On infinitary connectives in modal logics

Yoshihito Tanaka

Kyushu Sangyo University

January 27, 2011

Why infinitary connectives?

- Strength the expressive power of logics:
  - Infinite objects (e.g. canonical formulas for infinite structure).
  - Concepts which cannot be expressed by first order theory (e.g. finiteness).
  - Concepts which is defined by nesting (e.g. common knowledge property).
- As a tool to show various properties of standard proof systems.

Table of contents

Cut free system

We present cut free systems for:

- infinitary modal logic characterised by Kripke frames;
- common knowledge logic characterised by Kripke frames.

♠ Common knowledge logic does not have infinitary conjunction nor disjunction, but allows a non-compact inference rule for common knowledge operator.

Syntax

Syntax for infinitary modal logic:

- Language:
  - p, $\bigwedge$, $\bigvee$, $\supset$, $\neg$, $\Box$
- Formulas
  - For any countable set Θ of formulas, $\bigwedge \Theta$ and $\bigvee \Theta$ are well defined formulas.

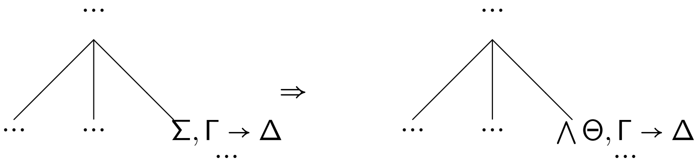
System TLM for infinitary modal logic

We introduce a system TLM for infinitary modal logic, based on sequent calculus.

- A sequent of TLM is a finite tree of usual sequents.
- Any inference rule of TLM, except for those for modality, is an application of usual rule to one of a node of the tree.

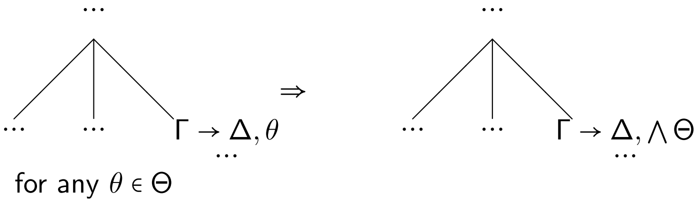
System TLM for infinitary modal logic

♠ The left introduction rule of $\bigwedge$ of TLM: Here, Σ is a countable subset of Θ.



System TLM for infinitary modal logic

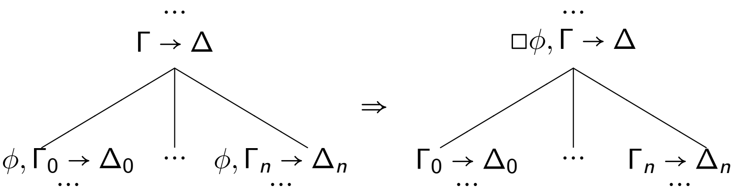
♠ The set of upper sequents of the right introduction rule of $\bigwedge$ of TLM is countable.



System TLM for infinitary modal logic

♠ Left introduction of BOX:

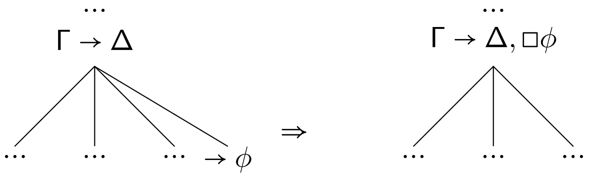
$\Box\phi$ can be introduced to the left hand side of a node if ϕ is in the left hand side of any successor of the node.



System TLM for infinitary modal logic

♠ Right introduction of BOX:

$\Box\phi$ can be introduced to the right hand side of a node if it has a leaf of the form $\to \phi$. The leaf will be cut off.



System TLM for infinitary modal logic

♠ A formula ϕ is *derivable* in TLM if a tree $\rightarrow \phi$, which consists only one node, is derivable.

Theorem

TLM *is sound and complete with respect to the class of all Kripke frames.*

Theorem

If ϕ *is derivable in* TLM, *then it is derivable without using cut rule.*

Concept of common knowledge

♠ By using tree sequent system, we can obtain a cut free system for common knowledge logic, which has neither infinitary conjunction nor disjunction, but does not satisfy compactness.

Concept of common knowledge:
Let I be a fixed class of agents and A be an idea. Suppose A belongs to the common knowledge of the class I of agents and i and j are some members of I. Then

1. i and j know A.
2. i knows that j knows A and j knows that i knows A, as well.
3. i also knows that j knows that i knows A, and vice versa...

Concept of common knowledge

♠ By using tree sequent system, we can obtain a cut free system for common knowledge logic, which has neither infinitary conjunction nor disjunction, but does not satisfy compactness.

Concept of common knowledge:
Let I be a fixed class of agents and A be an idea. Suppose A belongs to the common knowledge of the class I of agents and i and j are some members of I. Then

1. i and j know A.
2. i knows that j knows A and j knows that i knows A, as well.
3. i also knows that j knows that i knows A, and vice versa...

Concept of common knowledge

♠ By using tree sequent system, we can obtain a cut free system for common knowledge logic, which has neither infinitary conjunction nor disjunction, but does not satisfy compactness.

Concept of common knowledge:
Let I be a fixed class of agents and A be an idea. Suppose A belongs to the common knowledge of the class I of agents and i and j are some members of I. Then

1. i and j know A.
2. i knows that j knows A and j knows that i knows A, as well.
3. i also knows that j knows that i knows A, and vice versa...

Concept of common knowledge

♠ By using tree sequent system, we can obtain a cut free system for common knowledge logic, which has neither infinitary conjunction nor disjunction, but does not satisfy compactness.

Concept of common knowledge:
Let I be a fixed class of agents and A be an idea. Suppose A belongs to the common knowledge of the class I of agents and i and j are some members of I. Then

1. i and j know A.
2. i knows that j knows A and j knows that i knows A, as well.
3. i also knows that j knows that i knows A, and vice versa...

Common knowledge logic

Express the concept of common knowledge in modal logic:

- $\Box_i \phi$: i knows ϕ (for each $i \in I$)
- $\Box_c \phi$: ϕ belongs to the common knowledge
- $\models \Box_c \phi \iff \forall n \in \omega \forall i_1, \ldots, i_n \in I (\models \Box_{i_1} \cdots \Box_{i_n} \phi)$.

It is known that the class of Kripke frames which satisfy the above condition is not axiomatizable by any modal logic with compactness.

Common knowledge logic

Express the concept of common knowledge in modal logic:

- $\Box_i \phi$: i knows ϕ (for each $i \in I$)
- $\Box_c \phi$: ϕ belongs to the common knowledge
- $\models \Box_c \phi \iff \forall n \in \omega \forall i_1, \ldots, i_n \in I (\models \Box_{i_1} \cdots \Box_{i_n} \phi)$.

It is known that the class of Kripke frames which satisfy the above condition is not axiomatizable by any modal logic with compactness.

Common knowledge logic

Express the concept of common knowledge in modal logic:

- $\Box_i \phi$: i knows ϕ (for each $i \in I$)
- $\Box_c \phi$: ϕ belongs to the common knowledge
- $\models \Box_c \phi \iff \forall n \in \omega \forall i_1, \ldots, i_n \in I (\models \Box_{i_1} \cdots \Box_{i_n} \phi)$.

It is known that the class of Kripke frames which satisfy the above condition is not axiomatizable by any modal logic with compactness.

Common knowledge logic

Express the concept of common knowledge in modal logic:

- $\Box_i \phi$: i knows ϕ (for each $i \in I$)
- $\Box_c \phi$: ϕ belongs to the common knowledge
- $\models \Box_c \phi \iff \forall n \in \omega \forall i_1, \ldots, i_n \in I (\models \Box_{i_1} \cdots \Box_{i_n} \phi)$.

It is known that the class of Kripke frames which satisfy the above condition is not axiomatizable by any modal logic with compactness.

Cut free system for common knowledge logic

Define a tree sequent system TCK by the same way as TLM in such way that

- allow only finite conjunction and disjunction;
- left introduction of $\Box_c$ to a node:

$$\forall n \forall i_1, \ldots, i_n (\vdash \Box_{i_1} \cdots \Box_{i_n} \phi, \Gamma \to \Delta) \;\Rightarrow\; \vdash \Box_c \phi, \Gamma \to \Delta;$$

- right introduction of $\Box_c$ to a node:

$$\exists n \exists i_1, \ldots, i_n (\vdash \Gamma \to \Delta, \Box_{i_1} \cdots \Box_{i_n} \phi) \;\Rightarrow\; \vdash \Gamma \to \Delta, \Box_c \phi.$$

♠ The inference rules for $\Box_c$ in TCK is an application of the rules for $\bigwedge$ in TLM to a formula $\bigwedge\{\Box_{i_1} \cdots \Box_{i_n} \phi \mid n \in \omega,\; i_1, \ldots, i_n \in I\}$.

Cut free system for common knowledge logic

Theorem

TLM *is sound and complete with respect to the class of Kripke frames.*

♠ We can define a cut free system for predicate common knowledge logic in the same way.

Representation and Kripke completeness

♠ Jónsson-Tarski representation provide Kripke completeness of modal logics in a modular way.

♠ We show a representation theorem of modal algebras which preserves countably many infinitary meets and joins.

♠ From the extended representation theorem, Kripke completeness of infinitary, predicate and non-compact logics follows in a modular way.

Modal algebra

- An algebra $\langle A, \wedge, \vee, -, \Box, 0, 1\rangle$ is called a *Modal algebra* if $\langle A, \wedge, \vee, -, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operator such that
$$\Box 1 = 1$$
$$\Box(x \wedge y) = \Box x \wedge \Box y$$
for any x and y in A.
- We write $\mathcal{F}_p(A)$ for the set of all prime filters of A and A_+ for the frame $\langle \mathcal{F}_p(A), <_R\rangle$, where
$$F <_R G \iff \Box^{-1}[F] \subseteq G.$$

Modal algebra

Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame. We write $\mathcal{F}^+$ for the modal algebra

$$\langle \wp(W), \cap, \cup, W \setminus, \Box, \varnothing, W \rangle,$$

where $\Box X$ is defined by

$$\Box X = W \setminus \downarrow_R (W \setminus X)$$

for any $X \subseteq W$.

Jónsson-Tarski representation

Let $\eta : A \to (A_+)^+$ be

$$\eta(x) = \{F \in \mathcal{F}_p(A) \mid x \in F\},$$

for any $x \in A$. Then, η is an embedding of modal algebras.

Here,

$$(A_+)^+ = \langle \wp(\mathcal{F}_p(A)), \cap, \cup, \mathcal{F}_p(A) \setminus, \Box, \varnothing, \mathcal{F}_p(A) \rangle$$

where

$$\Box X = \mathcal{F}_p(A) \setminus \downarrow_R (\mathcal{F}_p(A) \setminus X)$$

for any $X \in \wp(\mathcal{F}_p(A))$.

Q-filter

Let A be a modal algebra and Q a countable subset of $\wp(A)$.

A prime filter F of A is called a *Q-filter*, if it satisfies the following properties:

1. for any $X \in Q$, if $X \subseteq F$ and $\bigwedge X \in A$ then $\bigwedge X \in F$;
2. for any $X \in Q$, if $\bigvee X \in F$ then there exists $x \in X$ such that $x \in F$.

A homomorphism $f : A \to B$ of modal algebras is said to be Q-complete, if for any $X \in Q$

$$f(\bigwedge X) = \bigwedge f[X], \; f(\bigvee X) = \bigvee f[X]$$

whenever, $\bigwedge X \in A$ or $\bigvee X \in A$, respectively.

Q-filter

Let A be a modal algebra and Q a countable subset of $\wp(A)$.

A prime filter F of A is called a *Q-filter*, if it satisfies the following properties:

1. for any $X \in Q$, if $X \subseteq F$ and $\bigwedge X \in A$ then $\bigwedge X \in F$;
2. for any $X \in Q$, if $\bigvee X \in F$ then there exists $x \in X$ such that $x \in F$.

A homomorphism $f : A \to B$ of modal algebras is said to be Q-complete, if for any $X \in Q$

$$f(\bigwedge X) = \bigwedge f[X], \; f(\bigvee X) = \bigvee f[X]$$

whenever, $\bigwedge X \in A$ or $\bigvee X \in A$, respectively.

Q-filter

Let A be a modal algebra and Q a countable subset of $\wp(A)$.

A prime filter F of A is called a *Q-filter*, if it satisfies the following properties:

1. for any $X \in Q$, if $X \subseteq F$ and $\bigwedge X \in A$ then $\bigwedge X \in F$;
2. for any $X \in Q$, if $\bigvee X \in F$ then there exists $x \in X$ such that $x \in F$.

A homomorphism $f : A \to B$ of modal algebras is said to be Q-complete, if for any $X \in Q$

$$f(\bigwedge X) = \bigwedge f[X], \; f(\bigvee X) = \bigvee f[X]$$

whenever, $\bigwedge X \in A$ or $\bigvee X \in A$, respectively.

Q-filter

We write $\mathcal{F}_Q(A)$ for the set of Q-filters of A, $A_\sharp$ for the frame $\langle \mathcal{F}_Q(A), <_R \rangle$.

♠ We show that for certain Q, there exists a Q-complete embedding of modal algebras from A to $(A_\sharp)^+$. Here, $(A_\sharp)^+$ is

$$(A_\sharp)^+ = \langle \wp(\mathcal{F}_Q(A)), \cap, \cup, \mathcal{F}_Q(A) \setminus, \Box, \varnothing, \mathcal{F}_Q(A) \rangle.$$

Extension of Jónsson-Tarski representation

Theorem

Let A be a modal algebra and Q a countable subset of $\wp(A)$ which satisfies the following, for any $X \in Q$:

1. *if $\bigwedge X \in A$ then $\bigwedge \Box X = \Box \bigwedge X$;*
2. *if $\bigwedge X \in A$ then $Y = \{\Box(z \to x) \mid x \in X\} \in Q$ and $\bigwedge Y \in A$ for any $z \in A$;*
3. *if $\bigvee X \in A$ then $Y = \{\Box(x \to z) \mid x \in X\} \in Q$ and $\bigwedge Y \in A$ for any $z \in A$.*

Then, $\eta : x \mapsto \{F \in \mathcal{F}_Q(A) \mid x \in F\}$ is a Q-complete embedding of modal algebras from A to $(A_\sharp)^+$.

Variant of Barcan formula

♠ From extended Jónsson-Tarski representation, we obtain Kripke completeness of some infinitary logics.

We write BF_{ω_1} for the formula

$$\bigwedge_{i \in \omega} \Box p_i \supset \Box \bigwedge_{i \in \omega} p_i .$$

Note that BF_{ω_1} is a translation of a formula

$$\mathrm{BF} = \forall x \Box \phi \supset \Box \forall x \phi$$

of predicate modal logic, which is known as Barcan formula, into infinitary logic.

Universal logic

- A class C of Kripke frames is said to be *universal*, if there exists a set Φ of first order closed formulas of the shape $\forall x_1 \cdots \forall x_n \psi$, where ψ is constructed from variables, predicates R and $=$ and connectives $\wedge$, $\vee$, $\neg$ and $\supset$, such that

$$C = \{\mathcal{F} \mid \mathcal{F} \text{ satisfies } \Phi \text{ as a first order structure}\}.$$

- A normal modal logic $\mathbf{L}$ is said to be *universal*, if the class

$$C = \{\mathcal{F} \mid \mathcal{F} \models \mathbf{L}\}$$

of Kripke frames is universal.

Kripke completeness

Let **L** be a normal modal logic. We write $\mathbf{L}_{\omega_1}$ for the infinitary logic defined by **L** and BF_{ω_1}.

Theorem

Let **L** *be a normal modal logic. If* **L** *is universal,* $\mathbf{L}_{\omega_1}$ *is complete with respect to the class* $C = \{\mathcal{F} \mid \mathcal{F} \models \mathbf{L}\}$ *of Kripke frames.*

Application to predicate modal logics

♠ Same argument can be applied to predicate modal logics.

Let $\mathbf{K}_*$ be the least predicate extension of $\mathbf{K}$.

It is known that $\mathrm{BF} = \forall x \Box \phi \supset \Box \forall x \phi$ axiomatizes the class of Kripke frames with constant domain.

Then, completeness theorem follows from extended representation theorem, immediately:

Theorem

Let $\mathbf{L}$ be a normal modal logic. If $\mathbf{L}$ is universal, $\mathbf{K}_* \oplus \mathbf{L} \oplus \mathrm{BF}$ is complete with respect to the class $C = \{\mathcal{F} \mid \mathcal{F} \models \mathbf{L}\}$ of Kripke frames with constant domain.

Language of non-compact modal logic

♠ We apply extended representation theorem to non-compact modal logics.

We consider propositional modal logics such that:

- with a countable set $\{\Box_i \mid i \in \omega\}$ of modal operators;
- without infinitary connectives;
- with non-compact axiom.

Language of non-compact modal logic

♠ We apply extended representation theorem to non-compact modal logics.

We consider propositional modal logics such that:

- with a countable set $\{\Box_i \mid i \in \omega\}$ of modal operators;
- without infinitary connectives;
- with non-compact axiom.

Language of non-compact modal logic

♠ We apply extended representation theorem to non-compact modal logics.

We consider propositional modal logics such that:

- with a countable set $\{\Box_i \mid i \in \omega\}$ of modal operators;
- without infinitary connectives;
- with non-compact axiom.

Language of non-compact modal logic

♠ We apply extended representation theorem to non-compact modal logics.

We consider propositional modal logics such that:

- with a countable set $\{\Box_i \mid i \in \omega\}$ of modal operators;
- without infinitary connectives;
- with non-compact axiom.

Non-compact theory of modal logic

Non-compact theory:

- A theory T is said to be *satisfiable*, if there exists a Kripke model $\mathcal{M}$ and a world w in $\mathcal{M}$ such that any formula in T is valid at w.
- A theory T is said to be *compact*, if T is satisfiable if and only if every finite subset of T is satisfiable.

Non-compact theory of modal logic

Example

Let $\Box_c\phi$ be the formula which expresses that ϕ is a common knowledge among the agent I and $\Box_i$ the formula which says that $i \in I$ knows ϕ. Now, the common knowledge condition is expressed in a Kripke frame in the following way:

$$w \vDash \Box_c\phi \iff \forall n \in \omega \forall i_1, \ldots, i_n \in I(w \vDash \Box_{i_1} \cdots \Box_{i_n} \phi)$$

Then, the theory

$$\{\Box_{i_1} \cdots \Box_{i_n} \phi \mid n \in \omega,\ i_1, \ldots, i_n \in I\} \cup \{\neg \Box_c \phi\}$$

is not compact.

Non-compact modal logic

Let Γ be a countable set of formulas. We say that an axiom $\Gamma \to \phi$ is compact, if the theory $T = \Gamma \cup \{\neg\phi\}$ is compact.

Let α be a set $\{\Gamma_i \to \phi_i \mid i \in \omega\}$ of axiom schemata. The symbol $\mathbf{K}(\alpha)$ denotes the logic defined by $\mathbf{K}$ and the set α of axiom schemata.

We say that $\mathbf{K}(\alpha)$ is non-compact if some of the axioms in α is non-compact.

♠ We assume the following condition:

($\sharp$) The set α is consistent and all instances of $\Gamma_i \to \phi_i$ in α is countable for each $i \in \omega$.

Non-compact modal logic

Example

Axiom schemata CK for common knowledge logic is:

1. $\bigwedge\{\Box_{i_1}\cdots\Box_{i_n} p \mid n \in \omega,\ i_1, \ldots, i_n \in I\} \to \Box_c p$;
2. $\Box_c p \to \Box_{i_1}\cdots\Box_{i_n} p\ (n \in \omega,\ i_1, \ldots, i_n \in I)$.

CK satisfies $(\sharp)$.

Non-compact modal logic

Example

Axiom schemata DL for dynamic logic is:

1. $[a+b]p \to [a]p$, $[a+b]p \to [b]p$;
2. $[a]p, [b]p \to [a+b]p$;
3. $[a;b]p \to [a][b]p$;
4. $[a][b]p \to [a;b]p$;
5. $[a^*]p \to [a]^n p$ for any $n \in \omega$;
6. $\{[a]^n p \mid n \in \omega\} \to [a^*]p$.

DL satisfies $(\sharp)$.

Model-existence theorem for non-compact logic

Theorem

Let α be a set of axiom schemata which satisfies ($\sharp$). Then, there exists a Kripke model $\mathcal{M}$ such that α is valid in $\mathcal{M}$, and $\phi \in \mathbf{K}(\alpha)$ if and only if $\mathcal{M} \models \phi$ for any formula ϕ.

Proof.

For each axiom $\Gamma \to \phi$ in α, we have

$$\bigvee -\Gamma \cup \{\phi\} = 1$$

in Lindenbaum algebra. Apply extended representation theorem so as to preserve these infinite joins. ❑

♠ We can also discuss predicate non-compact modal logics in the same way.

Jankov's theorem for canonical formula

♠ We extend Jankov's theorem of canonical formula for finite Heyting algebras to infinite Heyting algebras.

Theorem

(Jankov 69). For any subdirectly irreducible finite Heyting algebra A, there exists a canonical formula χ_A with such a property that for any Heyting algebra B, $B \not\models \chi_A$ if and only if there exists a Heyting algebra D, a monomorphism $e : A \to D$ and an epimorphism $f : B \to D$.

Jankov formula

Let A be a subdirectly irreducible complete Heyting algebra and β the second greatest element of A. Then, *the extended Jankov's formula* χ_A for A is defined by

$$\begin{aligned}\chi_A \quad = \quad & \Big(\bigwedge_{X \subseteq A} (p_{\bigwedge X} \equiv \bigwedge_{x \in X} p_x) \wedge \bigwedge_{X \subseteq A} (p_{\bigvee X} \equiv \bigvee_{x \in X} p_x) \\ & \wedge \bigwedge_{x,y \in A} (p_{x \to y} \equiv p_x \supset p_y) \wedge \bigwedge_{x \in A} (p_{x \to 0_A} \equiv \neg p_x) \Big) \\ & \supset p_\beta ,\end{aligned}$$

where $\{p_x \mid x \in A\}$ is a set of pairwise distinct propositional variables corresponding to the elements of A.

- It is straightforward to show that for any CHA B, if there exists a CHA D, a continuous monomorphism $e : A \to D$ and a continuous epimorphism $f : B \to D$, then $B \nvDash \chi_A$.
- However, we cannot prove the converse by the method developed by Jankov. To show it by Jankov's method, we have to show that if $v(\chi_A) \neq 1_B$, then there exists a continuous morphism $f : B \to D$ such that $|v(\chi_A)|$ is the second greatest element of $B/f^{-1}[1_D]$. But, we have a counter example of this.
- So, we need a notion which is weaker than continuous morphisms, to show it by the method developed by Jankov.

Pseudo continuous homomorphism

Definition

Let A and B be complete Heyting algebras, $f : A \to B$ a homomorphism of Heyting algebras, $F = f^{-1}[1_B]$ and $b = \bigvee(A \setminus F)$. Then, f is said to be *pseudo continuous*, if

$$\bigwedge\{b \to x \in F \mid x \in A\} \in F.$$

Let $a \in A$ and $a \neq 1_A$. A pseudo continuous homomorphism $f : A \to B$ is said to be a *pseudo continuous homomorphism for* a, if $|a|$ is the second greatest element of A/F and $F = A \setminus (\downarrow b)$.

Pseudo continuous homomorphism

♠ A pseudo continuous homomorphism $f : A \to B$ preserves all meets and joins in $A \setminus F$, where $F = \{x \in A \mid f(x) = 1_B\}$.

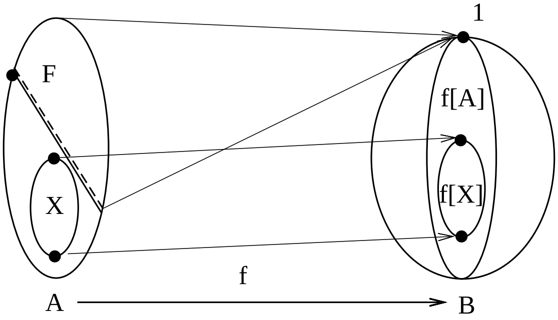


Figure: Pseudo continuous homomorphism

T-regular Heyting algebra

Definition

Let A be a complete Heyting algebra. An element $a \neq 1_A$ in A is said to be *T-regular*, if for any $c \in A$ with $c \nleq a$, there exists $b \in A$ such that

1. $F_a = A \setminus (\downarrow b)$ is a filter of A (i.e., b is meet irreducible);
2. $|a|$ is the second greatest element of the quotient algebra A/F_a;
3. $\bigwedge\{b \to x \in F_a \mid x \in A\} \in F_a$;
4. $c \in F_a$.

If every $a \in A$ but 1_A is T-regular, A is said to be T-regular.

T-regular Heyting algebra

♠ For any $a \not\geq c$, there is a prime filter F such that $a \notin F$, $c \in F$ and the projection $p : A \to A/F$ preserves all meets and joins in $A \setminus F$.

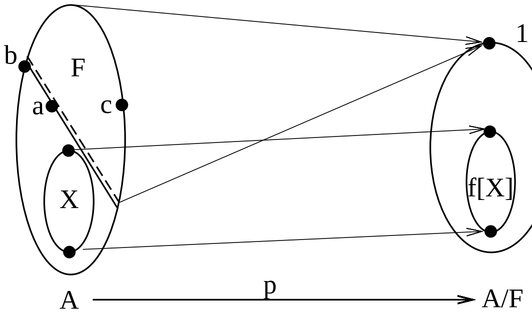


Figure: T-regular algebra

Main theorem

Theorem

Let A be a subdirectly irreducible complete Heyting algebra and β the second greatest element of A. Suppose B is a complete T-regular Heyting algebra. Then, the following two conditions are equivalent:

1. *There exists a complete Heyting algebra D, a continuous monomorphism $e : A \to D$ and a pseudo continuous epimorphism $f : B \to D$ for some $\overline{\beta} \in f^{-1}[\{e(\beta)\}]$.*
2. $B \not\models \chi_A$.