Universality Issues in Reversible Computing Systems and Cellular Automata

Kenichi Morita
Hiroshima University

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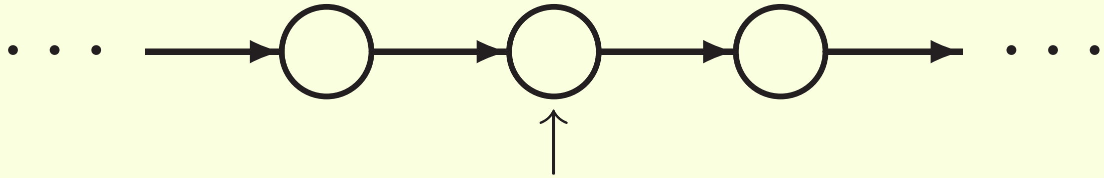
Even very simple reversible systems have universal computing ability!

1. Introduction

Reversible Computing

- Roughly speaking, it is a "backward deterministic" computing; i.e., every computational configuration has at most one predecessor.



- Though its definition is rather simple, it reflects physical reversibility well.

Reversible Computing

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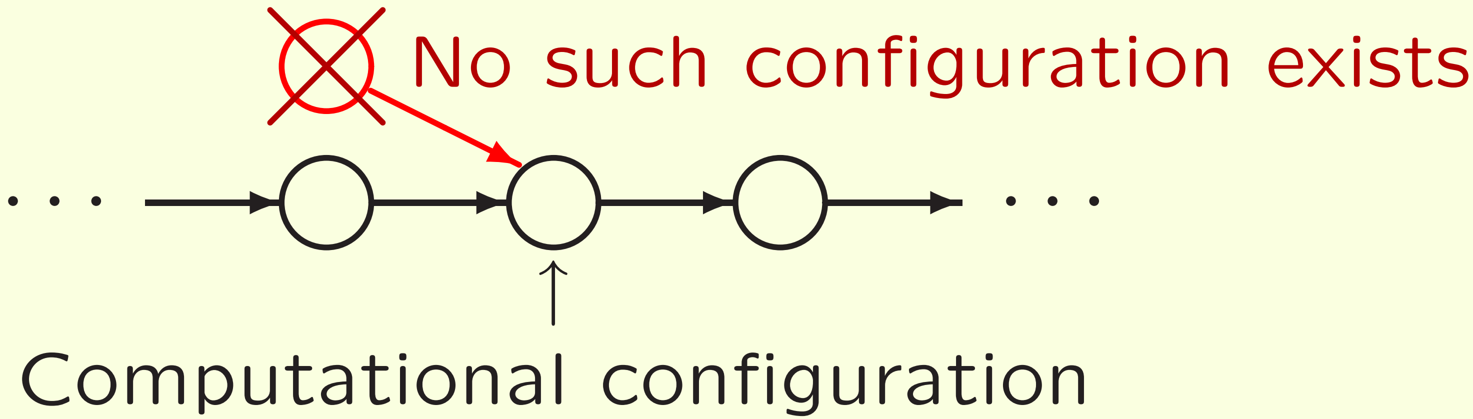

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Several Models of Reversible Computing

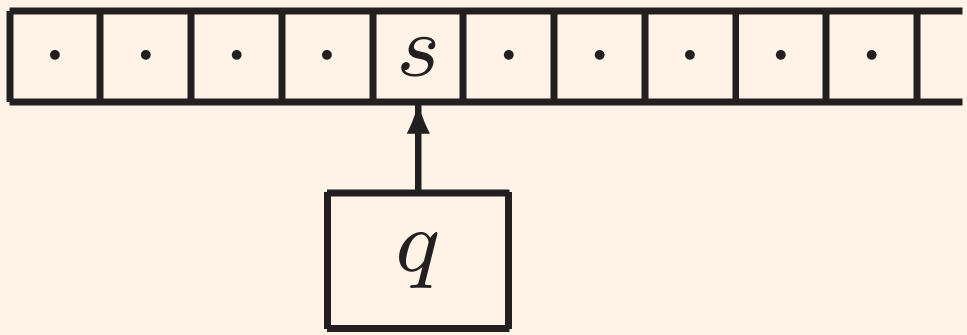
- Reversible Turing machines (RTMs)
- Reversible logic elements and circuits
- Reversible cellular automata (RCAs)
- Reversible counter machines (RCMs)
- Others

- These models are closely related each other.
- Reversible computers work in a very different fashion from classical computers!

2. Reversible Turing Machines

Reversible Turing Machines (RTMs)

A "backward deterministic" TM.



Definition of a TM

$$T = (Q, S, q_0, q_f, s_0, \delta)$$

Q: a finite set of states.
S: a finite set of tape symbols.
q_0: an initial state $q_0 \in Q$.
q_f: a final state $q_f \in Q$.
s_0: a blank symbol $s_0 \in S$.
δ: a move relation given by a set of quintuples $[p, s, s', d, q] \in Q \times S \times S \times \{-, 0, +\} \times Q$.

Definition of an RTM

A TM $T = (Q, S, q_0, q_f, s_0, \delta)$ is called *reversible* iff the following condition holds for any pair of distinct quintuples $[p_1, s_1, s'_1, d_1, q_1]$ and $[p_2, s_2, s'_2, d_2, q_2]$.

If $q_1 = q_2$, then $s'_1 \neq s'_2 \wedge d_1 = d_2$

(If the next states are the same, then the written symbols must be different and the shift directions must be the same.)

Universality of RTMs

Theorem [Bennett, 1973]
For any one-tape (irreversible) TM T, there is a garbage-less 3-tape reversible TM which simulates the former.

A Small Universal RTM (URTM)

A URTM is an RTM that can compute *any* recursive function.

Theorem The following URTMs exist:
17-state 5-symbol URTM [Morita and Yamaguchi, 2007]
15-state 6-symbol URTM [Morita, 2008]

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These URTMs can simulate any cyclic tag system [Cook, 2004], which is proved to be universal.

Cyclic Tag System (CTAG) [Cook, 2004]

$$C = (k, \{Y, N\}, (\mathsf{halt}, p_1, \cdots, p_{k-1}))$$

- k: the length of a cycle (positive integer).
- $\{Y, N\}$: the alphabet used in a CTAG.
- $(p_1, \cdots, p_{k-1}) \in (\{Y, N\}^*)^{k-1}$: production rules.

An *instantaneous description* (ID) is a pair (v, i), where $v \in \{Y, N\}^*$ and $i \in \{0, \cdots, k-1\}$.

For any $(v, i), (w, j) \in \{Y, N\}^* \times \{0, \cdots, k-1\}$,

$$\begin{aligned}
(Yv, i) \Rightarrow (w, j) \text{ iff } & [m \neq 0] \wedge [j = i + 1 \bmod k] \\
& \wedge [w = vp_i], \\
(Nv, i) \Rightarrow (w, j) \text{ iff } & [j = i + 1 \bmod k] \wedge [w = v].
\end{aligned}$$

A Simple Example of a CTAG System

$$C_1 = (3, \{Y, N\}, (\text{halt}, \; YN, \; YY))$$

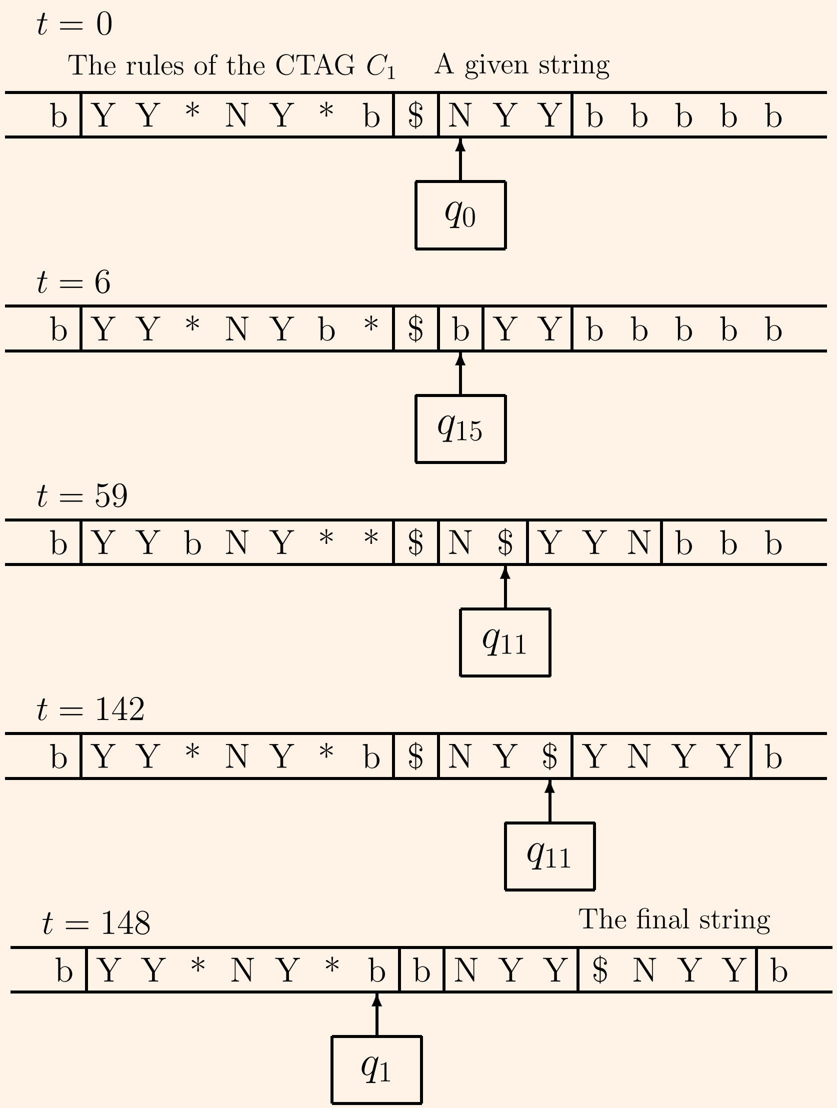
If an initial word NYY is given, the computing on C_1 proceeds as follows:

$$
\begin{aligned}
& (\; N\,Y\,Y \;, \quad 0\;) \\
\Rightarrow & (\; Y\,Y \;, \quad 1\;) \\
\Rightarrow & (\; Y\,Y\,N \;, \quad 2\;) \\
\Rightarrow & (\; Y\,N\,Y\,Y \;, \quad 0\;)
\end{aligned}
$$

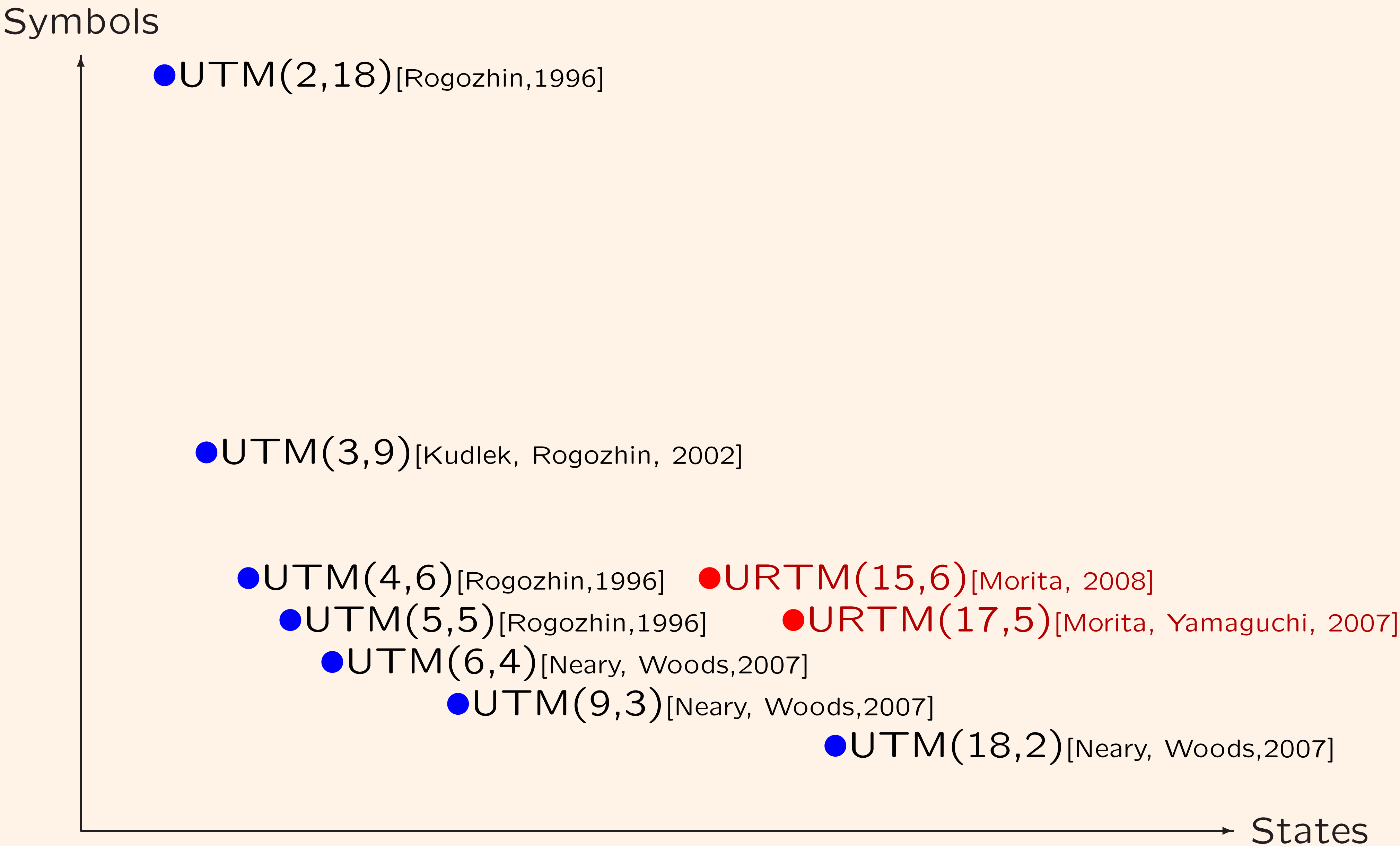
The quintuple set of the URTM(17,5)

	b	Y	N	$*$	$
q_0	$\$ - q_2$	$\$ - q_1$	$b - q_{13}$		
q_1	halt	$Y - q_1$	$N - q_1$	$* + q_0$	$b - q_1$
q_2	$* - q_3$	$Y - q_2$	$N - q_2$	$* - q_2$	null
q_3	$b + q_{12}$	$b + q_4$	$b + q_7$	$b + q_{10}$	
q_4	$Y + q_5$	$Y + q_4$	$N + q_4$	$* + q_4$	$\$ + q_4$
q_5	$b - q_6$				
q_6	$Y - q_3$	$Y - q_6$	$N - q_6$	$* - q_6$	$\$ - q_6$
q_7	$N + q_8$	$Y + q_7$	$N + q_7$	$* + q_7$	$\$ + q_7$
q_8	$b - q_9$				
q_9	$N - q_3$	$Y - q_9$	$N - q_9$	$* - q_9$	$\$ - q_9$
q_{10}		$Y + q_{10}$	$N + q_{10}$	$* + q_{10}$	$\$ + q_{11}$
q_{11}		$Y + q_{11}$	$N + q_{11}$	$* + q_{11}$	$Y + q_0$
q_{12}		$Y + q_{12}$	$N + q_{12}$	$* + q_{12}$	$\$ - q_3$
q_{13}	$* - q_{14}$	$Y - q_{13}$	$N - q_{13}$	$* - q_{13}$	$\$ - q_{13}$
q_{14}	$b + q_{16}$	$Y - q_{14}$	$N - q_{14}$	$b + q_{15}$	
q_{15}	$N + q_0$	$Y + q_{15}$	$N + q_{15}$	$* + q_{15}$	$\$ + q_{15}$
q_{16}		$Y + q_{16}$	$N + q_{16}$	$* + q_{16}$	$\$ - q_{14}$

Simulating the CTAG C_1 by the URTM(17,5)



Small UTMs and URTMs



3. Reversible Logic Elements

Reversible Logic Element

A logic element whose function is described by a one-to-one mapping.

(1) Reversible logic elements without memory
(i.e., reversible logic gates):

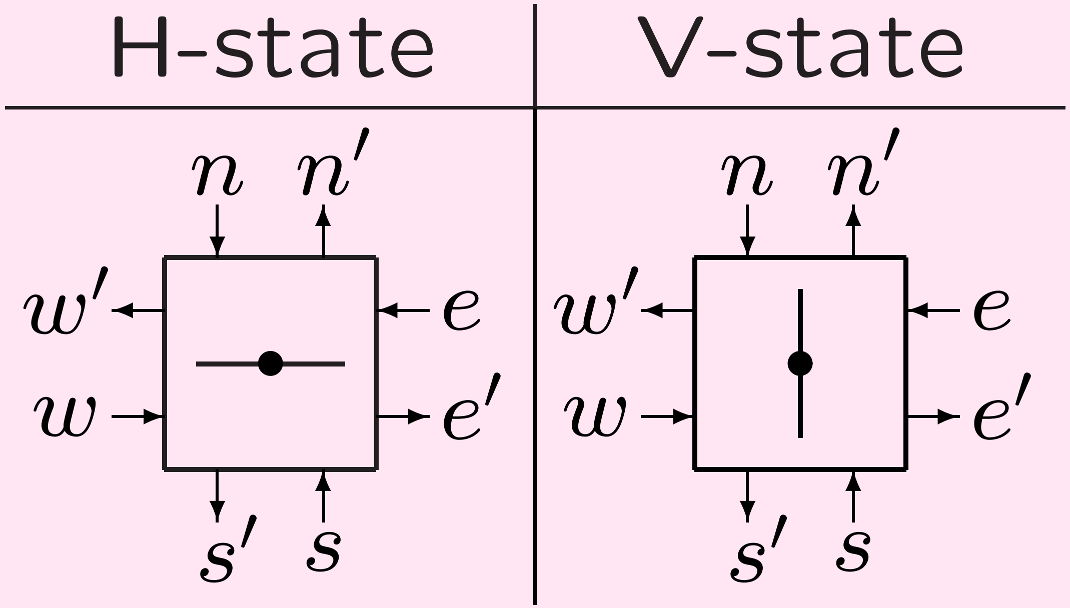
- Toffoli gate [Toffoli, 1980]
- Fredkin gate [Fredkin and Toffoli, 1982]
- etc.

(2) Reversible logic elements with memory:

- Rotary element (RE) [Morita, 2001]
- etc.

Rotary element (RE)

A 2-state 4-input-line 4-output-line element.



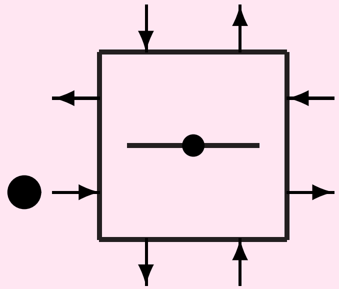
(Remark)

We assume signal "1" is given at most one input line.

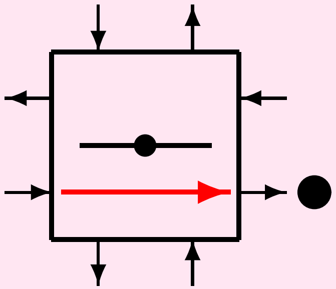
Operations of an RE

- Parallel case:

$t = 0$

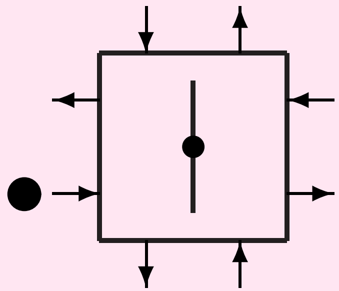


$t = 1$

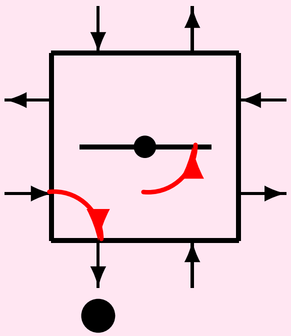


- Orthogonal case:

$t = 0$

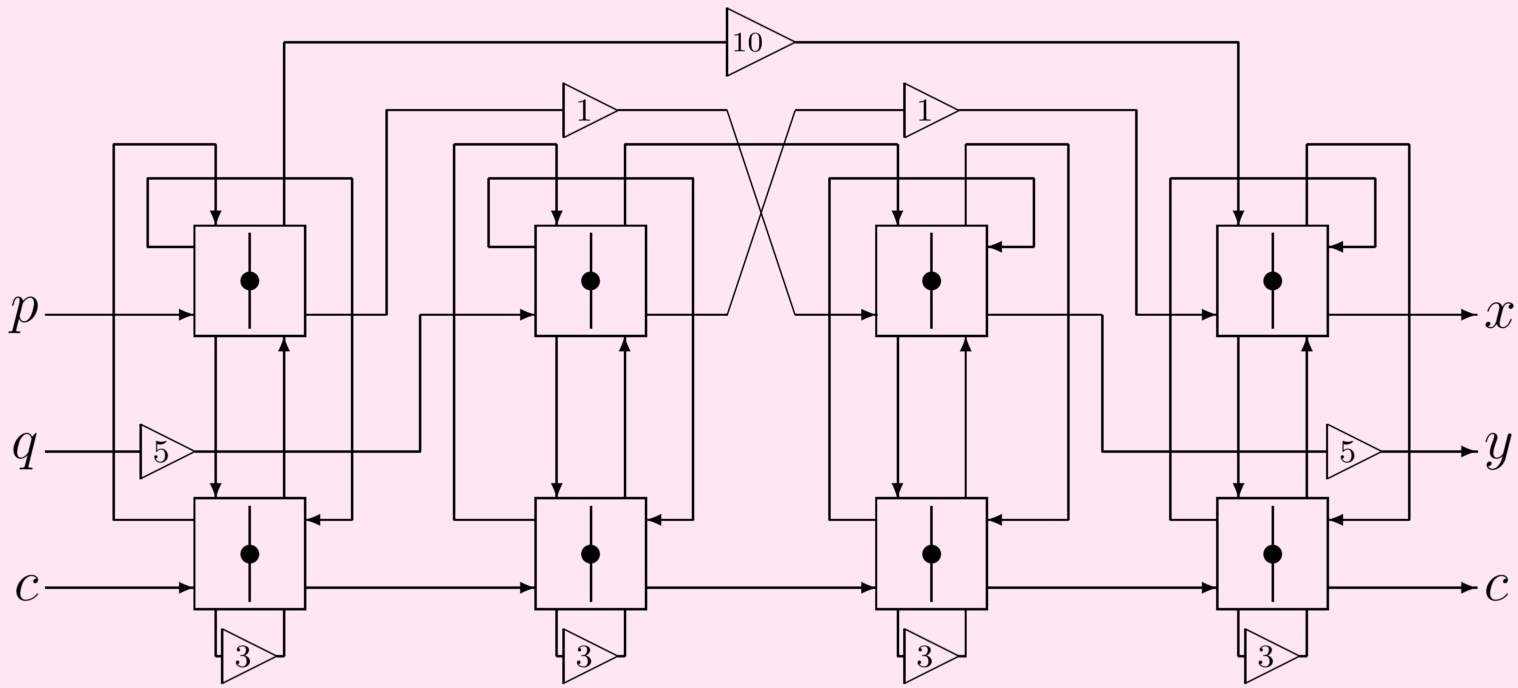


$t = 1$



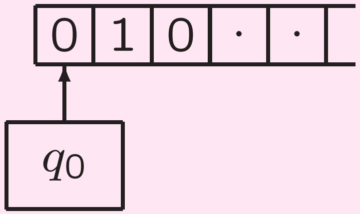
Logical Universality of a Rotary Element

A Fredkin gate can be composed of REs and delay elements.

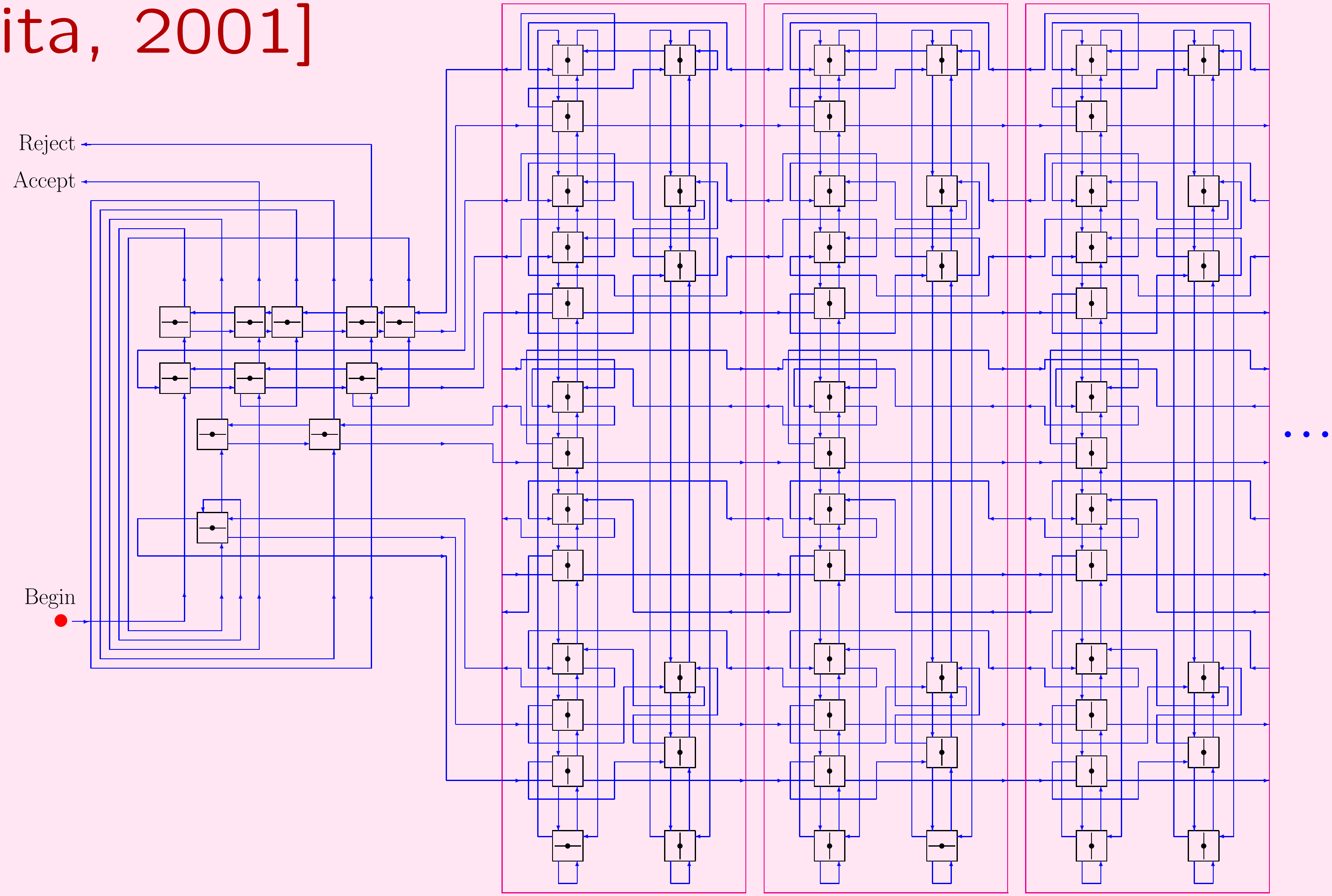


(Remark) But, this is not a good method to use REs.

Any Reversible Turing Machine Can Be Composed Only of REs



[Morita, 2001]



A Simple Example of an RTM T_{parity}

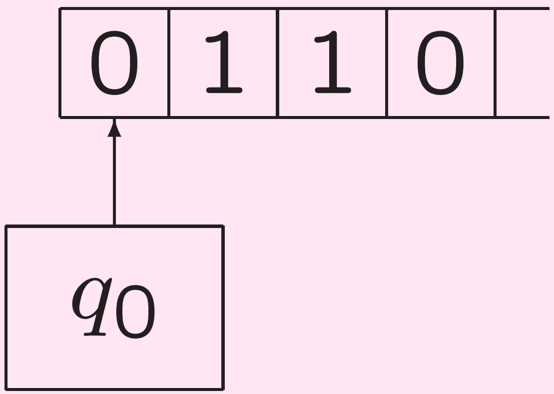
$$T_{\mathsf{parity}} = (Q, \{0, 1\}, q_0, q_{\mathsf{acc}}, 0, \delta)$$

$$Q = \{q_0, q_1, q_2, q_{\mathsf{acc}}, q_{\mathsf{rej}}\}$$

$$\begin{aligned}\delta = \{ & [\, q_0, 0, 1, R, q_1 \,], \\ & [\, q_1, 0, 1, N, q_{\mathsf{acc}} \,], \\ & [\, q_1, 1, 0, R, q_2 \,], \\ & [\, q_2, 0, 1, N, q_{\mathsf{rej}} \,], \\ & [\, q_2, 1, 0, R, q_1 \,] \ \}.\end{aligned}$$

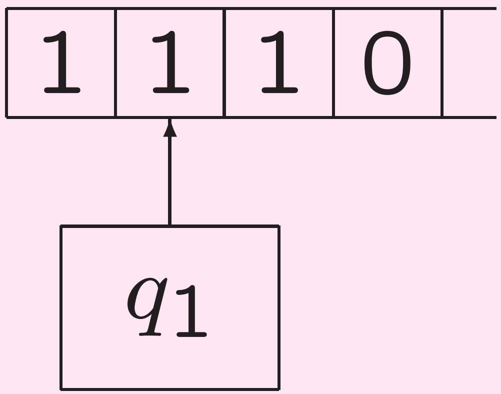
A Simple Example of an RTM T_{parity}

$$t = 0$$



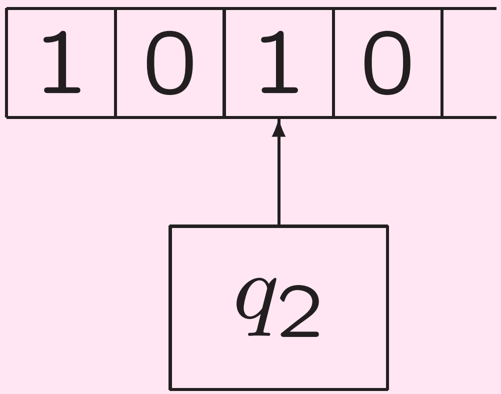
A Simple Example of an RTM T_{parity}

$t = 1$



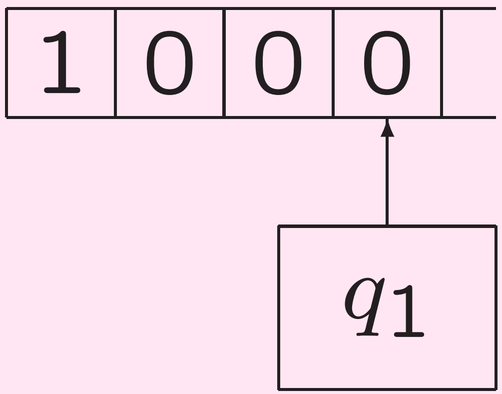
A Simple Example of an RTM T_{parity}

$$t = 2$$



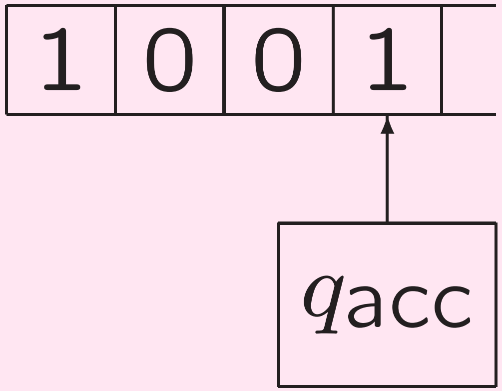
A Simple Example of an RTM T_{parity}

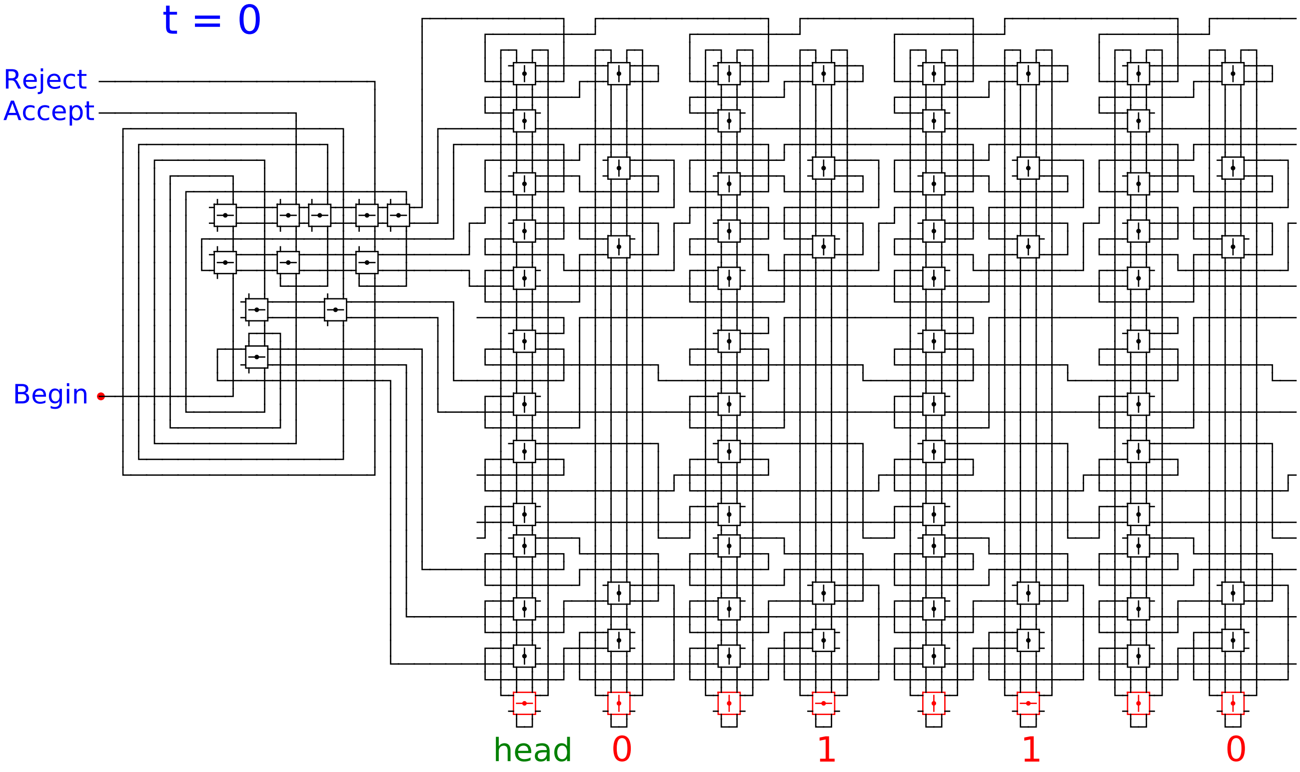
$t = 3$

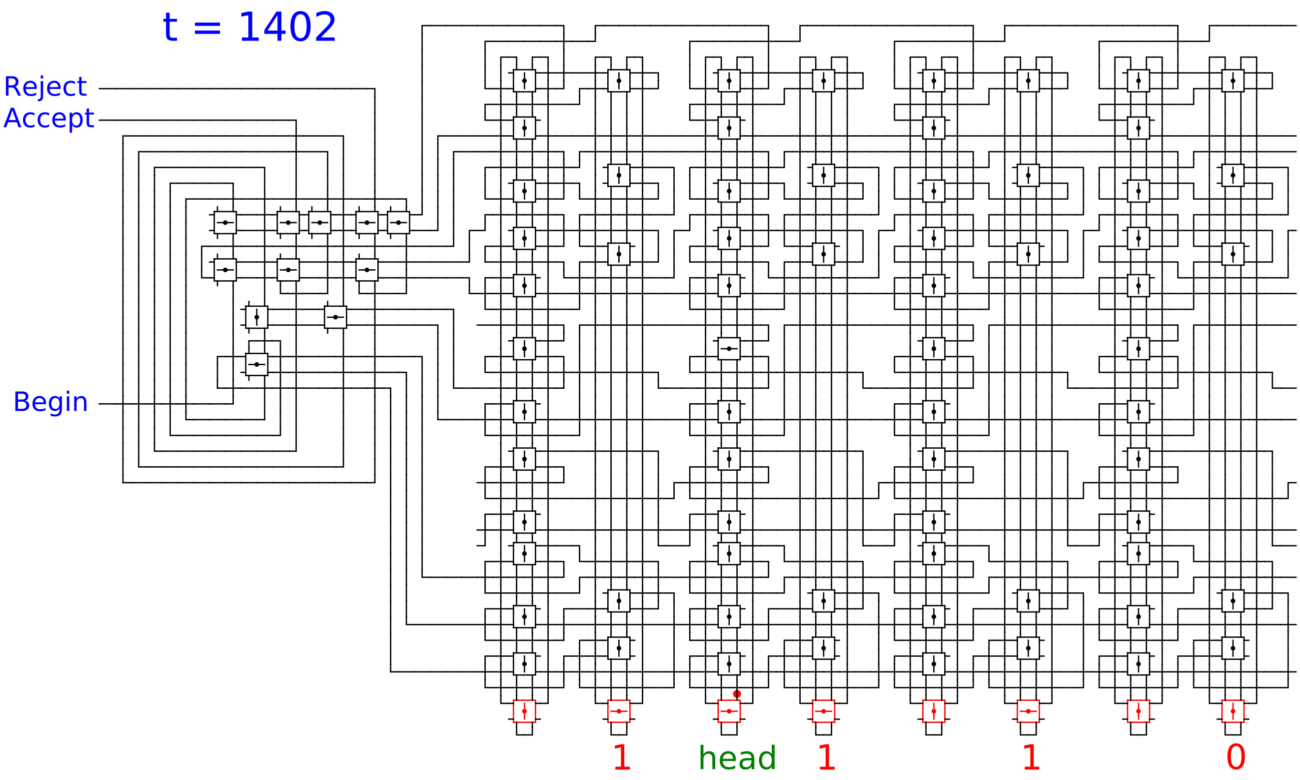


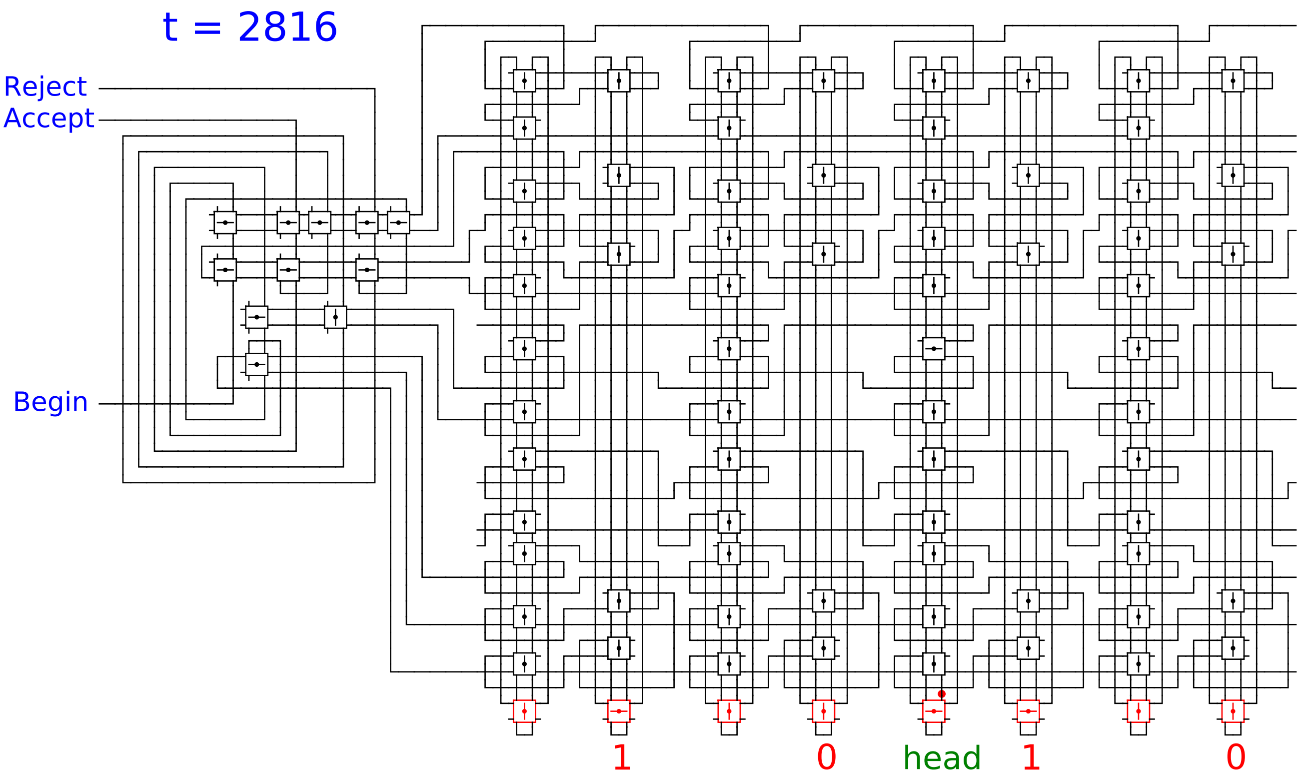
A Simple Example of an RTM T_{parity}

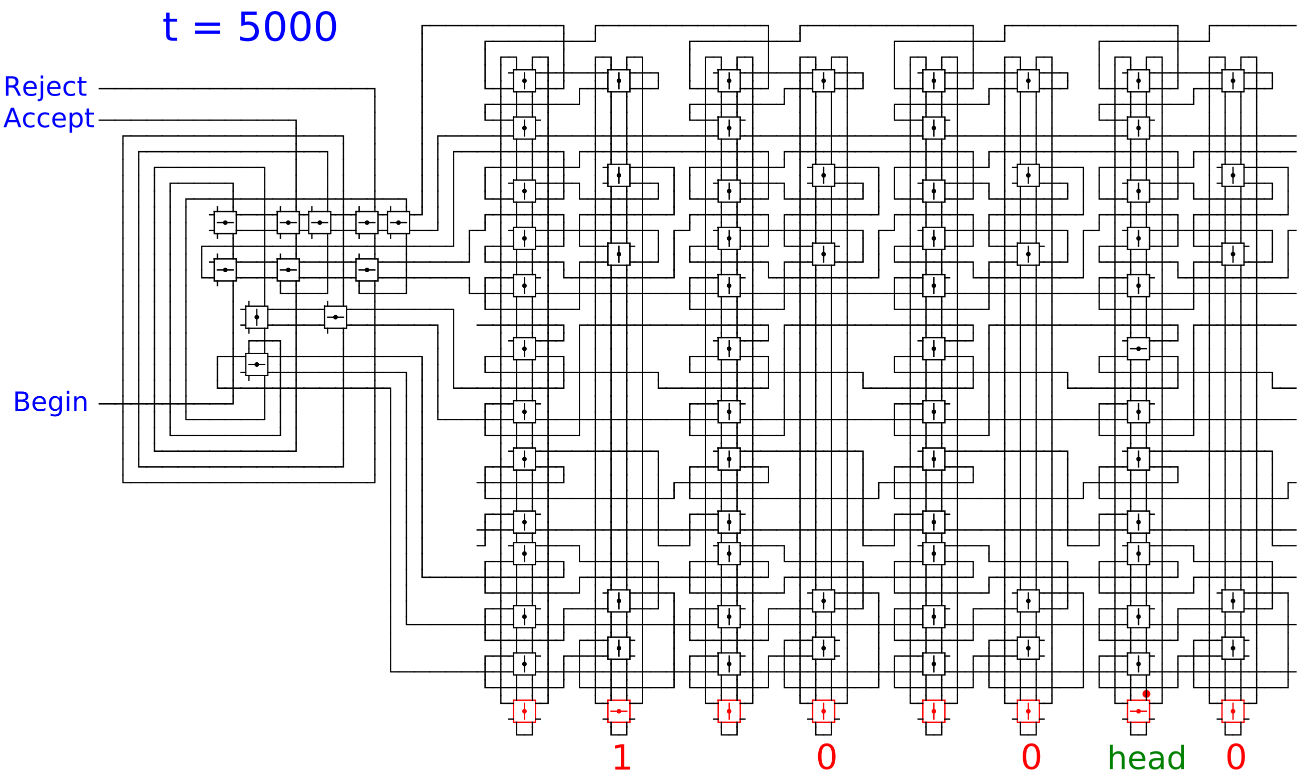
$$t = 3$$



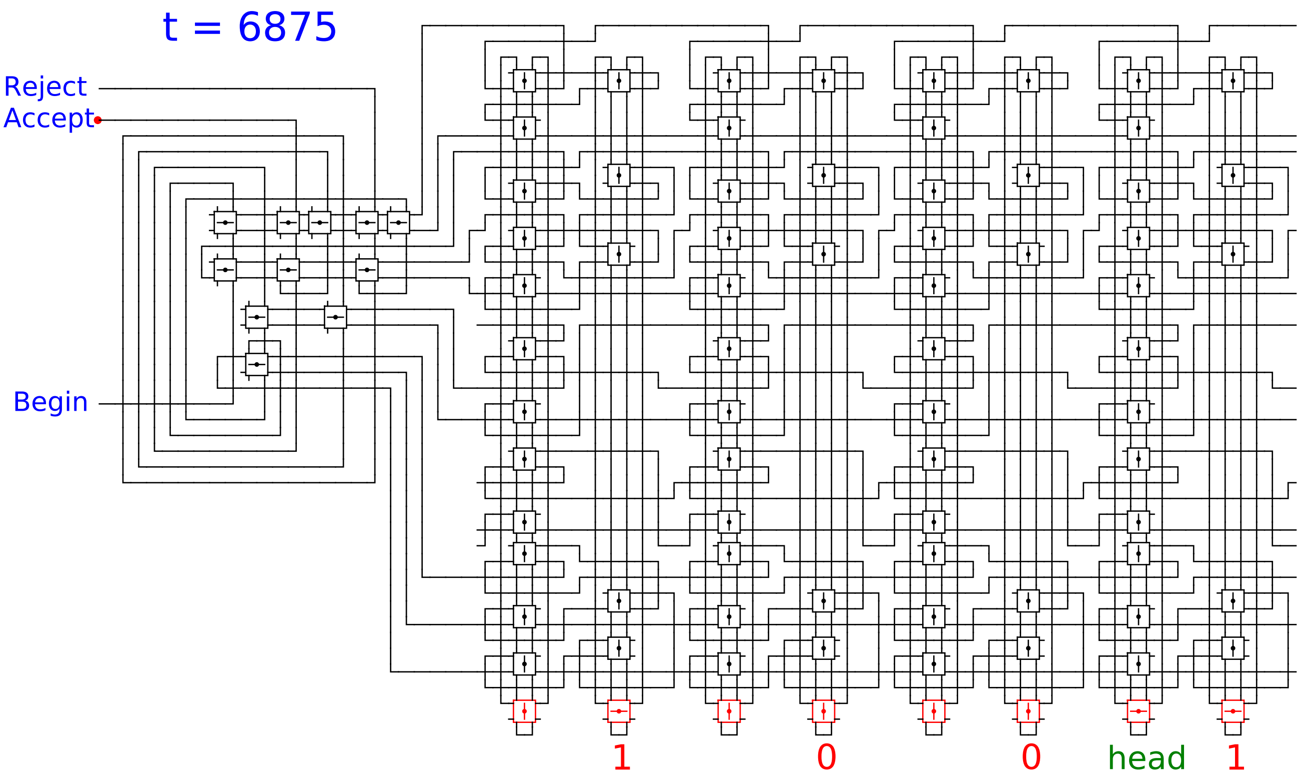

t = 0
Reject
Accept
Begin
head
0
1
1
0


t = 1402
Reject
Accept
Begin
1
head
1
1
0


t = 2816
Reject
Accept
Begin
1
0
head
1
0

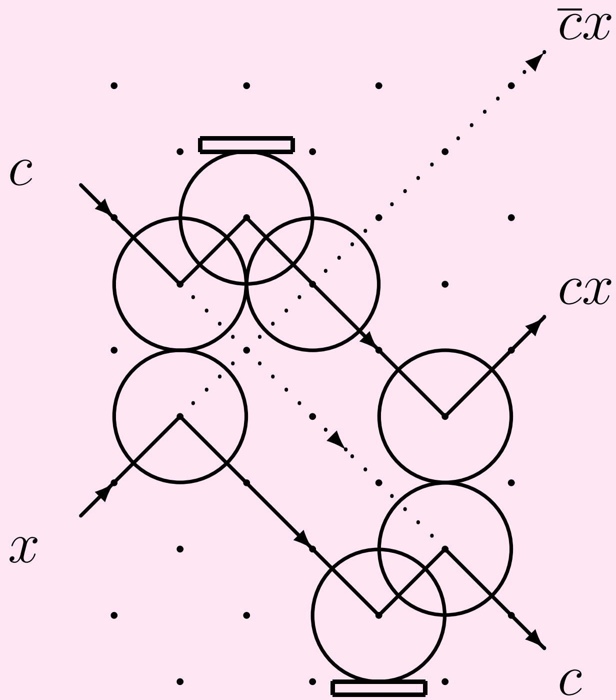


t = 5000
Reject
Accept
Begin
1
0
0
head
0

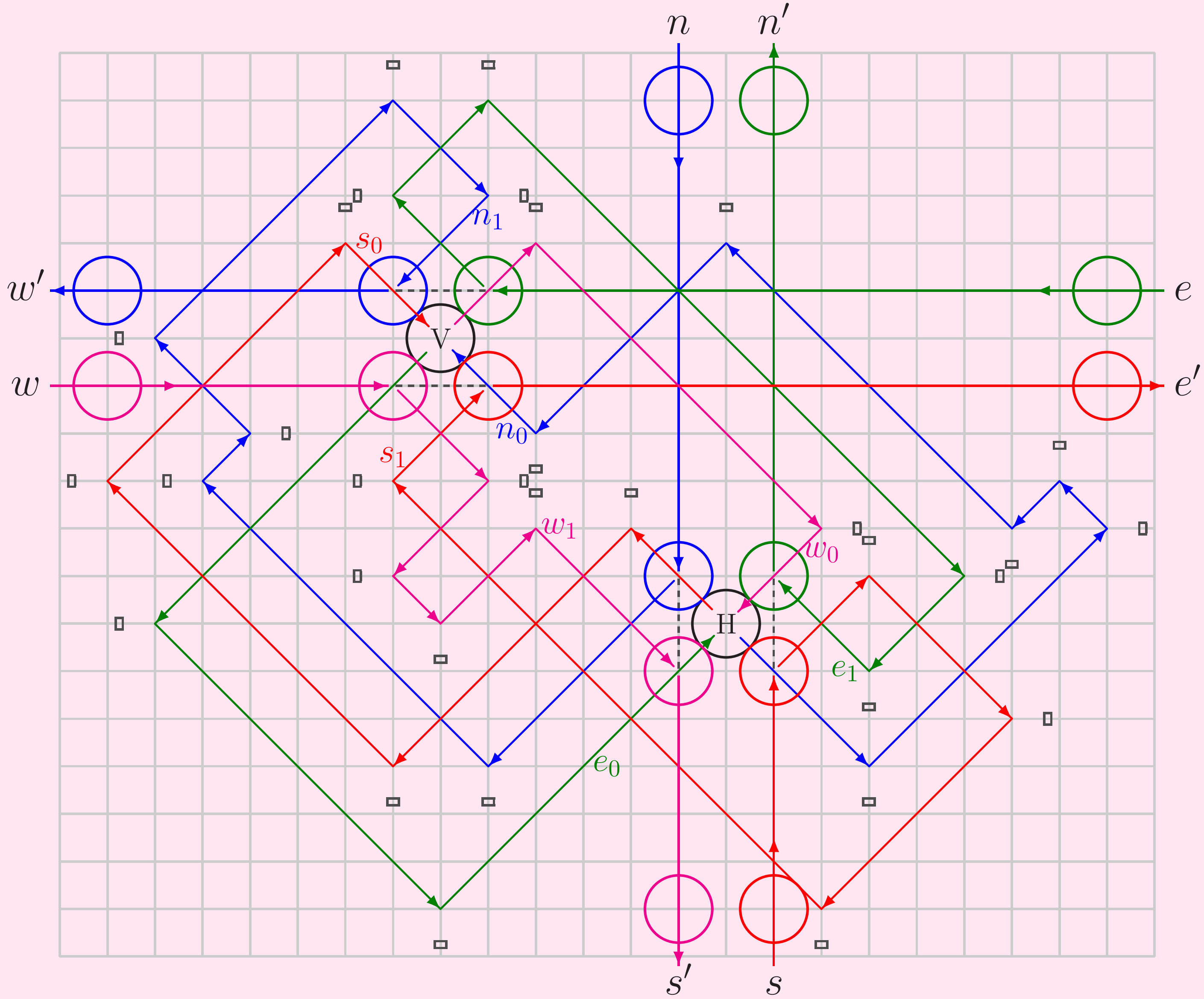

t = 6875
Reject
Accept
Begin
1
0
0
head
1

Billiard Ball Model (BBM)
– A reversible physical model of computing –

[Fredkin and Toffoli, 1982]

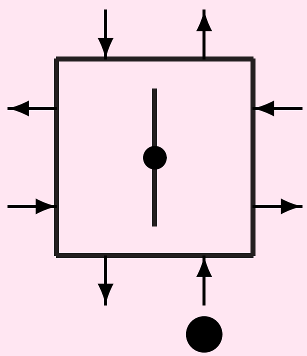


Realization of an RE by BBM [Morita, 2008]

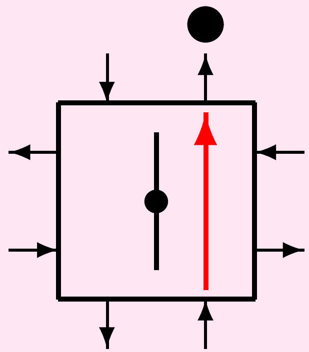


Parallel Case

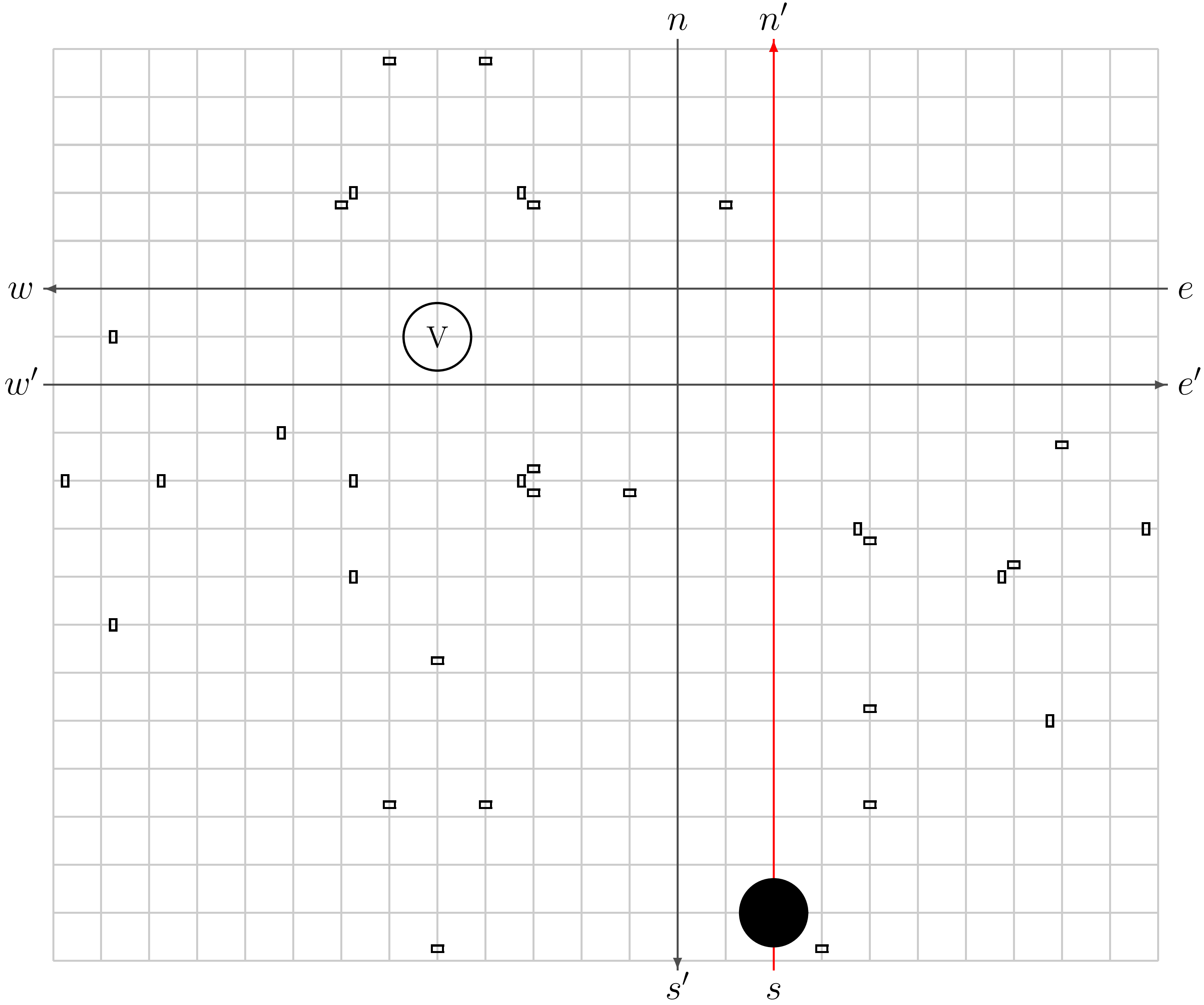
$t = 0$



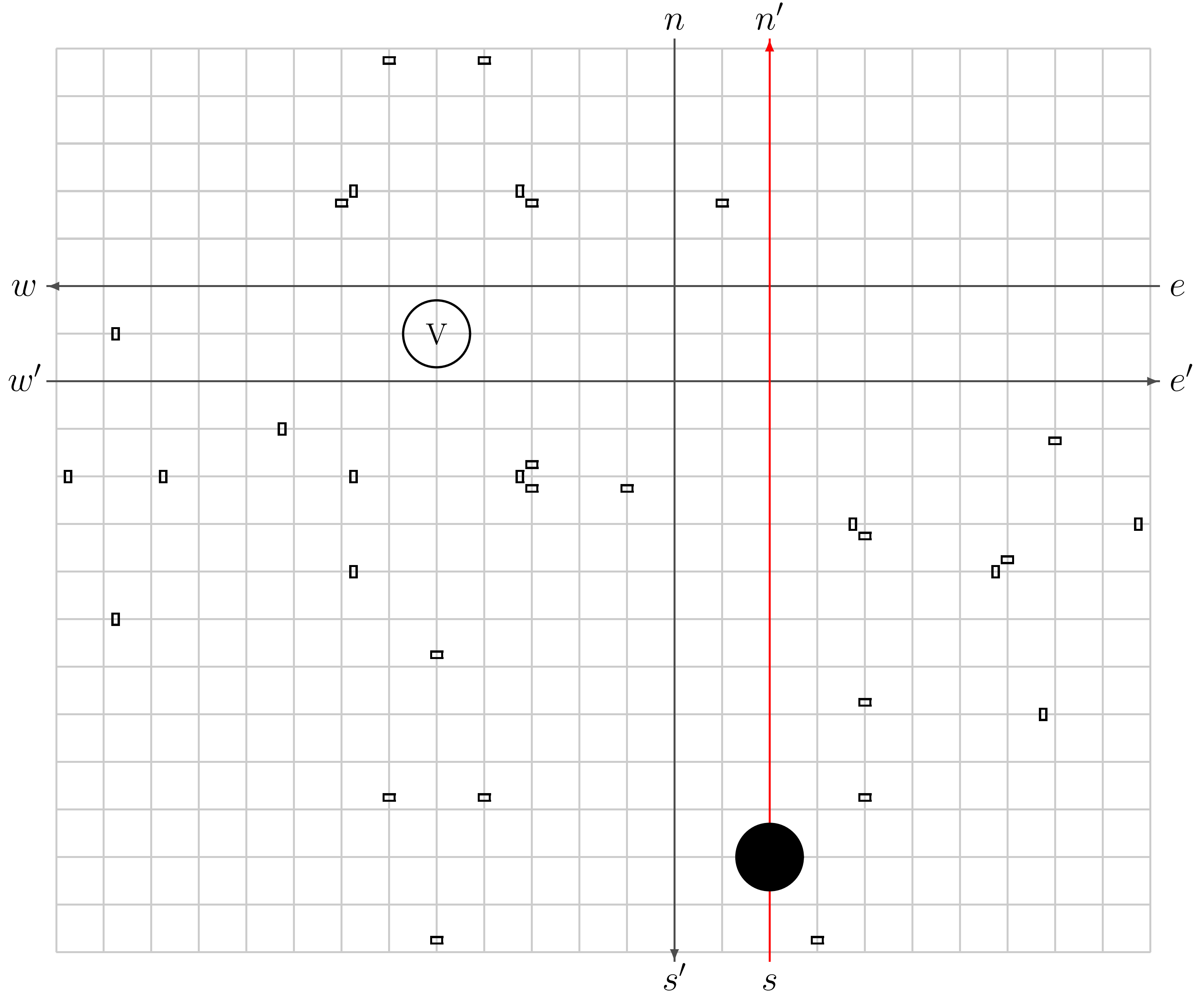
$t = 1$



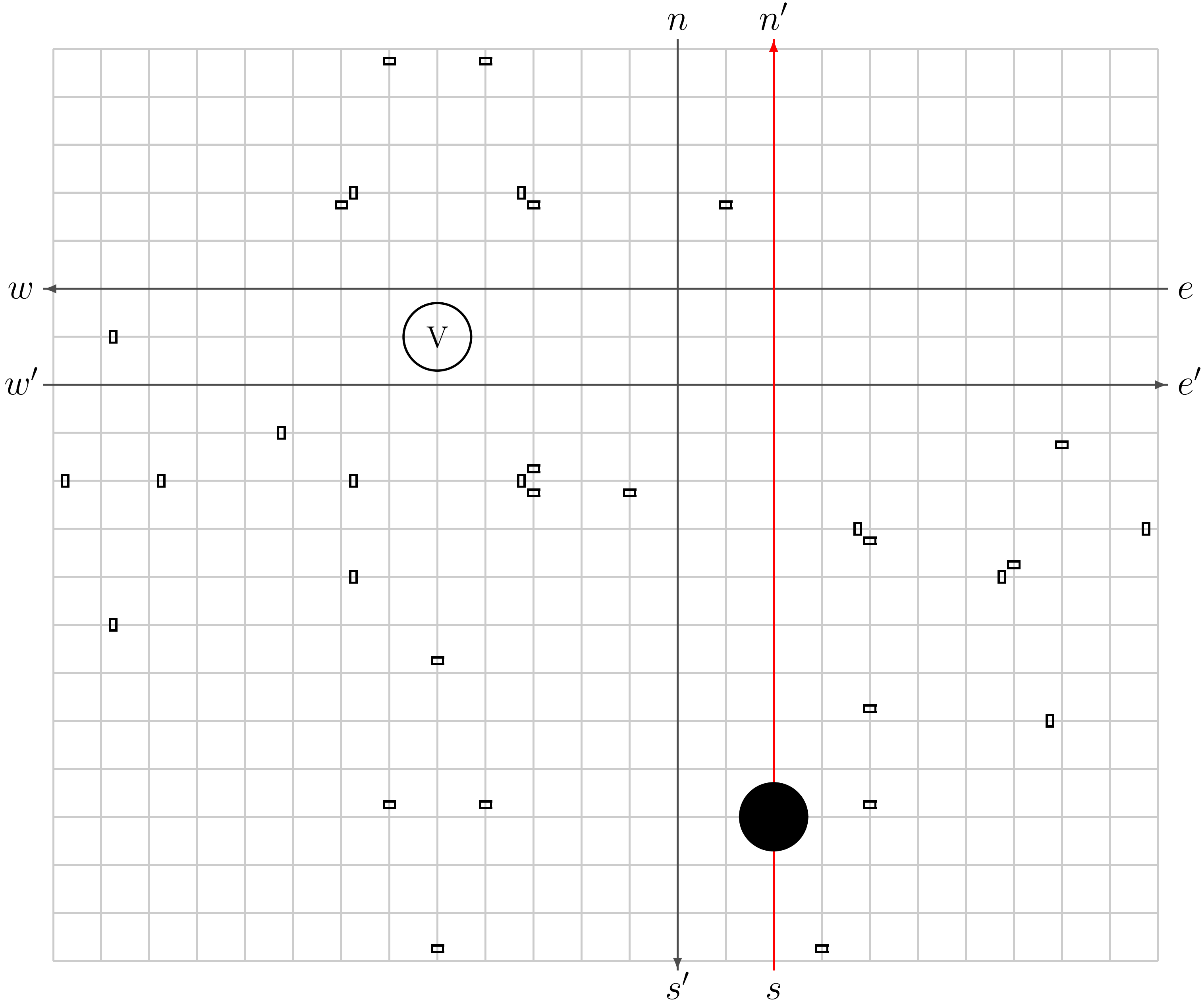
Movements of Balls (State: V, Input: s)



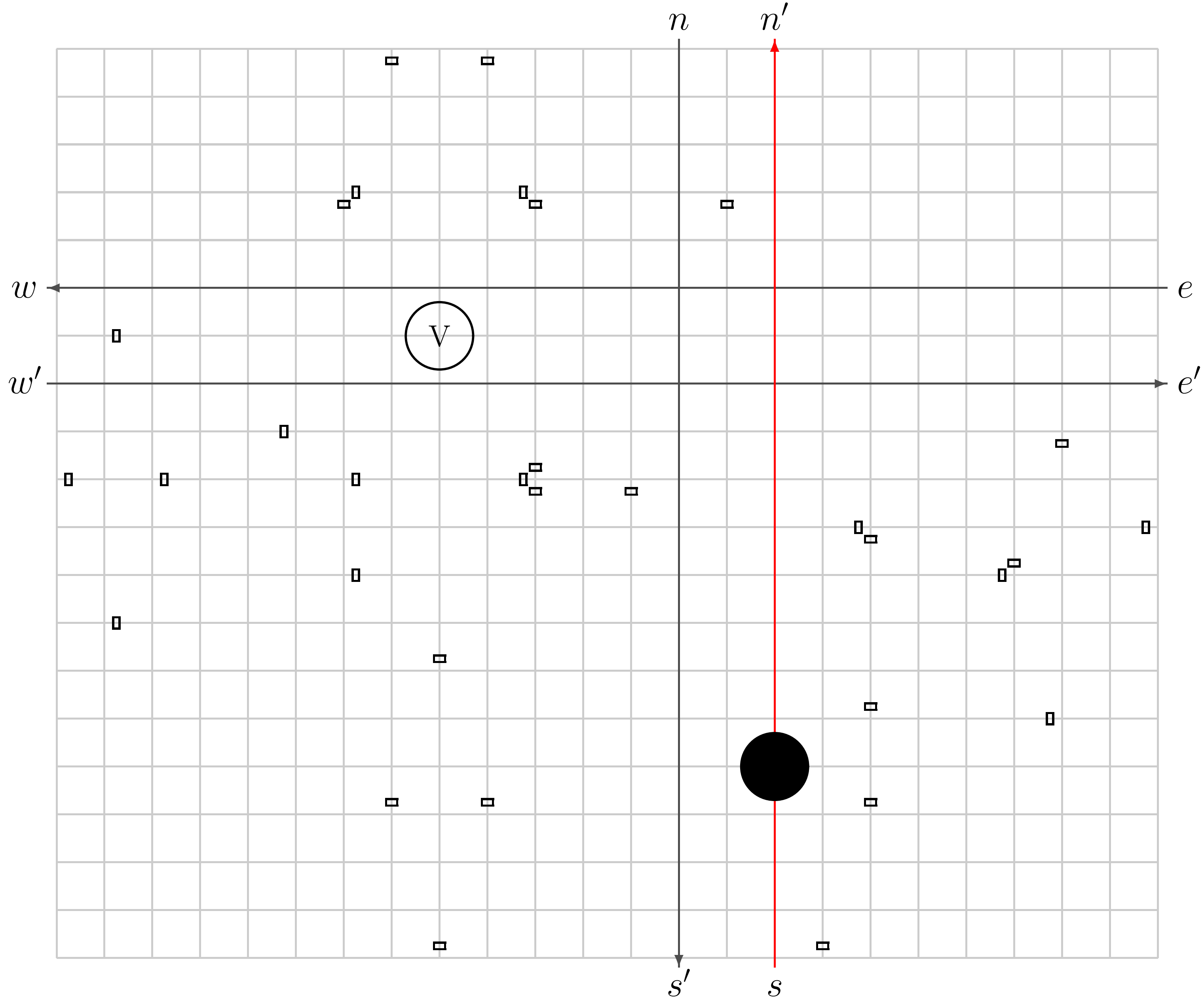
Movements of Balls (State: V, Input: s)



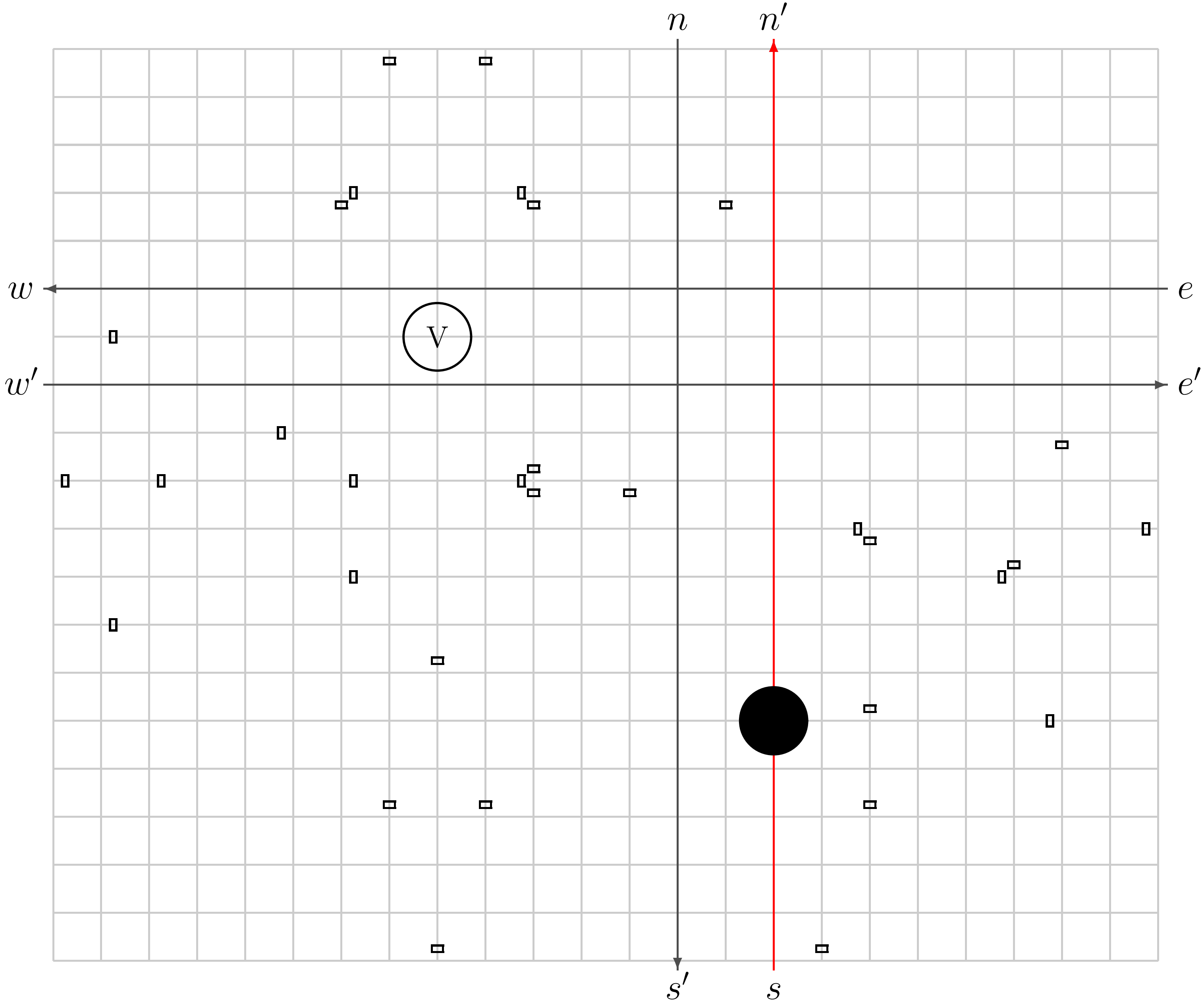
Movements of Balls (State: V, Input: s)



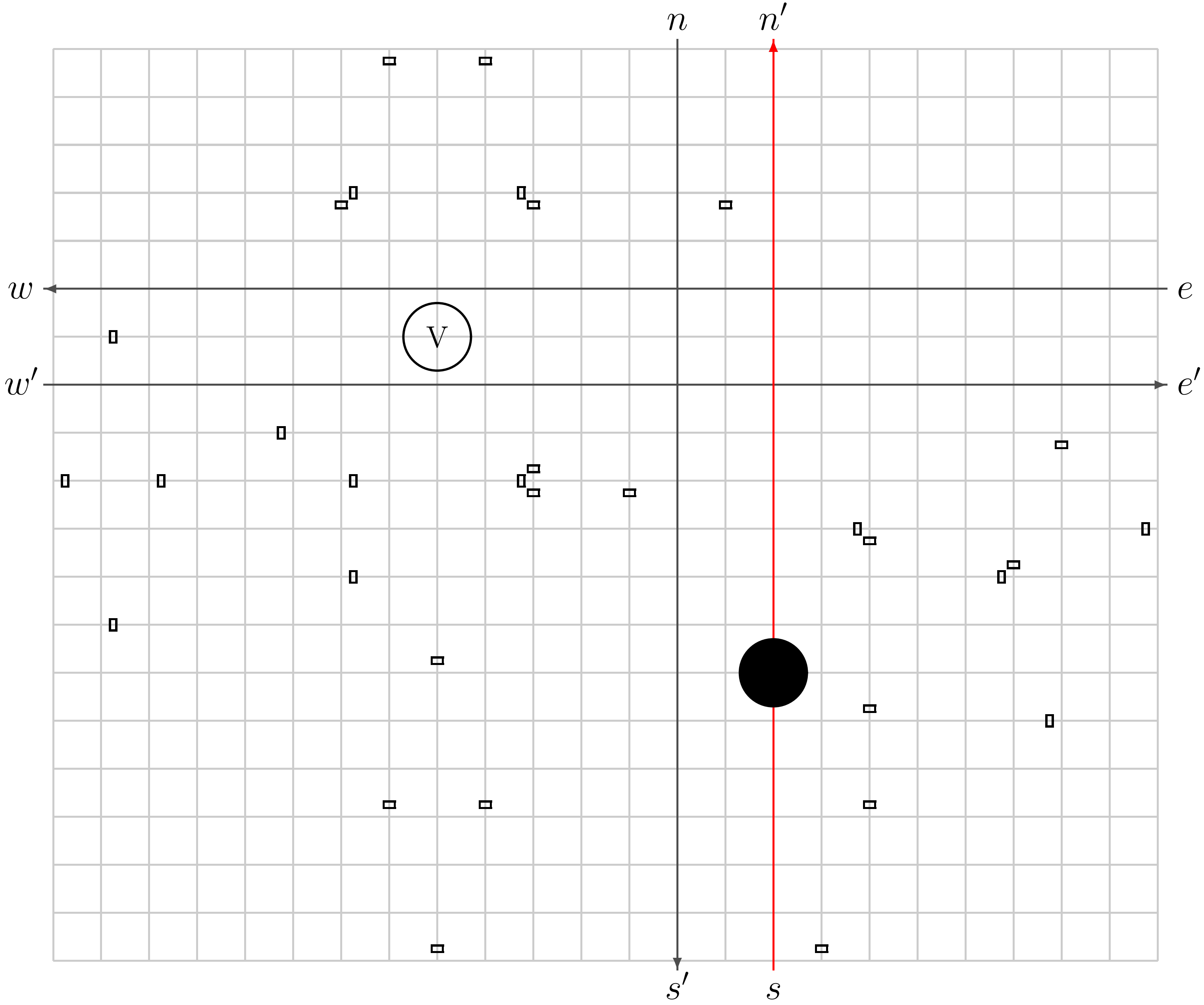
Movements of Balls (State: V, Input: s)



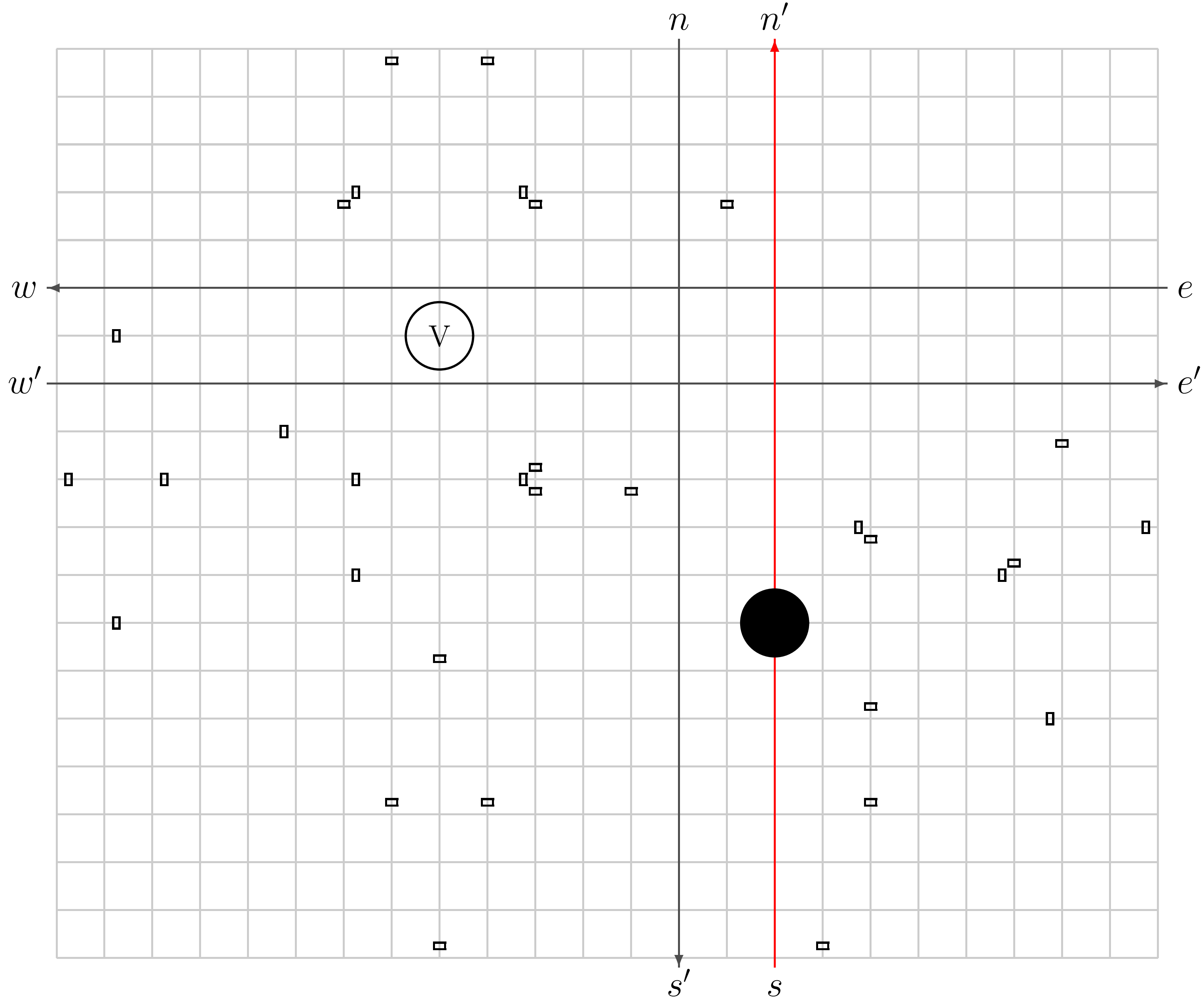
Movements of Balls (State: V, Input: s)



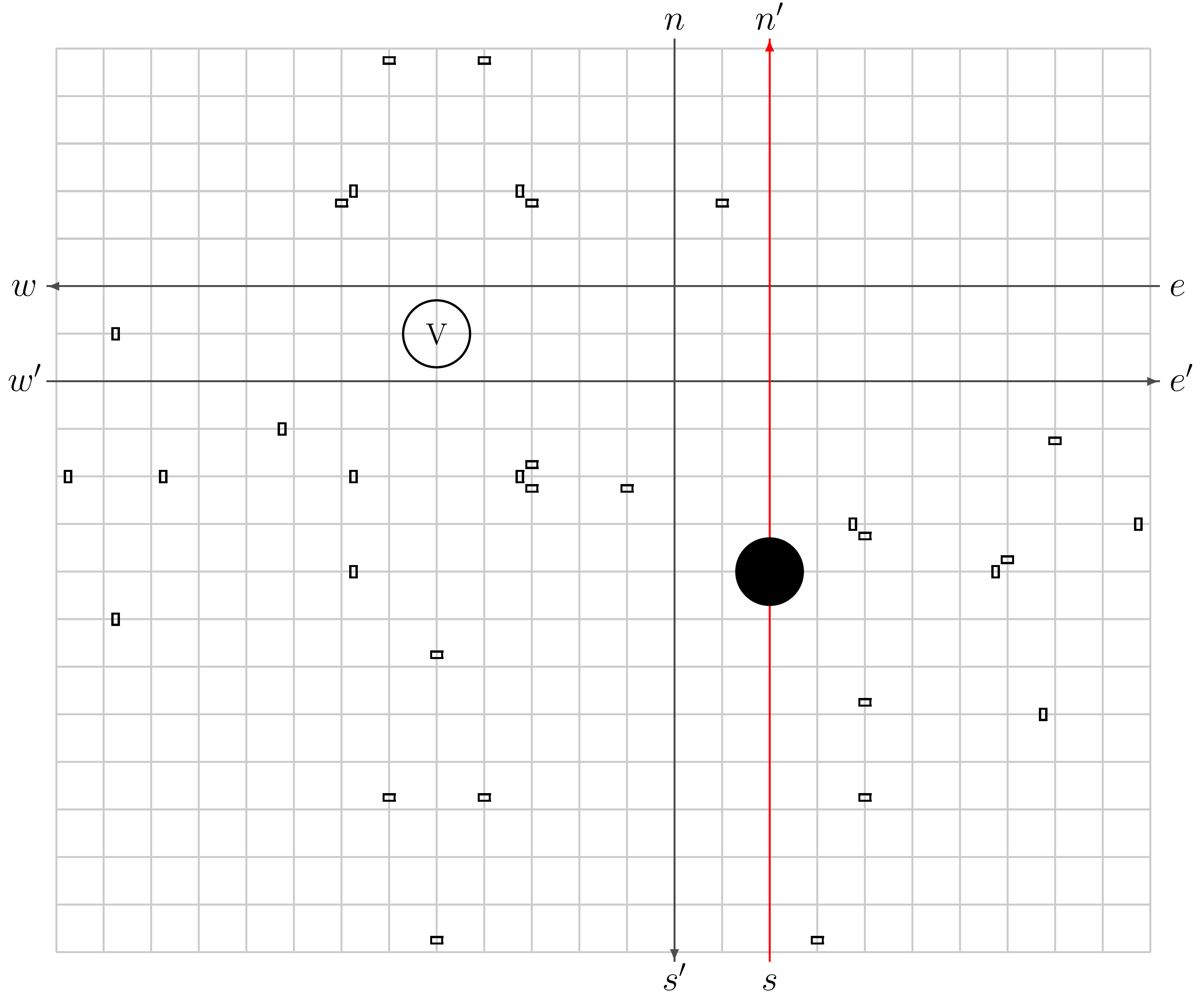
Movements of Balls (State: V, Input: s)



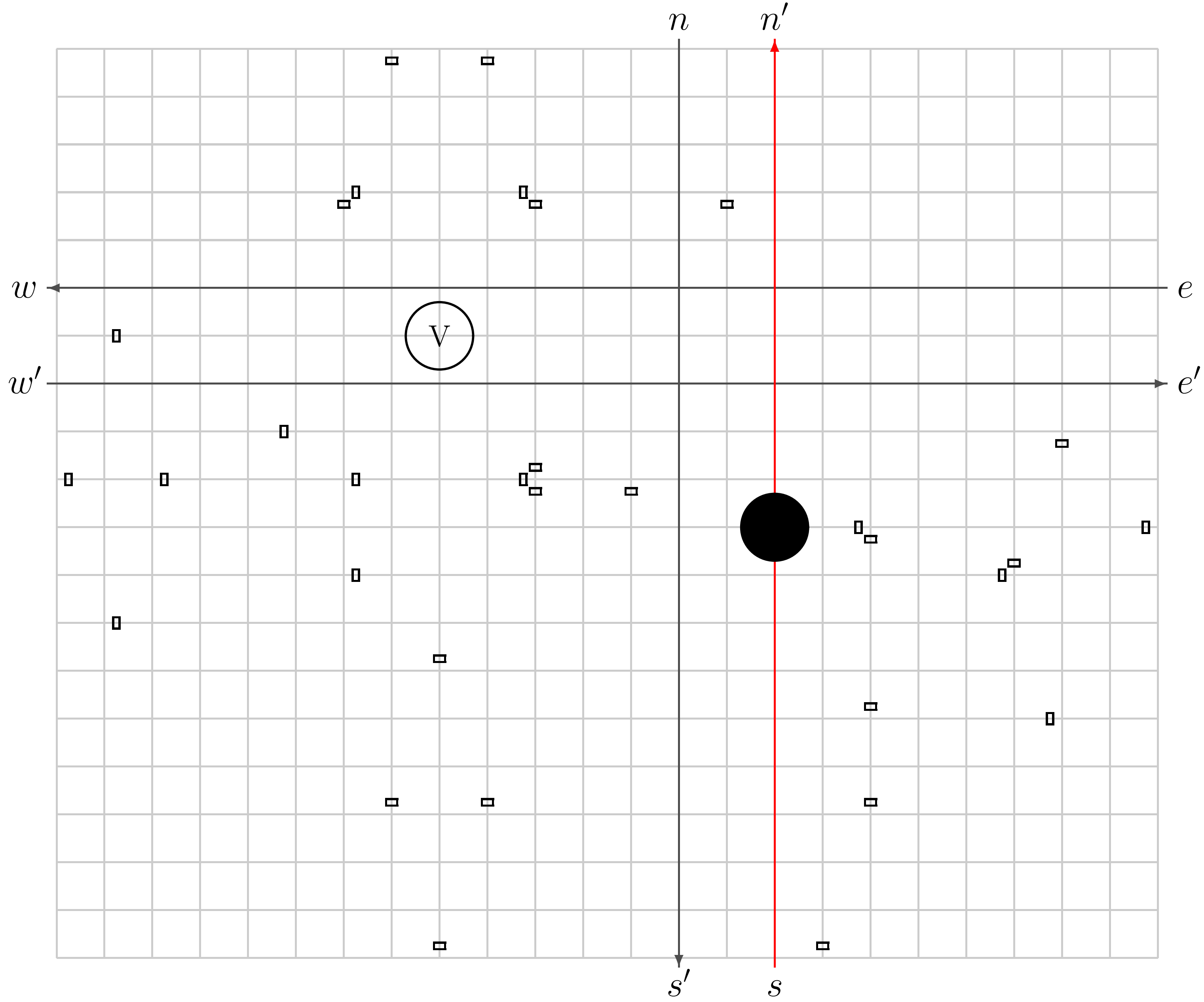
Movements of Balls (State: V, Input: s)



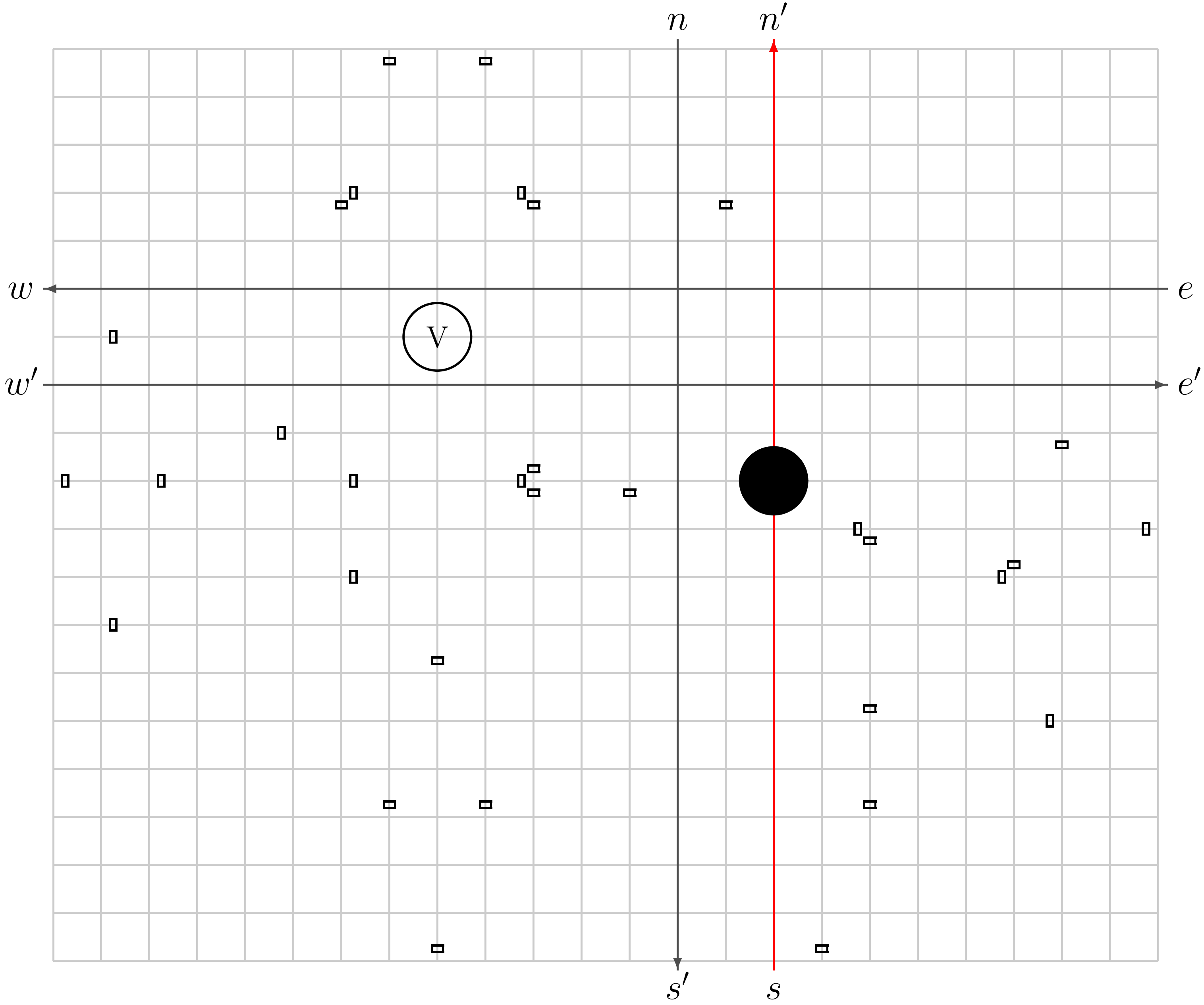
Movements of Balls (State: V, Input: s)



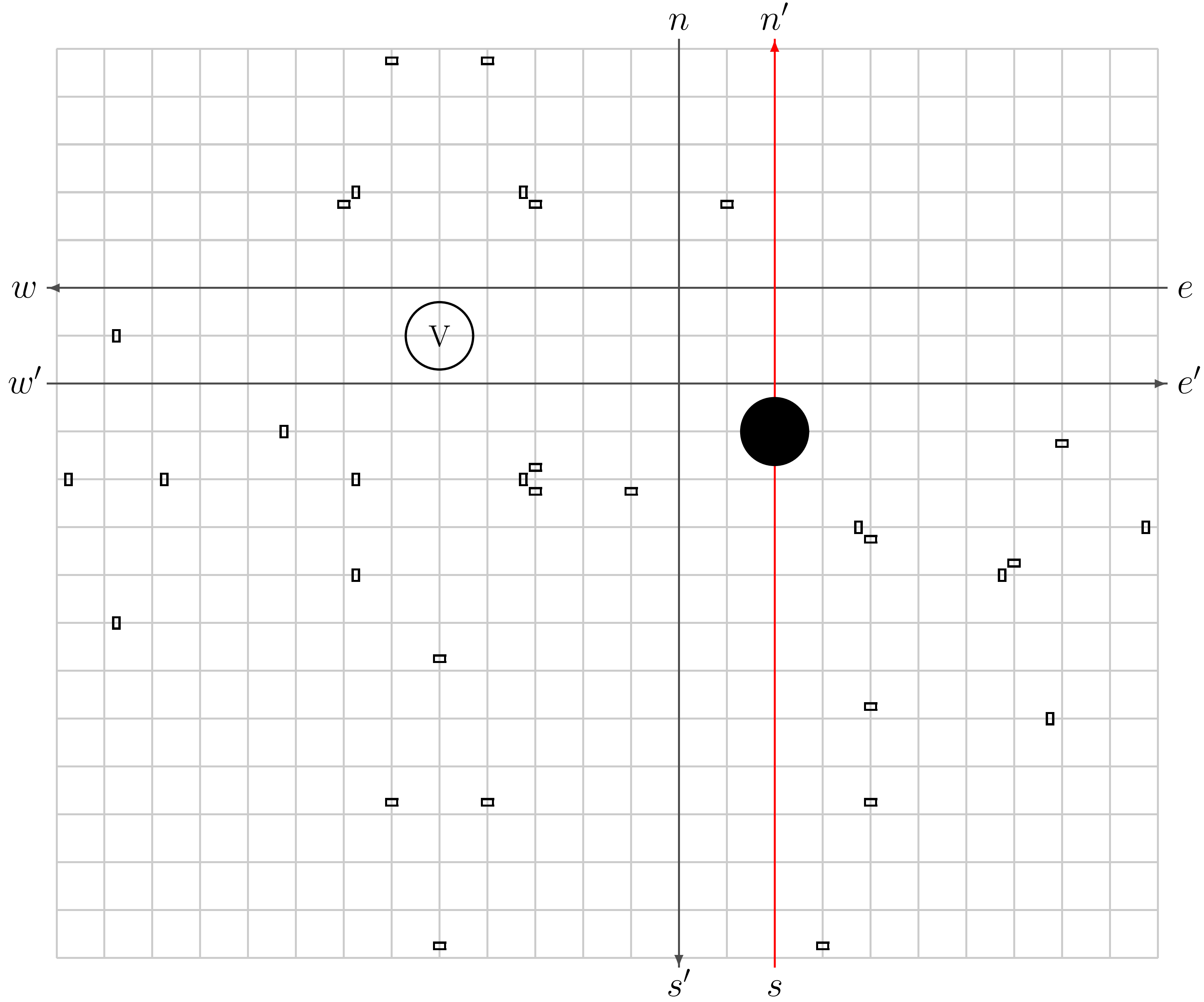
Movements of Balls (State: V, Input: s)



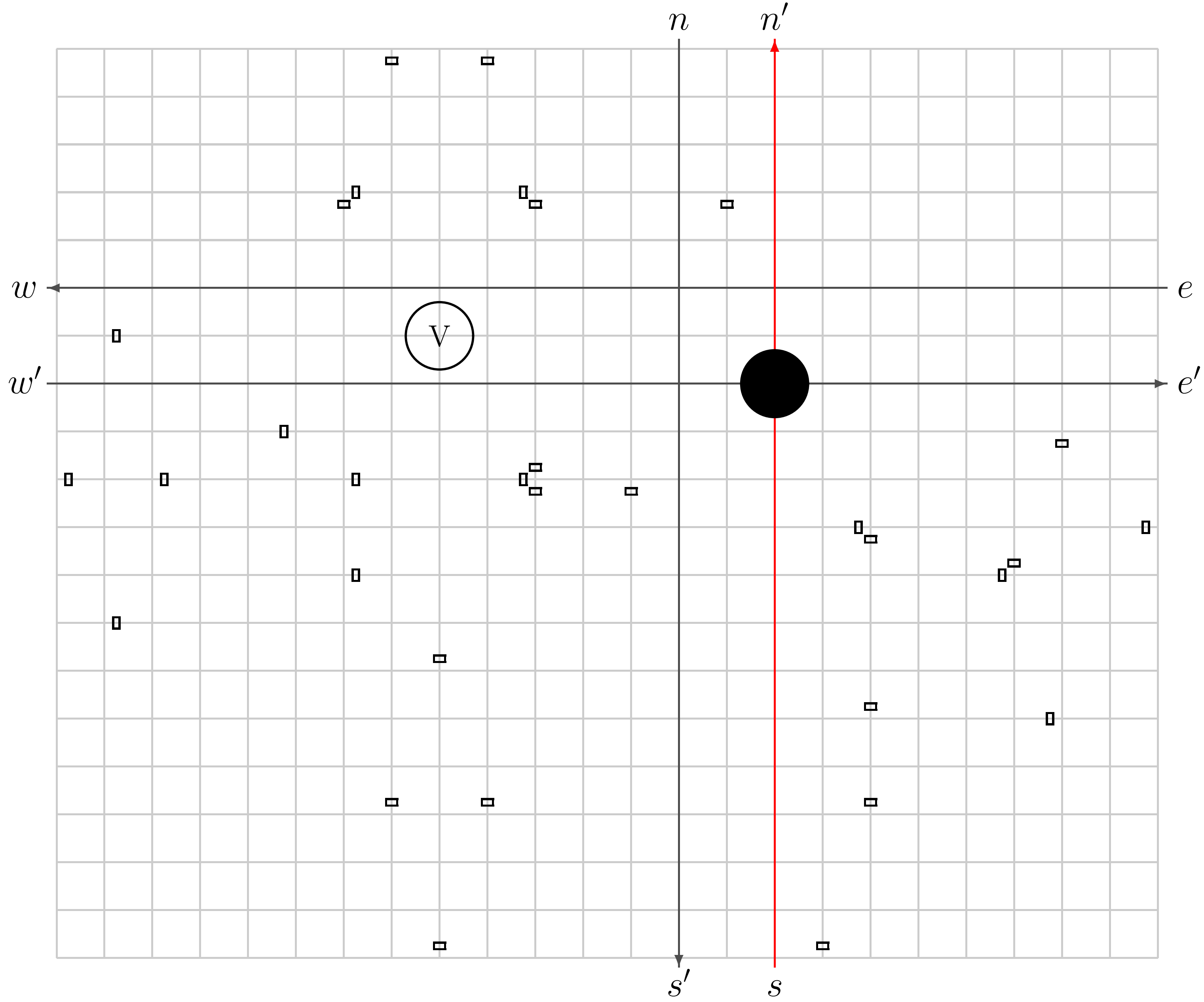
Movements of Balls (State: V, Input: s)



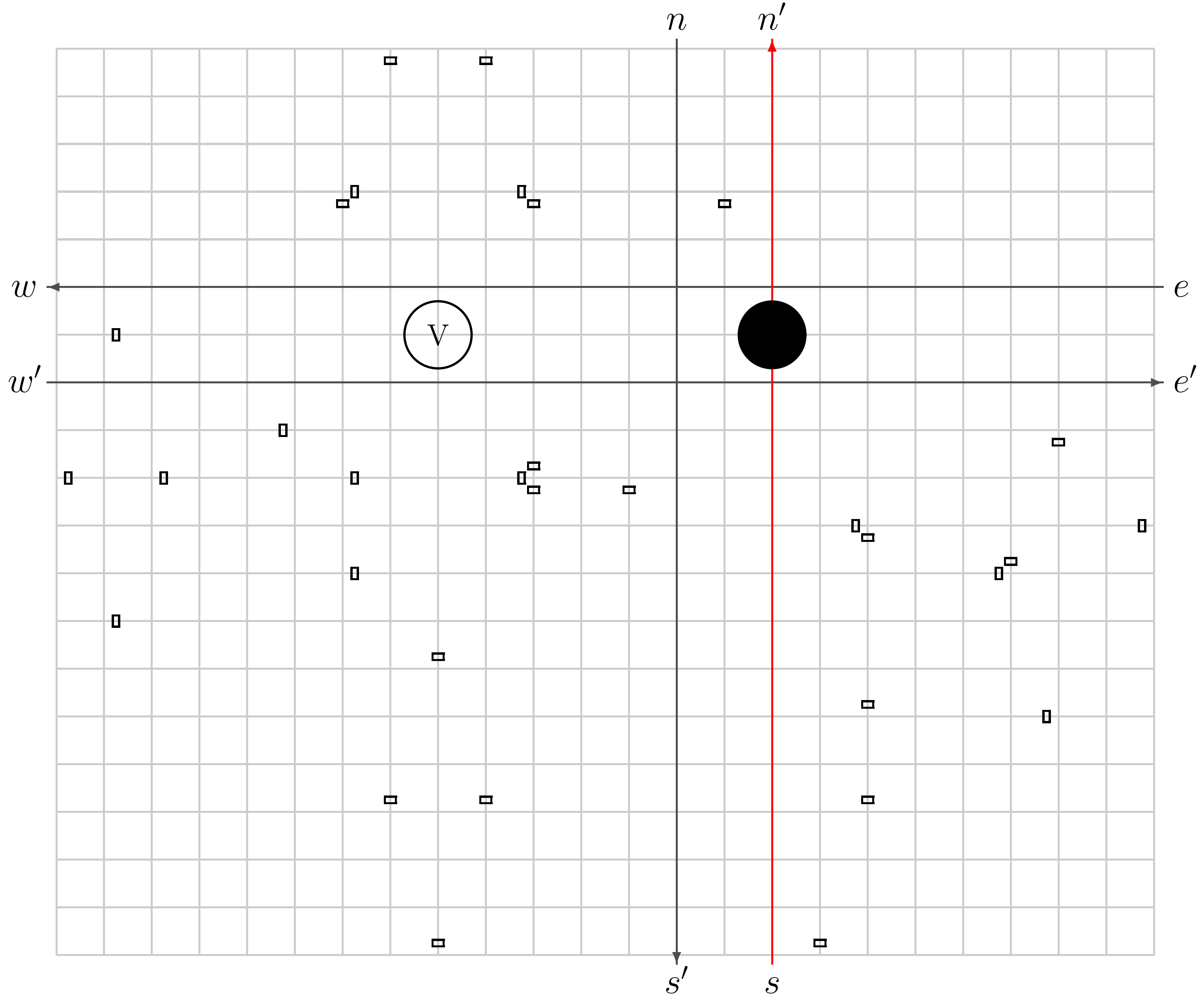
Movements of Balls (State: V, Input: s)



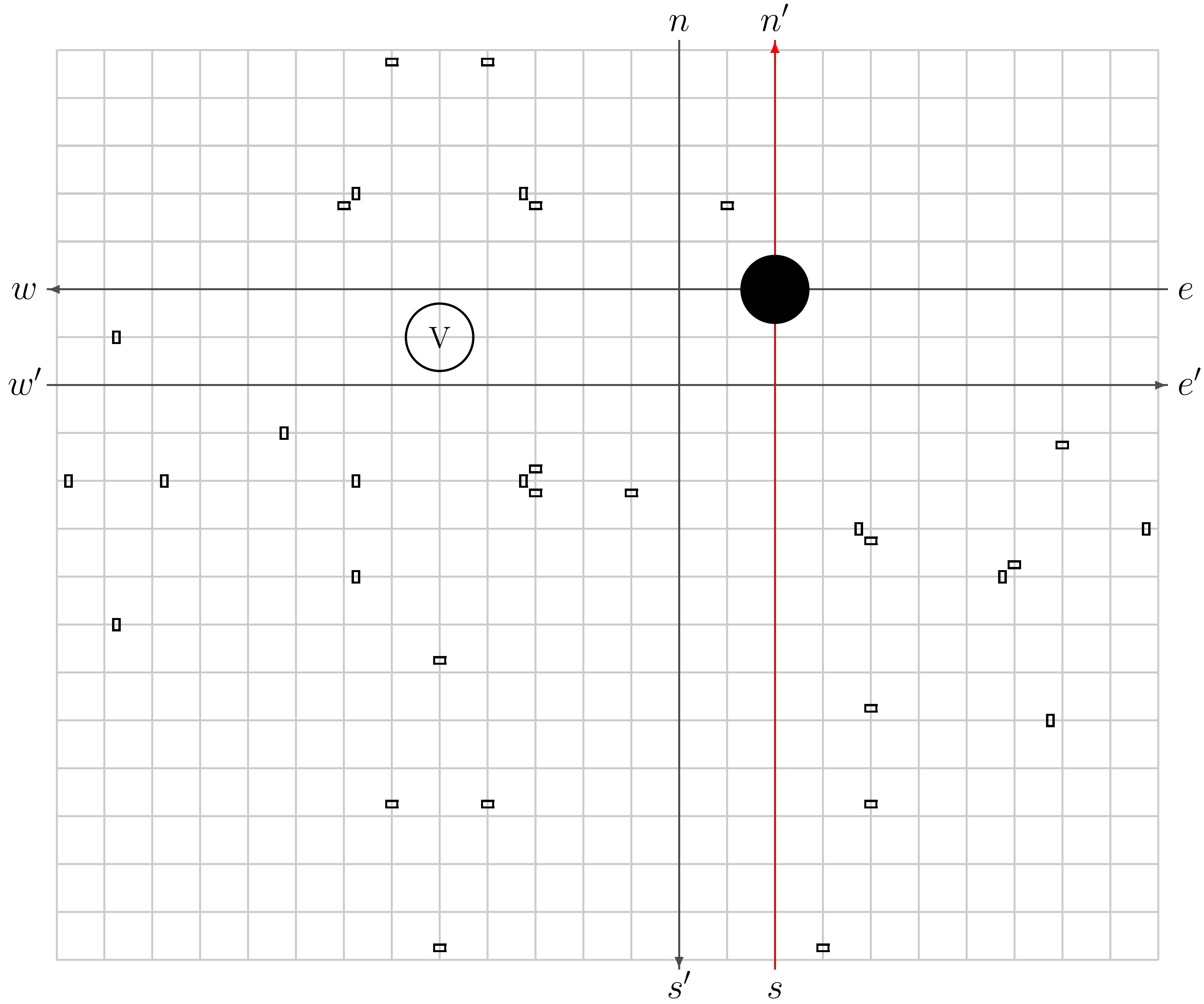
Movements of Balls (State: V, Input: s)



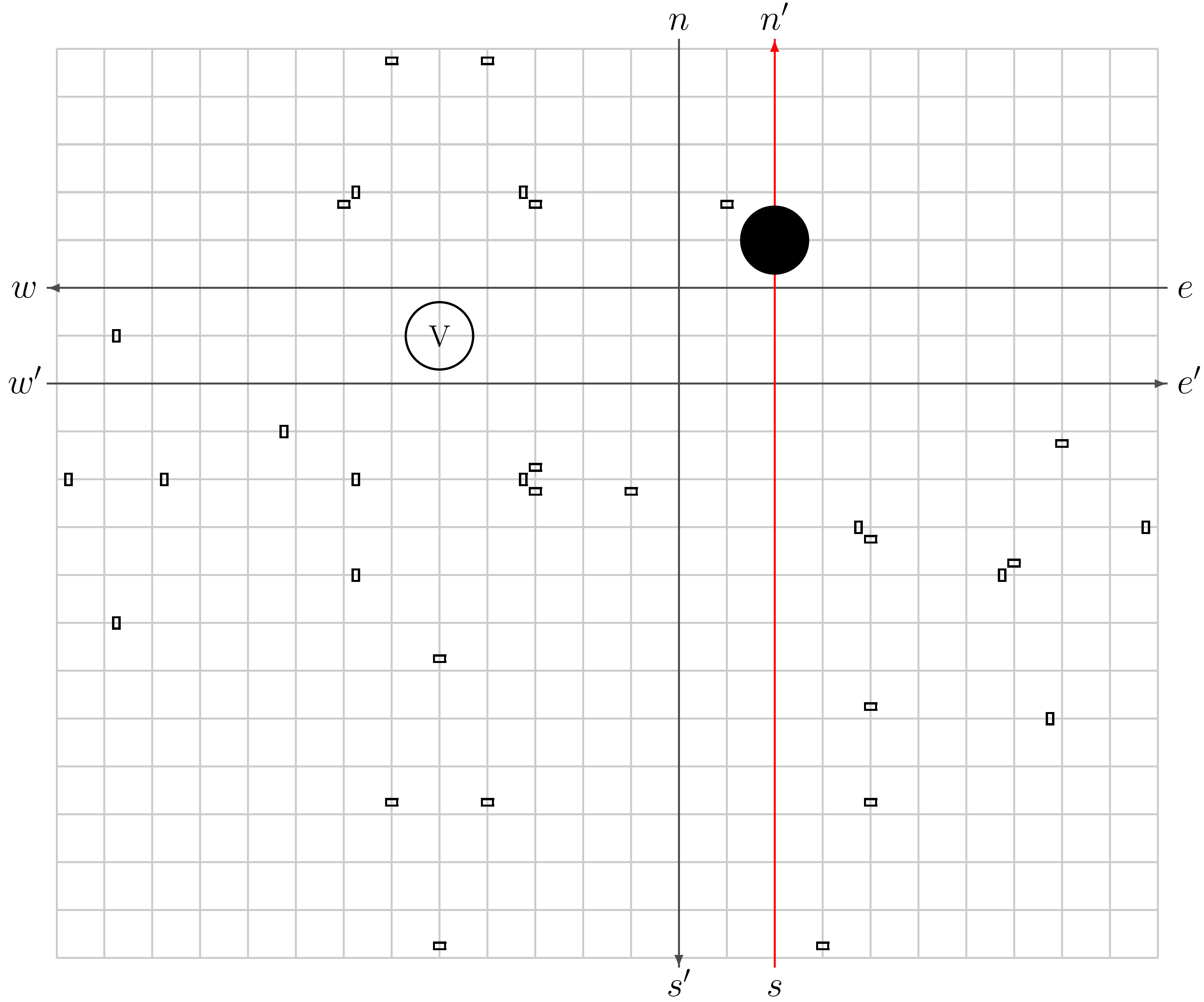
Movements of Balls (State: V, Input: s)



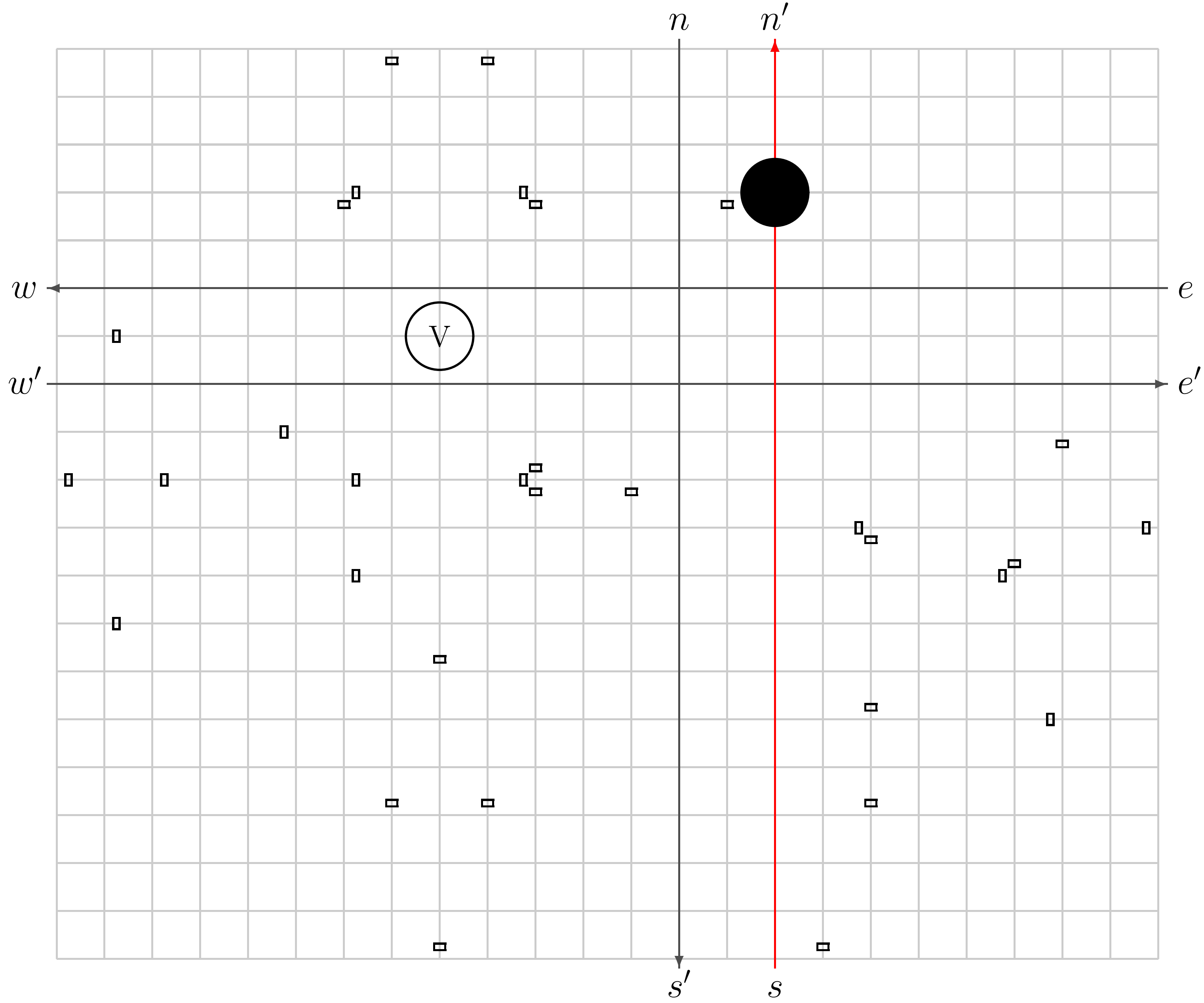
Movements of Balls (State: V, Input: s)



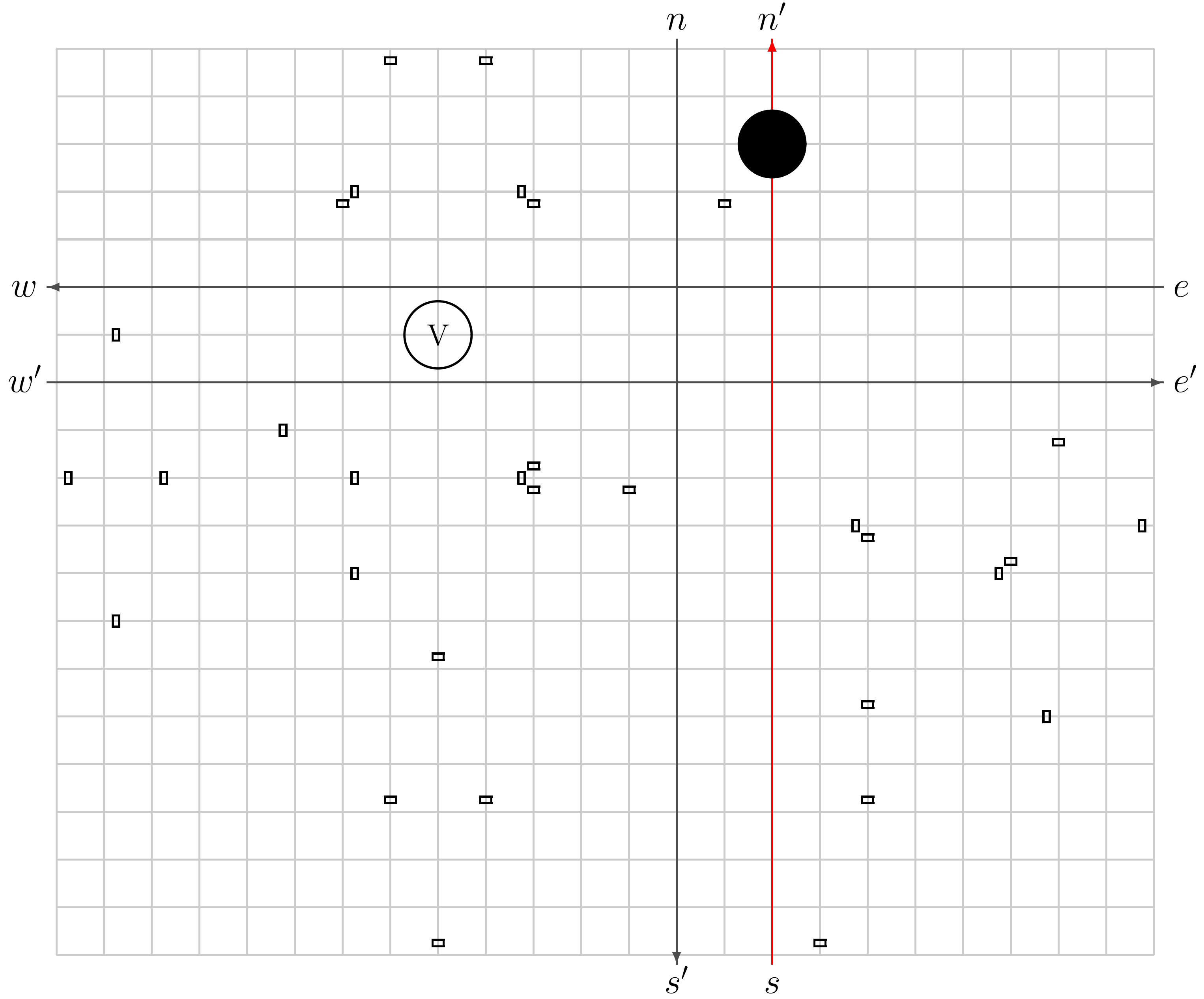
Movements of Balls (State: V, Input: s)



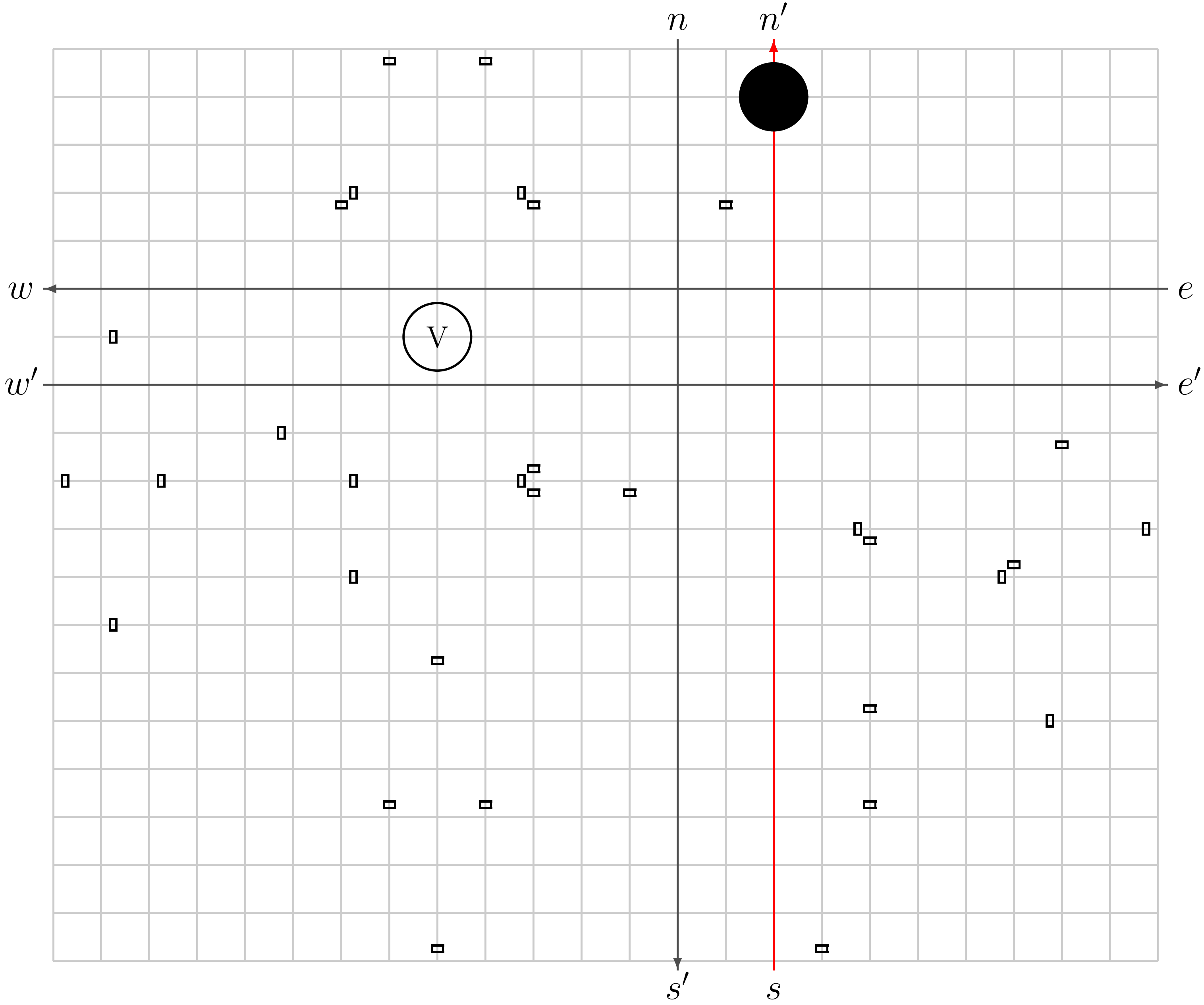
Movements of Balls (State: V, Input: s)



Movements of Balls (State: V, Input: s)

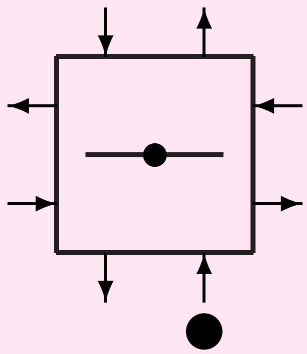


Movements of Balls (State: V, Input: s)

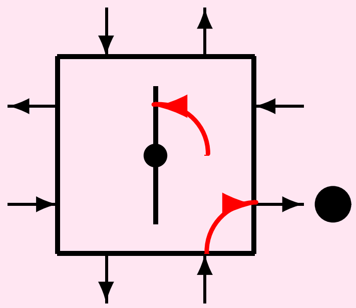


Orthogonal Case

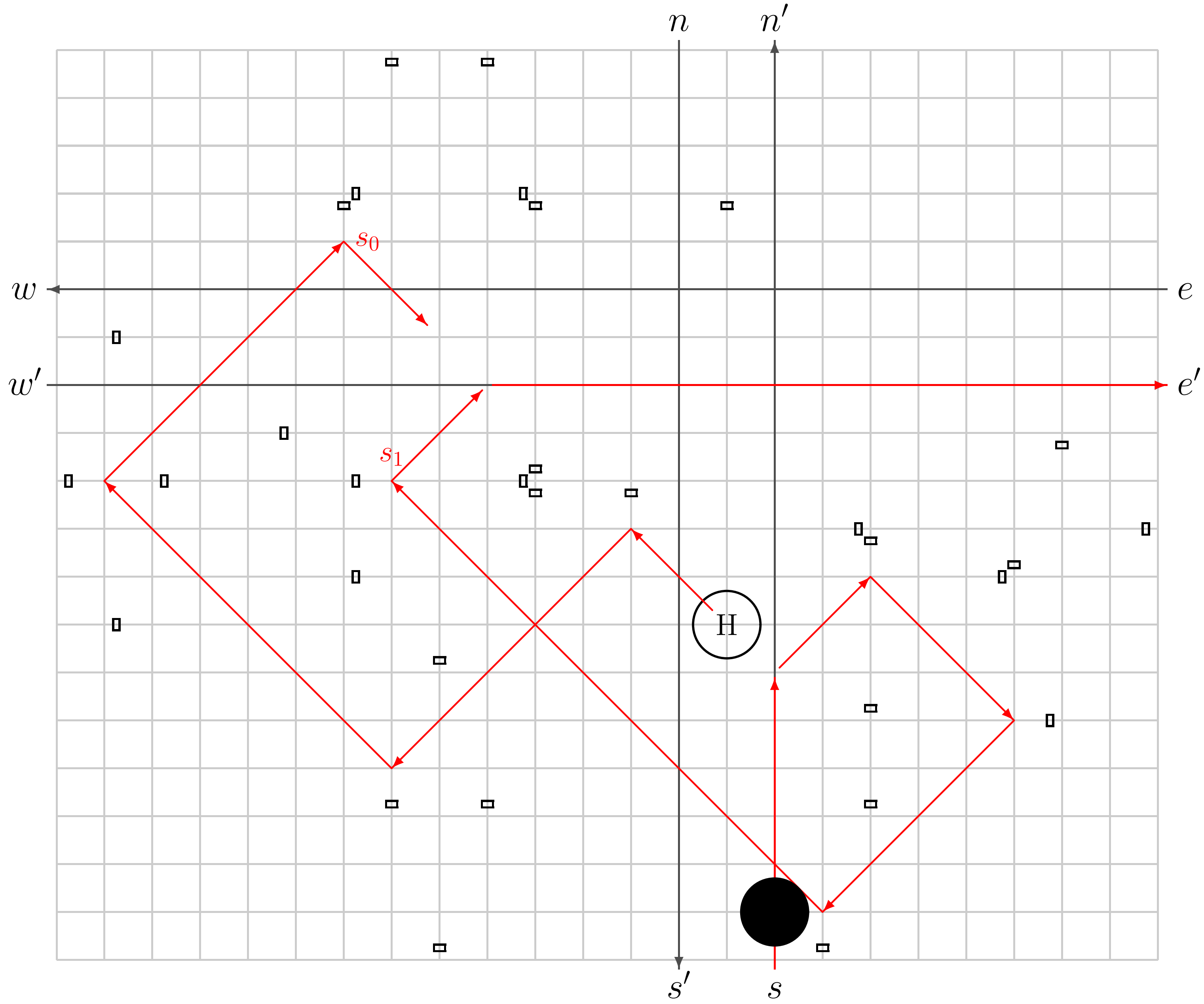
$t = 0$



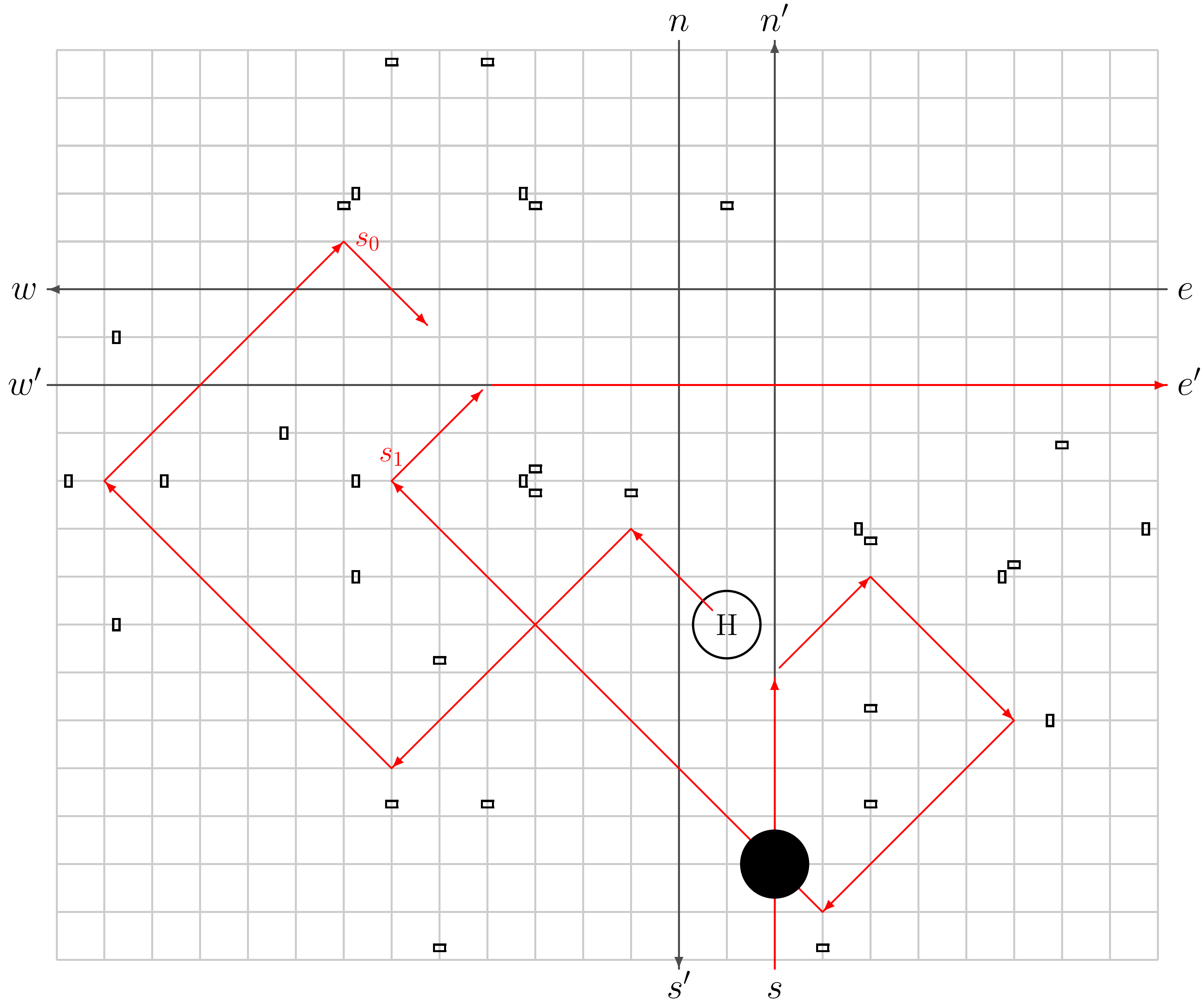
$t = 1$



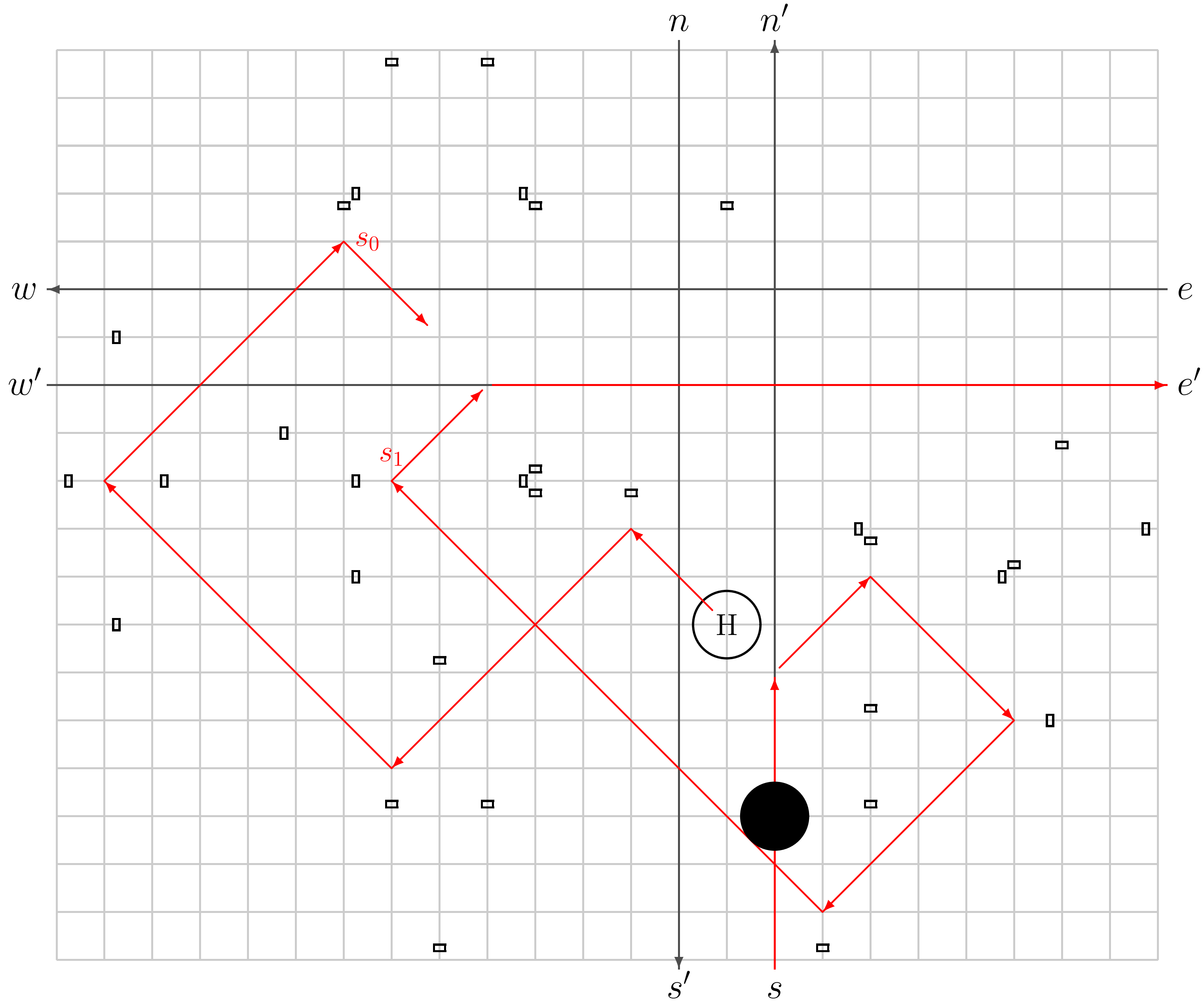
Movements of Balls (State: H, Input: s)



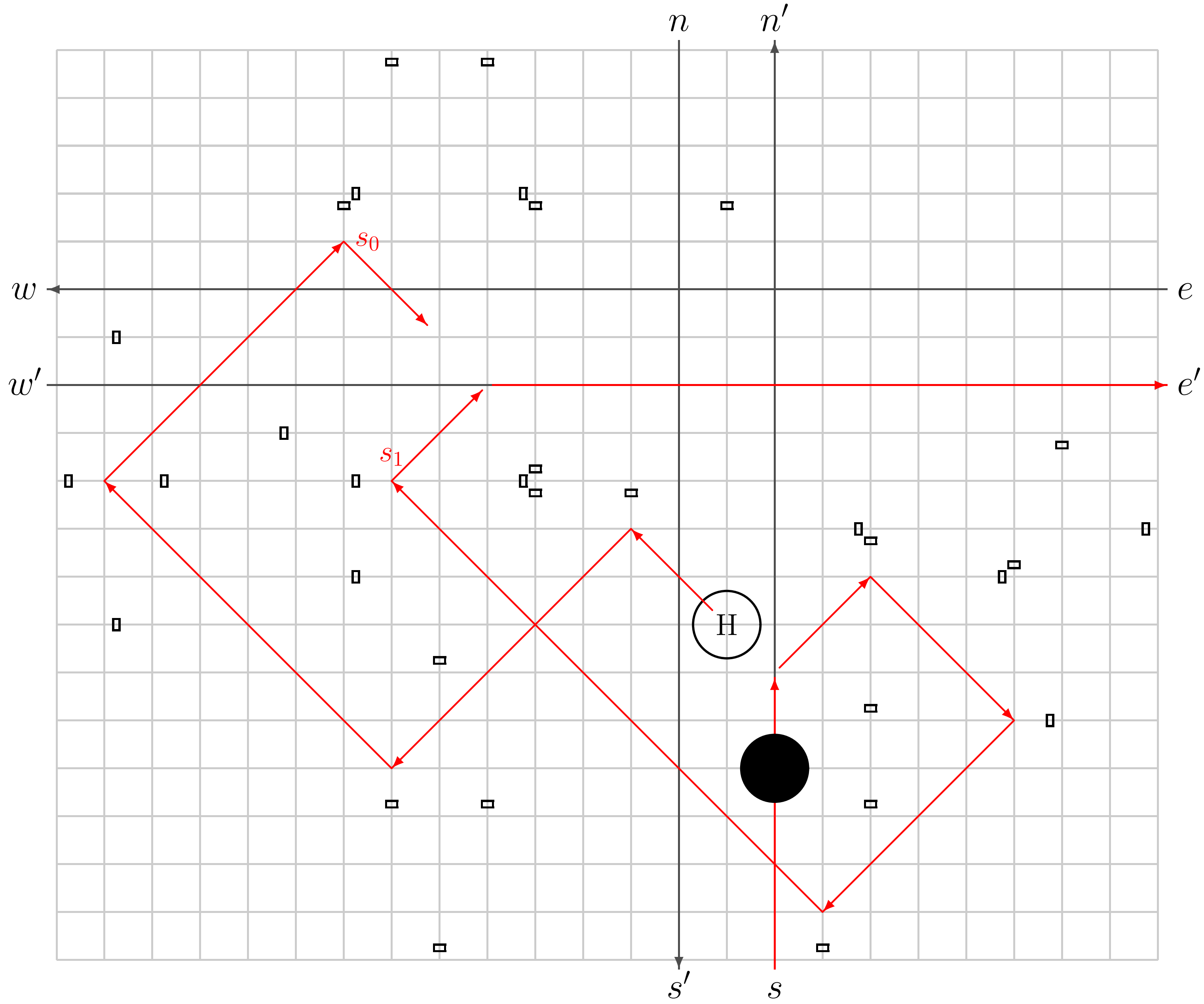
Movements of Balls (State: H, Input: s)



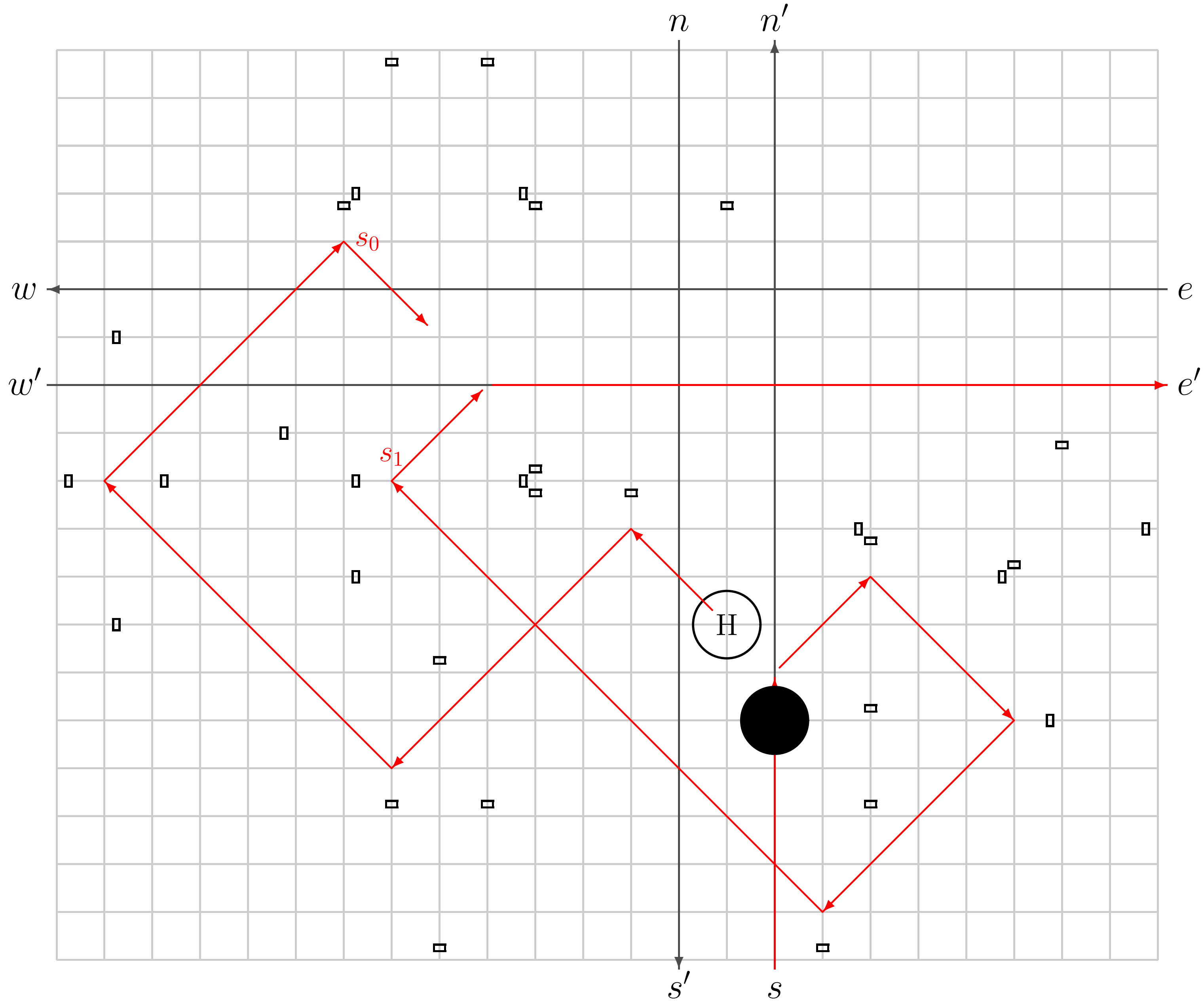
Movements of Balls (State: H, Input: s)



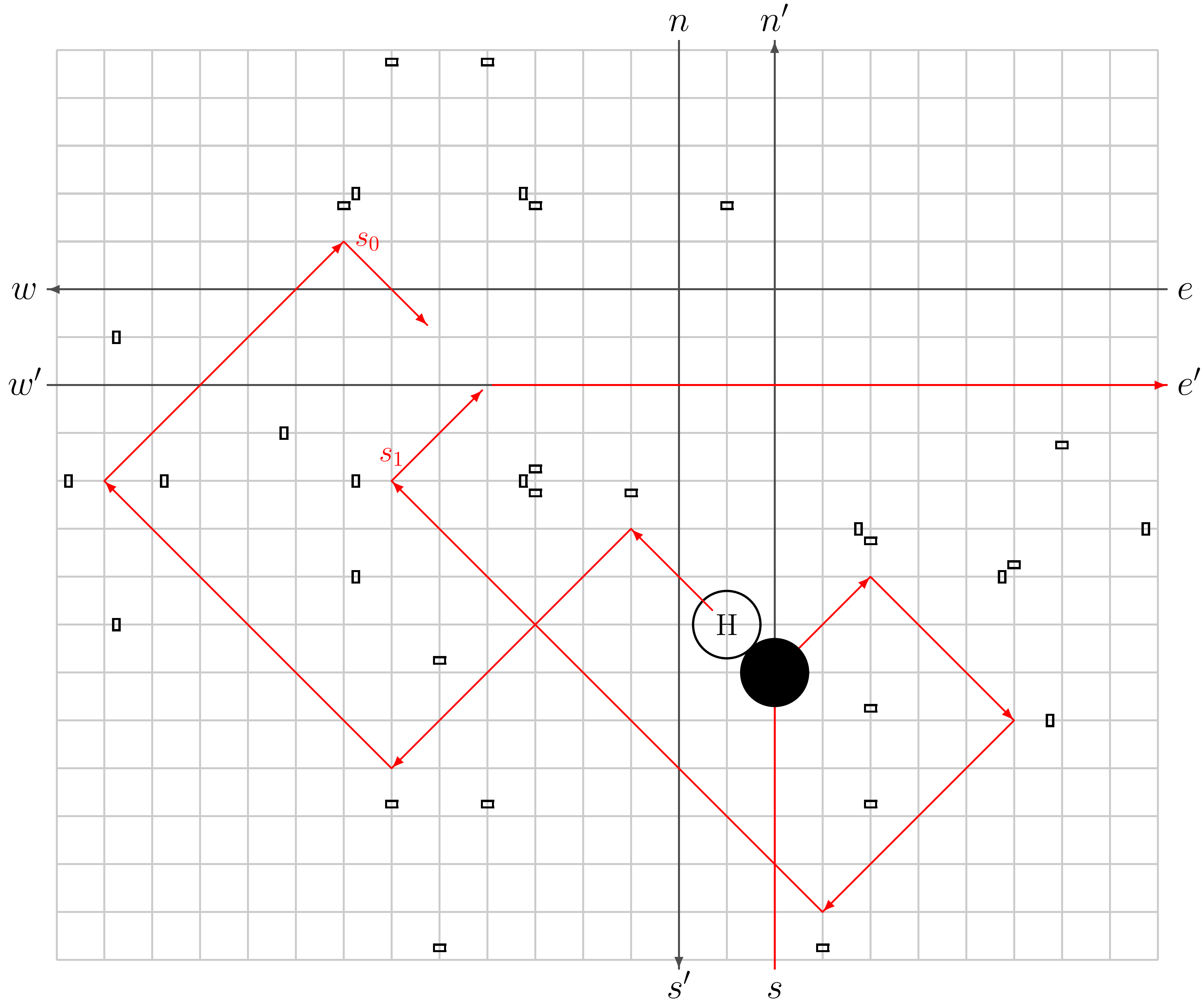
Movements of Balls (State: H, Input: s)



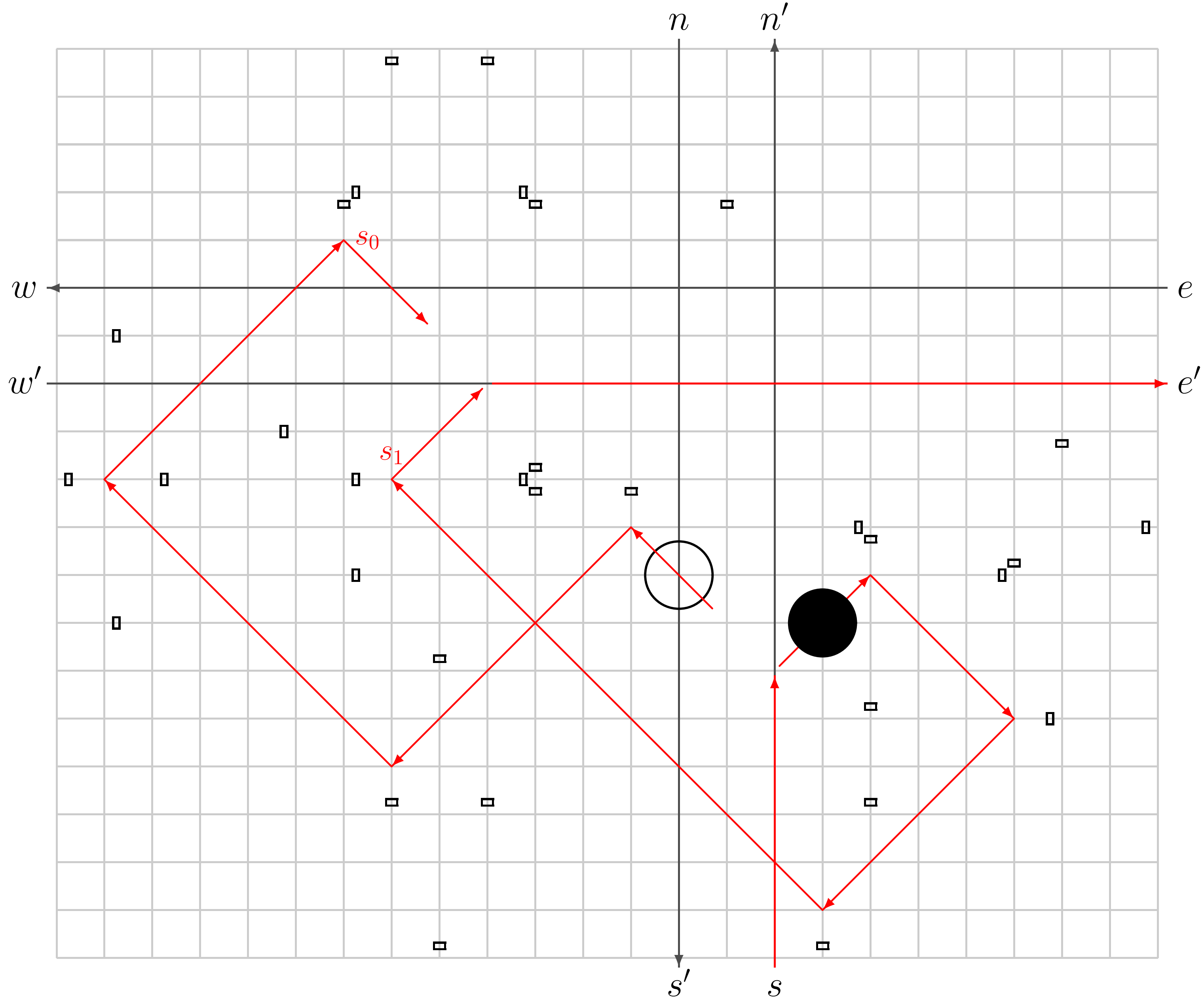
Movements of Balls (State: H, Input: s)



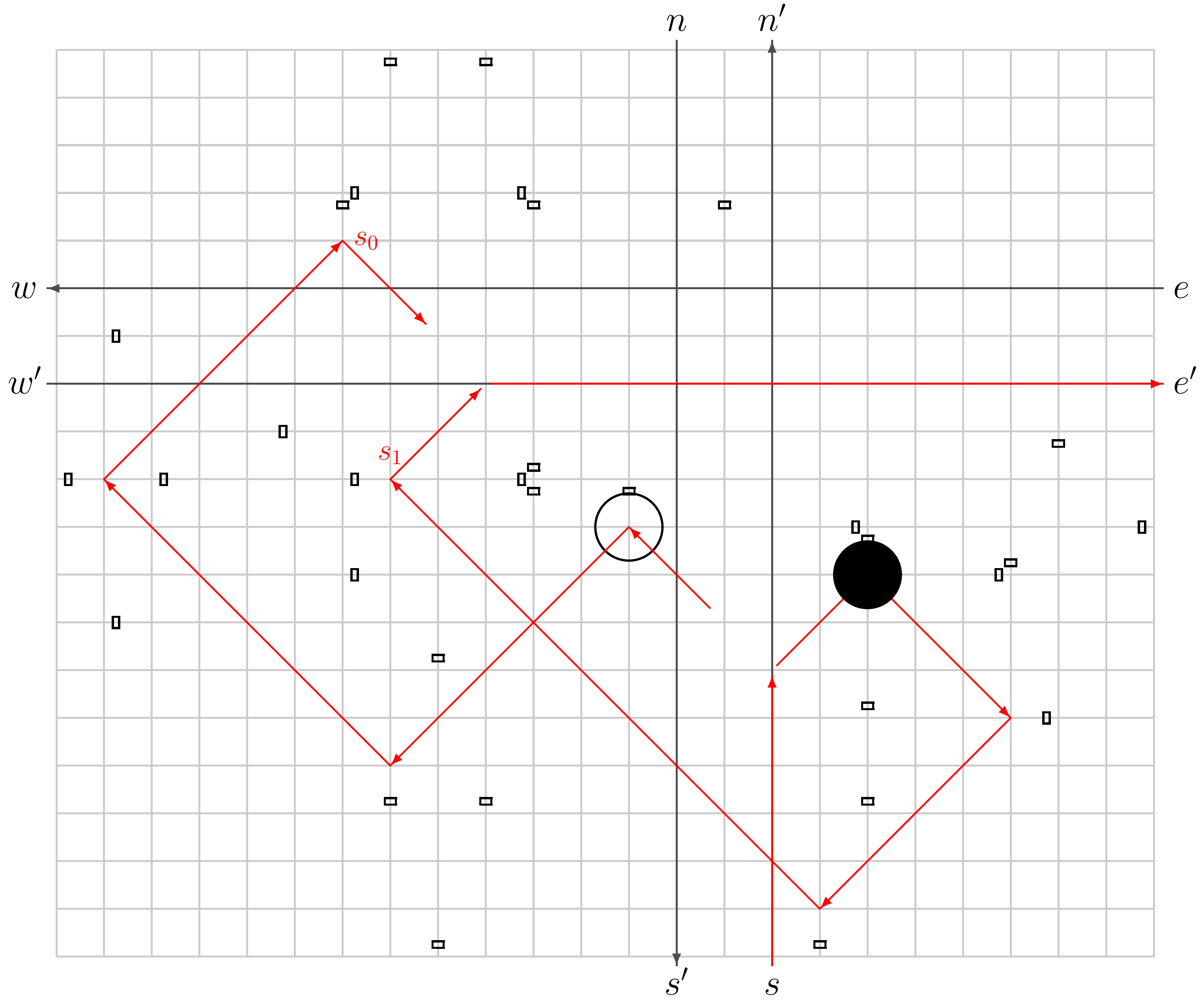
Movements of Balls (State: H, Input: s)



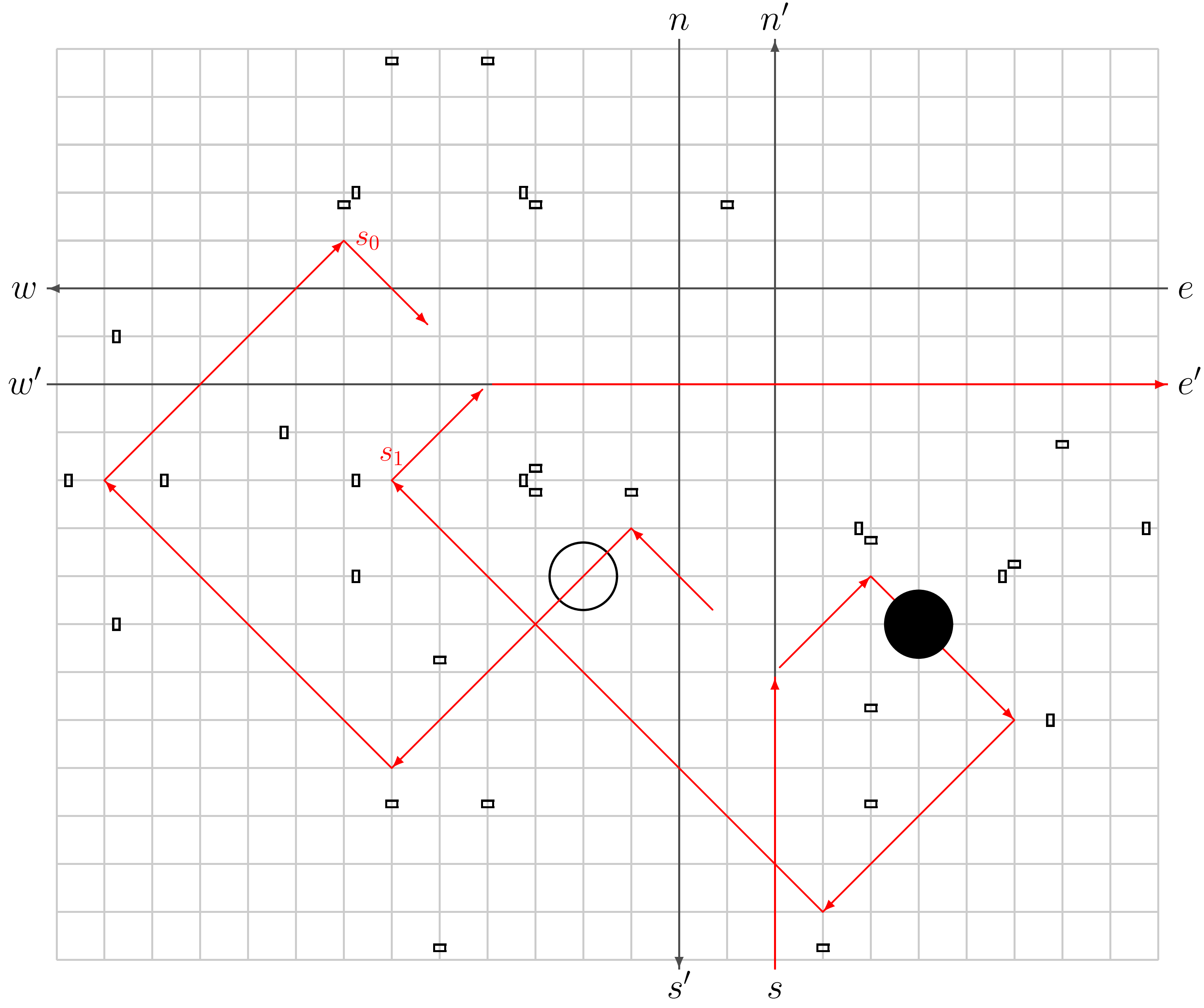
Movements of Balls (State: H, Input: s)



Movements of Balls (State: H, Input: s)



Movements of Balls (State: H, Input: s)



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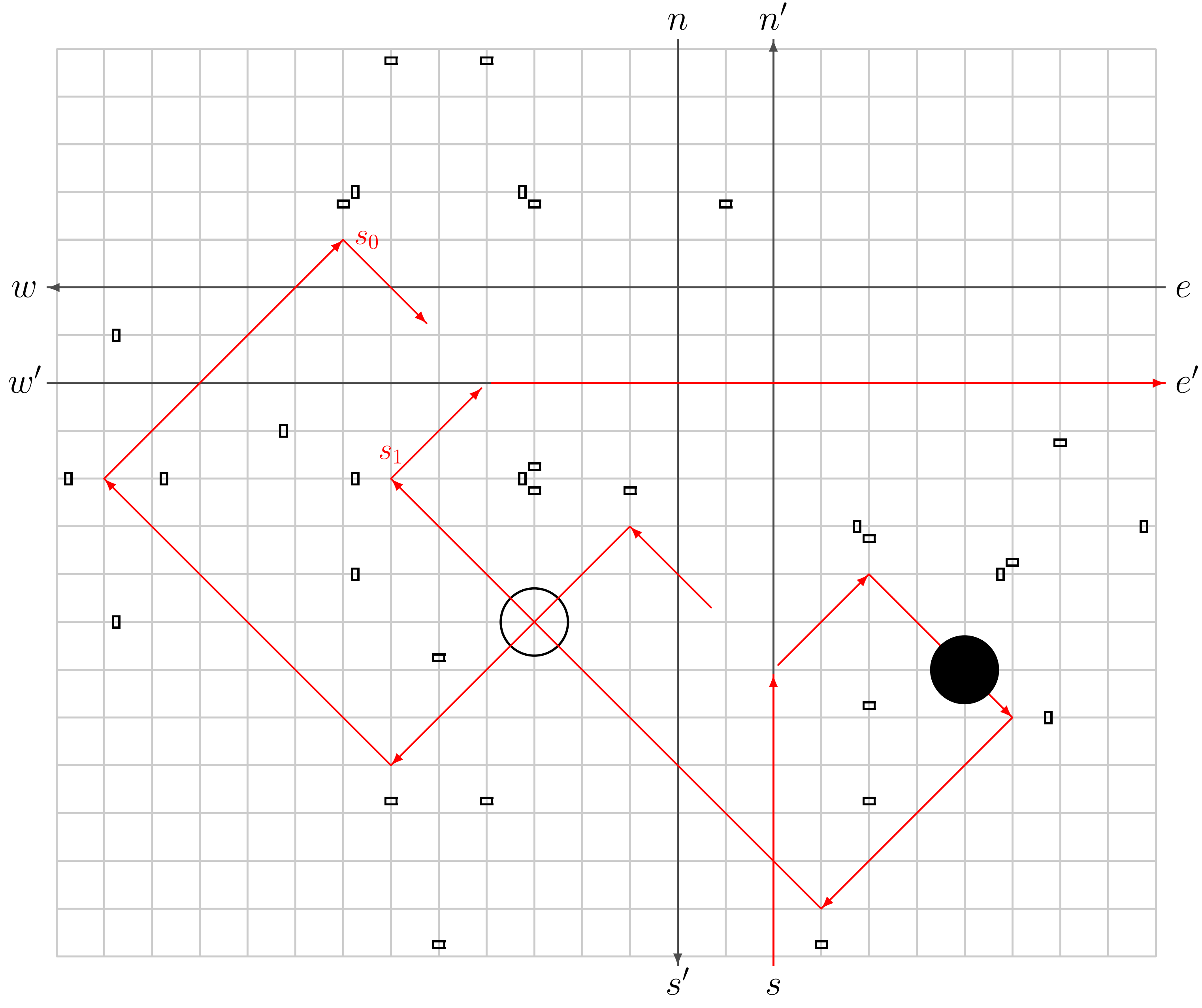

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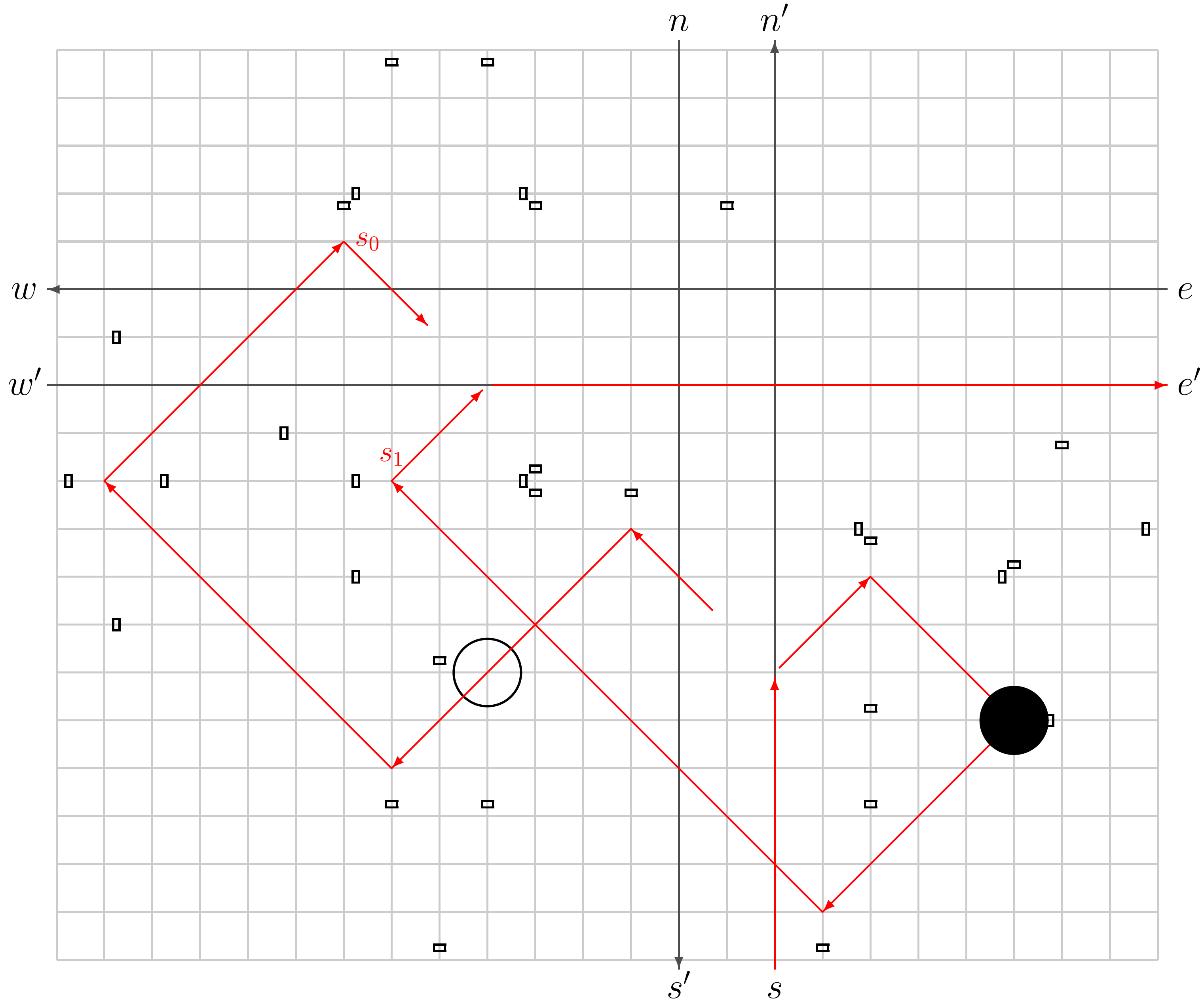
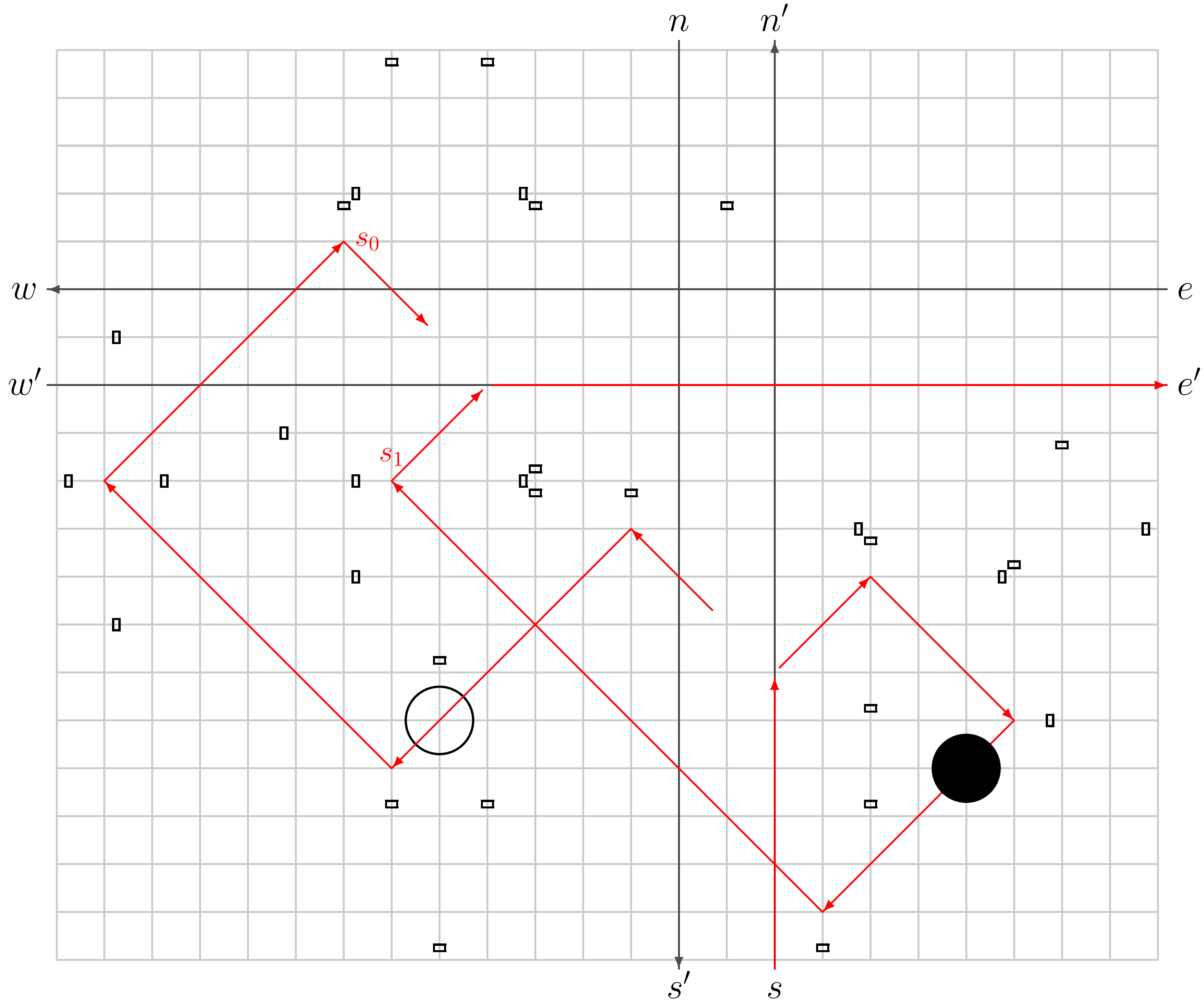


Movements of Balls (State: H, Input: s)



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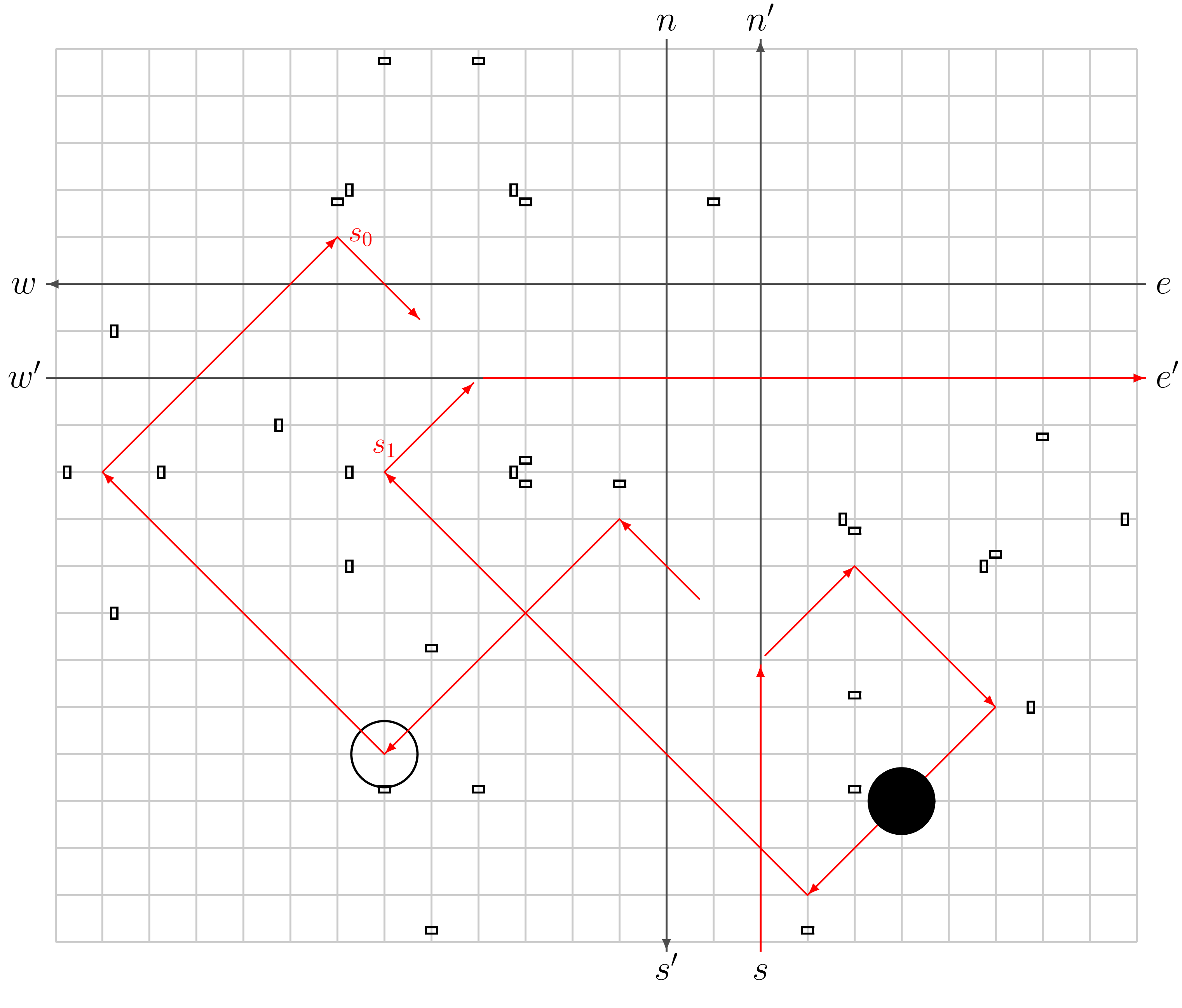
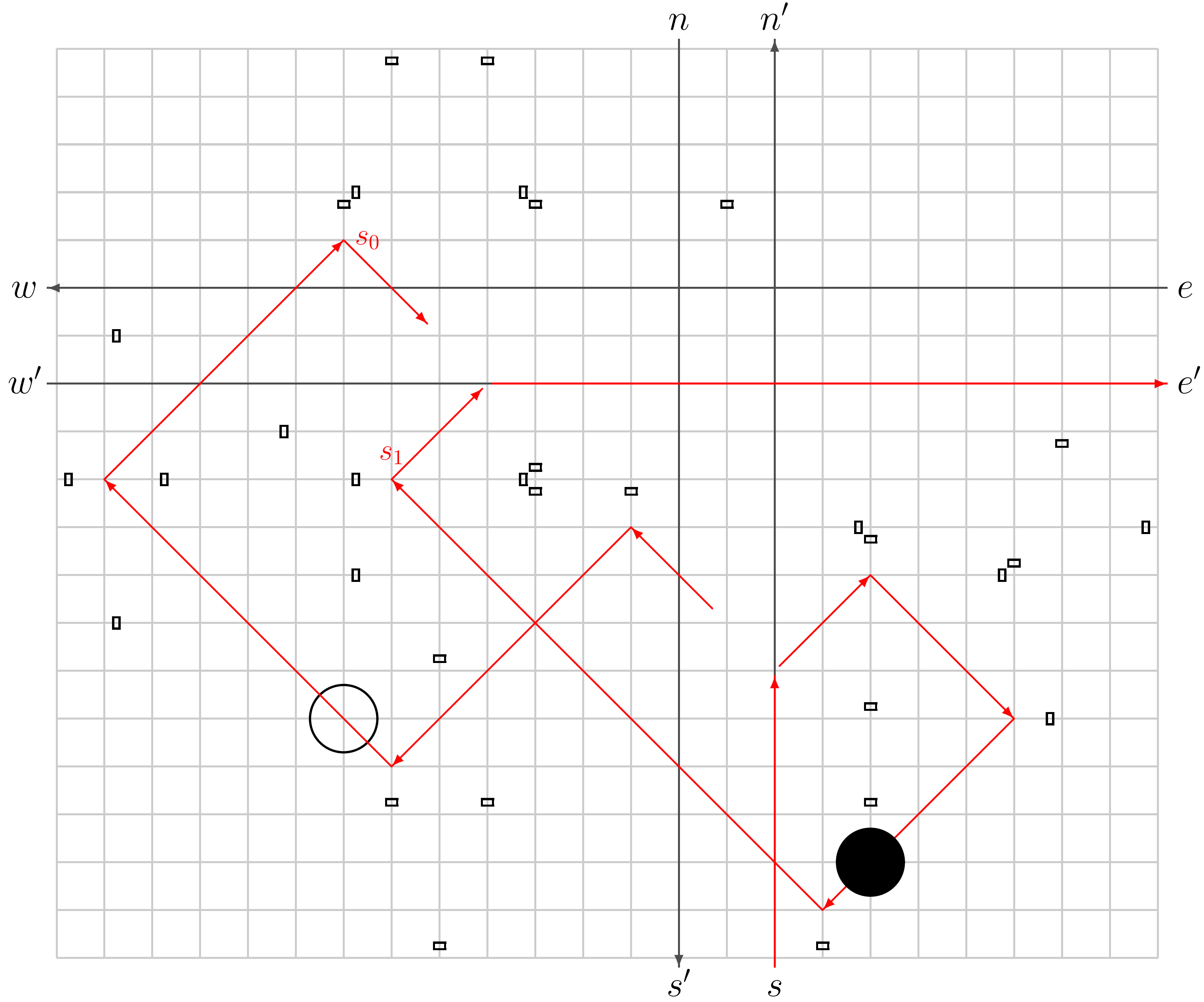


Movements of Balls (State: H, Input: s)



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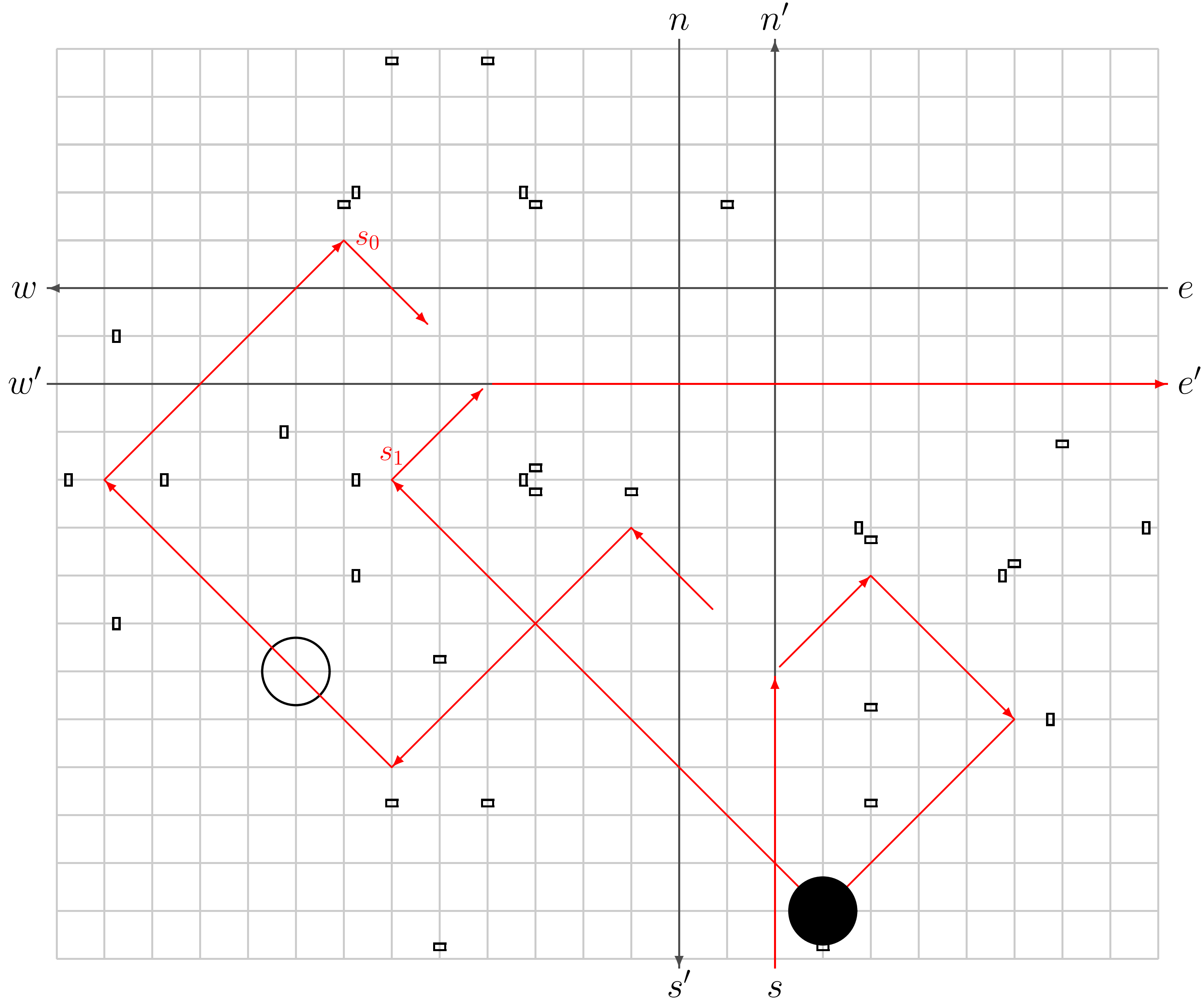

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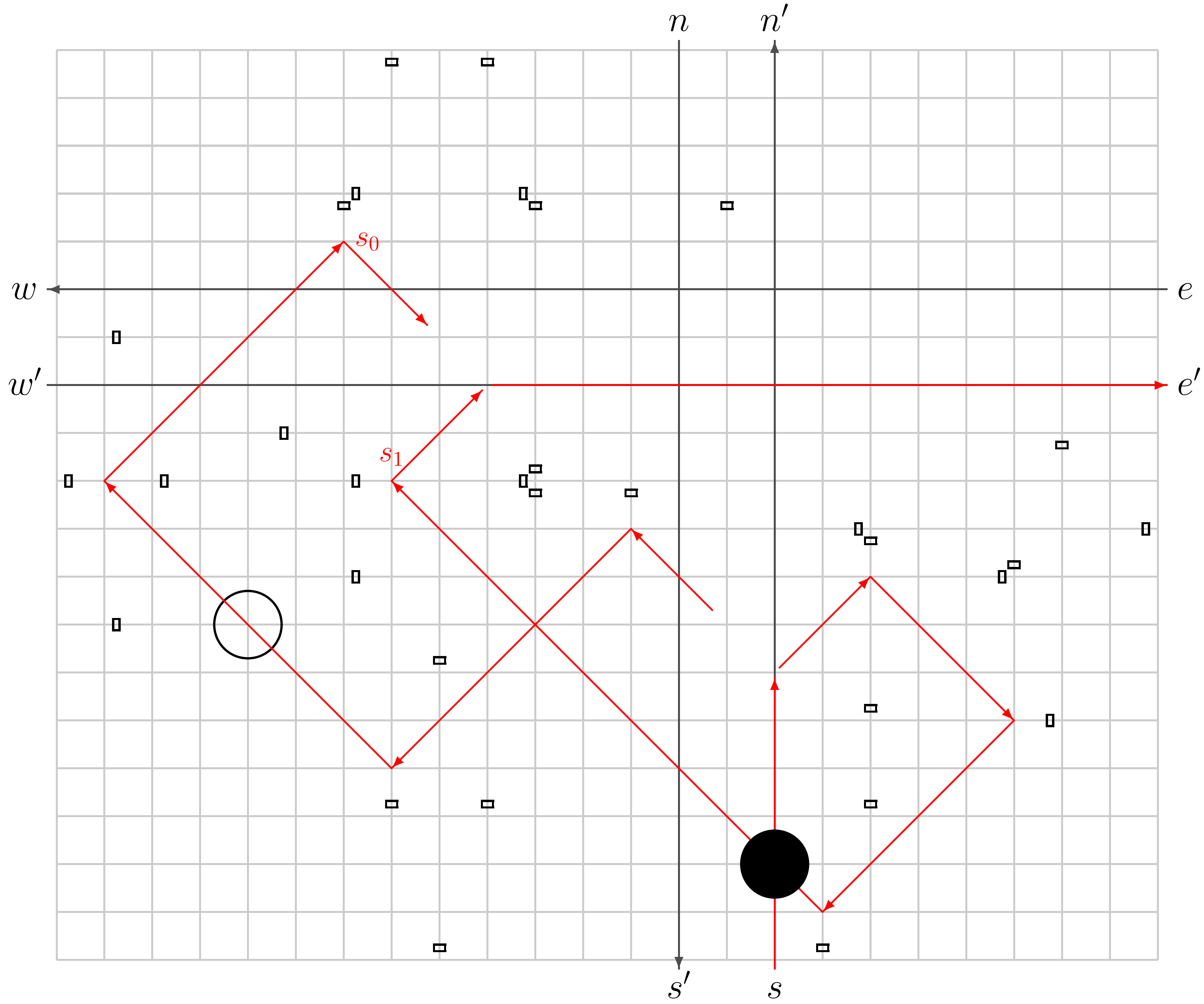
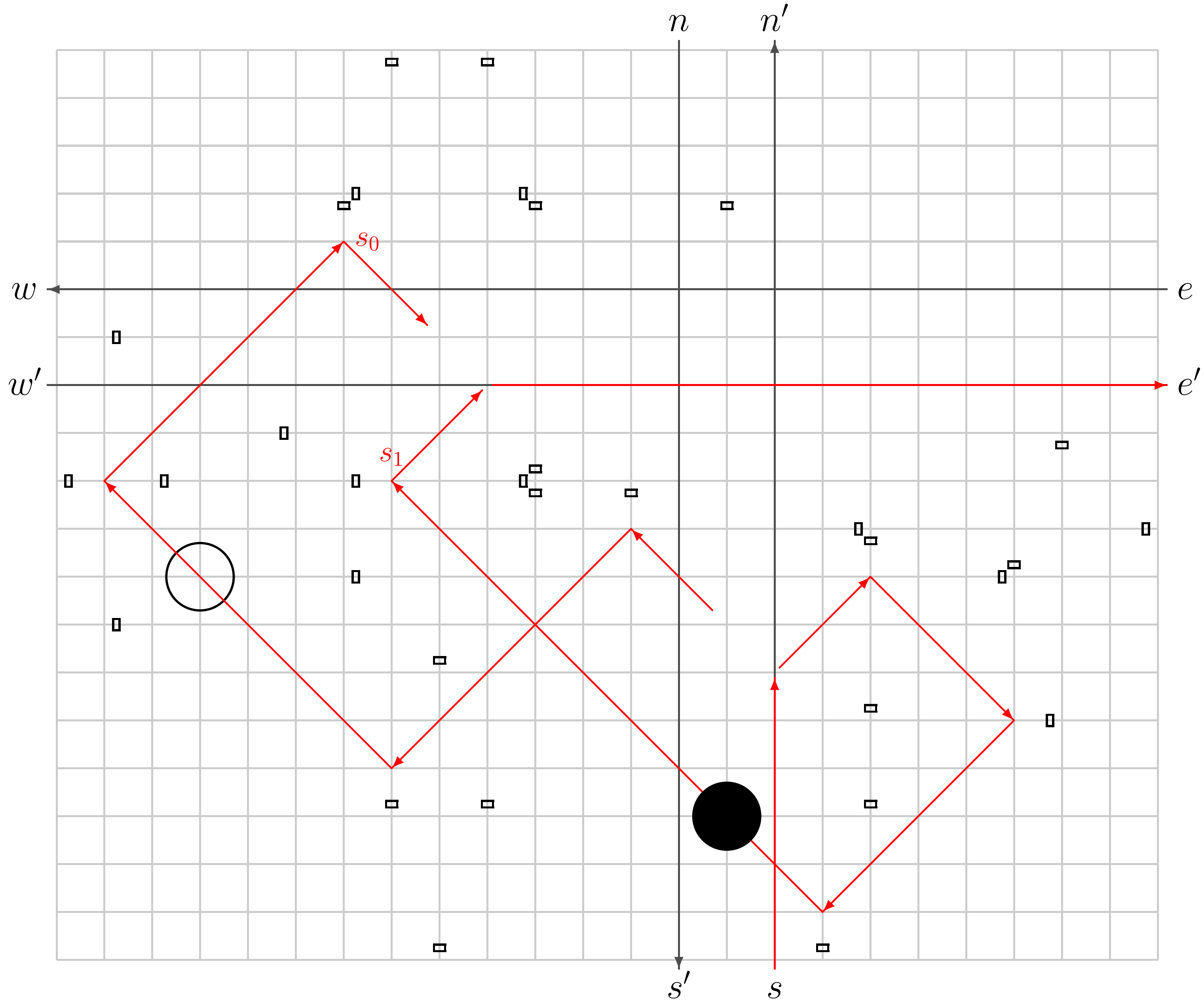
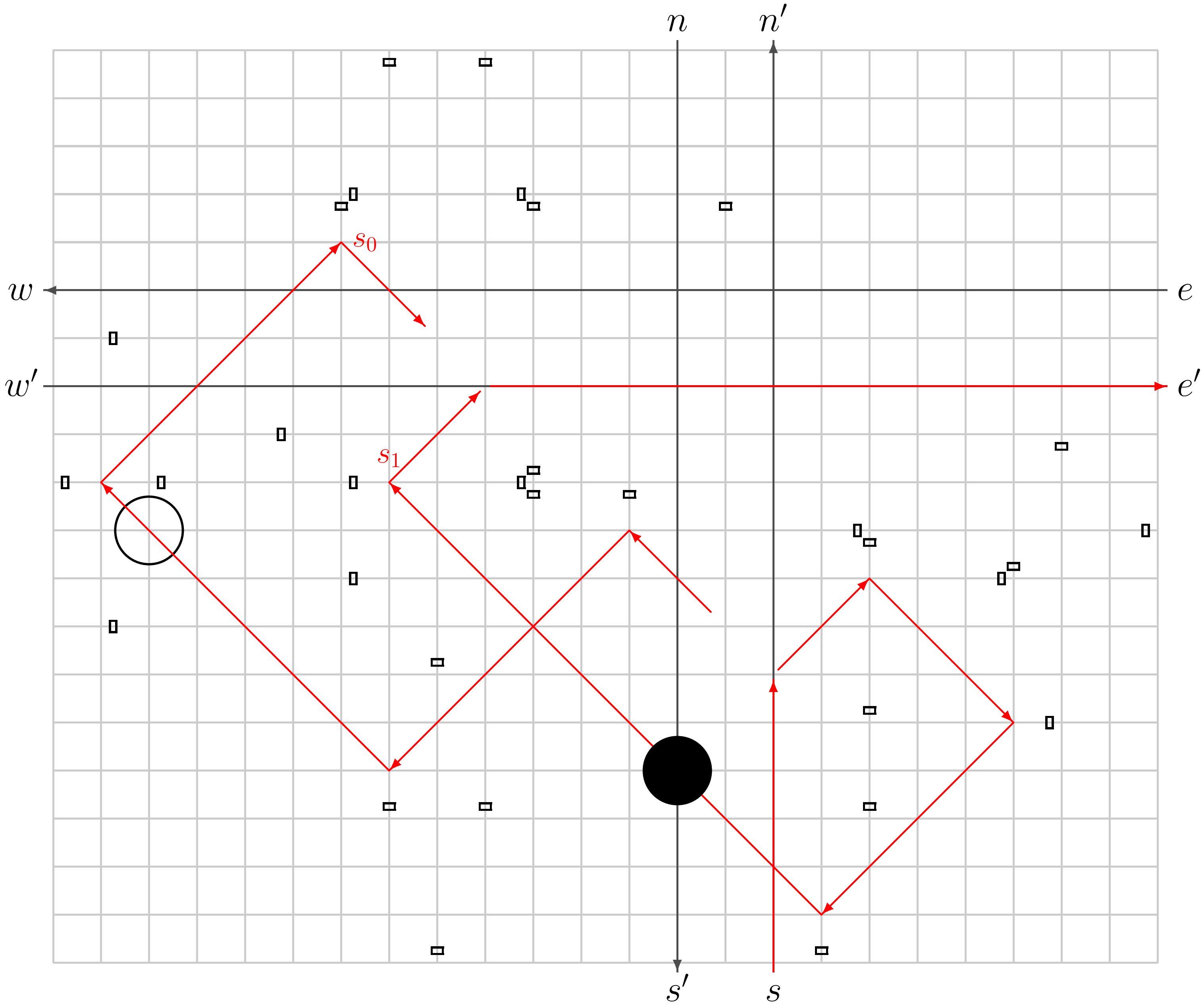


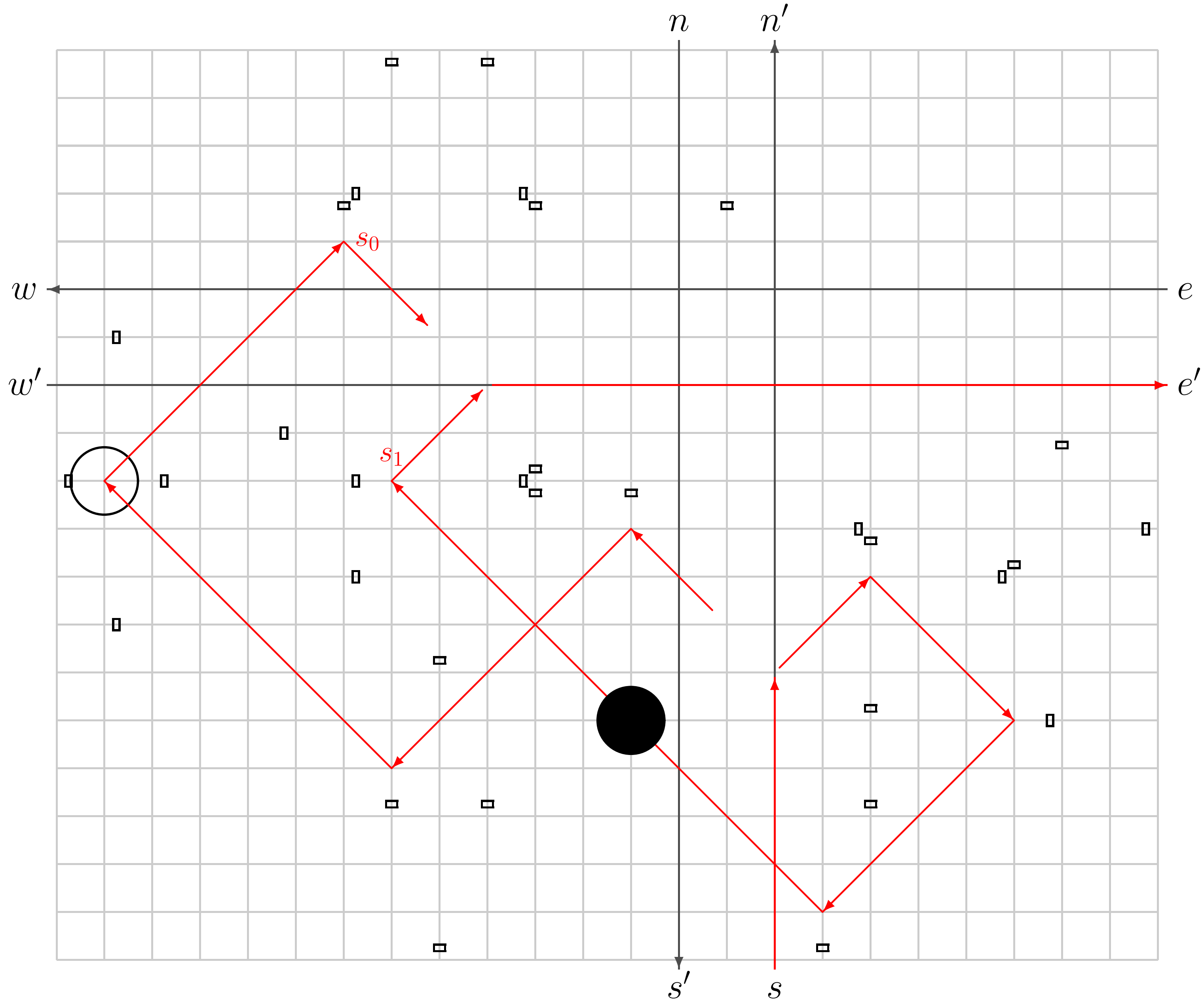
Movements of Balls (State: H, Input: s)



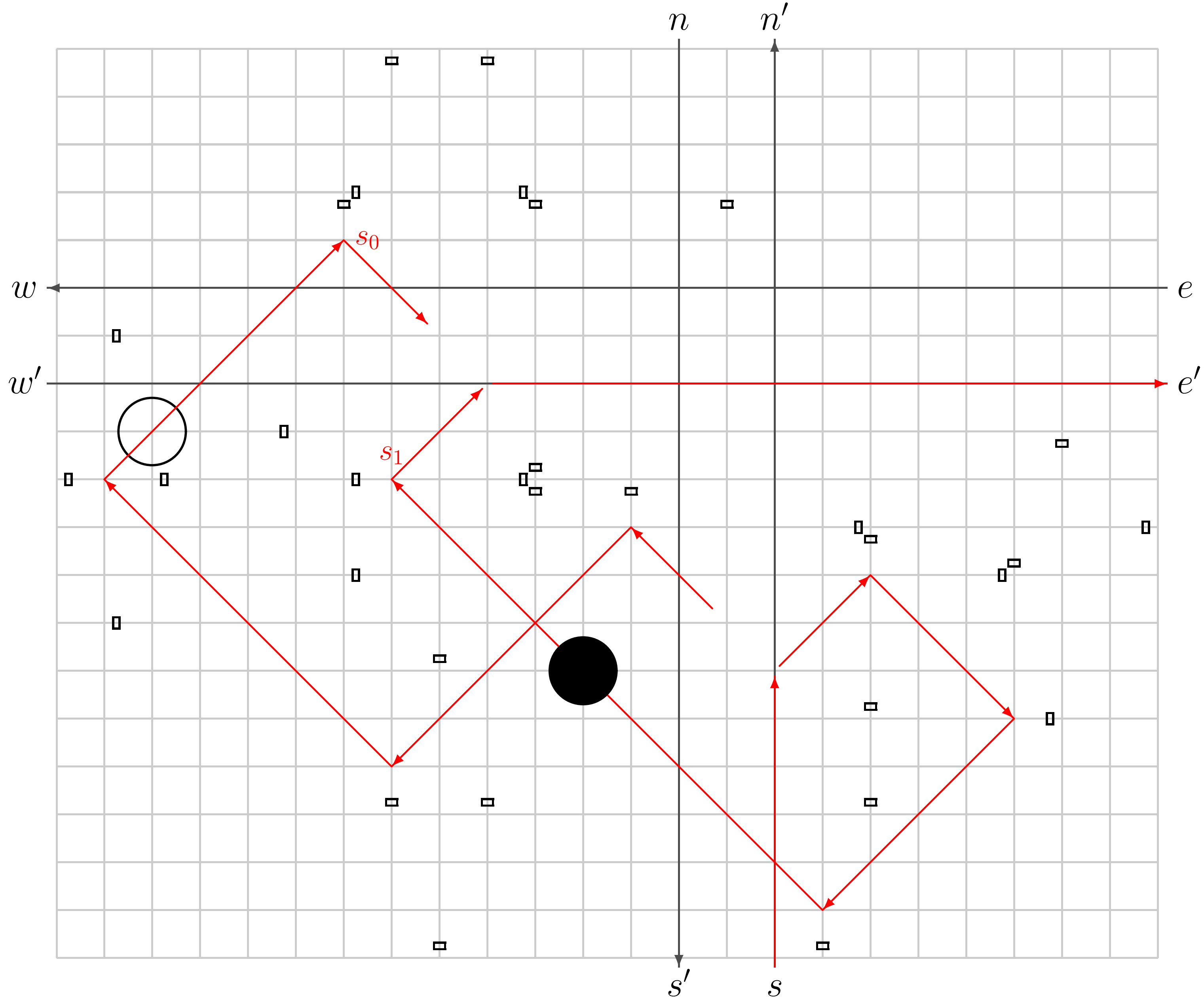
Movements of Balls (State: H, Input: s)



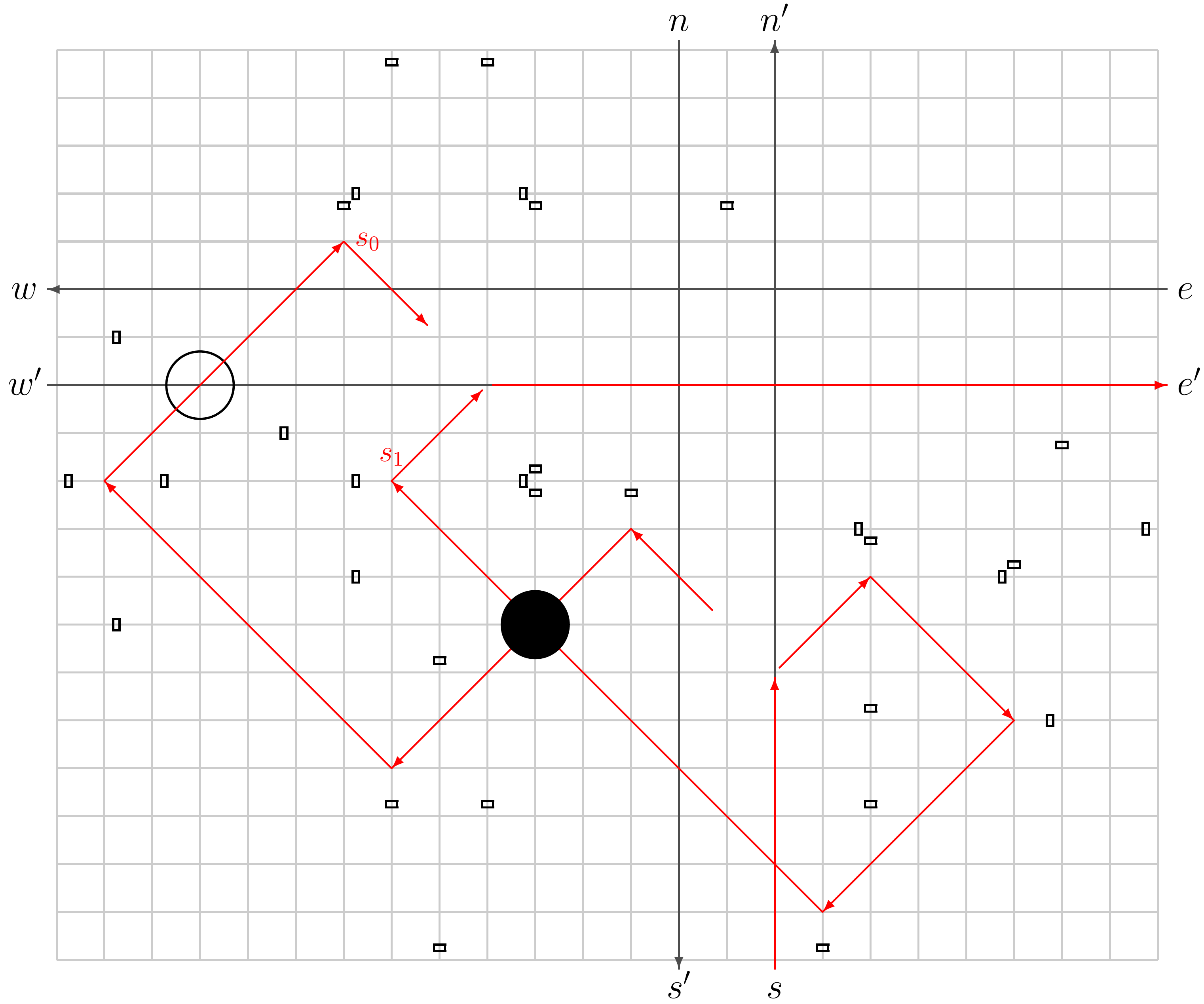
Movements of Balls (State: H, Input: s)



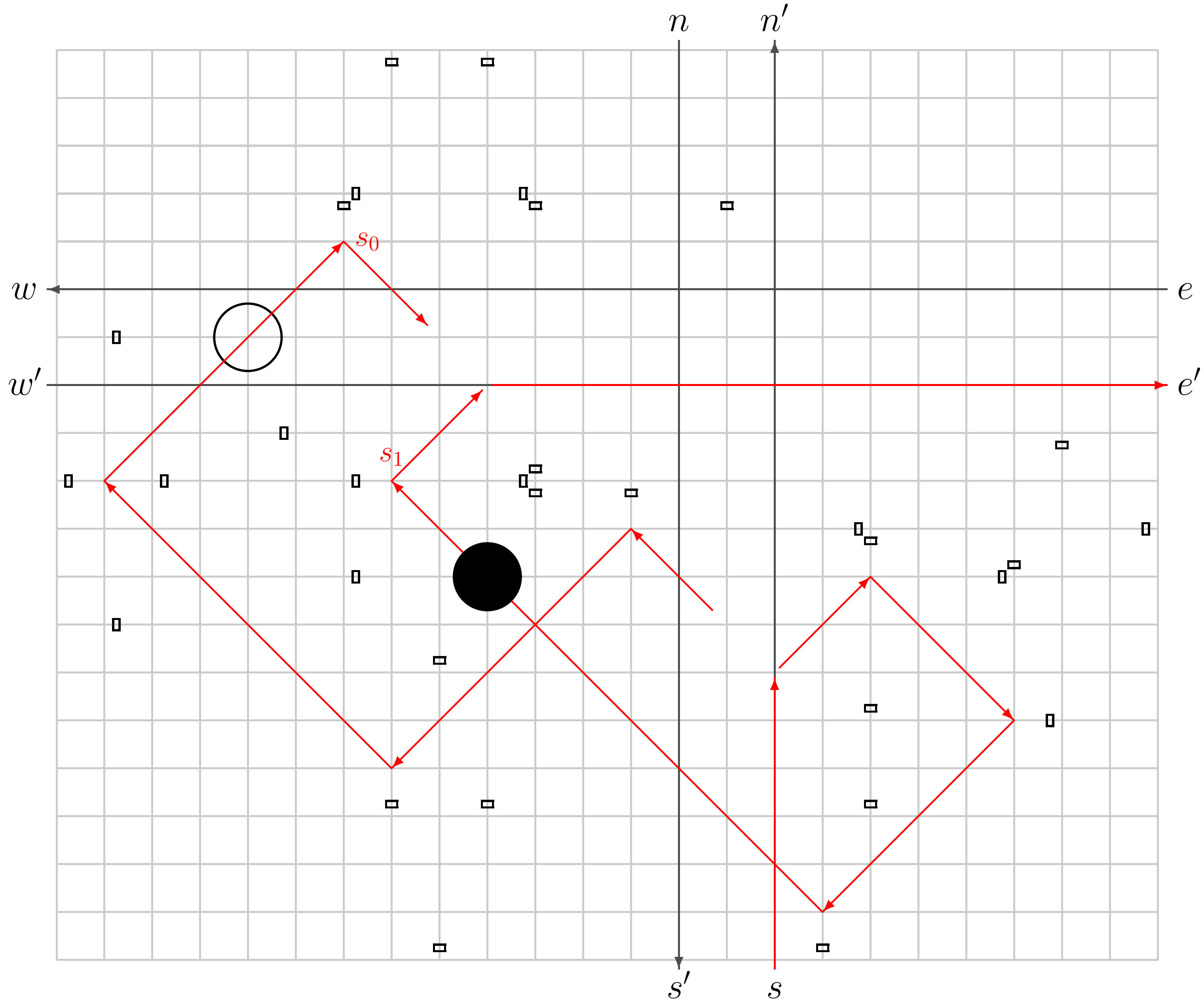
Movements of Balls (State: H, Input: s)



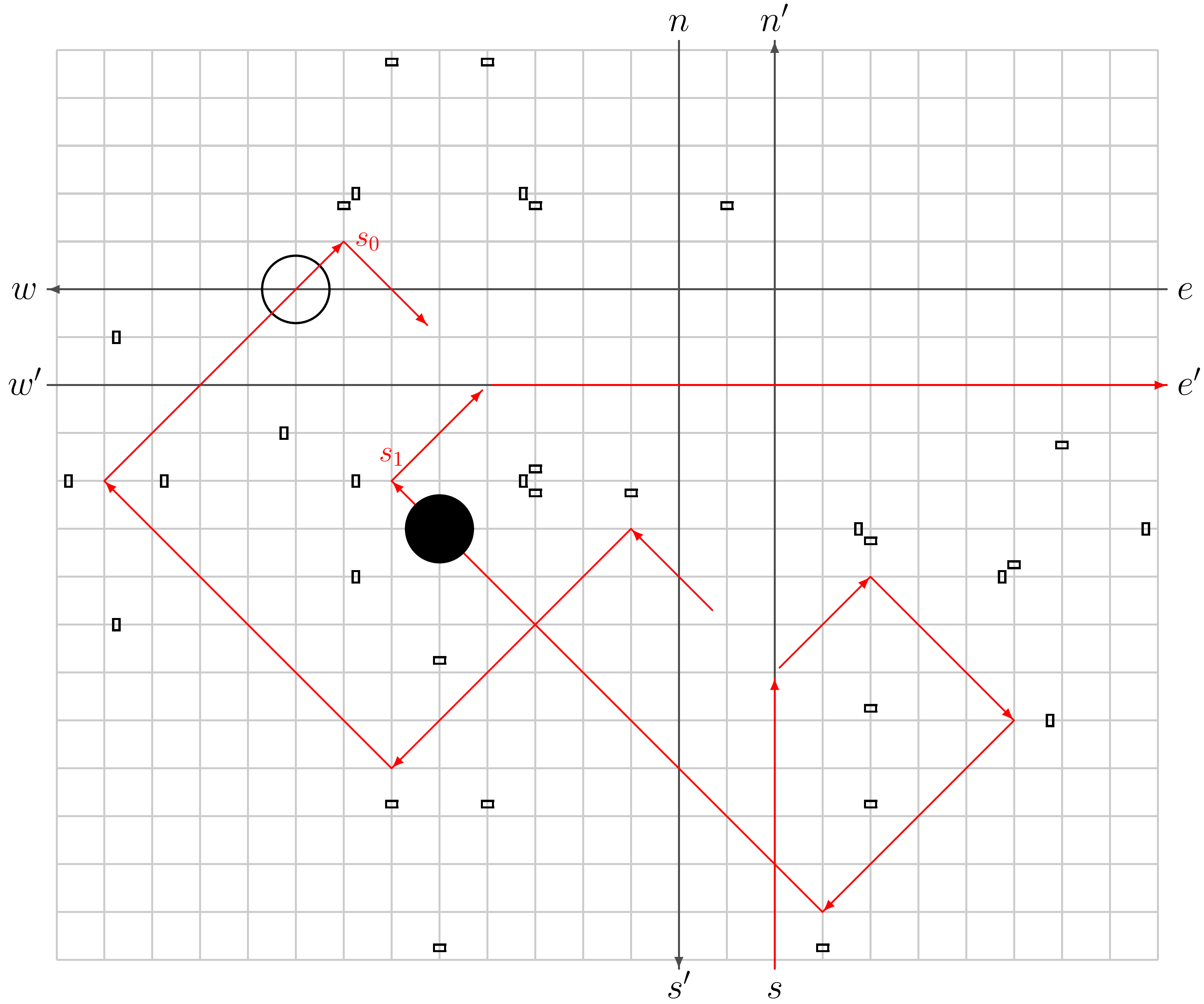
Movements of Balls (State: H, Input: s)



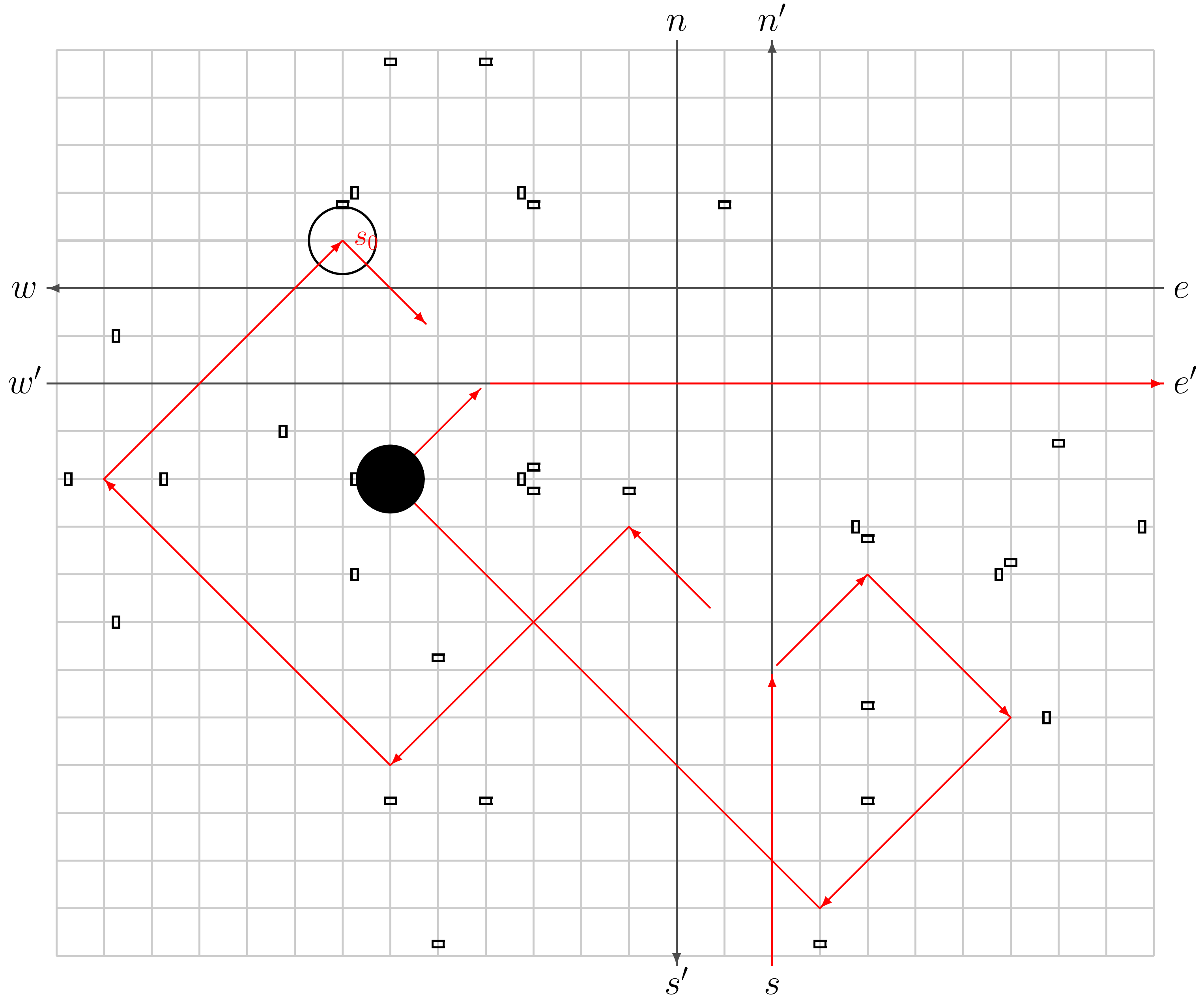
Movements of Balls (State: H, Input: s)



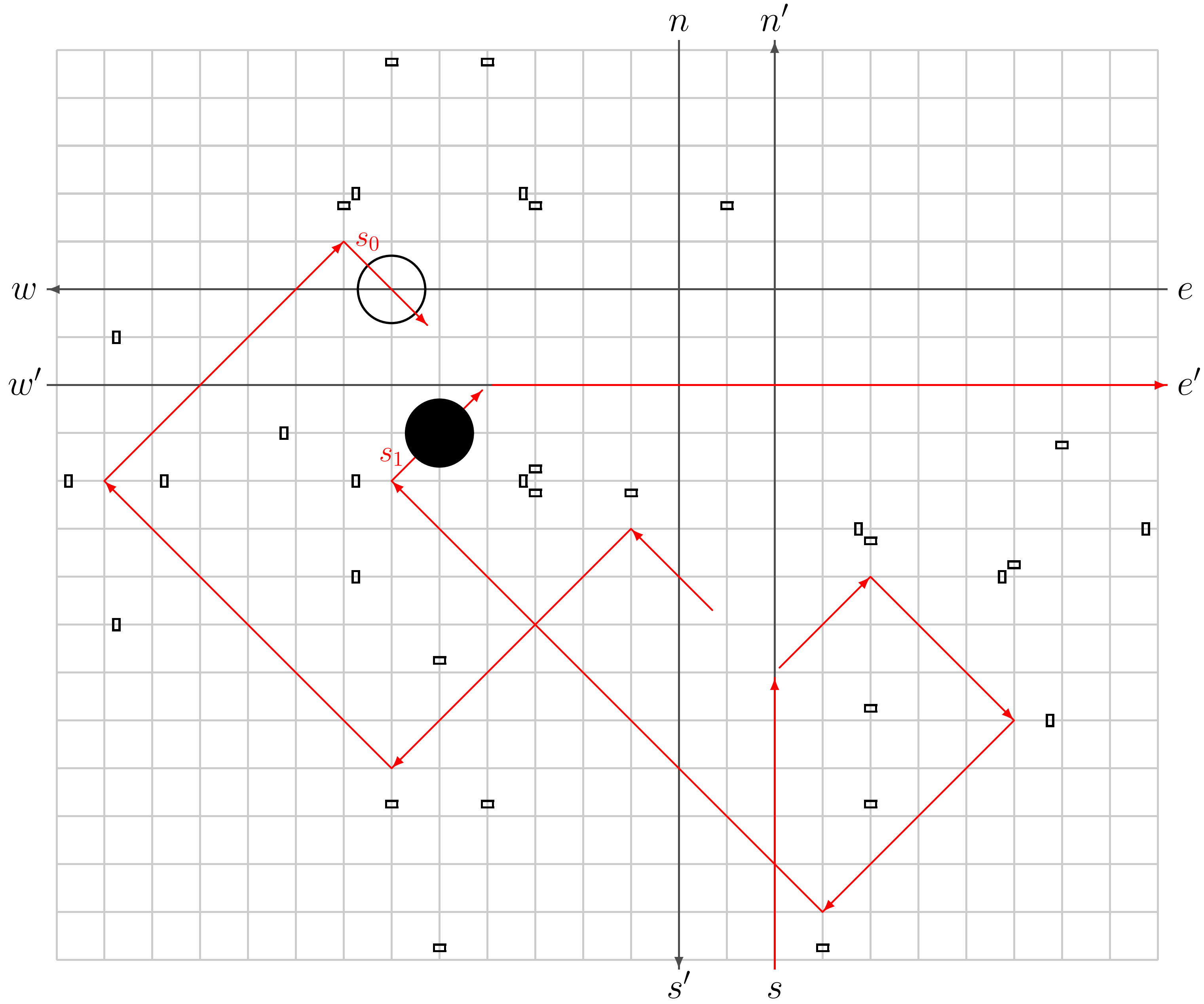
Movements of Balls (State: H, Input: s)



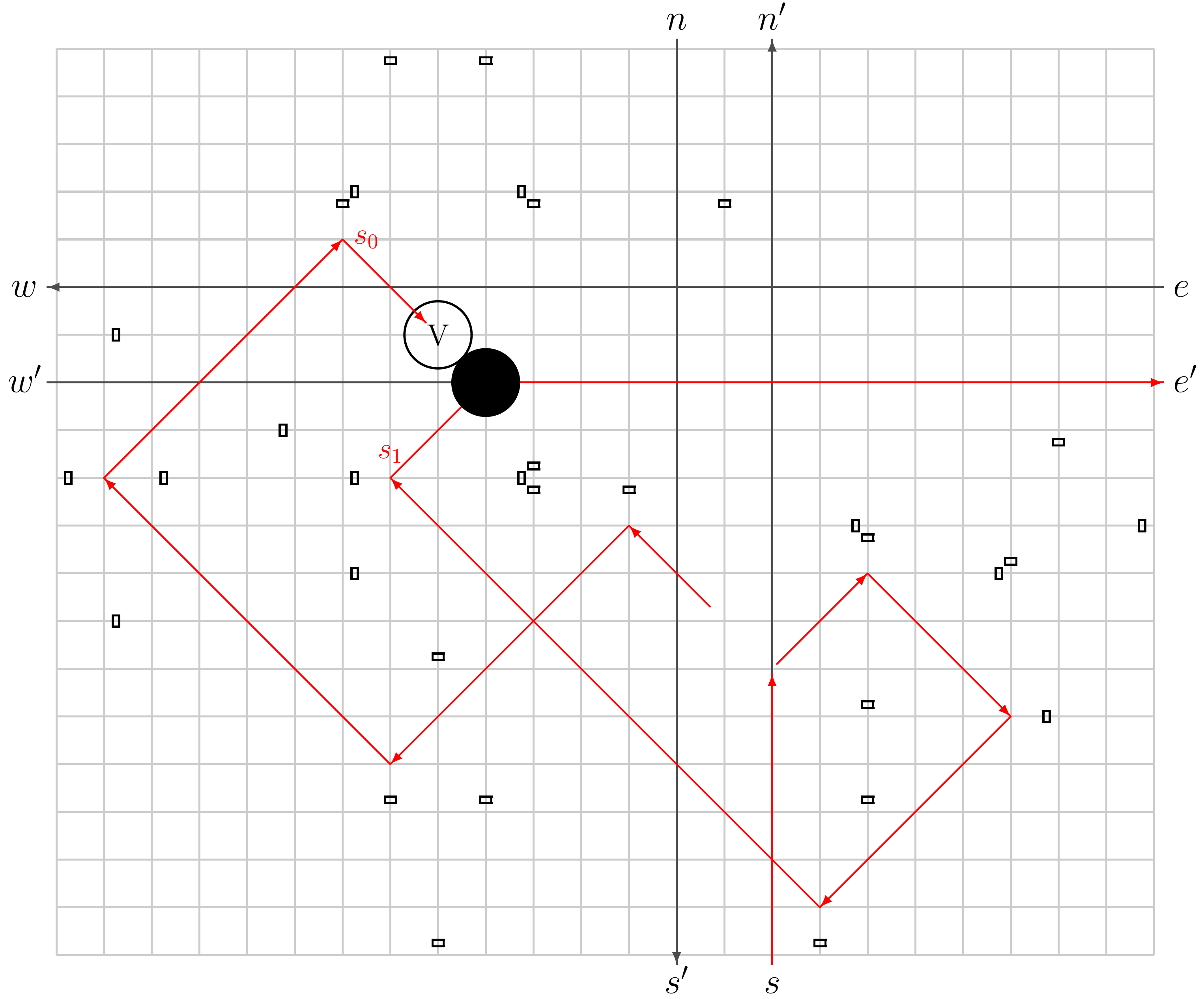
Movements of Balls (State: H, Input: s)



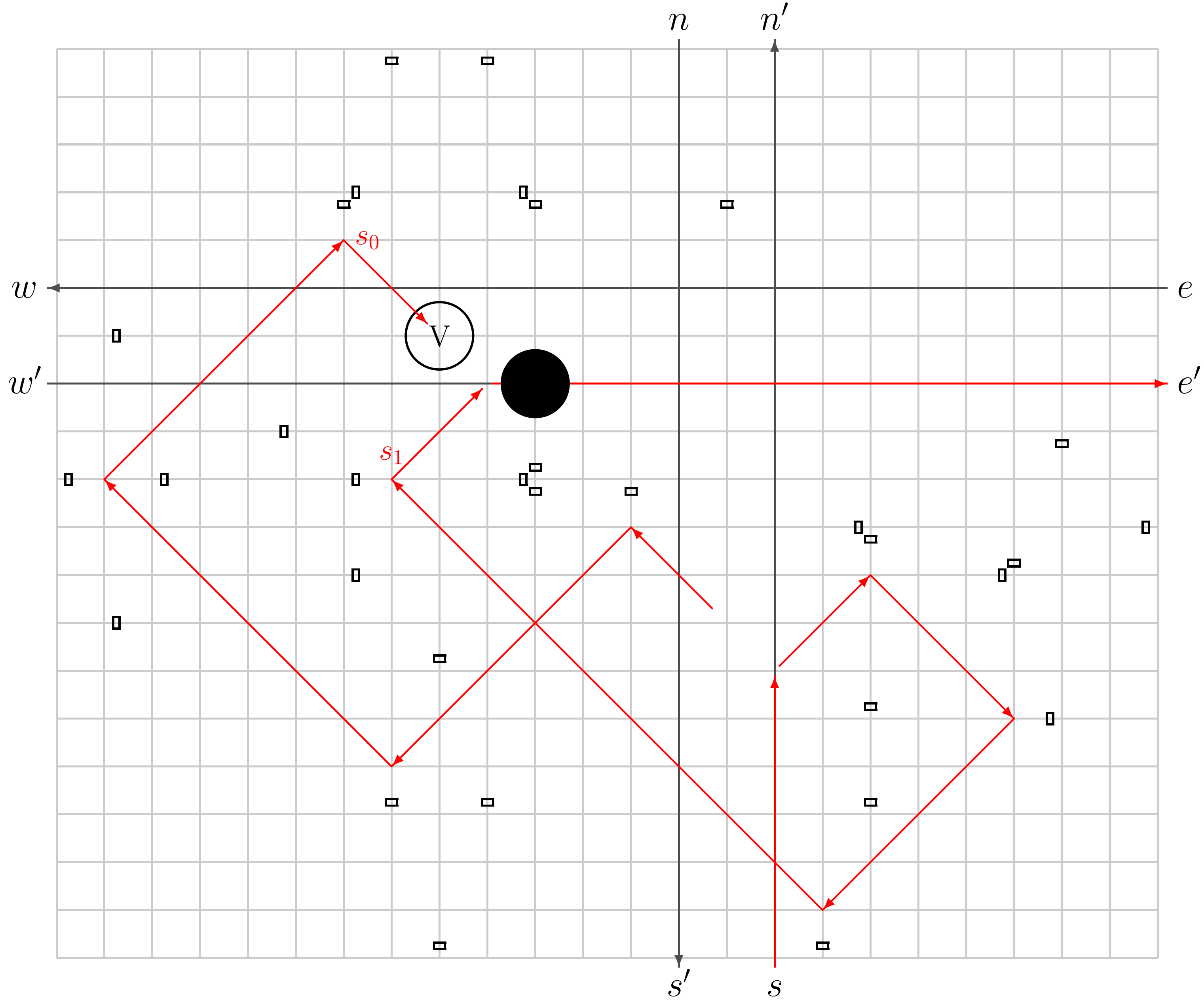
Movements of Balls (State: H, Input: s)



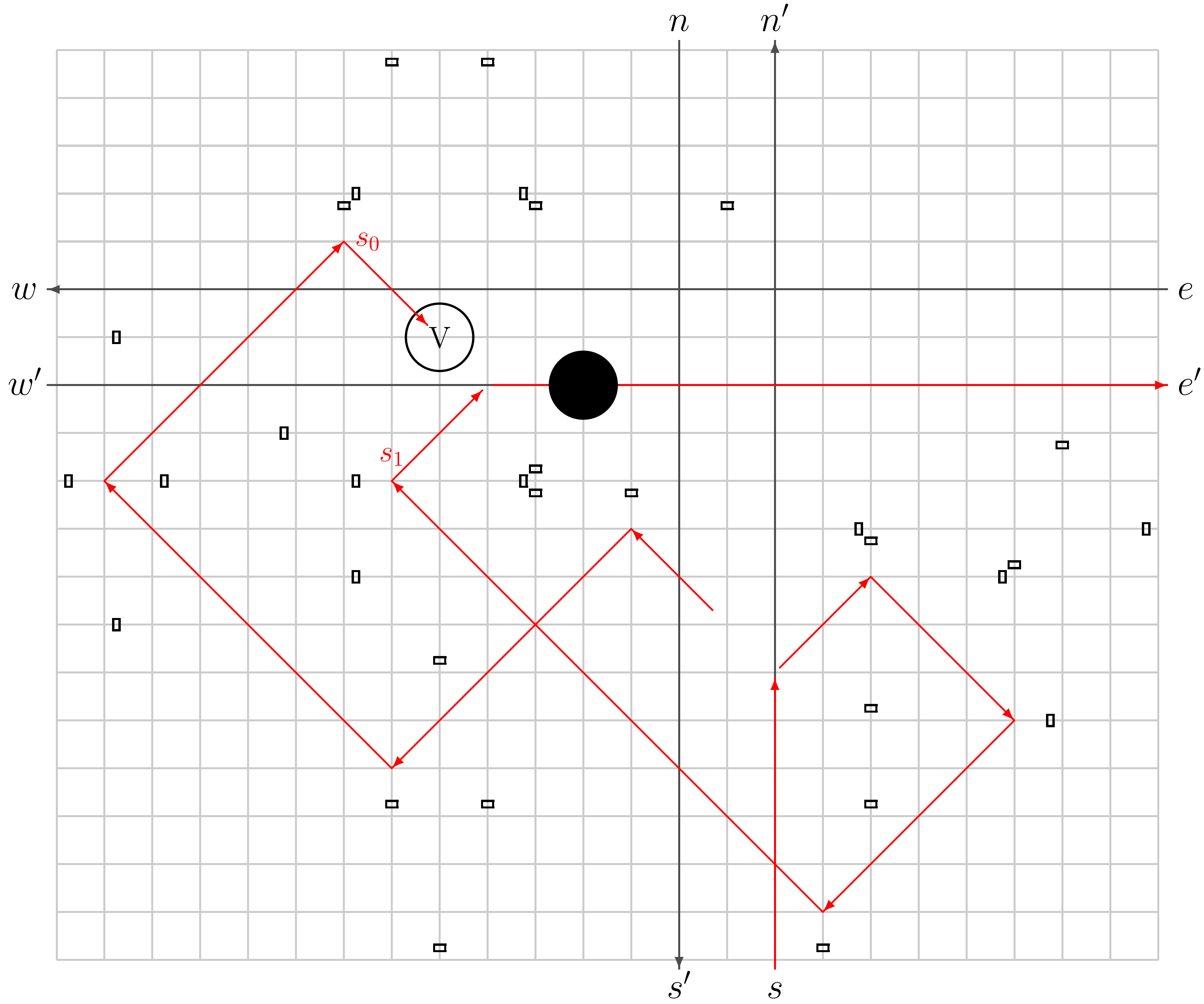
Movements of Balls (State: H, Input: s)



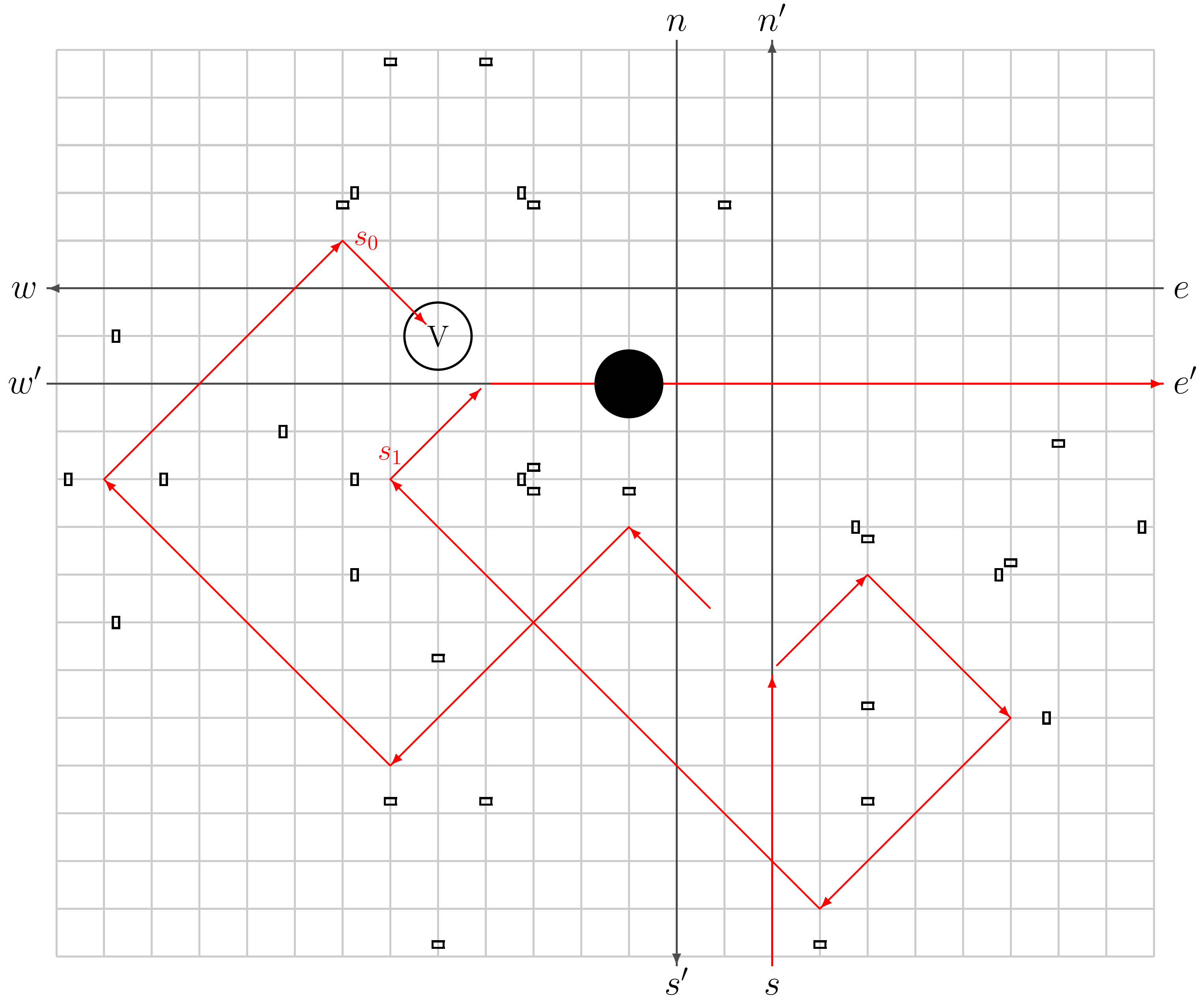
Movements of Balls (State: H, Input: s)



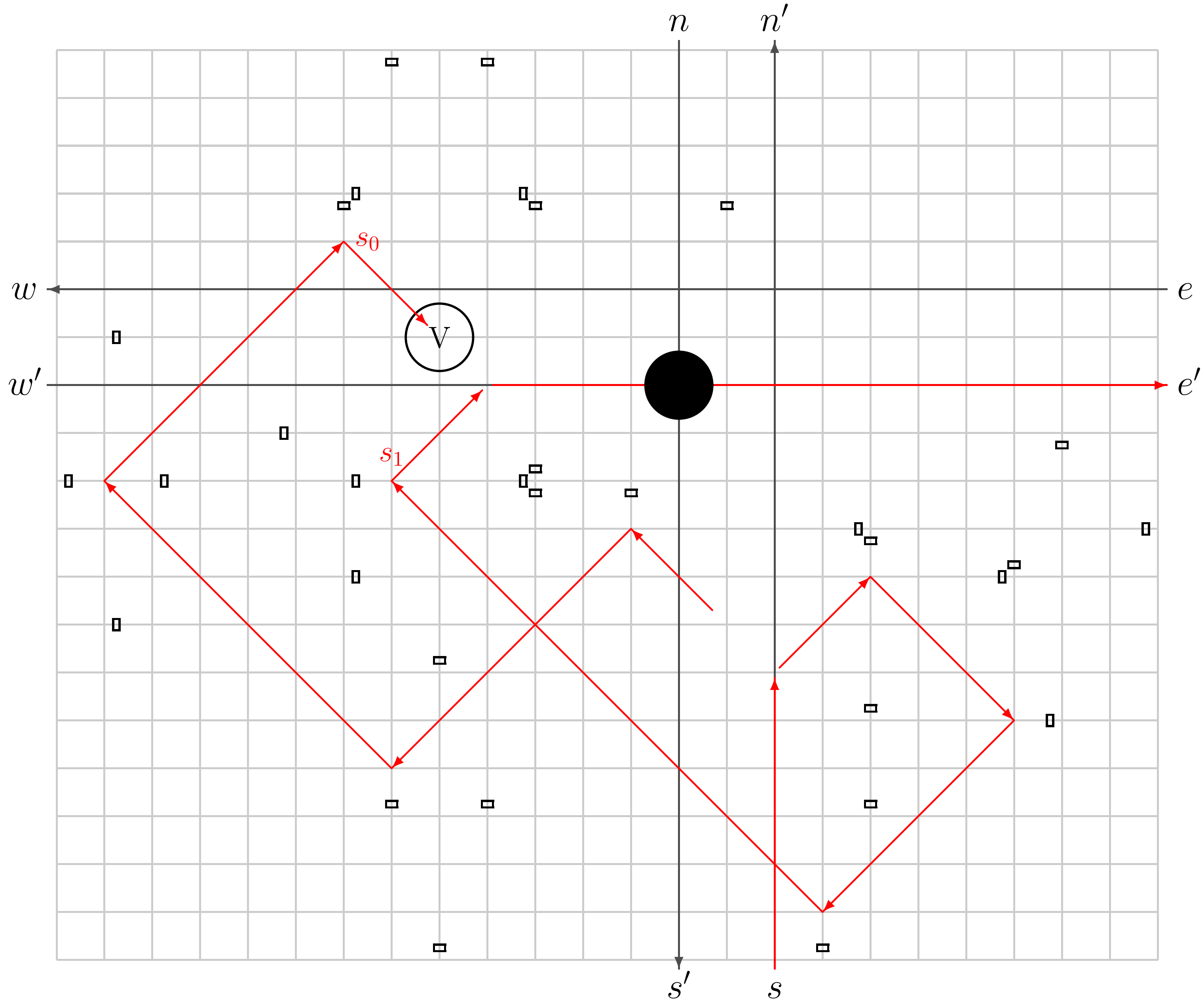
Movements of Balls (State: H, Input: s)



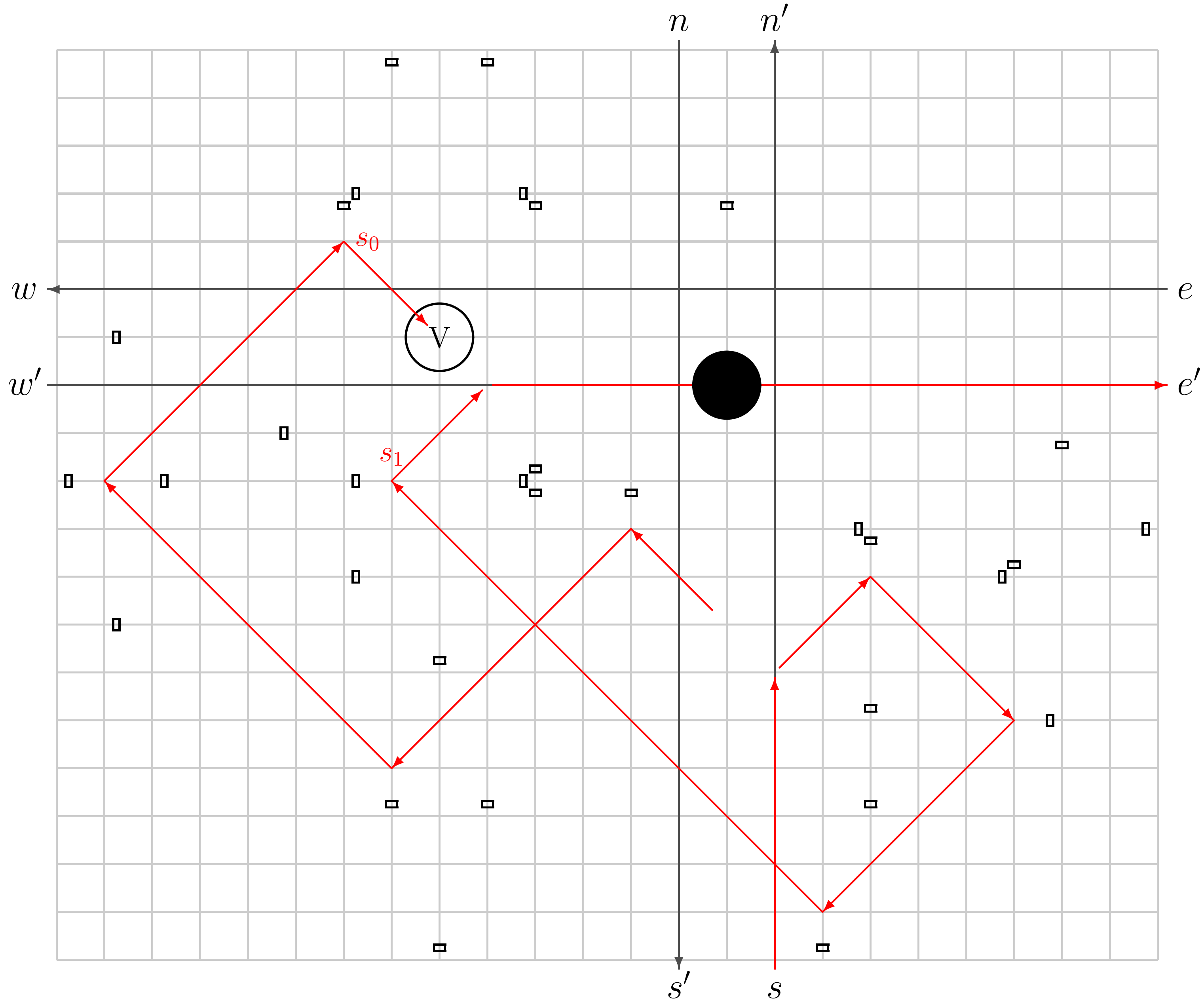
Movements of Balls (State: H, Input: s)



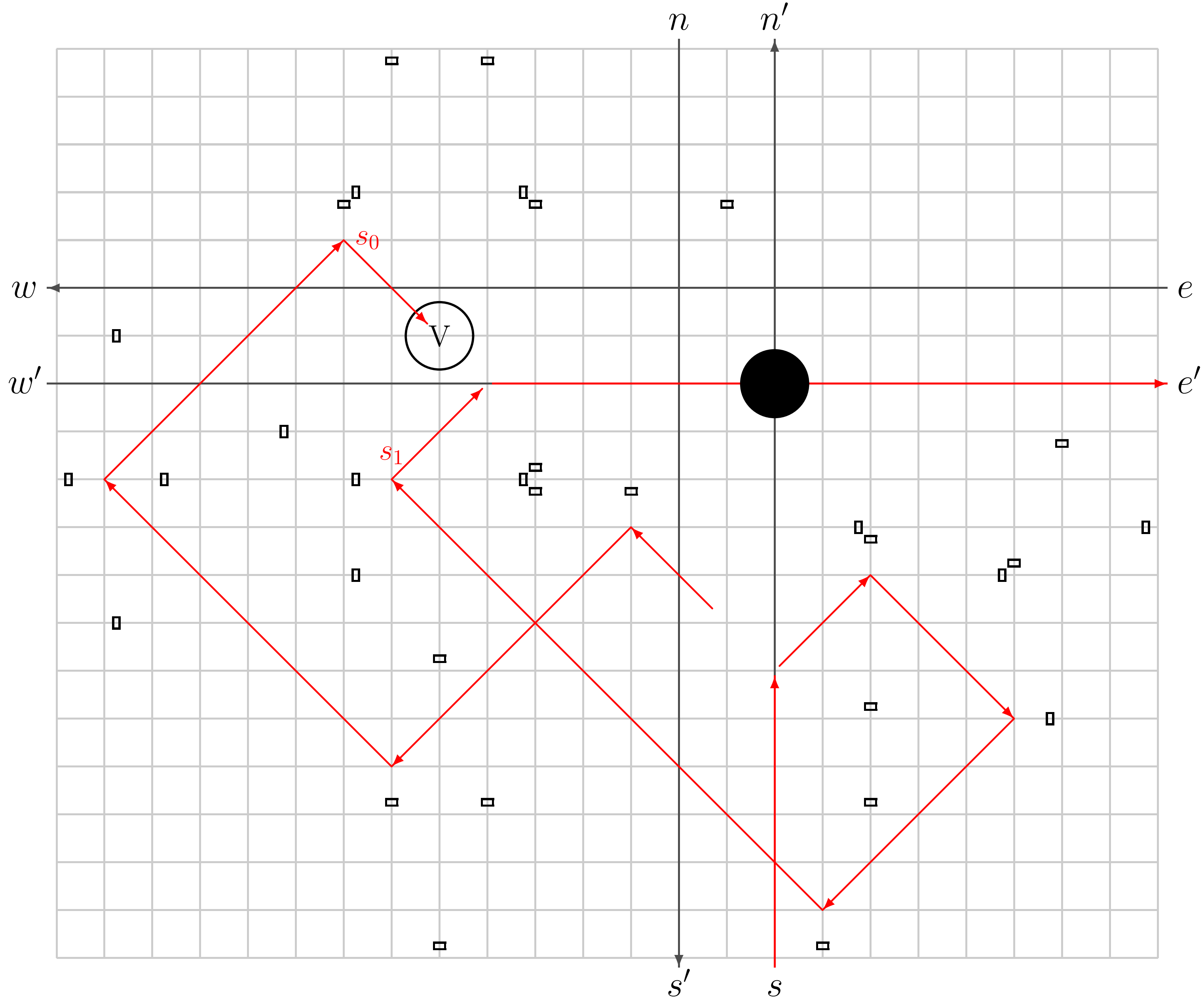
Movements of Balls (State: H, Input: s)



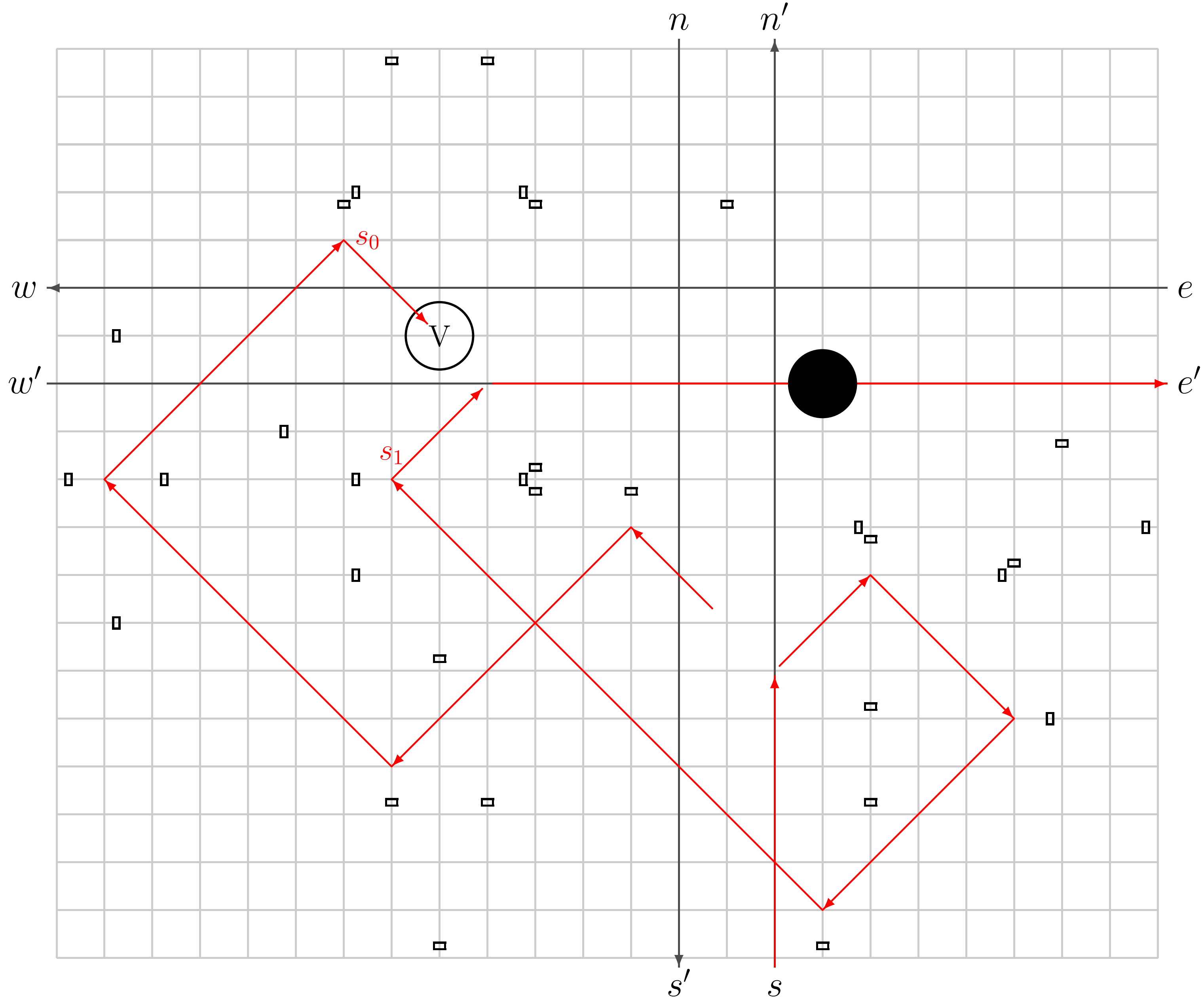
Movements of Balls (State: H, Input: s)



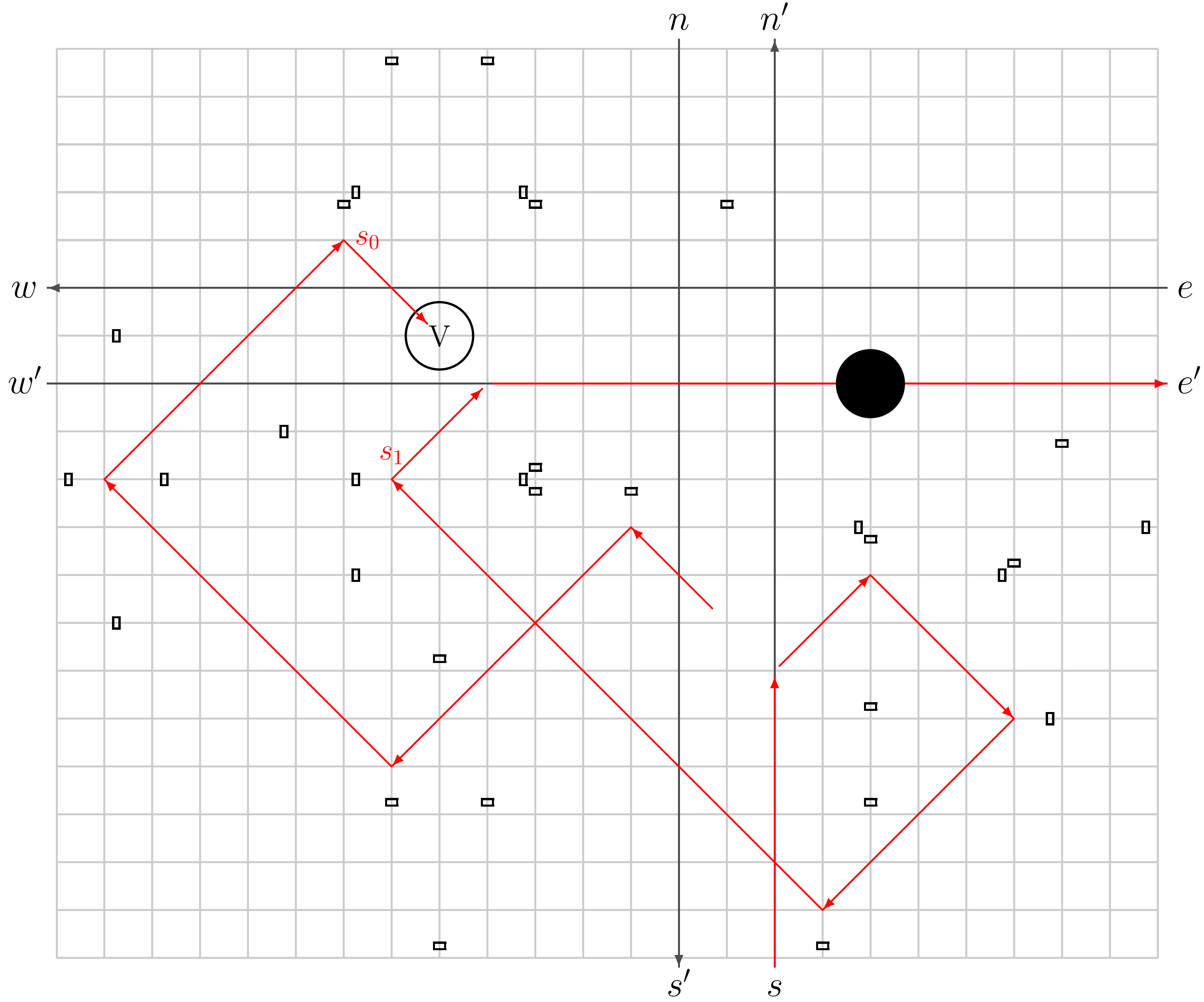
Movements of Balls (State: H, Input: s)



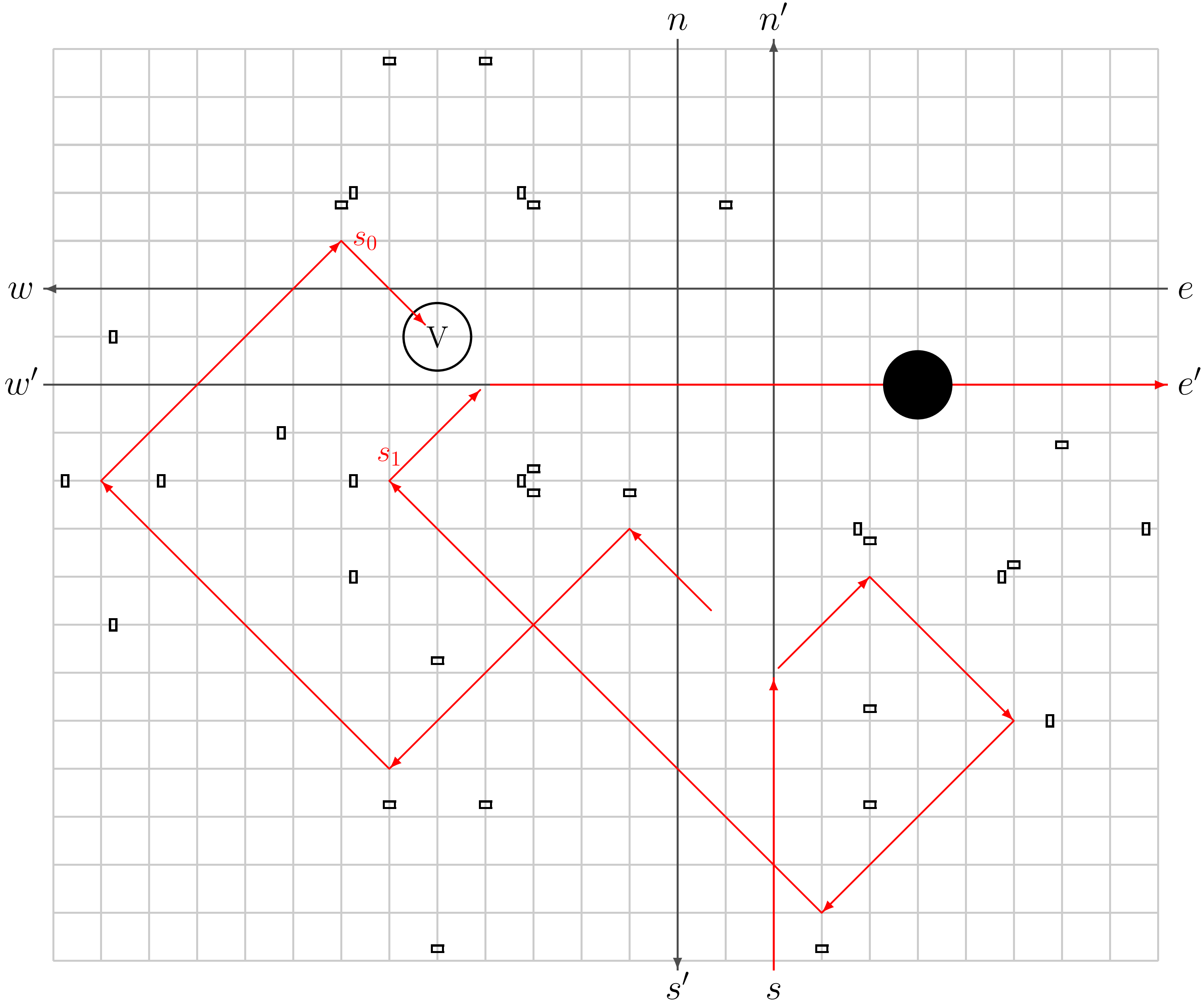
Movements of Balls (State: H, Input: s)



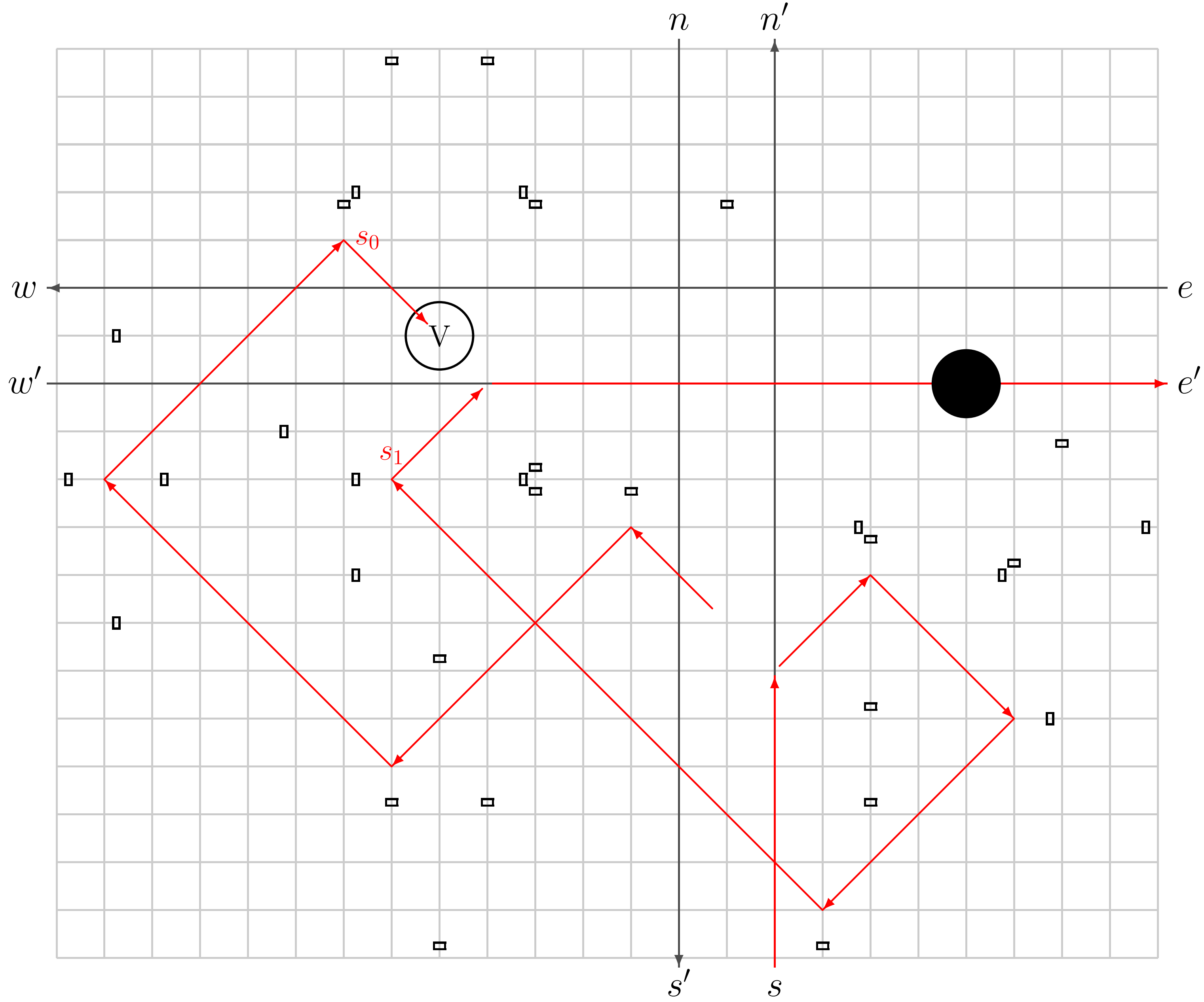
Movements of Balls (State: H, Input: s)



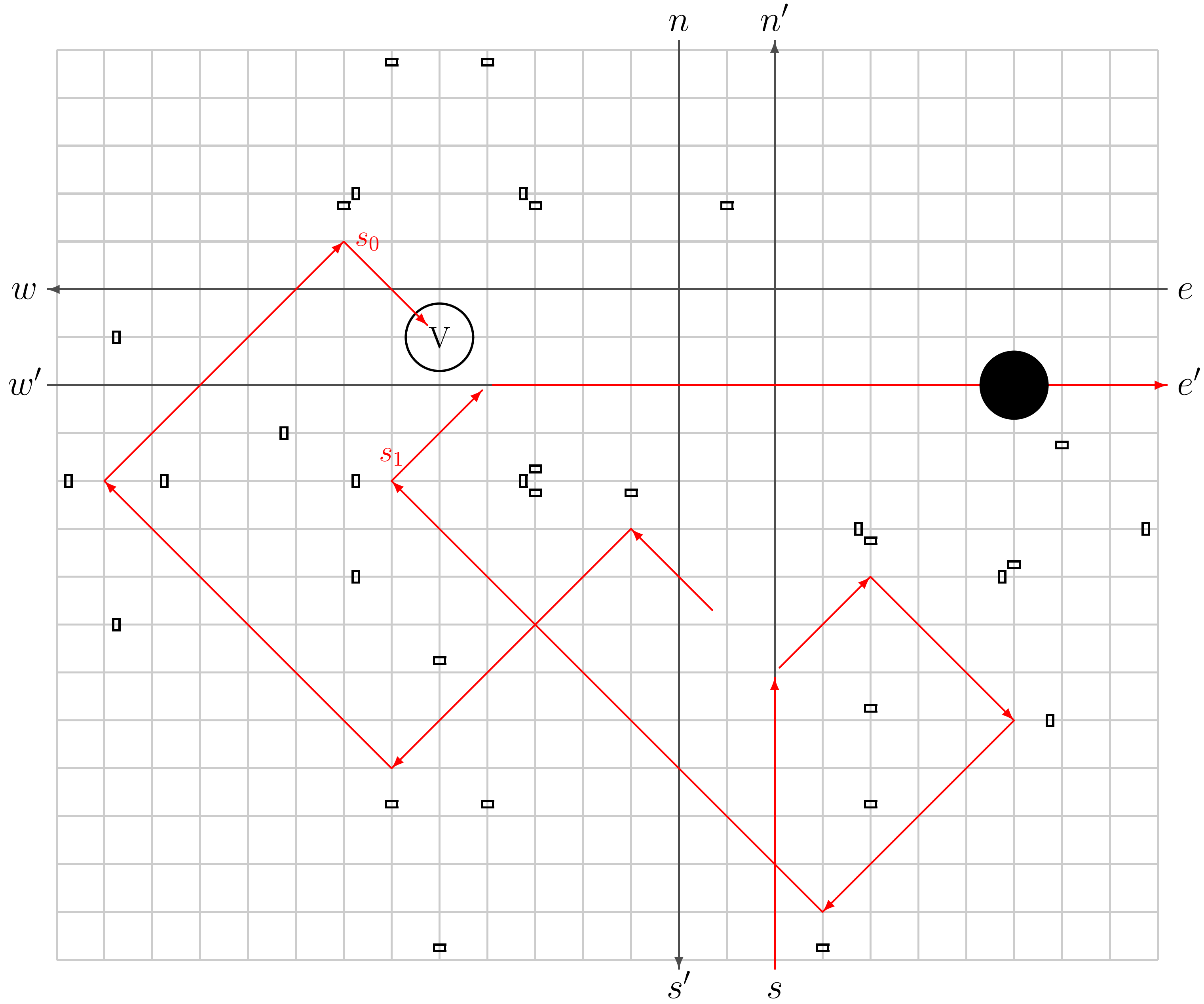
Movements of Balls (State: H, Input: s)



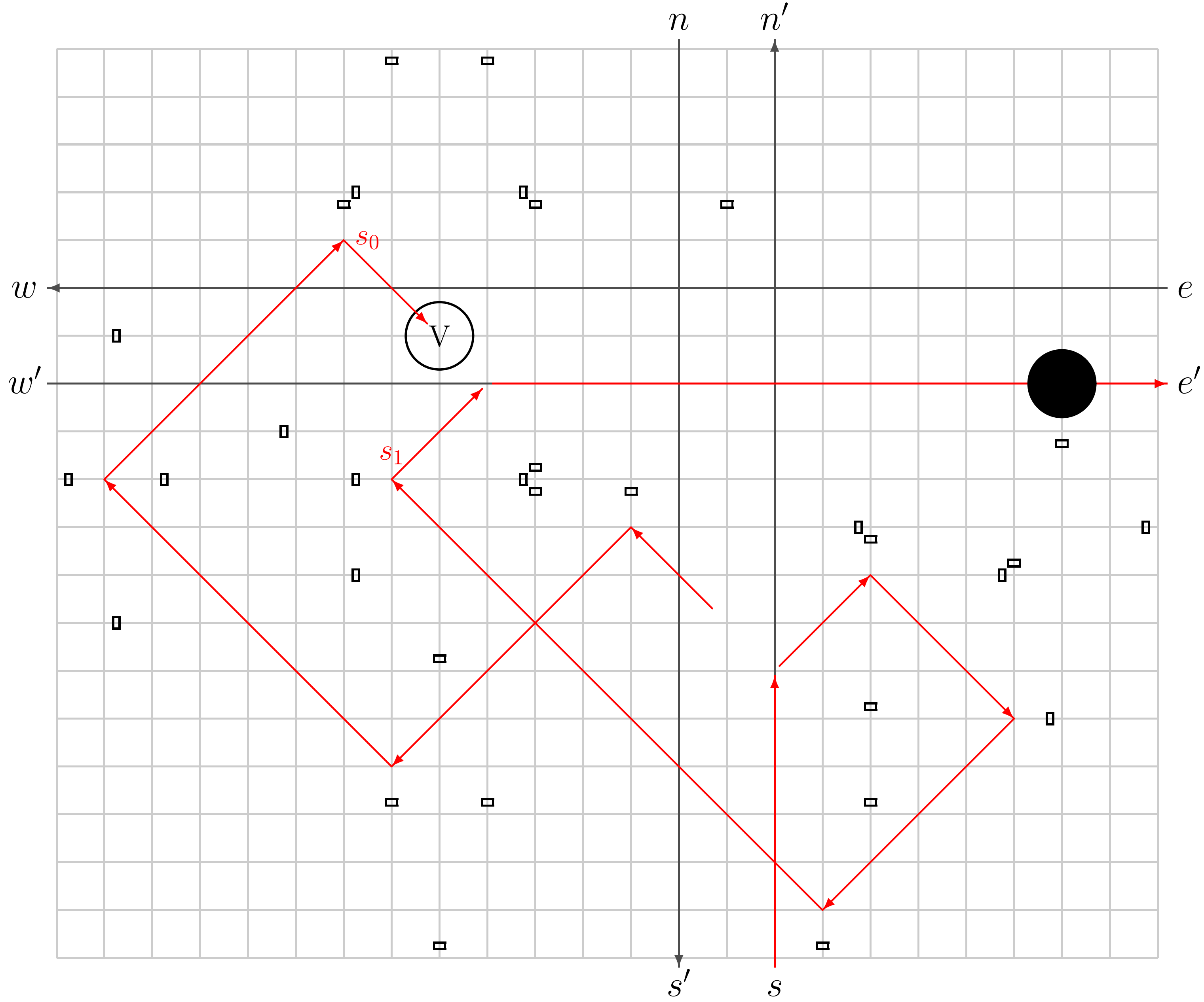
Movements of Balls (State: H, Input: s)



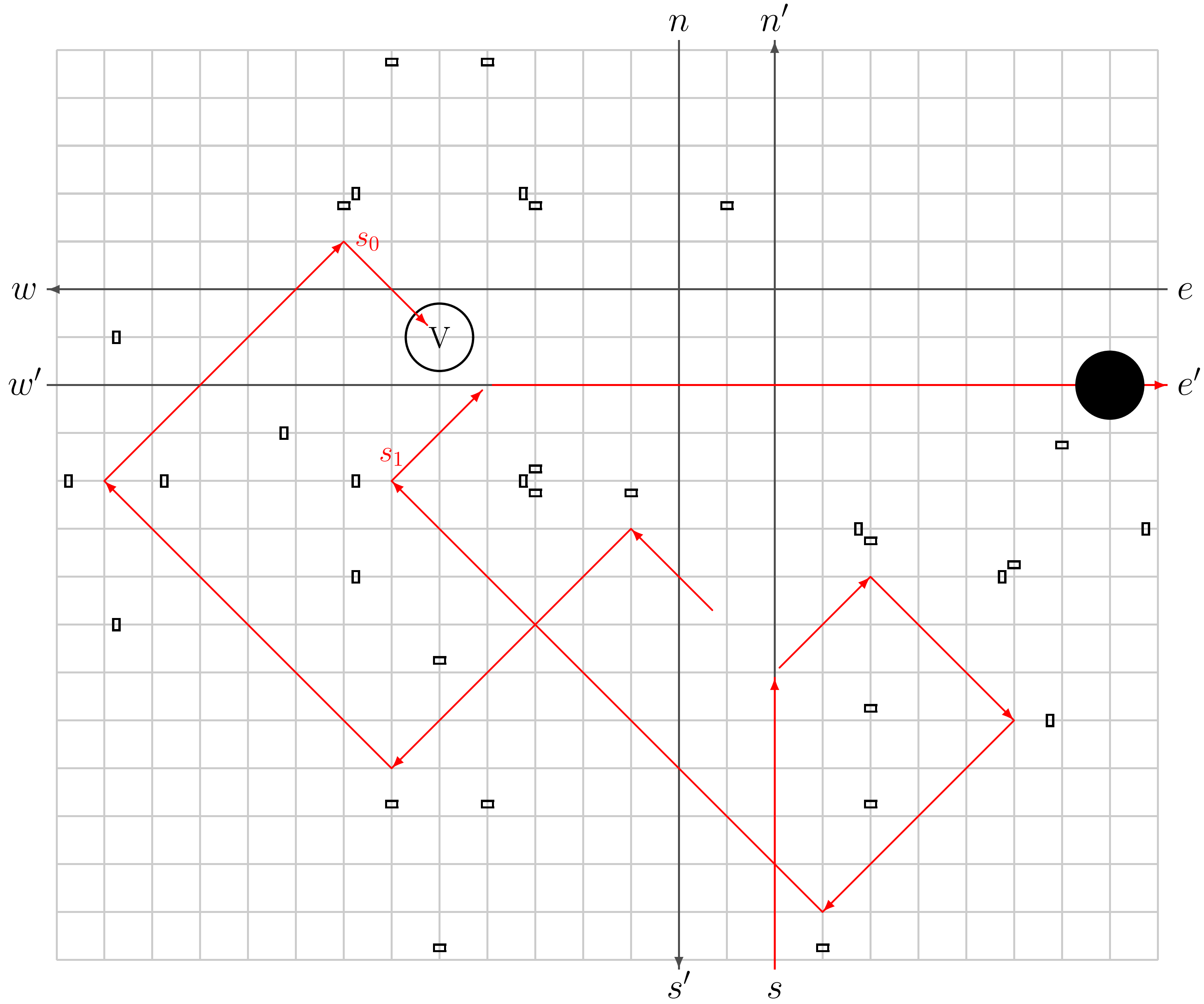
Movements of Balls (State: H, Input: s)



Movements of Balls (State: H, Input: s)



Movements of Balls (State: H, Input: s)



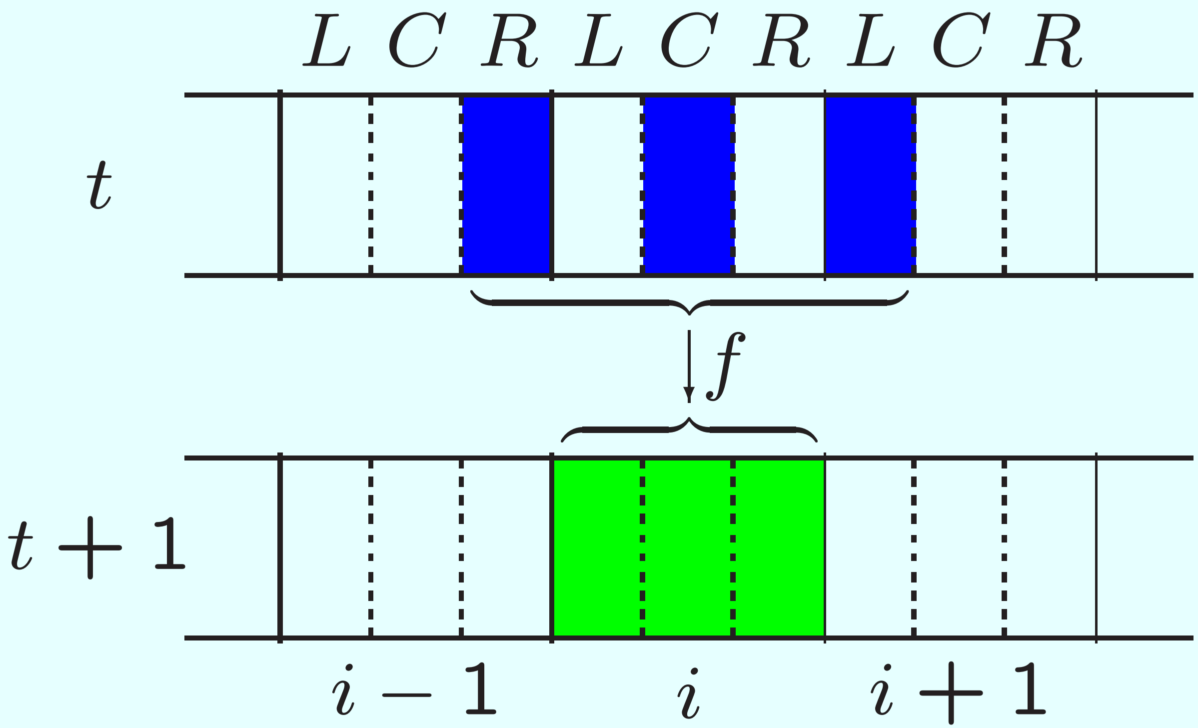
3. Reversible Cellular Automata

Reversible Cellular Automata (RCAs)

- It is a CA whose global function is one-to-one.
- A kind of spatio-temporal model of a physically reversible space.
- In spite of the strong restriction of reversibility, they have rich ability of computing.
  - Computation-universality
  - Self-reproduction
  - Synchronization
  - etc.

Partitioned Cellular Automata

- 1D Partitioned CA (PCA)



A local function f of a 1D PCA.

- We can design RCAs easily using PCAs.

Universal Reversible CAs
— 1D Case —

- On infinite configurations:
  24-state RPCA [Morita, 2008]
- On finite configurations:
  98-state RPCA [Morita, 2007]

cf. 1D Universal Irreversible CAs:

- On infinite configurations:
  2-state CA (ECA of rule 110) [Cook, 2004]
- On finite configurations:
  7-state CA (a modified model) [Lindgren et al., 1990]

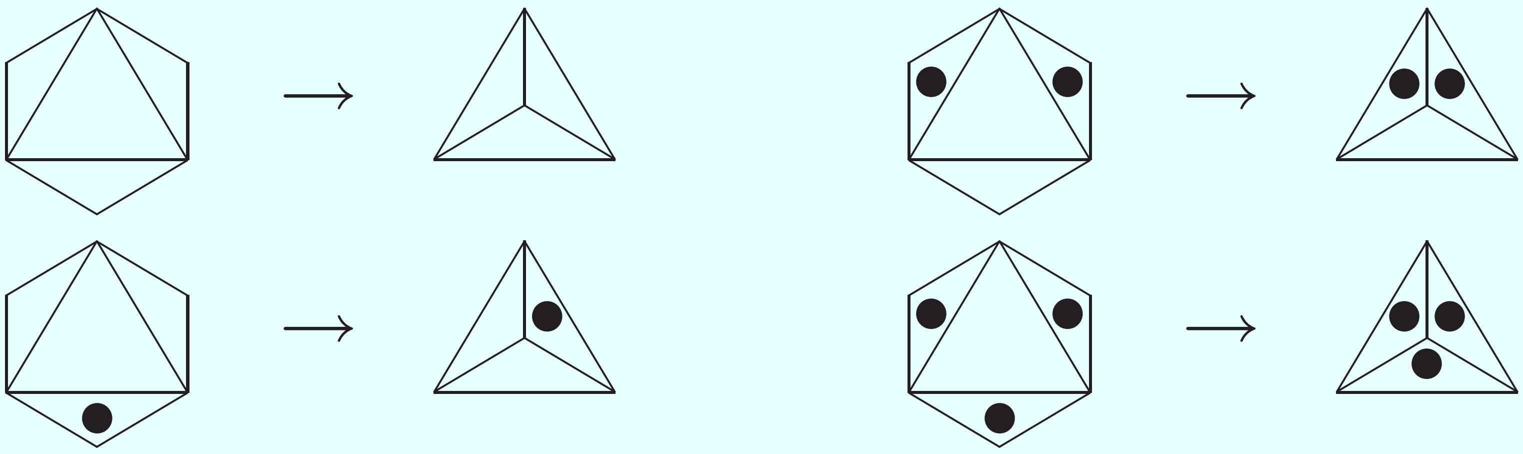
Universal Reversible CAs
— 2D Case —

- On infinite configurations:
  - 2-state Margolus-neighbor RCA [Margolus, 1984]
  - 16-state RPCAs [Morita and Ueno, 1992]
  - 8-state triangular RPCA [Imai and Morita, 1998]

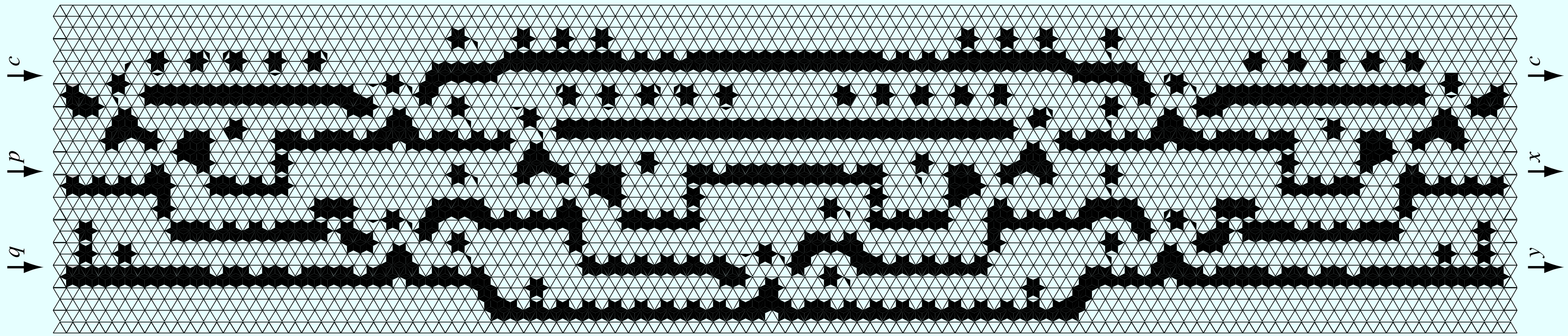
An 8-State Triangular RPCA T_1

[Imai and Morita, 1998]

- It has an extremely simple local function:



A Fredkin Gate in a Triangular 8-State RPCA T_1

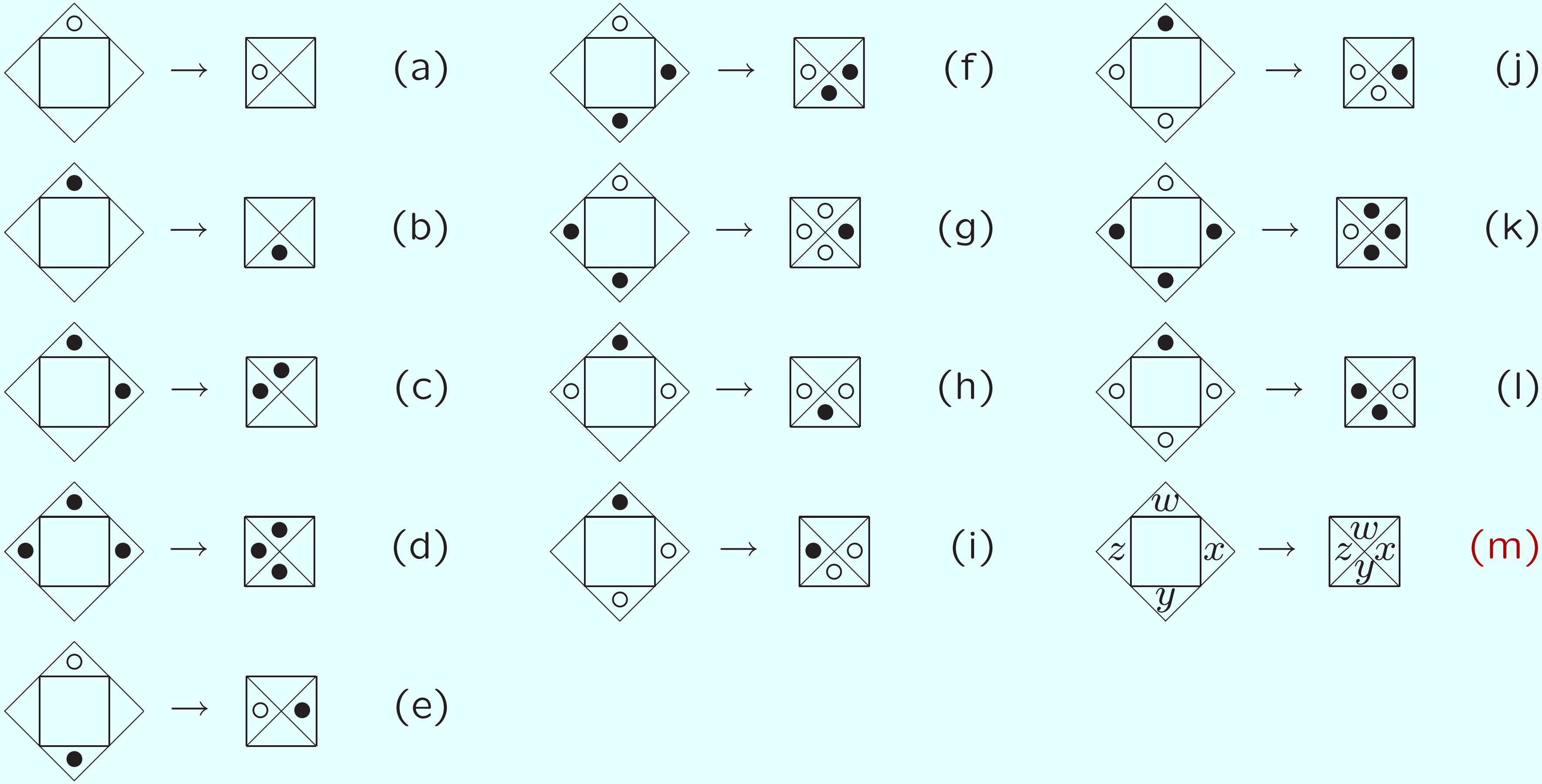


Universal Reversible CAs
— 2D Case —

- On infinite configurations:
  - 2-state Margolus-neighbor RCA [Margolus, 1984]
  - 16-state RPCAs [Morita and Ueno, 1992]
  - 8-state triangular RPCA [Imai and Morita, 1998]
- On finite configurations:
  - 81-state RPCA [Morita and Ogiro, 2001]

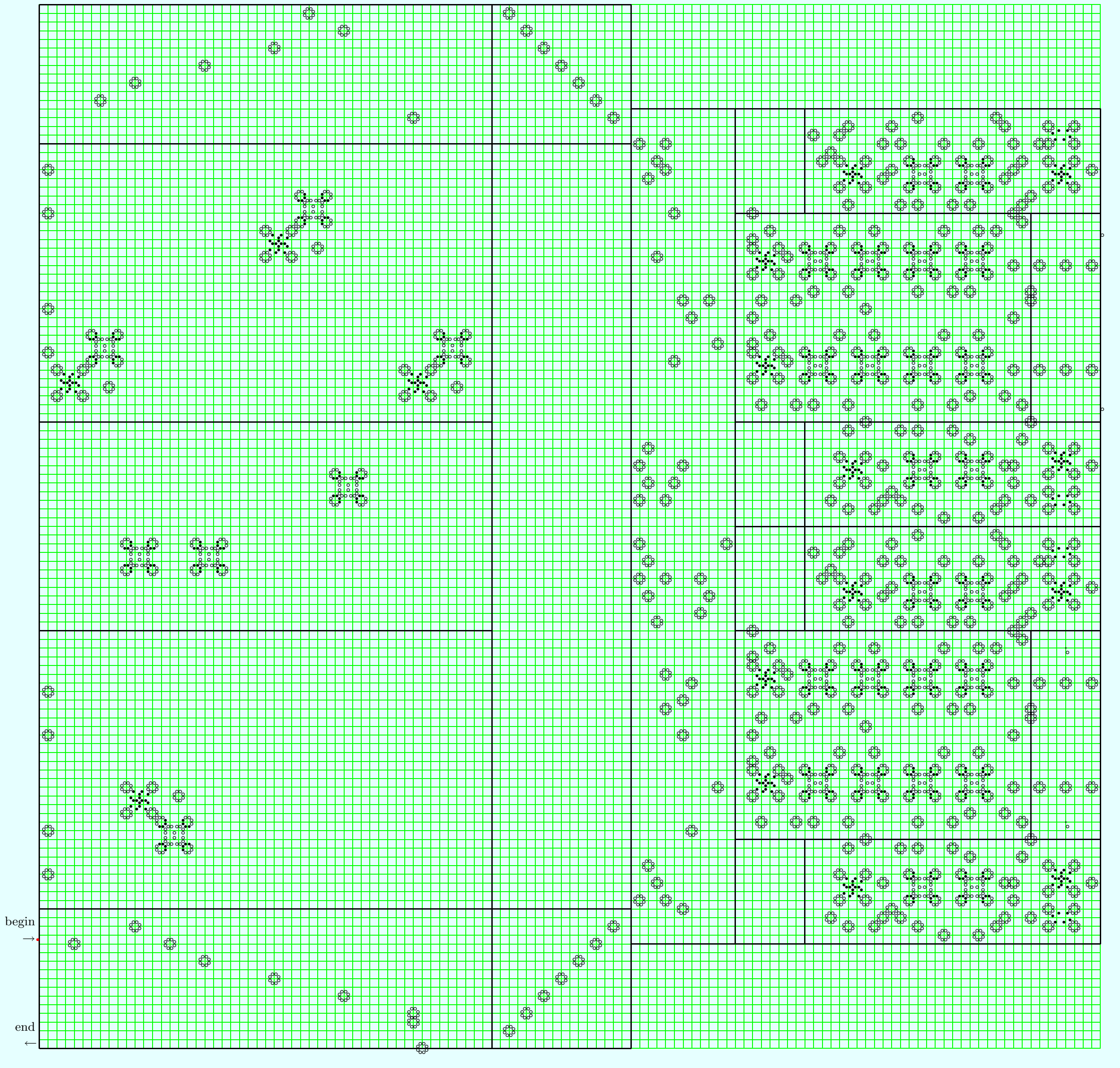
A 3^4-State Universal RPCA P_3

$$P_3 = (\mathrm{Z}^2, \{0, 1, 2\}^4, g_3, (0, 0, 0, 0))$$



The rule scheme (m) represents 33 rules not specified by (a)–(l)
$(w, x, y, z \in \{$ blank, ○, ● $\} = \{0, 1, 2\})$.

Reversible Counter Machine in P_3 Space



Movie of an RCM(2) in P_3

Self-Reproduction of a Worm in 2D RCA

[Morita and Imai, 1996]

Self-Reproduction of a Loop in 3D RCA

[Imai, Hori and Morita, 2002]

Concluding Remarks

- We saw even very simple reversible systems have computation-universality.
- Computation can be carried out in a very different way from that of conventional computers.
- We expect that further studies on them will give new insights for future computing.

Thank you for your attention!