Similarity quotients as final coalgebras

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Outline

Coalgebras and Bisimulation

Consider the countable powerset functor $\mathcal{P}^{[0,\aleph_0]}$.

A countably branching transition system M is a $\mathcal{P}^{[0,\aleph_0]}$-coalgebra $(M^{\cdot}, \zeta_M)$.

Bisimulations

Let M and N be countably branching transition systems.

A *bisimulation* is a relation $M^{\cdot} \xrightarrow{\mathcal{R}} N^{\cdot}$ such that $x \mathrel{\mathcal{R}} x'$ implies $\zeta_M x \ \mathrm{Bisim}\mathcal{R}\ \zeta_N x'$.

$U \ \mathrm{Bisim}\mathcal{R}\ V$ means $\forall y \in U.\ \exists y' \in V.\ y \mathrel{\mathcal{R}} y' \wedge \forall y' \in V.\ \exists y \in U.\ y \mathrel{\mathcal{R}} y'$.

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The greatest bisimulation is *bisimilarity*.
This is closely related to final $\mathcal{P}^{[0,\aleph_0]}$-coalgebras.

Bisimilarity: Using A Final Coalgebra

Let P be a final F-coalgebra, and σ_M the anamorphism from M.

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Theorem: characterizing bisimilarity

$x \in M$ is bisimilar to $y \in N$ iff $\sigma_M x = \sigma_N y$.

Encompassment

We define a preorder on countably branching transition systems.

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The Theorem

Let M be a countably branching transition system that is all-encompassing.
Then M modulo bisimilarity is a final $\mathcal{P}^{[0,\aleph_0]}$-coalgebra.

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All-Encompassing Example

Take the disjoint union of every transition system carried by a countable cardinal.
It's all-encompassing because every node of a countably branching transition system has countably many successors.

Simulation

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Let M and N be countably branching transition systems.

A simulation is a relation $M^{\cdot} \xrightarrow{\mathcal{R}} N^{\cdot}$ such that $x \ \mathcal{R} \ x'$ implies $\zeta_M x \ \mathrm{Sim}\mathcal{R} \ \zeta_N x'$.
$U \ \mathrm{Sim}\mathcal{R} \ V$ means $\forall y \in U. \ \exists y' \in V. \ y \ \mathcal{R} \ y'$.

The greatest simulation is similarity.

Relators

Bisim and Sim are both $\mathcal{P}^{[0,\aleph_0]}$-relators.

Let F be an endofunctor on **Set**.

An F-relator maps each relation $X \xrightarrow{\mathcal{R}} Y$ to a relation $FX \xrightarrow{\Gamma\mathcal{R}} FY$.

Monotonicity

$$X \xrightarrow{\mathcal{R},\mathcal{R}'} Y$$

$$\mathcal{R} \subseteq \mathcal{R}' \Rightarrow \Gamma\mathcal{R} \subseteq \Gamma\mathcal{R}'$$

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Stability (Hughes and Jacobs)

$$\begin{tikzcd} X' \arrow[r, "f"] & X \arrow[d, "\mathcal{R}", "+"'] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

$$\Gamma((f,g)^{-1}\mathcal{R}) = (Ff, Fg)^{-1}\Gamma\mathcal{R}$$

Properties of relational extension (2)

Lax functoriality

$$\begin{array}{rcl} \mathrm{id}_{\Gamma X} & \subseteq & \Gamma \mathrm{id}_X \\ (\Gamma \mathcal{R});(\Gamma \mathcal{S}) & \subseteq & \Gamma(\mathcal{R};\mathcal{S}) \end{array}$$

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Conversive Relator

$$\Gamma(\mathcal{R}^{\mathsf{c}}) = (\Gamma\mathcal{R})^{\mathsf{c}}$$

Γ-simulation

Definition of Γ-simulation

A Γ-simulation $M \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} N$ is a relation such that

$$\mathcal{R} \subseteq (\zeta_M, \zeta_N)^{-1} \Gamma \mathcal{R}$$

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Properties of Γ-similarity

- Reflexive and transitive.
- Symmetric if Γ is conversive.
- For $M \xrightarrow{f} N$ a coalgebra morphism, $x \in M$ and $f(x) \in N$ are mutually Γ-similar.

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This is reflexive and transitive.

If there is a coalgebra morphism $M \longrightarrow N$
then M is Γ-encompassed by N.

If there is a surjective coalgebra morphism $M \longrightarrow N$
then M and N are mutually Γ-encompassed.

Lower and Upper Simulations (Ulidowski)

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An upper simulation $M \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} N$ is a relation such that for $x \; \mathcal{R} \; x'$ with $x \not\Uparrow$

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A lower simulation $M \xrightarrow{\mathcal{R}} N$ is a relation such that for $x \mathrel{\mathcal{R}} x'$

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An upper simulation $M \xrightarrow{\mathcal{R}} N$ is a relation such that for $x \mathrel{\mathcal{R}} x'$ with $x \not\Uparrow$

- $x' \not\Uparrow$
- $x' \leadsto y'$ implies that there is y such that $x \leadsto y$ and $y \mathrel{\mathcal{R}} y'$.

Many variants, each given by a relator preserving binary composition.

Probabilistic Systems

DX is the set of discrete subprobability distributions on X, i.e. functions $d : X \to [0, 1]$ such that $\sum_{x \in X} d(x) \leqslant 1$.

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$$\zeta_M x(U) \quad \leqslant \quad \zeta_N y(\mathcal{R}(U)) \qquad \text{for all } U \subseteq X$$

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Given by relators $\mathrm{ProbSim}$ and conversive $\mathrm{ProbBisim}$.
They both preserve binary composition.

The endofunctor F_Γ

F_Γ is an endofunctor on **Preord**

- $A = (A_0, \leqslant_A)$ is mapped to $(FA_0, \Gamma(\leqslant_A))$
- $A \xrightarrow{f} B$ is mapped to Ff.

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Consider a final $\mathcal{P}^{[0,\aleph_0]}_{\mathrm{Sim}}$ coalgebra M.

- Equality on $M^{\cdot}$ is bisimilarity
- The preorder $\leqslant_{M^{\cdot}}$ is similarity.

What if we only care about similarity?

Preord and **Poset**

Poset is a replete subcategory of **Preord**, i.e. full and isomorphism-closed. It is also reflective.

Quotient of a Preordered Set A

The principal lower set of $x \in A$ is $[x] \stackrel{\mathrm{def}}{=} \{y \in A \mid y \leqslant_A x\}$.

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The quotienting map $A \xrightarrow{p_A} QA$ is $x \mapsto [x]$.

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Alternatively quotient A by the symmetrization of $\leqslant_A$.

The Endofunctor QF_Γ

We obtain a composite

$$\mathbf{Preord} \xrightarrow{F_\Gamma} \mathbf{Preord} \xrightarrow{Q} \mathbf{Poset}$$

In our example, it maps A to the set of countably generated lower sets, ordered by inclusion.

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If Γ is conversive, it restricts to **Setoid** or **DiscSetoid** $\cong$ **Set**.

Bimodules

Let A and B be preordered sets.

A bimodule $A \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\;} B$ is a relation such that

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We can quotient it to obtain a bimodule $QA \xrightarrow{Q\mathcal{R}} QB$.

$$[a] \; Q\mathcal{R} \; [b] \text{ when } a \; \mathcal{R} \; b$$

QF_Γ-coalgebras

Let M and N be QF_Γ-coalgebras.

A simulation $M \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;} N$ is a bimodule such that

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Also encompassment.

Extensional QF_Γ-coalgebras

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Key property

If M is encompassed by N, there is a unique coalgebra morphism
$M \longrightarrow N$.
Otherwise there is none.

Extensional quotient

Let M be a QF_{Γ}-coalgebra.

Let A be $M^{\cdot}$ quotiented by similarity.

Extensional quotient

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There is a unique QF_{Γ}-coalgebra N carried by $M^{\cdot} / \lesssim$,

such that $M \xrightarrow{p} N$ is a coalgebra morphism.

Moreover N is extensional.

From F-coalgebra to QF_Γ-coalgebra

We have a functor $\Delta : \mathbf{Set} \longrightarrow \mathbf{Preord}$ mapping X to $(X, (=_X))$.

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We have a functor $\Delta^\Gamma : \mathrm{Coalg}(\mathbf{Set}, F) \longrightarrow \mathrm{Coalg}(\mathbf{Preord}, QF_\Gamma)$

mapping $M = (M^\cdot, \zeta_M)$ to the coalgebra with

- carrier $\Delta M^\cdot$
- behaviour $\Delta M^\cdot \xrightarrow{\zeta_M} F_\Gamma \Delta M^\cdot \xrightarrow{p_{F_\Gamma \Delta M^\cdot}} QF_\Gamma \Delta M^\cdot$

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Simulation is unchanged

Let M and N be F-coalgebras. Then $x \in M$ is Γ-similar to $y \in N$ iff $(\Delta^\Gamma M)x$ is similar to $(\Delta^\Gamma N)y$.

Using A Final Coalgebra

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So we can use it to characterize Γ-similarity on F-coalgebras.

The only elements of P that matter for this task are anamorphic images of $(\Delta^\Gamma M)x$.

Are there any others?

From QF_Γ-coalgebra to F-coalgebra

Key Theorem

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there is an F-coalgebra M
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Beyond **Set**?

- Proof uses the Axiom of Choice.
- Nevertheless, we can generalize from **Set** to presheaf categories and sheaf categories, provided F preserves epimorphisms.

Constructing a Final Coalgebra

- If M is an all-Γ-encompassing F-coalgebra,

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  Any final QF_{Γ}-coalgebra must arise in this way.

2-nested simulation

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A 2-nested simulation from M to N is a simulation contained in the converse of a simulation.

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Can we characerize 2-nested similarity using a final coalgebra?

Nested Preordered Sets

A nested preordered set is a set X with two preorders

- $\leqslant_{\mathsf{n}}$ think 2-nested similarity
- $\leqslant_{\mathsf{o}}$ think converse of similarity

subject to the constraints

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NestPoset is a reflective replete subcategory of **NestPreord**.

Double relations

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We have a $\mathcal{P}^{[0,\aleph_0]}$-relator matrix TwoSim

$$\mathrm{TwoSim}_{\mathsf{n},\mathsf{n}} \stackrel{\mathrm{def}}{=} \mathrm{Sim} \qquad \mathrm{TwoSim}_{\mathsf{n},\mathsf{o}} \stackrel{\mathrm{def}}{=} \mathrm{Sim}^{\mathsf{c}}$$
$$\mathrm{TwoSim}_{\mathsf{o},\mathsf{n}} \stackrel{\mathrm{def}}{=} \top \qquad \mathrm{TwoSim}_{\mathsf{o},\mathsf{o}} \stackrel{\mathrm{def}}{=} \mathrm{Sim}^{\mathsf{c}}$$

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This maps a double relation $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$
to a double relation $\mathcal{P}^{[0,\aleph_0]}X \xrightarrow{\mathrm{TwoSim}\mathcal{R}} \mathcal{P}^{[0,\aleph_0]}Y$

$$\begin{aligned}(\mathrm{TwoSim}\mathcal{R})_{\mathsf{n}} &= \mathrm{Sim}\mathcal{R}_{\mathsf{n}} \cap \mathrm{Sim}^{\mathsf{c}}\mathcal{R}_{\mathsf{o}} \\ (\mathrm{TwoSim}\mathcal{R})_{\mathsf{o}} &= \mathrm{Sim}^{\mathsf{c}}_{\mathsf{o}}\end{aligned}$$

This operation behaves like a relator and does not preserve binary composition.

Endofunctor for Nested Simulation

We obtain a functor

$$\textbf{NestPreord} \xrightarrow{\mathcal{P}^{[0,\aleph_0]}_{\mathrm{TwoSim}}} \textbf{NestPreord} \xrightarrow{Q} \textbf{NestPoset}$$

giving an endofunctor on **NestPreord** and on **NestPoset**.
Its final coalgebra characterizes 2-nested simulation.

Multiple Relations

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Need positive and negative constraints on the preorders.

Multiple Relations

Instead of working with relations, we work with I-relations. I a fixed set

Instead of a relator, we use a relator I-matrix.

Need positive and negative constraints on the preorders.

More generally, we can work with $\mathcal{Q}$-relations, where $\mathcal{Q}$ is a quantale.

Conclusions

- We can treat bisimulation, many notions of simulation and nested simulation.
- We can use a final coalgebra to characterize similarity.
- Conversely, we can construct a final coalgebra by taking an all-encompassing transition system and quotienting by similarity.