Priestley duality for bilattices

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3 March, 2011

Outline

1. (Just) bilattices
2. Language expansions
3. Dualities

Bilattices: definitions

Pre-bilattices

A **pre-bilattice** is a structure $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ such that $\langle B, \leqslant_t, \wedge, \vee \rangle$ and $\langle B, \leqslant_k, \otimes, \oplus \rangle$ are both lattices.

Bilattices

A **bilattice** is a structure $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ such that $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ is a pre-bilattice and $\neg : B \to B$ is a function such that for all $a, b \in B$:

(i) if $a \leqslant_t b$ then $\neg b \leqslant_t \neg a$ (anti-monotone)

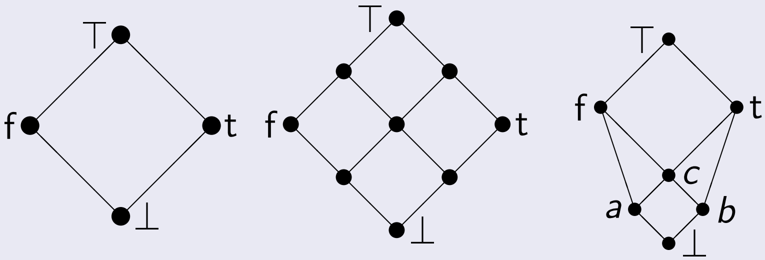
(ii) if $a \leqslant_k b$ then $\neg a \leqslant_k \neg b$ (monotone)

(iii) $a = \neg\neg a$ (involutive)

The smallest non-trivial bilattice is $\mathcal{FOUR}$.

Bilattices: definitions

Some (pre-)bilattices



$\mathcal{FOUR}$ $\mathcal{NINE}$ $\mathcal{SEVEN}$

Bilattices: interpretation

An interpretation of the two lattice orders

The truth–order $\leqslant_t$ is meant to reflect the degree of **truth** associated with a given sentence, while the knowledge–order $\leqslant_k$ should reflect the degree of **knowledge** (or, better: of **information**) associated with the sentence.

In case the elements of the bilattice are ordered pairs $\langle a, b \rangle$ where a, b are elements of two lattices (this holds for all interlaced (pre-)bilattices), then a may be thought as intuitively representing the evidence for (or degree of belief in) the truth of some proposition and b as representing the evidence against (or degree of doubt in) the truth of the same proposition.

Bilattices: subclasses

Main sublcasses of bilattices

- A pre-bilattice is **interlaced** when all the operations $\{\wedge, \vee, \otimes, \oplus\}$ are monotonic w.r.t. both lattice orders.
- A pre-bilattice is **distributive** when all twelve possible distributive laws concerning $\{\wedge, \vee, \otimes, \oplus\}$ hold.
- Interlaced pre-bilattices form a variety, and distributive pre-bilattices are a proper subvariety of the interlaced.
- A bilattice is interlaced (distributive) when its pre-bilattice reduct is interlaced (distributive). Both classes are varieties, and distributive bilattices are a proper subvariety of the interlaced.

Bilattices: bounded

Observation

In any bounded interlaced pre-bilattice $\langle B, \wedge, \vee, \otimes, \oplus, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ we can define $\{\otimes, \oplus\}$ as follows: for any $a, b \in B$,

$$a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b)$$

$$a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b).$$

Bilattices: bounded

Observation

Using the above fact, it is possible to prove that following varieties are termwise equivalent:

1. bounded interlaced (distributive) pre-bilattices presented in the language $\{\wedge, \vee, \otimes, \oplus, \mathrm{f}, \mathrm{t}, \bot, \top\}$
2. algebras $\langle B, \wedge, \vee, \mathrm{f}, \mathrm{t}, \bot, \top\rangle$ such that $\langle B, \wedge, \vee, \mathrm{f}, \mathrm{t}\rangle$ is a bounded (distributive) lattice and the constants $\bot$ and $\top$ satisfy:
   1. $\top \vee \bot = \mathrm{t}$
   2. $\top \wedge \bot = \mathrm{f}$
   3. if $\top \in \{a, b, c\}$ or $\bot \in \{a, b, c\}$, then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Bilattices: bounded

Observation

In particular, a bounded distributive (pre-)bilattice

$$\langle B, \wedge, \vee, \otimes, \oplus, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$$

can be seen as a De Morgan algebra

$$\langle B, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t} \rangle$$

having two extra constants $\bot, \top$ satisfying that

i. $\top \wedge \bot = \mathsf{f}$ $\qquad$ $\top \vee \bot = \mathsf{t}$

ii. $\top = \neg \top$ $\qquad$ $\bot = \neg \bot$.

Bilattices: products

Product pre-bilattice

Let $\mathbf{L_1} = \langle L_1, \sqcap_1, \sqcup_1 \rangle$ and $\mathbf{L_2} = \langle L_2, \sqcap_2, \sqcup_2 \rangle$ be lattices. The **product pre-bilattice** $\mathbf{L_1} \odot \mathbf{L_2} = \langle L_1 \times L_2, \wedge, \vee, \otimes, \oplus \rangle$ is defined as follows: for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L_1 \times L_2$,

$$\begin{aligned} \langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &= \langle a_1 \sqcap_1 b_1,\ a_2 \sqcup_2 b_2 \rangle \\ \langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &= \langle a_1 \sqcup_1 b_1,\ a_2 \sqcap_2 b_2 \rangle \\ \langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle &= \langle a_1 \sqcap_1 b_1,\ a_2 \sqcap_2 b_2 \rangle \\ \langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle &= \langle a_1 \sqcup_1 b_1,\ a_2 \sqcup_2 b_2 \rangle . \end{aligned}$$

Remark: Any product pre-bilattice $\mathbf{L_1} \odot \mathbf{L_2}$ is interlaced, and it is distributive if and only if both $\mathbf{L_1}$ and $\mathbf{L_2}$ are distributive.

Bilattices: products

Product bilattice

If $h : L_1 \cong L_2$ is a lattice isomorphism, we define, for any $\langle a_1, a_2 \rangle \in L_1 \times L_2$: $\neg \langle a_1, a_2 \rangle = \langle h^{-1}(a_2), h(a_1) \rangle$.

In particular, for any lattice $\mathbf{L} = \langle L, \sqcap, \sqcup \rangle$, the **product bilattice** $\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is defined as follows: for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$,

$$\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle = \langle a_1 \sqcap b_1,\ a_2 \sqcup b_2 \rangle$$
$$\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle = \langle a_1 \sqcup b_1,\ a_2 \sqcap b_2 \rangle$$
$$\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle = \langle a_1 \sqcap b_1,\ a_2 \sqcap b_2 \rangle$$
$$\langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle = \langle a_1 \sqcup b_1,\ a_2 \sqcup b_2 \rangle$$
$$\neg \langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle .$$

Bilattices: representation

Representation theorem

Let $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ be an interlaced (pre-)bilattice. Then:

- there are two lattices $\mathbf{L_1}$ and $\mathbf{L_2}$ such that $\mathbf{B} \cong \mathbf{L_1} \odot \mathbf{L_2}$
- $\mathbf{B}$ is distributive iff both $\mathbf{L_1}$ and $\mathbf{L_2}$ are distributive
- $\mathrm{Con}(\mathbf{B}) = \mathrm{Con}(\langle B, \wedge, \vee \rangle) = \mathrm{Con}(\langle B, \otimes, \oplus \rangle)$
- the lattice $\mathrm{Con}(\mathbf{B})$ is isomorphic to $\mathrm{Con}(\mathbf{L_1}) \times \mathrm{Con}(\mathbf{L_2})$
- if $\mathbf{B}$ is a bilattice, then $\mathbf{L_1} \cong \mathbf{L_2}$ and $\mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{L_1})$.

Bilattices: representation

Representation theorem

One way to prove the representation theorem is to use (one of) the bounds to make projections as follows:

$$a \longmapsto \langle a \vee \top, a \wedge \top \rangle$$
$$\mathbf{L_1} = \langle \{a \in B : a \geqslant_t \top\}, \otimes, \oplus \rangle$$
$$\mathbf{L_2} = \langle \{a \in B : a \leqslant_t \top\}, \otimes, \oplus \rangle$$

Brouwerian bilattices: definition

Definition (Bou-Jansana-R.)

A **Brouwerian bilattice** is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ such that $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and $\mathbf{B}$ satisfies:

(B1) $(x \supset x) \supset y \approx y$

(B2) $x \supset (y \supset z) \approx (x \wedge y) \supset z \approx (x \otimes y) \supset z$

(B3) $(x \vee y) \supset z \approx (x \supset z) \wedge (y \supset z) \approx (x \oplus y) \supset z$

(B4) $x \wedge ((x \supset y) \supset (x \otimes y)) \approx x$

(B5) $\neg(x \supset y) \supset z \approx (x \wedge \neg y) \supset z.$

An **implicative bilattice** is a Brouwerian bilattice that additionally satisfies: $((x \supset y) \supset x) \supset x \approx x \supset x.$

Brouwerian bilattices: representation

Product Brouwerian bilattice

- Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \rightarrow \rangle$ be Brouwerian lattice (i.e. a Heyting algebra that need not have a bottom element)
- $\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ is the Brouwerian bilattice whose bilattice reduct is defined as before and for all $a_1, a_2, b_1, b_2 \in L$,

$$\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle := \langle a_1 \rightarrow b_1, a_1 \sqcap b_2 \rangle.$$

Representation theorem

For any Brouwerian bilattice $\mathbf{B} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$

- $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$ for some Brouwerian lattice $\mathbf{L}$
- $\mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{L})$.

Brouwerian bilattices: bounded

Some facts

- Any Brouwerian bilattice **B** has a maximum w.r.t. $\leqslant_k$ defined by $\top = (a \supset a) \oplus \neg(a \supset a)$
- the underlying pre-bilattice of a Brouwerian bilattice is distributive.

N4-lattices

An *N4-lattice* is a $\{\wedge, \vee, \supset, \neg, \mathrm{f}, \mathrm{t}\}$-subreduct of a bounded Browerian bilattice.

(Any N4-lattice can be embedded into a bilattice product $\mathbf{H} \odot \mathbf{H}$ where $\mathbf{H}$ is a Heyting algebra).

Brouwerian bilattices: bounded

As before

A bounded Brouwerian bilattice

$$\langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$$

can be seen as an N4-lattice

$$\langle B, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$$

having extra constants $\bot$ and $\top$ such that

- $\langle B, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ is a bounded bilattice
- $\top \supset \mathsf{f} = \mathsf{f}$ and $\bot \supset \bot = \mathsf{t}$.

Towards duality

Resuming

We can then hope to extend known results on Priestley duality for

1. bounded distributive lattices
2. bounded distributive lattices with an involutive negation (aka De Morgan algebras)
3. Heyting algebras

to, respectively,

1. bounded distributive pre-bilattices
2. bounded distributive bilattices
3. bounded Brouwerian bilattices.

Priestley duality (Distributive lattices)

Priestley spaces

A Priestley space is a structure $\mathcal{X} = \langle X, \tau, \leqslant \rangle$ such that

1. $\langle X, \leqslant \rangle$ is a poset
2. $\langle X, \tau \rangle$ is a compact (Hausdorff) space
3. for all $x, y \in X$ with $x \nleqslant y$, there is a clopen up-set U s.t. $x \in U$ and $y \notin U$ ($\mathcal{X}$ is totally order-disconnected).

The distributive lattice of a Priestley space

Denoting by $D(\mathcal{X})$ be the set of clopen up-sets of $\mathcal{X}$, we have that $\langle D(\mathcal{X}), \cap, \cup, \emptyset, X \rangle$ is a bounded distributive lattice.

Priestley duality (Distributive lattices)

The Priestley space of a distributive lattice

Let $\mathbf{D}$ be a bounded distributive lattice and $X(\mathbf{D})$ the set of prime filters of $\mathbf{D}$.

- Define, for all $a \in D$, $\Phi(a) := \{P \in X(\mathbf{D}) : a \in P\}$
- the topology τ given by the sub-base

$$\{\Phi(a) : a \in D\} \cup \{X \backslash \Phi(a) : a \in D\}.$$

- Then $\langle X(\mathbf{D}), \tau, \subseteq \rangle$ is a Priestley space
- $\Phi \colon \mathbf{D} \cong D(X(\mathbf{D}))$.

Conversely, for any Priestley space $\mathcal{X}$, $\mathcal{X} \cong X(D(\mathcal{X}))$.

Priestley duality (Distributive lattices)

Categories

- $\mathcal{PS}$ is the category having as objects Priestley spaces and as morphisms continuous order-preserving maps.
- $\mathcal{DL}$ is the category having as objects bounded distributive lattices and as morphisms algebraic bounded lattice homomorphisms.

Priestley duality (Distributive lattices)

Functors

- For any $\mathcal{X} \in \mathcal{PS}$, we have that $D(\mathcal{X}) \in \mathcal{DL}$.
- For $\mathcal{X}, \mathcal{X}' \in \mathcal{PS}$ and $f \colon \mathcal{X} \to \mathcal{X}'$, define $D(f) \colon D(\mathcal{X}') \to D(\mathcal{X})$ as $D(f)(U) := f^{-1}(U)$ for all $U \in D(\mathcal{X}')$.
- For any $\mathbf{D} \in \mathcal{PS}$, we have that $X(\mathbf{D}) \in \mathcal{PS}$.
- For $\mathbf{D}, \mathbf{D}' \in \mathcal{DL}$ and $h \colon \mathbf{D} \to \mathbf{D}'$, define $X(h) \colon X(\mathbf{D}') \to X(\mathbf{D})$ as $X(h)(P) := h^{-1}(P)$ for all $P \in X(\mathbf{D}')$.

Equivalence

The categories $\mathcal{PS}$ and $\mathcal{DL}$ are dually equivalent via functors D and X.

Priestley duality (Distributive pre-bilattices)

Filters and ideals on pre-bilattices

In a bounded pre-bilattice $\langle B, \leqslant_t, \leqslant_k \rangle$, we can consider subsets $F \subseteq B$ that are

(1) lattice filters of both $\leqslant_t$ and $\leqslant_k$

(2) filters of $\leqslant_t$ and ideals of $\leqslant_k$

(3) ideals of $\leqslant_t$ and filters of $\leqslant_k$

(4) ideals of both $\leqslant_t$ and $\leqslant_k$.

We shall focus on $\leqslant_t$-filters.

Proposition

Any prime $\leqslant_t$-filter is either (1) or (2).

Priestley duality (Distributive pre-bilattices)

Filters and ideals on pre-bilattices

Let $\mathbf{B} = \mathbf{L_1} \odot \mathbf{L_2}$, where $\mathbf{L_1}, \mathbf{L_2}$ are bounded distributive lattices.
Define:

- $\mathcal{F}_1(\mathbf{B}) := \{F \subseteq B : F \text{ is a } \leqslant_t\text{-}\leqslant_k\text{-filter}\}$
- $\mathcal{F}_2(\mathbf{B}) := \{F \subseteq B : F \text{ is a } \leqslant_t\text{-filter and } \leqslant_k\text{-ideal}\}$
- $X(\mathbf{B}) := \{P \subseteq B : P \text{ is a prime } \leqslant_t\text{-filter}\}$
- $X_1(\mathbf{B}) := \mathcal{F}_1(\mathbf{B}) \cap X(\mathbf{B})$
- $X_2(\mathbf{B}) := \mathcal{F}_2(\mathbf{B}) \cap X(\mathbf{B})$

Then,

- $\mathcal{F}_1(\mathbf{B}) \cong \mathcal{F}(\mathbf{L_1})$ and $X_1(\mathbf{B}) \cong X(\mathbf{L_1})$
- $\mathcal{F}_2(\mathbf{B}) \cong \mathcal{I}(\mathbf{L_2})$ and $X_2(\mathbf{B}) \cong X(\mathbf{L_2^{op}})$

Priestley duality (Distributive pre-bilattices)

The Priestley space of a distributive pre-bilattice

Let $\mathbf{B} = \langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ be a bounded distributive pre-bilattice and $X(\mathbf{B})$ the set of prime $\leqslant_t$-filters of $\mathbf{B}$. Then

- $\Phi(a) := \{P \in X(\mathbf{B}) : a \in P\}$
- $X(\mathbf{B}) = X_1(\mathbf{B}) \cup X_2(\mathbf{B})$, where

$$X_1(\mathbf{B}) := \{P \in X(\mathbf{B}) : P \text{ is a prime } \leqslant_k\text{-filter}\}$$
$$X_2(\mathbf{B}) := \{P \in X(\mathbf{B}) : P \text{ is a prime } \leqslant_k\text{-ideal}\}.$$

- $X_1(\mathbf{B}) \cap X_2(\mathbf{B}) = \emptyset$
- $X_1(\mathbf{B}) = \Phi(\top)$ and $X_2(\mathbf{B}) = \Phi(\bot)$.

Priestley duality (Distributive pre-bilattices)

The Priestley space of a distributive pre-bilattice

Let $\mathbf{B} = \langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ be a bounded distributive pre-bilattice and $X(\mathbf{B})$ the set of prime $\leqslant_t$-filters of $\mathbf{B}$. Then

- $\Phi(a) := \{P \in X(\mathbf{B}) : a \in P\}$
- $X(\mathbf{B}) = X_1(\mathbf{B}) \cup X_2(\mathbf{B})$, where

$$X_1(\mathbf{B}) := \{P \in X(\mathbf{B}) : P \text{ is a prime } \leqslant_k\text{-filter}\}$$
$$X_2(\mathbf{B}) := \{P \in X(\mathbf{B}) : P \text{ is a prime } \leqslant_k\text{-ideal}\}.$$

- $X_1(\mathbf{B}) \cap X_2(\mathbf{B}) = \varnothing$
- $X_1(\mathbf{B}) = \Phi(\top)$ and $X_2(\mathbf{B}) = \Phi(\bot)$.

Priestley duality (Distributive pre-bilattices)

Remark

If $\mathbf{B} = \mathbf{L_1} \odot \mathbf{L_2}$, then

- the subspace $\langle X_1(\mathbf{B}), \tau, \subseteq \rangle$ is homeomorphic to the Priestley space of $\mathbf{L_1}$
- $\langle X_2(\mathbf{B}), \tau, \subseteq \rangle$ is homeomorphic to the Priestley space of $\mathbf{L_2^{op}}$.

Priestley duality (Distributive pre-bilattices)

Pre-bilattice spaces

Define a *pre-bilattice space* to be a tuple $\mathcal{X} = \langle X, \tau, \leqslant, X_1, X_2 \rangle$ s.t.:

- $\langle X, \tau, \leqslant \rangle$ is a Priestley space
- $X_1, X_2 \subseteq X$ are clopen up-sets such that $X_1 \cap X_2 = \varnothing$ and $X_1 \cup X_2 = X$.

A *pre-bilattice function* $f : X \to X'$ is a continuous order-preserving map such that $f(X_1) \subseteq X'_1$ and $f(X_2) \subseteq X'_2$.

Proposition

Denoting by $D(\mathcal{X})$ the set of clopen up-sets of $\mathcal{X}$, we have that $\langle D(\mathcal{X}), \cap, \cup, \varnothing, X, X_1, X_2 \rangle$ is a bounded distributive pre-bilattice and, for any bounded distributive pre-bilattice **B**,

$$\mathbf{B} \cong \langle D(X(\mathbf{B})), \cap, \cup, \varnothing, X(\mathbf{B}), X_1(\mathbf{B}), X_2(\mathbf{B}) \rangle.$$

Priestley duality (Distributive pre-bilattices)

Proposition

For any pre-bilattice space $\mathcal{X}$, $\mathcal{X} \cong \langle X(D(\mathcal{X})), \Phi(X_1), \Phi(X_2)\rangle$.

Priestley duality extends to a duality between:

- pre-bilattices with $\{\wedge, \vee, \mathsf{f}, \mathsf{t}, \top, \bot\}$-homomorphisms as morphisms
- pre-bilattice spaces with pre-bilattice functions as morphisms.

Priestley duality (De Morgan algebras)

De Morgan algebras

A *De Morgan algebra* is a bounded distributive lattice $\langle A, \wedge, \vee, \neg, \mathrm{f}, \mathrm{t} \rangle$ with a unary operation $\neg : A \to A$ that satisfies:

- $x \leqslant y$ iff $\neg y \leqslant \neg x$
- $\neg\neg x = x$.

Relation with bilattices

The $\{\wedge, \vee, \neg, \mathrm{f}, \mathrm{t}\}$-reduct of any bounded distributive bilattice is a De Morgan algebra. Conversely, any bilattice can be seen as a De Morgan algebra having two extra constants $\bot, \top$ satisfying that

i. $\top \wedge \bot = \mathrm{f}$ $\quad\quad$ $\top \vee \bot = \mathrm{t}$

ii. $\top = \neg\top$ $\quad\quad$ $\bot = \neg\bot$.

Priestley duality (De Morgan algebras)

De Morgan algebras

Let $\mathbf{A}$ be a De Morgan algebra and $P \in X(\mathbf{A})$ a prime filter. Define:

- $\neg P := \{a \in A : \neg a \in P\}$
- $g(P) := A \setminus \neg P$.

Then:

- $\neg P$ is a prime ideal
- $g(P)$ is a prime filter
- $g : X(\mathbf{A}) \to X(\mathbf{A})$ is an order-reversing involution, i.e. $g^2 = id_{X(\mathbf{A})}$.

Priestley duality (De Morgan algebras)

The Priestley space of a De Morgan algebra

For any De Morgan algebra $\mathbf{A}$,

- $\langle X(\mathbf{A}), \tau, \subseteq \rangle$ is a Priestley space
- $g \colon X(\mathbf{A}) \to X(\mathbf{A})$ is an order-reversing homeomorphism s.t. $g^2 = id_{X(\mathbf{A})}$.
- $\Phi \colon \mathbf{A} \cong \langle D(X(\mathbf{A})), \cap, \cup, \neg, \varnothing, X(\mathbf{A}) \rangle$, where

$$\neg U := X(\mathbf{A}) \backslash g(U)$$

for any $U \in D(X(\mathbf{A}))$.

Priestley duality (De Morgan algebras)

De Morgan spaces

A *De Morgan space* to is a tuple $\mathcal{X} = \langle X, \tau, \leqslant, g \rangle$ such that

- $\langle X, \tau, \leqslant \rangle$ is a Priestley space
- $g \colon X \to X$ is an order-reversing homeomorphism s.t. $g^2 = id_{\mathcal{X}}$.

Proposition

For any De Morgan space $\mathcal{X}$, $\mathcal{X} \cong \langle X(D(\mathcal{X})), g \rangle$. Priestley duality extends to a duality between:

- De Morgan algebras with $\{\wedge, \vee, \neg, \mathrm{f}, \mathrm{t}\}$-homomorphisms as morphisms
- De Morgan spaces with *De Morgan functions* as morphisms, i.e. monotonic and continuous functions that commute with g.

Priestley duality (Distributive bilattices)

The Priestley space of a distributive bilattice

For any distributive bilattice $\mathbf{B} = \langle B, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}, \top, \bot \rangle$,

(1) $\langle X(\mathbf{B}), \tau, \subseteq, X_1(\mathbf{B}), X_2(\mathbf{B}) \rangle$ is a pre-bilattice space

(2) $\langle X(\mathbf{A}), \tau, g \subseteq \rangle$ is a De Morgan space

(3) $g(X_1(\mathbf{B})) = X_2(\mathbf{B})$

(4) $\Phi \colon \mathbf{B} \cong \langle D(X(\mathbf{B})), \cap, \cup, \neg, \emptyset, X(\mathbf{B}), X_1(\mathbf{B}), X_2(\mathbf{B}) \rangle$.

Bilattice spaces

We define $\mathcal{X} = \langle X, \tau, \leqslant, g, X_1, X_2 \rangle$ to be a *bilattice space* if it satisfies (1) to (3).
A *bilattice function* $f \colon \mathcal{X} \to \mathcal{X}'$ is a De Morgan function that satisfies $f(X_1) \subseteq X'_1$.

Priestley duality (Distributive bilattices)

Proposition

For any bilattice space $\mathcal{X}$, $\mathcal{X} \cong \langle X(D(\mathcal{X})), g \rangle$. Priestley duality extends to a duality between:

- bilattices with $\{\wedge, \vee, \neg, \mathsf{f}, \mathsf{t}, \top, \bot\}$-homomorphisms as morphisms
- bilattice spaces with bilattice functions as morphisms.

Priestley duality (Brouwerian bilattices)

Heyting spaces

A *Heyting space* is a Priestley space $\mathcal{X} = \langle X, \tau, \leqslant \rangle$ such that, for any open $O \in \tau$, the down-set $O{\downarrow}$ is also open.

A *Heyting function* is a continuous monotone map $f : X \to X'$ such that, for any open $O \in X'$,

$$f^{-1}(O{\downarrow}) = (f^{-1}(O)){\downarrow}$$

The Priestley space of a Brouwerian bilattice

For any Brouwerian bilattice $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \top, \bot \rangle$,

(1) $\langle X(\mathbf{B}), \tau, \subseteq, g, X_1(\mathbf{B}), X_2(\mathbf{B}) \rangle$ is a bilattice space

(2) $\langle X_1(\mathbf{B}), \tau, \subseteq \rangle$ is a Heyting space

We take this as definition of *Brouwerian bilattice space*.

Priestley duality (Brouwerian bilattices)

Brouwerian bilattice spaces

Given a Brouwerian bilattice space $\mathcal{X} = \langle X, \tau, \subseteq, g, X_1, X_2 \rangle$ and $U, V \subseteq X$, define:

$$U \supset V := \big(X_1 \setminus [(U \setminus V) \cap X_1]{\downarrow}\big) \cup \big(X_2 \setminus (g(U) \setminus V)\big).$$

Then

$$\langle D(\mathcal{X}), \cap, \cup, \supset, \neg, \emptyset, X, X_1, X_2 \rangle$$

is a Brouwerian bilattice.

Moreover, for any Brouwerian bilattice $\mathbf{B}$,

$$\Phi \colon \mathbf{B} \cong \langle D(X(\mathbf{B})), \cap, \cup, \supset, \neg, \emptyset, X(\mathbf{B}), X_1(\mathbf{B}), X_2(\mathbf{B}) \rangle.$$

Conversely, any Brouwerian bilattice space $\mathcal{X}$ is homeomorphic to $X(D(\mathcal{X}))$.

Priestley duality (Brouwerian bilattices)

Brouwerian bilattice functions

Define a *Brouwerian bilattice function* to be a bilattice function $f : X \to X'$ such that $f : X_1 \to X'_1$ is a Heyting fuction, i.e. for any open $O \subseteq X'$,

$$f^{-1}([O \cap X'_1]\downarrow) \cap X_1 = [f^{-1}(O \cap X'_1)]\downarrow \cap X_1.$$

Duality

We obtain thus a duality between:

- Brouwerian bilattices with $\{\wedge, \vee, \supset, \neg, \mathrm{f}, \mathrm{t}\}$-homomorphisms as morphisms
- Brouwerian bilattice spaces with Brouwerian bilattice functions as morphisms.