\} \text{ by induction hypothesis} \\ &= \{A \mid \textstyle\bigcap_{i=1}^{n}(B_i \to C_i) \leq A \& \Gamma_\rho[x := \uparrow B_i] \vdash^{\mathcal{T}} N : C_i\} \\ &= \{A \mid \Gamma_\rho \vdash^{\mathcal{T}} \lambda x.N : A\} \text{ by Lemma 2.16} \end{aligned}$$

(3) The case $M = NP$ is similar to the previous one.

□

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