Lambda Calculus between Algebra and Topology

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Lambda Calculus between Algebra and Topology

$$\begin{tikzcd} & \lambda-\text{calculus} \arrow[dl] \arrow[dr] & \\ \text{Algebra} & {} \arrow[l] \arrow[r] & \text{Topology} \end{tikzcd}$$

- From Lambda calculus to Universal Algebra:

(i) Lambda abstraction algebras

(ii) Church algebras

(iii) Boolean-like-algebras

(iv) n-subtractive algebras

- From Universal Algebra to Lambda calculus:

(i) The structure of the lattice of the λ-theories

(ii) Boolean algebras, Stone representation theorem and the indecomposable semantics

(iii) The order-incompleteness problem

Lambda Calculus between Algebra and Topology

Topology refines partial orderings through the separation axioms:

A space (X, τ) is T_0 iff the specialization preorder $\leq_\tau$ is a partial order.
Every partial order is the specialization order of a space.

- From Lambda calculus to Topology:

(i) new axioms of separation

(ii) topological algebras

(iii) Visser spaces and Priestley spaces

- From Topology to Lambda calculus:

(i) Topological incompleteness/completeness theorems

(ii) Topological models

(iii) The equational completeness problem for Scott semantics

Part 0
Lambda Calculus: Church, Curry, Scott

Scott

- Church (around 1930): Lambda calculus

- λ-theory = congruence w.r.t. application and λ-abstractions containing $\alpha\beta$-conversion

- Scott: First model and Continuous Semantics (1969) A *continuous model* $\mathcal{D}$ is a reflexive object in the category **CPO** of complete partial orderings.

- All known models of λ-calculus admits a compatible (w.r.t. application) partial order and are topological algebras (w.r.t. Scott topology).

- Each model $\mathcal{D}$ defines an equational theory and an order theory:

$$Eq(\mathcal{D}) = \{(M,N) : |M|^{\mathcal{D}} = |N|^{\mathcal{D}}\}; \quad Ord(\mathcal{D}) = \{(M,N) : |M|^{\mathcal{D}} \leq |N|^{\mathcal{D}}\}$$

Theory (In)Completeness Problem

- A class $\mathbb{C}$ of models of λ-calculus is theory complete if

$$(\forall \text{ consistent } \lambda\text{-theory } T)(\exists \mathcal{M} \in \mathbb{C}) \;\; Th_{=}(\mathcal{M}) = T.$$

  Theory incomplete, otherwise.

  Theorem 1 *(Theory incompleteness) All known semantics are theory incomplete. Honsell-Ronchi: Scott semantics; Bastonero-Gouy: stable semantics; Salibra: strongly stable semantics and all pointed po-models.*

- Selinger (1996) asked: Are partial orderings intrinsic to computations?

- A *po-model* is a pair $(\mathcal{M}, \leq)$, where $\mathcal{M}$ is a model and $\leq$ is a nontrivial partial ordering on $\mathcal{M}$ making the application monotone.

- The order-completeness problem by Selinger asks whether the class $\mathbb{PO}$ of po-models is theory complete or not.

  ANSWER: Unknown.

  The best we know about order-incompleteness:

  Theorem 2 (Carraro-S. 2013) *There exists a λ-theory T such that, for every po-model $(\mathcal{M}, \leq)$,*

$$Th_{=}(\mathcal{M}) \supseteq T \Rightarrow (\mathcal{M}, \leq) \textit{ has infinite connected components}$$

and the connected component of the looping term Ω is a singleton set.

Two Other Open Problems

- Equational Completeness Problem asks whether there exists a Scott continuous model whose equational theory is the least λ-theory $\lambda\beta$:

$$Eq(\mathcal{D}) = \lambda\beta, \quad \text{for some Scott continuous model } \mathcal{D}$$

  ANSWER: Unknown

- Equational Consistency Problem (Honsell-Plotkin 2006) asks whether, for every finite set E of equations between λ-terms consistent with the λ-calculus, there exists a Scott continuous model contemporaneously satisfying all equations of E.

  $(\forall E$ finite set of identities$)[(E \cup \lambda\beta$ consistent$) \to (\exists \mathcal{D} \in$ Scott$)\ \mathcal{D} \models E]$?

  ANSWER: No (Carraro-S. 2013).

- These problems and the order incompleteness problem are interconnected

Part I
Topology: Scott, Selinger, Visser, Priestley

The technique for the Equational Completeness Problem

Remember that the Equational Completeness Problem asks whether there exists a Scott continuous model whose equational theory is the least λ-theory $\lambda\beta$:

$$Eq(\mathcal{D}) = \lambda\beta, \quad \text{for some Scott continuous model } \mathcal{D}.$$

Given a class $\mathbb{C}$ of po-models, we sometimes are able to construct an "effective" po-model $\mathcal{E}$ (maybe not in the class $\mathbb{C}$) such that

$$Ord(\mathcal{E}) \subseteq Ord(\mathcal{D}), \quad \text{for all } \mathcal{D} \in \mathbb{C}.$$

Lemma 1 *If $\mathcal{E}$ is an effective po-model, then, after encoding,*

1. *$|M|^{\mathcal{E}}$ is an r.e. element of the model for every closed λ-term M;*
2. *$|M|^{\mathcal{E}}$ is a decidable element for every closed normal form M.*
3. *$\{N : \mathcal{E} \models N \leq \lambda x.x\}$ is co-r.e.*

Visser

Theorem 3 *(Berline-Manzonetto-S. 2007) Given a class $\mathbb{C}$ of models, if there exists an "effective" model $\mathcal{E}$ such that*

$$(\forall \mathcal{D} \in \mathbb{C})\ Ord(\mathcal{E}) \subseteq Ord(\mathcal{D}),$$

then, for every model $\mathcal{D} \in \mathbb{C}$, we have:

(i) $Ord(\mathcal{D})$ is not r.e.

(ii) $Eq(\mathcal{D}) \neq \lambda\beta$.

Proof. Define Visser topology over the set Λ of λ-terms (modulo $\lambda\beta$):

$X \subseteq \Lambda$ is Visser base open if it is β-closed and co-r.e.

Theorem 4 *(Visser 1980) Visser topology is hyperconnected on Λ.*

(i) Assume $Ord(\mathcal{D})$ to be r.e. for some $\mathcal{D} \in \mathbb{C}$.

$$\underbrace{\{N : \mathcal{E} \models N \leq \lambda x.x\} \text{ co-r.e.}}_{\text{Visser open}} \subseteq \underbrace{\{N : \mathcal{D} \models N \leq \lambda x.x\} \text{ r.e.}}_{\text{Visser closed}}$$

$$\Downarrow$$

$$\{N : \mathcal{D} \models N \leq \lambda x.x\} = \Lambda.$$

(ii) (Selinger 1996) If $Eq(\mathcal{D}) = \lambda\beta$ for a po-model $\mathcal{D}$, then the term denotations are an antichain. Consequence: $Eq(\mathcal{D}) = Ord(\mathcal{D}) = \lambda\beta$ are r.e.

The Equational Completeness Problem

SOME RESULTS (Carraro-S. 2009):

- $\lambda\beta\eta$ is not the theory of a model living in the category of Scott domains.
- $\lambda\beta$ is not the theory of a filter model living in **CPO**.

Leaving Lambda Calculus Towards Computability Theory (Work in Progress)

- An enumerated set is a pair $X = (|X|, \phi_X)$, where $\phi_X : \omega \to |X|$ is an onto map (Mal'cev 1964)

- Recursive Functions: $\phi : \omega \to \mathsf{RecFun}$ mapping the "program" n into the function ϕ_n computed by the program "n".

- Lambda Calculus: $\phi_\Lambda : \Lambda \to \Lambda/\lambda\beta$ mapping a λ-term M into its equivalence class $[M]_\beta$.

- Given an enumerated set X, we define
  $Y \subseteq X$ r.e. (co-r.e., decidable) if $\phi_X^{-1}(Y)$ is r.e. (co-r.e., decidable).

The Visser topology

- The r.e. sets of X are a ring $\mathcal{R}_X$ of sets generating a topology τ_E on $|X|$.

- The co-r.e. sets of X are a ring co-$\mathcal{R}_X$ of sets generating a topology τ_V on $|X|$.

$(|X|, \tau_E, \tau_V)$ is a bitopological space

τ_E is the Ershov topology and τ_V is the Visser topology

Lemma 2 *τ_E is T_0 iff τ_V is T_0 ($a \leq_E b$ iff $b \leq_V a$).*

- Recursive Functions: τ_E on RecFun is the Scott topology; while τ_V is T_0 with $f{\downarrow}$ τ_V-open iff $graph(f)$ is decidable.

- Lambda Calculus: τ_E on $\Lambda/\lambda\beta$ is the discrete topology, while τ_V is non-trivial, hyperconnected and T_1.

The Visser topology

Definition 1 *(Visser) An enumerated set* $X = (|X|, \phi)$ *is* pre-complete *if, for every partial recursive function* f*, there exists a total recursive function* g *such that*

$$f{\downarrow}n \;\Rightarrow\; \phi_{f(n)} = \phi_{g(n)}.$$

Proposition 1 *(i) The set of computable functions is precomplete.*
(ii) (Visser) Λ/T *(*T *a* λ*-theory) is precomplete.*

Proof. (i) Define

$$\phi_{g(x)}(y) = \begin{cases} \phi_{f(x)}(y), & \text{if } x \in dom(f) \\ \uparrow, & \text{otherwise} \end{cases}$$

(ii) Let $\ulcorner M \urcorner$ be the Godel number of λ-term M and let n_λ be the λ-term denoted by the number n. Barendregt has shown that there exists a λ-term E such that, for every M, $E\ulcorner M \urcorner =_{\lambda\beta} M$. Let F be a λ-term representing the computable function f. Define

$$g(n) = \ulcorner E(F\underline{n})\urcorner.$$

$$g(n)_\lambda = E(F\underline{n}) =_{\lambda\beta} E(\underline{f(n)}) =_{\lambda\beta} f(n)_\lambda.$$

Proposition 2 *(Visser) If the enumerated set $X = (|X|, \phi)$ is pre-complete then*

1. *τ_V is hyperconnected;*

2. *τ_E is compact iff $\leq_{\tau_E}$ has a bottom element.*

Proof: (1) If $V \cup U = \omega$, where V and U are r.e. and ϕ-closed sets of natural numbers, then either $V = \omega$ or $U = \omega$.
By contraposition assume that neither V nor U is ω. Let $a \in V \setminus U$ and $b \in U \setminus V$. Let A and B be two recursively inseparable sets of natural numbers. Define the partial function

$$f(x) = \begin{cases} a, & \text{if } x \in A \\ b, & \text{if } x \in B \\ \uparrow, & \text{otherwise} \end{cases}$$

Consider a total recursive function g completing f up to ϕ-equivalence. We have $g^{-1}(V) \cup g^{-1}(U) = \omega$, $A \subseteq g^{-1}(V) \setminus g^{-1}(U)$ and $B \subseteq g^{-1}(U) \setminus g^{-1}(V)$. Then A and B are recursively separable. Contradiction.

(2) By (1) every finite covering of $|X|$ must contain $|X|$.

The Priestley space of computability

Hereafter we always assume that τ_E is T_0.

Proposition 3 *$\tau_E \vee \tau_V$ is zero-dimensional (i.e., it has a base of clopens), Hausdorff and satisfies the Priestley separation axiom (w.r.t. $\leq_E$).*

Proof: Let $a, b \in X$. Since τ_E is T_0, either $a \not\leq_E b$ or $b \not\leq_E a$. In the first case there is an r.e. open U such that $a \in U$ but $b \notin U$. $X \setminus U$ is a co-r-e open such that $b \in X \setminus U$ and $a \notin X \setminus U$.

Let $x \not\leq_E y$. Then there is an r.e. set U such that $x \in U$ but $y \notin U$. U is E-upper. The complement is E-down which contains y but not x.

Proposition 4 *Let X be an enumerated set. If $(|X|, \tau_E)$ is a T_0-space, then the compactfication of $(|X|, \tau_E \vee \tau_V)$ is a Priestly space and $(|X|, \tau_E \vee \tau_V)$ is a dense subspace of this compactification.*

Proof: We consider the ring $\mathcal{R}_X$ of r.e. subsets of $|X|$ and consider the product topology on $2^{\mathcal{R}_X}$. Consider the closed subspace of lattice homomorphisms $HOM(\mathcal{R}_X^E, 2)$. It is Priestley (because closed), and $(|X|, \tau_E \vee \tau_V)$ embeddes into $HOM(\mathcal{R}_X, 2)$ as a dense subspace.

Consider $2 = \{0, 1\}$ with three topologies:

- The discrete top τ_d; The top τ_0 with $0 < 1$; The top τ_1 with $1 < 0$.

We have $\tau_d = \tau_0 \vee \tau_1$. We consider the ring $\mathcal{R}_X^E$ of r.e. subsets of $|X|$ and consider the product topology on $2^{\mathcal{R}_X^E}$. We have:

- The topology $\prod \tau_0$ on $2^{\mathcal{R}_X^E}$ is the Scott topology w.r.t. $\subseteq$;

- The topology $\prod \tau_1$ on $2^{\mathcal{R}_X^E}$ is the Scott topology w.r.t. $\supseteq$;

- The topology $\prod \tau_d = \prod \tau_0 \vee \prod \tau_1$ on $2^{\mathcal{R}_X^E}$ is a Priestley space.

Consider the closed subspace of lattice homomorphisms $HOM(\mathcal{R}_X^E, 2)$. It is Priestley (because closed), and $(|X|, \tau_E \vee \tau_V)$ embeddes into $HOM(\mathcal{R}_X^E, 2)$ as a dense subspace.

We consider a map $e : X \to \mathrm{Hom}(\mathcal{R}X, 2)$ defined as follows, for every r.e. set Y and every $x \in X$: $e(x)(Y) = 1$ iff $x \in Y$.

The map e is bi-continuous because, for every r.e. set Y, $Y = e^{-1}(\{f : f(Y) = 1\})$ and $X \setminus Y = e^{-1}(\{f : f(Y) = 0\})$.

The codomain of X is a dense subspace Y of Hom$(\mathcal{R}X, 2)$.

X is homeomorphic to Y iff the ring $\mathcal{R}X$ distinguishes the points of X.

Remark: What is the compactfication of lambda calculus? We extend the application operator and the lambda-abstractions to its compactification.

Part II
Algebras: Stone, Boole and Church

Stone and Boole

Theorem 5 *(Stone Representation Theorem)*

- *Every Boolean algebra is isomorphic to a field of sets.*
- *Every Boolean algebra can be embedded into a Boolean product of indecomposable Boolean algebras (*$\mathbf{2}$ *is the unique indecomposable Boolean algebra!).*

Then every Boolean algebra is isomorphic to a subalgebra of $\mathbf{2}^I = \mathcal{P}(I)$ for a suitable set I.

Generalisations to other classes of algebras by Pierce (rings with unit) Comer and Vaggione.

Combinatory algebras (CA) and λ-abstraction algebras (LAA) satisfy an analogous theorem...

Church algebras

The untyped λ-calculus has truth values $0, 1$ and "if-then-else" construct $q(x, y, z)$ of programming:

- λ-calculus (LAA): $1 \equiv \lambda xy.x$; $\quad 0 \equiv \lambda xy.y$; $\quad q(e, x, y) = (ex)y$
- Combinatory logic (CA): $1 \equiv \mathbf{k}$; $\quad 0 \equiv \mathbf{sk}$; $\quad q(e, x, y) = (ex)y$
- Boolean algebras: $q(e, x, y) = (e \wedge x) \vee (\neg e \wedge y)$
- Rings with unit 1: $q(e, x, y) \equiv ex + (1 - e)y$.

Definition 2 *(Manzonetto-Salibra 2008) An algebra* $\mathbf{A}$ *is a* Church algebra *if it admits two constants* $0, 1$ *and a ternary term* $q(x, y, z)$ *satisfying:*

$$q(1, x, y) = x; \qquad q(0, x, y) = y.$$

There are equations which are contemporaneously satisfied by 0 and 1: for example,

$$q(1, x, x) = x; \qquad q(0, x, x) = x.$$

Central elements

An element e of a Church algebra $\mathbf{A}$ is *central* if

$$\mathbf{A} \cong \mathbf{A}/Cong(e=1) \times \mathbf{A}/Cong(e=0).$$

Lemma 3 *Let $\mathbf{A}$ be a Church algebra et $e \in \mathbf{A}$. The following conditions are equivalent:*

- *e is central;*
- *e satisfies the following identities:*

$$\begin{array}{rl} (i) & q(e,x,x) = x. \\ (ii) & q(e,q(e,x,y),z) = q(e,x,z) = q(e,x,q(e,y,z)). \\ (iii) & q(e,f(\overline{x}),f(\overline{y})) = f(q(e,x_1,y_1),\dots,q(e,x_n,y_n)),\ \forall \text{ operation } f \\ (iv) & e = q(e,1,0). \end{array}$$

Central elements are the unique way to decompose the algebra as Cartesian product.

$\mathbf{A}$ is indecomposable if the unique central elements are $0, 1$.

Stone, Boole and Church

Theorem 6 • *The central elements of a Church algebra* $\mathbf{A}$ *constitute a Boolean algebra:*

$$e \vee d = q(e, 1, d); \quad e \wedge d = q(e, d, 0); \quad \neg e = q(e, 0, 1)$$

• *Let* $\mathcal{V}$ *be a variety of Church algebras,* $\mathbf{A} \in \mathcal{V}$ *and* $\mathcal{F}$ *be the Boolean space of maximal ideals of the Boolean algebra of central elements of* $\mathbf{A}$*. Then the map*

$$f : A \to \Pi_{I \in \mathcal{F}}(A/\theta_I),$$

defined by

$$f(x) = (x/\theta_I : I \in \mathcal{F}),$$

gives a **weak** *Boolean product representation of* $\mathbf{A}$*. The quotient algebras* $\mathbf{A}/\theta_I$ *are directly indecomposable if the indecomposable members of* $\mathcal{V}$ *constitute a universal class. (True for CA and LAA!)*

Central elements at work in lambda calculus!

The indecomposable CAs (models of λ-calculus) are the building blocks of CA.

The **indecomposable semantics** is the class of models which are indecomposable as combinatory algebras.

Theorem 7 *Scott is always simple!*

Proof: Scott continuous semantics (and the other known semantics of λ-calculus) are included within the indecomposable semantics, because every Scott model is simple (i.e., it admits only trivial congruences) as a combinatory algebra.

Central elements at work in lambda calculus!

Theorem 8 *The algebraic incompleteness theorem: There exists a continuum of λ-theories which are not equational theories of indecomposable models.*

Proof:

1.Decomposable CAs are closed under expansion.
2. $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ is a non-trivial central element in the term algebra of a suitable λ-theory ϕ, because

- the λ-theory ψ_1 generated by $\Omega = \lambda xy.x$ is consistent;

- the λ-theory ψ_2 generated by $\Omega = \lambda xy.y$ is consistent;

- Ω is central in the term algebra of $\phi = \psi_1 \cap \psi_2$.

3. All models of ϕ are decomposables!

The algebraic incompleteness theorem encompasses all known theory incompleteness theorems:

(Honsell-Ronchi 1992) Scott continuous semantics;

(Bastonero-Guy 1999) Stable semantics;

(Salibra 2001) Strongly stable semantics.

Central elements at work in universal algebra!

1. *Boolean-like algebras*: Church algebras (of any algebraic type), where all elements are central.

2. *Semi-Boolean-like algebras*: Church algebras (of any algebraic type), where all elements are semi-central.

   Theorem 9 *A double pointed variety is discriminator iff it is idempotent semi-Boolean-like and* 0*-regular.*

3. Lattices of equational theories

Part III: The λ-calculus is algebraic

Lambda terms

- Algebraic similarity type Σ:
  - Nullary operators: $x, y, z, \ldots$ (names = variables of λ-calculus)
  - Binary operator: $\bullet$ (application)
  - Unary operators: $\lambda x, \lambda y, \lambda z, \ldots$ (λ-abstractions)

- A λ-term is a ground Σ-term (no algebraic variable)

$$\lambda x.xy$$

- A context is just a term of type Σ; algebraic variables $a, b, c, \ldots$ (holes in Barendregt's terminology) may be involved

$$\lambda x.xa$$

Two substitutions

- Substitution for names (with α-conversion)

$$(\lambda x.xy)[y := x] = \lambda z.zx$$

We care...

- Substitution for variables (without α-conversion)

$$(\lambda x.xa)[a := x] = \lambda x.xx$$

We do not care...

The untyped λ-calculus

- Let

$$\mathbf{\Lambda} = (\Lambda, \cdot, \lambda x, x)_{x \in \mathsf{Names}}$$

be the absolutely free $\mathbf{\Sigma}$-algebra over an empty set of generators.

A λ-theory is any congruence on $\mathbf{\Lambda}$ (i.e., equivalence relation compatible w.r.t. application and λ-abstractions) including $\alpha\beta$-conversion

It seems that Universal Algebra cannot be applied to λ-calculus because $\alpha\beta$-conversion does not involve algebraic variables!

Two starting points for the algebraic λ-calculus

- The lattice λT of λ-theories

$$\cong$$

The lattice of congruences of the term algebra $\mathbf{\Lambda}/\lambda\beta$
($\lambda\beta$ is the least congruence on $\mathbf{\Lambda}$ including α- and β-conversion)

- The variety (equational class) generated by $\mathbf{\Lambda}/\lambda\beta$

$$=$$

Class of Σ-algebras satisfying all identities between contexts satisfied by $\mathbf{\Lambda}/\lambda\beta$

- CA $=$ class of combinatory algebras (Curry-Schönfinkel 1920-30)
- LAA $=$ class of λ-abstraction algebras (Pigozzi-Salibra 1993)

Theorem 10 *(Salibra 2000)* LAA $=$ *Variety*($\mathbf{\Lambda}/\lambda\beta$)

The algebraic lambda calculus

Theorem 11 *(S. 2000) The variety generated by $\mathbf{\Lambda}/\lambda\beta$ is axiomatized by:*

(β_1) $(\lambda a.a)x = x$

(β_2) $(\lambda a.b)x = b \qquad (b \neq a)$

(β_3) $(\lambda a.x)a = x$

(β_4) $(\lambda aa.x)y = \lambda a.x$

(β_5) $(\lambda a.xy)z = (\lambda a.x)z \cdot (\lambda a.y)z$

(β_6) $(\lambda b.y)c = y \;\Rightarrow\; (\lambda ab.x)y = \lambda b.(\lambda a.x)y \qquad (c \neq b,\ a \neq b)$

(α) $(\lambda b.x)c = x \;\Rightarrow\; \lambda a.x = \lambda b.(\lambda a.x)b \qquad (a \neq b)$

Algebras satisfying (β_1)-(β_6) and (α) are called lambda abstraction algebras (LAAs, for brevity) and were introduced by Pigozzi-S. (1993)

The algebraic lambda calculus

- An element of a LAA may depend on all possible names in $Na = \{x, y, z, x_1, y_1, z_1, \dots\}$
  - Cartesian product: $\langle x, y, z, x_1, y_1, z_1, \dots\rangle \in (\mathbf{\Lambda}/\lambda\beta)^{Na}$
  - Lambda theories of infinitary λ-calculus: $\lambda x.x(y(z(x_1(y_1(x_1(\dots) .$
- Examples of LAAs:
  - The term algebra $\mathbf{\Lambda}/\phi$ of a (infinitary) λ-theory ϕ
  - Algebras of functions obtained by the models of λ-calculus

LAA **and Universal Algebra**

Universal Algebra: A variety $\mathcal{V}$ is studied by means of the lattice identities contemporaneously satisfied by all congruence lattices of the algebras in $\mathcal{V}$.

A priori we can apply the last 30 years of Universal Algebra to the variety LAA:

Theorem 12 (Lusin-Salibra 2004) *Every lattice identity holding in (all congruence lattices of algebras in)* LAA *is trivial.*

But Universal Algebra is at work!

Universal Algebra at work

Theorem 13 *(Salibra 2000) The lattice λT of λ-theories is isomorphic to the lattice of equational theories of* LAA*'s.*

Corollary 1 *Every variety of* LAAs *is generated by the term algebra $\mathbf{\Lambda}/\phi$ of a suitable λ-theory ϕ.*

- λ-theory $\phi \Leftrightarrow$ Variety generated by term algebra $\mathbf{\Lambda}/\phi$.
- λ-theory $\phi \Leftrightarrow$ Lattice interval $\{\psi : \psi \geq \phi\} \cong$ congruence lattice of $\mathbf{\Lambda}/\phi$
- λ-calculus problem $=$ Problem of existence of a subvariety of LAA.

Example: Order-incompleteness problem (by Selinger)

The lattice λT of λ-theories

Conjecture: Every nontrivial lattice identity fails in λT

- (Visser 1980)
  - Every countable poset embeds into λT by an order-preserving map.
  - Every lattice interval $[\phi, \psi]$ (ϕ, ψ r.e.) λ-theories has a continuum of elements.

- (Lusin-Salibra 2004) λT satisfies the Zipper condition:

$$\phi \vee \psi = 1 \text{ and } \delta \wedge \phi = \delta \wedge \psi \;\Rightarrow\; \delta \leq \phi \wedge \psi.$$

- (Salibra 2001) λT is not modular.

- (Berline-Salibra 2006) $\exists$ a finite axiomatisable λ-theory ϕ such that the lattice interval $[\phi) = \{\psi : \psi \geq \phi\}$ is distributive.

- (Statman 2001) The meet of all coatoms of λT is $\neq \lambda\beta$. (i.e., there exist equations $M = N$ such that $\phi \cup \{M = N\}$ is consistent for every consistent λ-theory ϕ).

- (Manzonetto-Salibra 2008)
  ($\forall$natural number n)($\exists\lambda$-theory ϕ_n) such that the interval sublattice
  $[\phi_n) = \{\psi : \psi \geq \phi_n\}$ is isomorphic to the finite Boolean lattice 2^n.

Part IV
Separability: Selinger, Coleman, Kearnes, Sequeira

Theory (In)Completeness Problem

- A class $\mathbb{C}$ of models of λ-calculus is theory complete if

  $$(\forall \text{ consistent } \lambda\text{-theory } T)(\exists \mathcal{D} \in \mathbb{C}) \quad Eq(\mathcal{D}) = T.$$

  Theory incomplete, otherwise.

  Theorem 14 *(Theory incompleteness) All known semantics are theory incomplete.*
  Honsell-Ronchi (1984): Scott semantics;
  Bastonero-Gouy (1996): stable semantics;
  S. (2001): strongly stable semantics and all pointed po-models.

- A *po-model* is a pair $(\mathcal{D}, \leq)$, where $\mathcal{D}$ is a model and $\leq$ is a nontrivial partial ordering on $\mathcal{D}$ making monotone the application operator.

- Selinger (1996) The order-completeness problem asks whether the class $\mathbb{PO}$ of po-models is theory complete or not.

  ANSWER: Unknown.

  The best we know about order-incompleteness:

  Theorem 15 (Carraro-S. 2013) *There exists a finitely axiomatizable λ-theory T such that, for every po-model $(\mathcal{D}, \leq)$,*

  $$Eq(\mathcal{D}) \supseteq T \Rightarrow (\mathcal{D}, \leq) \text{ has infinite connected components}$$

  and the connected component of the looping term Ω is a singleton set.

The Order-Incompleteness Problem

Theorem 16 (Hagemann 73, Selinger 96, Coleman 96-97) *Let $\mathcal{V}$ be a variety of algebras. Then the following conditions are equivalent:*

1. *$\mathcal{V}$ is n-permutable for some $n \geq 2$ (i.e., $\theta \vee \phi = \theta \circ \phi \circ \theta \circ \cdots \circ \phi$ (n-times)).*
2. *There exist a natural number $n \geq 2$ and ternary terms $p_1, \ldots, p_{n-1}$ in the type of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the following Mal'cev identities:*
$$\begin{array}{rcl} x & = & p_1(x,y,y); \\ p_i(x,x,y) & = & p_{i+1}(x,y,y) \quad (i = 1, \ldots, n-2); \\ p_{n-1}(x,x,y) & = & y. \end{array}$$
3. *Every T_0-topological algebra in $\mathcal{V}$ is T_1.*
4. *Every T_0-topological algebra in $\mathcal{V}$ is T_1 and sober.*
5. *Every algebra in $\mathcal{V}$ is unorderable.*
6. *Every compatible preorder on an algebra in $\mathcal{V}$ is symmetric (and thus a congruence).*

The order incompleteness problem is equivalent to find an n-permutable variety of combinatory algebras.

The Order-Incompleteness Problem

In the case a variety $\mathcal{V}$ has two constants 0 and 1, the Mal'cev identities give:

$$\begin{aligned} 0 &= p_1(0,1,1); \\ p_i(0,0,1) &= p_{i+1}(0,1,1) \quad (i = 1, \ldots, n-2); \\ p_{n-1}(0,0,1) &= 1. \end{aligned}$$

If we define the unary term operations $f_i(x) = p_i(0, x, 1)$, then the above identities can be written as follows:

$$0 = f_1(1); \qquad f_i(0) = f_{i+1}(1) \quad (i = 1, \ldots, n-2); \qquad f_{n-1}(0) = 1. \tag{1}$$

This suggests the following theorem:

Theorem 17 *Let $\mathcal{V}$ be a variety with two constants 0 and 1. Then the constants 0 and 1 are incomparable in all ordered algebras in $\mathcal{V}$ if, and only if, there exist a natural number $n \geq 2$ and unary terms $f_1, \ldots, f_{n-1}$ in the type of $\mathcal{V}$ such that the identities (1) hold in $\mathcal{V}$.*

The Order-Incompleteness Problem

In the case a variety $\mathcal{V}$ has a constant 0, then we can relativise the Mal'cev identities of Theorem **??** as follows:

$$\begin{aligned} 0 &= p_1(0,y,y); \\ p_i(0,0,y) &= p_{i+1}(0,y,y) \quad (i=1,\ldots,n-2); \\ p_{n-1}(0,0,y) &= y. \end{aligned}$$

If we define the binary term operations $s_i(y,x) = p_i(0,x,y)$, then the above identities can be written as follows:

$$\begin{aligned} 0 &= s_1(x,x) \\ s_i(x,0) &= s_{i+1}(x,x) \quad (i=1,\ldots,n-2); \\ s_{n-1}(x,0) &= x. \end{aligned} \tag{2}$$

Relaxing the Order-Incompleteness Problem

Theorem 18 (Carraro-S. 2013) *Let $\mathcal{V}$ be a variety of algebras with* 0*. Then the following conditions are equivalent:*

1. *$\mathcal{V}$ is n-subtractive for some $n \geq 2$, that is, there exist $n \geq 2$ and binary terms $s_1, \ldots, s_{n-1}$ such that $\mathcal{V}$ satisfies the Mal'cev identities:*

$$\begin{array}{rcll} 0 & = & s_1(y, y); & \\ s_i(y, 0) & = & s_{i+1}(y, y) & (i = 1, \ldots, n-2); \\ s_{n-1}(y, 0) & = & y. & \end{array}$$

2. *Every T_0-topological algebra in $\mathcal{V}$ is T_1-separated in* 0*.*
3. *Every algebra in $\mathcal{V}$ is* 0*-unorderable.*

Theorem 19 *2-subtractivity is consistent with lambda calculus.*

Equational Consistency Problem (Honsell-Plotkin 2006) asks whether

$(\forall E$ finite set of identities$)[(E \cup \lambda\beta$ consistent$) \rightarrow (\exists \mathcal{D} \in$ Scott$)\ \mathcal{D} \models E]$?

ANSWER: No

Separability in n-permutable varieties: n-step Hausdorff

Theorem 20 Kearnes-Sequeira (2002) *Every n-permutable variety of algebras is $\lfloor n/2 \rfloor$-step Hausdorff.*

Let X be a topological space. For every $a \in X$, we define:

1. $\Gamma_0^a = \emptyset$;

2. $\Gamma_{i+1}^a = \{b : \exists$ open U, V with $a \in U$, $b \in V$ and $U \cap V \subseteq \Gamma_i^a\}$.

Definition 3 Coleman(1997) *X is n-*step Hausdorff* if $\Gamma_n^a = A/\{a\}$ for all $a \in X$.*

n-step Hausdorff implies T_1.

n-step Hausdorff implies k-step Hausdorff for every $k \geq n$.

1-step Hausdorff is equivalent to T_2.

A variety of algebras is n-step Hausdorff if every topological algebra in the variety is n-step Hausdorff.

Separability in n-subtractive varieties

Axioms of n-subtractivity:

$$\begin{array}{rcl} 0 & = & s_1(y,y); \\ s_i(y,0) & = & s_{i+1}(y,y) \quad (i = 1,\ldots,n-2); \\ s_{n-1}(y,0) & = & y. \end{array}$$

The *rank* $r(y) = min\{k : s_k(y,0) \neq 0\} \qquad (y \neq 0)$.

The rank $r(y)$ exists and $1 \leq r(y) \leq n-1$.

Let $\mathbf{A}$ be an n-subtractive T_0-topological algebra. Define the opens

$$R_i = \{a : \kappa(a) \leq i\} = \bigcup_{1\leq j\leq i} \{a : s_j(a,0) \neq 0\}$$

Then

$$R_0 = \emptyset; \qquad R_i \subseteq R_{i+1}; \qquad R_{n-1} = A/\{0\}.$$

Separability in n-subtractive varieties

Define the opens

$$\Sigma_i = \{a : (\exists U, V \text{ opens})\ a \in U,\ 0 \in V \text{ and } U \cap V \subseteq R_{i-1}\} \quad (1 \leq i \leq n)$$

Then we have:

- $\Sigma_1 = \{a : a \text{ and } 0 \text{ are } T_2\text{-separated}\}$;
- $R_{i-1} \subseteq \Sigma_i \subseteq \Sigma_{i+1}$;
- $\Sigma_n = A \setminus \{0\}$.

Separability in n-subtractive varieties

Theorem 21 *Let $\mathbf{A}$ be an n-subtractive T_0-topological algebra. Then we have:*

$$R_i \subseteq \Sigma_i = \{a : (\exists U, V \text{ opens})\ a \in U,\ 0 \in V \text{ and } U \cap V \subseteq R_{i-1}\}.$$

Proof: We show that $a \in \Sigma_{r(a)}$. Since $s_{r(a)}(a, 0) \neq 0$, then there exists an open neighbourhood W of $s_{r(a)}(a, 0)$ such that $0 \notin W$. By the continuity of $s_{r(a)}$ there exist two open neighbourhoods U and V of a and 0 respectively such that

$$s_{r(a)}(U, V) \subseteq W.$$

If $r(a) = 1$ and there exists $b \in U \cap V$, then $0 = s_1(b, b) \in W$, contradicting the hypothesis on W. Then $V \cap U = \emptyset$; thus a and 0 are T_2-separated, and $a \in \Sigma_1$.

If $r(a) > 1$, for every $b \in U \cap V$ we have that $s_{r(a)}(b, b) \in W$, that implies

$$s_{r(a)-1}(b, 0) = s_{r(a)}(b, b) \neq 0.$$

This means that the rank of b is less than the rank of a for every $b \in U \cap V$. Then $U \cap V \subseteq R_{r(a)-1}$, so that $a \in \Sigma_{r(a)}$.

Separability in n-subtractive varieties

Proposition 5 *Every n-subtractive T_0-topological algebra is $n-1$-step Hausdorff in* 0*.*

Proof: We show by induction that

$$\Sigma_i \subseteq \Gamma_i^0 = \{b : \exists \text{ open } U, V \text{ with } b \in U,\ 0 \in V \text{ and } U \cap V \subseteq \Gamma_{i-1}^0\}$$

for all $1 \leq i \leq n$. For $i = 0$ the result is trivial.

$$\begin{array}{rcll} \Sigma_{i+1} & = & \{a : \exists \text{ open } U, V \text{ with } a \in U,\ 0 \in V \text{ and } U \cap V \subseteq R_i\} & \text{by definition} \\ & \subseteq & \{a : \exists \text{ open } U, V \text{ with } a \in U,\ 0 \in V \text{ and } U \cap V \subseteq \Sigma_i\} & \text{by } R_i \subseteq \Sigma_i \\ & \subseteq & \{a : \exists \text{ open } U, V \text{ with } a \in U,\ 0 \in V \text{ and } U \cap V \subseteq \Gamma_i^0\} & \text{by induction} \\ & = & \Gamma_{i+1}^0 & \text{by definition} \end{array}$$

The conclusion follows because $R_{n-1} \subseteq \Sigma_{n-1} \subseteq \Gamma_{n-1}^0$ and $R_{n-1} = A \setminus \{0\}$.

New separability axioms in topological spaces

Let (X, τ) be a topological space and $a \in X$.

A sequence Y of length an ordinal α of open sets is called an *a-sequence* if

$$Y_0 = \emptyset;\ Y_i \subseteq Y_{i+1} \text{ and } a \notin Y_i \text{ for every } i < \alpha.$$

For every $i \geq 1$ define the opens

$$\Sigma^a_{i,Y} = \{b : (\exists U, V \text{ opens})\ a \in U,\ b \in V \text{ and } U \cap V \subseteq Y_{i-1}\}$$

X is *β-step Y, a-Hausdorff* if β is the least ordinal satisfying $\Sigma^a_{\beta,Y} = X \setminus \{a\}$.

X is *α-step Y, a-Hausdorff* if $\bigcup_{\beta \geq 1} \Sigma^a_{\beta,Y} = X \setminus \{a\}$.

Proposition 6 *If $Y_i \subseteq \Sigma^a_{i,Y}$ then $\Sigma^a_{i,Y} \subseteq \Gamma^a_i$.*

Corollary 2 *If $Y_i \subseteq \Sigma^a_{i,Y}$ for every i, and X is n-step Y-Hausdorff, then X is n-step Hausdorff.*

What are the a-sequences Y satisfying $Y_i \subseteq \Sigma^a_{i,Y}$ for every i?