Bilattices with modal operators

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23 February, 2012
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Introduction

Modal four-valued logics

A number of authors have considered modal expansions of the four-valued Belnap logic (in the language $\{\wedge, \vee, \neg\}$, perhaps augmented with an implication $\rightarrow$), with different motivations:

- S. Odintsov & H. Wansing (2003): expanding paraconsistent Nelson logic to obtain an inconsistency-tolerant Description Logic
- G. Priest (2008): Belnap logic without implication, with philosophical motivations
- E. Sherkhonov (2008): different possible constructive (in the sense of Nelson) modal logics

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- S. Odintsov & H. Wansing (2010): the least normal modal logic over Belnap logic (with a 'classical' implication)
- N. Kamide & H. Wansing (2011): temporal operators on paraconsistent Nelson logic, for time-dependent and inconsistency-tolerant reasoning.

An example

Odintsov & Wansing's (2010) logic BK, semantically

The logic BK is defined over the language $\{\wedge, \vee, \rightarrow, \neg, \Box, \mathrm{t}, \mathrm{f}\}$, with $\Diamond\varphi$ defined as $\neg\Box\neg\varphi$.
The semantics of the modal operator is very similar to the classical case: models are tuples $\langle W, R, V\rangle$ such that $R \subseteq W^2$ and

- $V : Fm \times W \rightarrow 4$, where 4 is the Belnap lattice
- $V(\varphi \wedge \psi, w) = V(\varphi, w) \wedge V(\psi, w)$
- $V(\varphi \vee \psi, w) = V(\varphi, w) \vee V(\psi, w)$, etc.
- $V(\Box\varphi, w) = \bigwedge\{V(\varphi, u) : wRu\}$.

Thus, the $\{\wedge, \vee, \rightarrow, \neg, \mathrm{t}, \mathrm{f}\}$-fragment of BK coincides with Belnap's logic with a 'classical' implication.

An example

Odintsov & Wansing's (2010) logic BK, syntactically

The logic BK can be axiomatized by:

- the axioms of classical logic in the language $\{\wedge, \vee, \rightarrow, \mathrm{f}\}$
- strong negation axioms:

  $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q) \qquad \neg\neg p \leftrightarrow p$

  $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q) \qquad \neg(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$

- K axioms:

  $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q) \qquad \Box(p \rightarrow p)$

- modus ponens and monotonicity rules:

$$\frac{p, \;\; p \rightarrow q}{q} \qquad \frac{p \rightarrow q}{\Box p \rightarrow \Box q} \qquad \frac{\neg p \rightarrow \neg q}{\neg \Box p \rightarrow \neg \Box q}$$

An example

Odintsov & Wansing's (2010) logic BK, algebraically

The logic BK is algebraizable with respect to *BK-lattices*, i.e., algebras $\langle A, \wedge, \vee, \rightarrow, \neg, \Box, \mathrm{f}, \mathrm{t} \rangle$ such that $\langle A, \wedge, \vee, \rightarrow, \neg, \mathrm{f}, \mathrm{t} \rangle$ is a bounded *N4-lattice* and

(E1)	$(a \rightarrow b) \rightarrow a \leq a$
(E2)	$\Box a \wedge \Box b \leq \Box (a \wedge b)$
(E3)	$\Box (a \rightarrow a) = \Box (a \rightarrow a) \rightarrow \Box (a \rightarrow a)$
(E4)	$\Box a \rightarrow \mathrm{f} \leq\geq \Diamond (a \rightarrow \mathrm{f})$
(E5)	$\Diamond a \rightarrow \mathrm{f} \leq\geq \Box (a \rightarrow \mathrm{f})$
(Q1)	if $a \leq b$, then $\Box a \leq \Box b$
(Q2)	if $\neg a \leq \neg b$, then $\neg \Box a \leq \neg \Box b$

where $a \leq b$ abbreviates $a \rightarrow b = (a \rightarrow b) \rightarrow (a \rightarrow b)$.

An example

BK-lattices as twist-structures

Any BK-lattice **A** can be represented as a *twist-structure* over a modal algebra, i.e.:

- $\mathbf{B} = \langle B, \wedge, \vee, \neg, \Box, \Diamond, 0, 1 \rangle$ is a modal Boolean algebra
- $\mathbf{B}^{\bowtie} = \langle B \times B, \wedge^{\bowtie}, \vee^{\bowtie}, \to^{\bowtie}, \neg^{\bowtie}, \Box^{\bowtie}, \langle 0, 1 \rangle, \langle 1, 0 \rangle \rangle$
- $\Box^{\bowtie} \langle x, y \rangle = \langle \Box x, \Diamond y \rangle$
- **A** is a subalgebra of $\mathbf{B}^{\bowtie}$ and $\pi_1(A) = B$.

Bilattices

Modal bilattice logics

- Odintsov & Wansing's (2010) approach can be straightforwardly generalized to define, for instance, bilattice logics with modalities that are conservative over the Arieli & Avron (1996) "logic of logical bilattices" with implication
- the algebraic models of such logics are going to be bilattices with modal operators (whose properties are determined by the logical axioms and rules).

Bilattices

Bilattices with modal operators

Some possibilities:

1. bilattices $\langle B, \wedge_1, \vee_1, \wedge_2, \vee_2, (\rightarrow, \neg), \Box \rangle$ corresponding to twist-structures over modal Boolean algebras (or over weaker algebras with monotone operators)
2. bilattices where (one of) the two lattice reducts $\langle B, \wedge_1, \vee_1 \rangle$, $\langle B, \wedge_2, \vee_2 \rangle$ is an algebra (for instance, a distributive lattice) with modal operators.

Bilattices

(1) As twist-structures with modal operators

- The bilattice will have to satisfy properties such as:
  if $a \leq\geq b$, then $\Box a \leq\geq \Box b$
  if $\neg a \leq\geq \neg b$, then $\neg\Box a \leq\geq \neg\Box b$
- if $\Box_1, \Diamond_1$ correspond to $\bigwedge_1, \bigvee_1$, then one can define

$$\Box_2 a = (\Box_1 a \wedge_1 (\Box_1 a \vee_2 \Diamond_1 a)) \wedge_2 (\Diamond_1 a \vee_1 (\Box_1 a \vee_2 \Diamond_1 a))$$
$$\Diamond_2 a = (\Diamond_1 a \wedge_1 (\Box_1 a \vee_2 \Diamond_1 a)) \wedge_2 (\Box_1 a \vee_1 (\Box_1 a \vee_2 \Diamond_1 a))$$

which will correspond to $\bigwedge_2, \bigvee_2$.

Bilattices

(2) Focusing on one of the lattice reducts

- We can build on the known results on lattices with modal operators (e.g., Priestley duality-Kripke semantics) and this is going to be straightforward if the bilattice is bounded (90-degree lemma)
- if the modal operators are compatible with the relation $\leq\geq$, then we can represent the 'modal bilattice' as a twist-structure and we get the properties of (1) as well.

References

- N. Kamide & H. Wansing (2011): A paraconsistent linear-time temporal logic. *Fundamenta Informaticae*, 106, 1, pp. 1–23.
- S. P. Odintsov, & H. Wansing (2003): Inconsistency-tolerant description logic. Motivation and basic systems. In: V. Hendricks & J. Malinowski (eds), *50 Years of Studia Logica*, Kluwer, pp. 301–335.
- S. P. Odintsov & H. Wansing (2010): Modal logic with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20, pp. 279–301.
- G. Priest (2008): Many-valued modal logics: a simple approach. *Review of Symbolic Logic*, 1, pp. 190–203.
- E. Sherkhonov (2008): Modal operators over constructive logic. *Journal of Logic and Computation*, 18, pp. 815–829.