Some extensions of the Belnap-Dunn logic

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The Belnap-Dunn logic

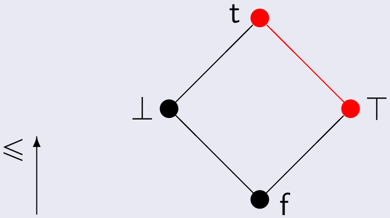
Introduction

The Belnap-Dunn four-valued logic (a.k.a. *first degree entailment*) is a well-known system related to relevant and paraconsistent logics, widely known and applied in computer science.
In recent years it has also been studied algebraically (Font, Pynko) and in connection with more general structures such as bilattices and generalized Kleene logics (Fitting, Arieli & Avron, Shramko & Wansing).

The Belnap-Dunn logic

Semantic definition of $\mathcal{B}$ (Dunn 1966, Belnap 1977)

The Belnap-Dunn logic (that we denote by $\mathcal{B}$) is the logic determined by the matrix $\langle \mathbf{M_4}, \{t, \top\} \rangle$, where $\mathbf{M_4}$ is the four-element diamond De Morgan lattice shown below.



$\neg$	
t	f
f	t
$\bot$	$\bot$
$\top$	$\top$

The Belnap-Dunn logic

Miscellaneous facts (Pynko 1995, Font 1997)

$\mathcal{B}$ is:

- a theoremless (a.k.a. "purely inferential") logic
- paraconsistent, in the sense that $\varphi \wedge \neg\varphi \not\models_{\mathcal{B}} \psi$
- relevant, in the sense that if $\varphi \models_{\mathcal{B}} \psi$, then $var(\varphi) \cap var(\psi) \neq \emptyset$
- the logic of the lattice order of De Morgan lattices, i.e.

  $$\Gamma \models_{\mathcal{B}} \varphi \text{ iff } \mathsf{DMLat} \models \bigwedge \Gamma \leq \varphi$$

- finitely axiomatized by Hilbert- or Gentzen-style calculi

The Belnap-Dunn logic

Miscellaneous facts (Pynko 1995, Font 1997)

Moreover

- $\mathcal{B}$ is non-protoalgebraic, self-extensional and non-Fregean
- $\mathcal{B}$ has an associated fully adequate Gentzen calculus that is algebraizable w.r.t. the variety DMLat of De Morgan lattices
- $\mathbf{Alg}^*\mathcal{B} \subsetneq \mathbf{Alg}\mathcal{B} = \mathrm{DMLat}$.

Remark

DMLat is not the equivalent algebraic semantics of any algebraizable logic; the same holds for any sub-quasi-variety of DMLat (except Boolean algebras).

The Belnap-Dunn logic

Algebraic models of $\mathcal{B}$ (Font 1997)

- Any matrix $\langle \mathbf{A}, D \rangle$ such that $\mathbf{A} \in$ DMLat and $D \subseteq A$ is a lattice filter (or is empty) is a model of $\mathcal{B}$.
- $\langle \mathbf{A}, \mathcal{C} \rangle$ is a reduced full model of $\mathcal{B}$ iff $\mathbf{A} \in$ DMLat and $\mathcal{C}$ is the set of all lattice filters of $\mathbf{A}$ plus the empty set.
- $\langle \mathbf{A}, D \rangle$ is a reduced matrix for $\mathcal{B}$ iff $\mathbf{A} \in$ DMLat and D is a lattice filter satisfying that for all $a, b \in A$ with $a \not\geq b$, at least one of the following conditions holds:
  - (i) there is $c \in A$ such that $a \vee c \notin D$ and $b \vee c \in D$
  - (ii) there is $c \in A$ such that $\neg a \vee c \in D$ and $\neg b \vee c \in D$.

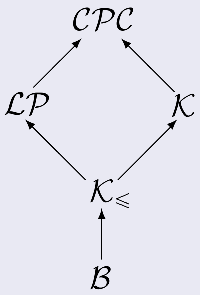
Extensions of $\mathcal{B}$

Well-known ones

- Priest's (1979) logic of paradox $\mathcal{LP}$
- Kleene's logic of order $\mathcal{K}_{\leqslant}$
  (a.k.a. the implicationless fragment of the relevant logic $\mathcal{RM}$)
- the strong three-valued (assertional) Kleene logic $\mathcal{K}$
- classical logic $\mathcal{CPC}$.

Extensions of $\mathcal{B}$

Inclusions



Extensions of $\mathcal{B}$

Axiomatizations (Hilbert-style)

- $\mathcal{LP} = \mathcal{B}$ plus $\vdash p \vee \neg p$ (Pynko 1995)
- $\mathcal{K}_{\leqslant} = \mathcal{B}$ plus $(p \wedge \neg p) \vee r \vdash q \vee \neg q \vee r$
- $\mathcal{K} = \mathcal{K}_{\leqslant}$ plus $p \wedge (\neg p \vee q) \vdash q$
- $\mathcal{CPC} = \mathcal{K}$ plus $\vdash p \vee \neg p$.

Algebraic models

- $\mathbf{Alg}\mathcal{LP} = \mathbf{Alg}\mathcal{K}_{\leqslant} = \mathbf{Alg}\mathcal{K} = \mathsf{KLat}$
- $\langle \mathbf{A}, D \rangle \in \mathbf{Matr}^{*}\mathcal{L}$ iff $\langle \mathbf{A}, D \rangle \in \mathbf{Matr}^{*}\mathcal{B}$ and D is closed under the corresponding additional rule(s) of $\mathcal{L}$ listed above.

Extensions of $\mathcal{B}$

Semantical presentations

- $\mathcal{LP}$ is defined by $\langle \mathbf{K_3}, \{t, \bot\} \rangle$
- $\mathcal{K}_{\leqslant}$ by $\{\langle \mathbf{K_3}, \{t, \bot\} \rangle, \langle \mathbf{K_3}, \{t\} \}$
- $\mathcal{K}$ by $\langle \mathbf{K_3}, \{t\} \rangle$
- $\mathcal{CPC}$ by $\langle \mathbf{B_2}, \{t\} \rangle$.

There are more...

Some more extensions of $\mathcal{B}$

Any (set of) matrices belonging to $\mathbf{Matr}^*\mathcal{B}$ defines an extension of $\mathcal{B}$. For instance:

- $\langle \mathbf{M_4}, \{t\} \rangle$
- $\langle \mathbf{K_6}, \{t, a\} \rangle$
- $\langle \mathbf{M_8}, \{t, a\} \rangle$
- $\{\langle \mathbf{M_4}, \{t\} \rangle, \langle \mathbf{K_3}, \{t, \bot\} \rangle\}$
- ...

There are more. . .

Some more extensions of $\mathcal{B}$

We are going to consider the following new "basic logics":

- $\mathcal{B}_4$ defined by $\langle \mathbf{M_4}, \{t\} \rangle$
- $\mathcal{B}_6$ defined by $\langle \mathbf{K_6}, \{t, a\} \rangle$
- $\mathcal{B}_8$ defined by $\langle \mathbf{M_8}, \{t, a\} \rangle$

together with the known ones:

- $\mathcal{K}$ defined by $\langle \mathbf{K_3}, \{t\} \rangle$
- $\mathcal{LP}$ defined by $\langle \mathbf{K_3}, \{t, \bot\} \rangle$.

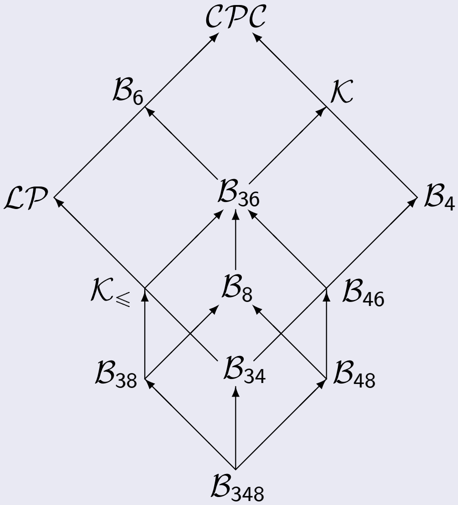
There are more...

Some more extensions of $\mathcal{B}$

The above-mentioned basic logics generate the following:

- $\mathcal{B}_{36} = \mathcal{B}_6 \cap \mathcal{K}$
- $\mathcal{B}_{46} = \mathcal{B}_{36} \cap \mathcal{B}_4$
- $\mathcal{K}_{\leqslant} = \mathcal{B}_{36} \cap \mathcal{LP}$
- $\mathcal{B}_{38} = \mathcal{K}_{\leqslant} \cap \mathcal{B}_8$
- $\mathcal{B}_{48} = \mathcal{B}_8 \cap \mathcal{B}_{46}$
- $\mathcal{B}_{34} = \mathcal{K}_{\leqslant} \cap \mathcal{B}_{46}$
- $\mathcal{B}_{348} = \mathcal{B}_{34} \cap \mathcal{B}_{38}$.

A hierarchy


$\mathcal{CPC}$
$\mathcal{B}_6$
$\mathcal{K}$
$\mathcal{LP}$
$\mathcal{B}_{36}$
$\mathcal{B}_4$
$\mathcal{K}_{\leqslant}$
$\mathcal{B}_8$
$\mathcal{B}_{46}$
$\mathcal{B}_{38}$
$\mathcal{B}_{34}$
$\mathcal{B}_{48}$
$\mathcal{B}_{348}$

What we do (not) know

Miscellaneous facts

- The above logics are obtained from $\mathcal{B}$ by adding some form of excluded middle and/or *ex contradictione quodlibet*.
- $\mathcal{B}_4$ has no axiomatic extensions of except $\mathcal{CPC}$.
- $\mathcal{K}$ has no finitary extensions of except $\mathcal{CPC}$.
- The only logics with theorems are $\mathcal{LP}$ and $\mathcal{B}_6$.
- All the logics (except $\mathcal{K}_{\leqslant}$) are non-selfextensional.
- All the logics are non-protoalgebraic.

What we do (not) know

Relations with the order

For any $\varphi, \psi \in Fm$, it holds that $\varphi \vDash_{\mathcal{L}} \psi$ if and only if:

- $\mathsf{DMLat} \vDash \varphi \preceq \psi$ for $\mathcal{L} = \mathcal{B}$
- $\mathsf{KLat} \vDash \varphi \preceq \psi$ for $\mathcal{L} = \mathcal{K}_{\leqslant}$
- $\mathsf{DMLat} \vDash \varphi \vee \neg\psi \preceq \neg\varphi \vee \psi$ for $\mathcal{L} = \mathcal{B}_{34}$
- $\mathsf{DMLat} \vDash \varphi \preceq \neg\varphi \vee \psi$ for $\mathcal{L} = \mathcal{B}_4$
- $\mathsf{KLat} \vDash \varphi \preceq \neg\varphi \vee \psi$ for $\mathcal{L} = \mathcal{K}$
- $\mathsf{KLat} \vDash \neg\varphi \vee \neg\psi \preceq \neg\varphi \vee \psi$ for $\mathcal{L} = \mathcal{LP}$
- $Q(\mathbf{K_6}) \vDash \neg\varphi \preceq \varphi \Rightarrow \neg\psi \preceq \psi$ for $\mathcal{L} = \mathcal{B}_6$.

What we do (not) know

Axiomatizations (Hilbert-style)

The previous results allow to prove that

- $\mathcal{B}_4 \; = \; \mathcal{B}$ plus $p \wedge (\neg p \vee q) \vdash q$
- $\mathcal{B}_6 \; = \; \mathcal{LP}$ plus $p \wedge \neg p \vdash q$
- $\mathcal{K}_{\leqslant} = \mathcal{B}$ plus $(p \wedge \neg p) \vee r \vdash q \vee \neg q \vee r$
- $\mathcal{K} = \mathcal{K}_{\leqslant}$ plus $p \wedge (\neg p \vee q) \vdash q$

What we do (not) know: algebraic models

Algebraic reducts of educed g-models

- $\mathbf{Alg}\mathcal{B}_{34} = \mathbf{Alg}\mathcal{B}_{4} = \mathbf{Alg}\mathcal{B}_{46} = \mathbf{Alg}\mathcal{B}_{48} = \mathbf{Alg}\mathcal{B}_{348} = \mathsf{DMLat}$
- $\mathbf{Alg}\mathcal{B}_{8},\ \mathbf{Alg}\mathcal{B}_{38} \subseteq \mathsf{DMLat}\quad (?)$
- $\mathbf{Alg}\mathcal{B}_{6},\ \mathbf{Alg}\mathcal{B}_{36} \subseteq \mathsf{KLat}\quad (?)$

What we do (not) know: algebraic models

Reduced models

For $\mathbf{A} \in$ DMLat, let $A^+ := \{a \in A : a \geqslant \neg a\}$. Let $\langle \mathbf{A}, D \rangle$ be a reduced matrix: if $\langle \mathbf{A}, D \rangle$ is a model of

- $\mathcal{B}$, then $D \subseteq A^+$
- $\mathcal{B}_{34}$ or $\mathcal{B}_{48}$ and $D \cap A^- \neq \emptyset$, then $D = A^+$
- $\mathcal{LP}$, then $D = A^+$
- $\mathcal{B}_4$ or $\mathcal{K}$, then $\mathbf{A}$ is bounded and $D = \{1\}$.

What we do (not) know: algebraic models

Reduced models

So, a matrix $\langle \mathbf{A}, D \rangle \in \mathbf{Matr}^*\mathcal{B}$ is a reduced model of

- $\mathcal{B}_{34}$ iff $\mathbf{A} \in$ DMLat and D is a lattice filter such that $b \in D$ whenever $a \in D$ and $a \vee \neg b \leqslant \neg a \vee b$
- $\mathcal{B}_4$ iff $\mathbf{A}$ is bounded De Morgan lattice and $D = \{1\}$
- $\mathcal{K}_{\leqslant}$ iff $\mathbf{A} \in$ KLat and D is a lattice filter
- $\mathcal{K}$ iff $\mathbf{A}$ is a bounded Kleene lattice and $D = \{1\}$
- $\mathcal{LP}$ iff $\mathbf{A} \in$ KLat and $D = \{a \in A : a \geqslant \neg a\}$.

What there is to know

Open problems

- Complete the picture of all (?) the extensions of $\mathcal{B}$ (for instance: is there some logic between $\mathcal{B}$ and $\mathcal{B}_{348}$?).
- Axiomatize $\mathcal{B}_{34}, \mathcal{B}_{48}, \mathcal{B}_{8}$ etc. and characterize their reduced models.
- Characterize the extensions of $\mathcal{B}$ in terms of their metalogical properties.
- Introduce and study Gentzen calculi associated with these logics (study their algebraizability etc.).

References

- J. M. Font (1997): Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the I.G.P.L.*, 5(3):413–440.
- A. Pynko (1995a): On Priest's logic of paradox. *Journal of Applied Non-Classical Logics*, 5(2):219–225.
- A. P. Pynko (1995b): Characterizing Belnap's logic via De Morgan's laws. *Mathematical Logic Quarterly*, 41(4):442–454.
- A. P. Pynko (1999): Implicational classes of De Morgan lattices. *Discrete Math.*, 205(1-3):171–181.

DMLat and KLat

De Morgan lattices

A **De Morgan lattice** is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a distributive lattice and the following equations are satisfied:

(i) $\neg(x \wedge y) \approx \neg x \vee \neg y$

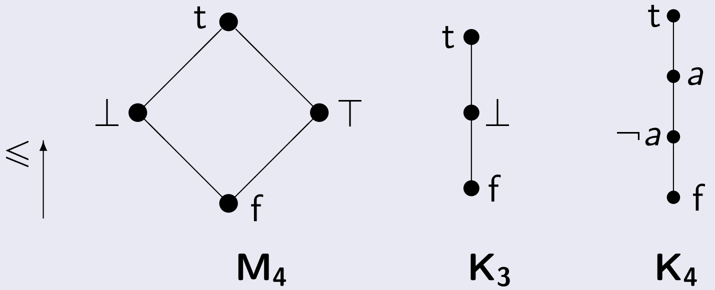
(ii) $\neg(x \vee y) \approx \neg x \wedge \neg y$

(iii) $x \approx \neg\neg x$

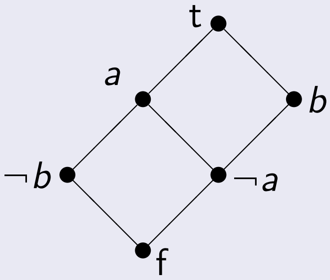
A **Kleene lattice** is a De Morgan lattice satisfying:

(iv) $x \wedge \neg x \leq y \vee \neg y$.

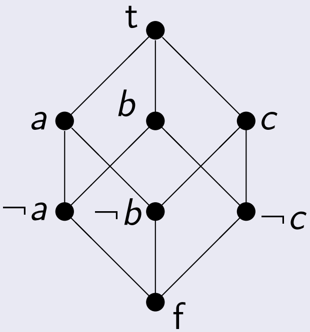
Some De Morgan lattices



Some De Morgan lattices



$\mathbf{K_6}$



$\mathbf{M_8}$

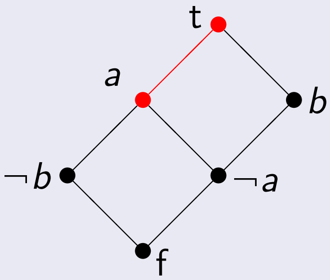
Sub-quasi-varieties of DMLat

Some facts

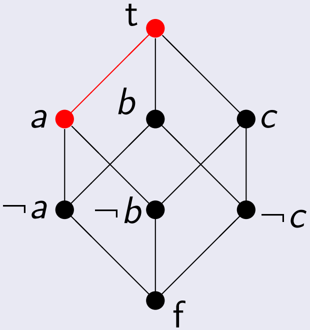
- The class DMLat is a variety, generated by $\mathbf{M_4}$.
- Besides $\mathbf{M_4}$, the the only subdirectly irreducible De Morgan lattices are $\mathbf{K_3}$ and $\mathbf{B_2}$.
- $\mathrm{V}(\mathbf{K_3}) = \mathrm{KLat}$ and $\mathrm{V}(\mathbf{B_2})$ are the only subvarieties of DMLat.
- (Pynko 1999): DMLat has four sub-quasi-varieties (that are not varieties), i.e.
  1. regular Kleene lattices $\mathrm{Q}(\mathbf{K_4})$
  2. non-idempotent Kleene lattices $\mathrm{Q}(\mathbf{K_6})$
  3. non-idempotent De Morgan lattices $\mathrm{Q}(\mathbf{M_8})$
  4. KLat $\cup$ non-idempotent De Morgan lattices $\mathrm{Q}(\mathbf{M_8}, \mathbf{K_3})$.

◂ Back

De Morgan matrices



$\langle \mathbf{K_6}, \{\mathrm{t}, \mathrm{a}\} \rangle$



$\langle \mathbf{M_8}, \{\mathrm{t}, \mathrm{a}\} \rangle$

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A Hilbert calculus for $\mathcal{B}$ (Font, 1997)

$$\text{(R1)}\ \frac{p \wedge q}{p} \qquad \text{(R2)}\ \frac{p \wedge q}{q} \qquad \text{(R3)}\ \frac{p \quad\quad q}{p \wedge q}$$

$$\text{(R4)}\ \frac{p}{p \vee q} \qquad \text{(R5)}\ \frac{p \vee q}{q \vee p} \qquad \text{(R6)}\ \frac{p \vee p}{p}$$

$$\text{(R7)}\ \frac{p \vee (q \vee r)}{(p \vee q) \vee r} \qquad \text{(R8)}\ \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)} \qquad \text{(R9)}\ \frac{(p \vee q) \wedge (p \vee r)}{p \vee (q \wedge r)}$$

$$\text{(R10)}\ \frac{p \vee r}{\neg\neg p \vee r} \qquad \text{(R11)}\ \frac{\neg\neg p \vee r}{p \vee r} \qquad \text{(R12)}\ \frac{\neg(p \vee q) \vee r}{(\neg p \wedge \neg q) \vee r}$$

$$\text{(R13)}\ \frac{(\neg p \wedge \neg q) \vee r}{\neg(p \vee q) \vee r} \qquad \text{(R14)}\ \frac{\neg(p \wedge q) \vee r}{(\neg p \vee \neg q) \vee r} \qquad \text{(R15)}\ \frac{(\neg p \vee \neg q) \vee r}{\neg(p \wedge q) \vee r}$$

Back