Graphical Linear Algebra

Midlands Graduate School 2017

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graphicallinearalgebra.net

What will you get out of this?

- An introduction to symmetric monoidal categories and *diagrammatic reasoning*, an important tool in
    - quantum computing and quantum information (Abramsky, Coecke, Pavlovic, Duncan, Kissinger, …)
    - asynchronous circuits (Ghica)
    - signal flow graphs (Bonchi, S., Zanasi)
    - linear time-invariant dynamical systems (Baez, Erbele, Fong, Rapisarda, S.)
- A different way to think about several concepts of linear algebra!
    - linear algebra is **everywhere** (quantum, machine learning, systems and control theory, graph theory, …)
    - a different language means different ways to think about applications, and it is sometimes more efficient than the classical concepts of matrices, bases, etc.
    - a nice calculus for recurrence relations
- An introduction to several algebraic structures that are quite common in computational models
    - e.g. monoids, comonoids, Frobenius monoids, bimonoids, Hopf algebras, …

More generally: resource sensitive syntax

- Ordinary, tree-like syntax, together with its operations (tuples, substitution) is good in situations where all operations have a coarity 1
  - i.e. operations of the form $X^n \to X$
- This often goes together with the implicit assumption that the resources in X are ***copyable*** and ***discardable***
- In many situations (e.g. quantum, many kinds of circuits) we have to be more careful with resources — here tree-like syntax can unnatural
  - the language of symmetric monoidal categories provides the algebra for this syntax
  - the language of string diagrams provides a nice notation

Plan

- **Lecture 1 - String diagrams and symmetric monoidal categories**

- **Lecture 2** - Resource-sensitive algebraic theories

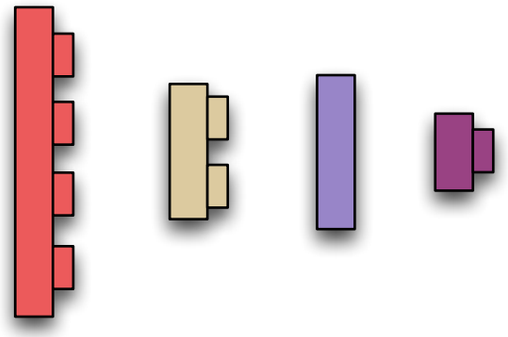
- **Lecture 3** - Interacting Hopf monoids and graphical linear algebra

- **Lecture 4** - Signal Flow Graphs and recurrence relations

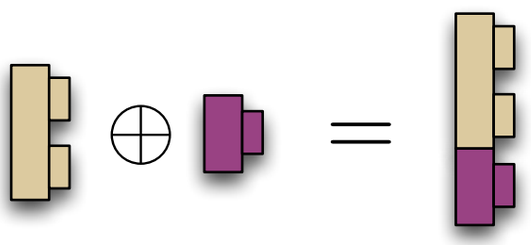
Lecture 1

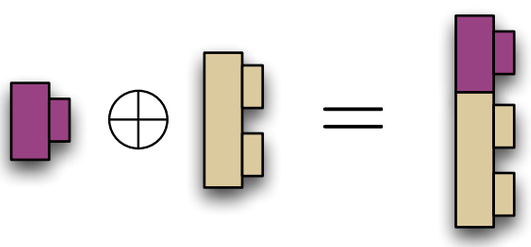
String diagrams and symmetric monoidal categories

“stacking things and plugging them together”



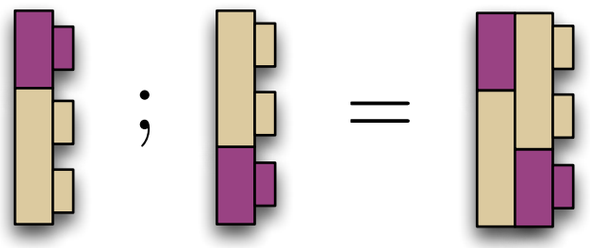
⊕ - stacking one on top of the other

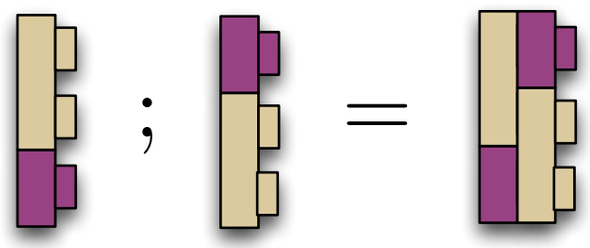




associative but not commutative

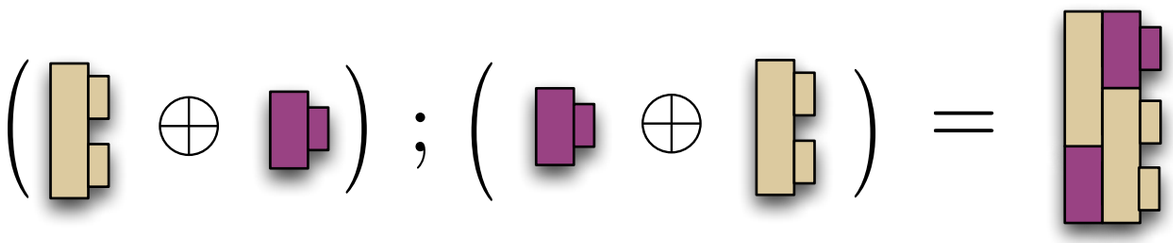
; - plugging the first into the second

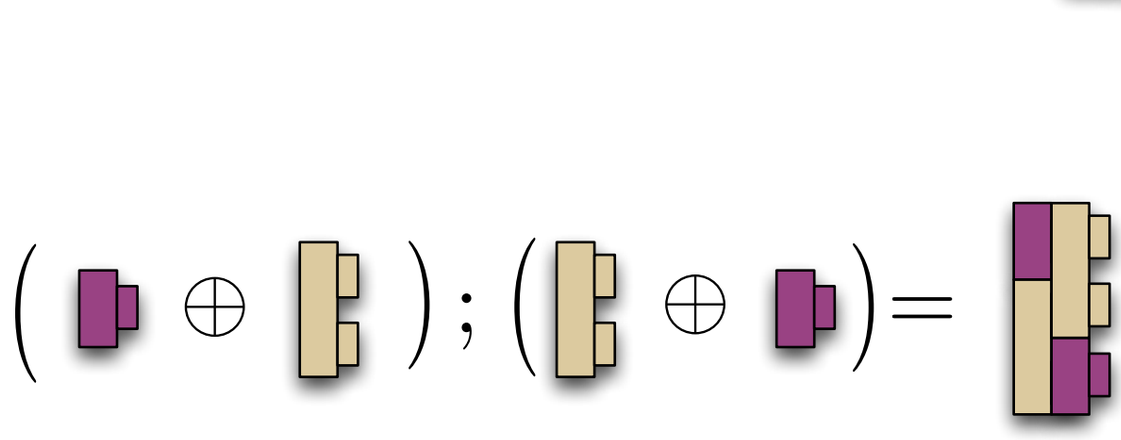




associative but not commutative

Combining operations

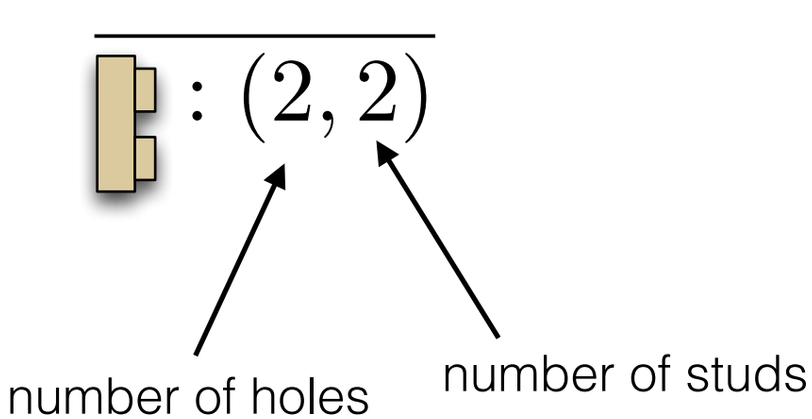


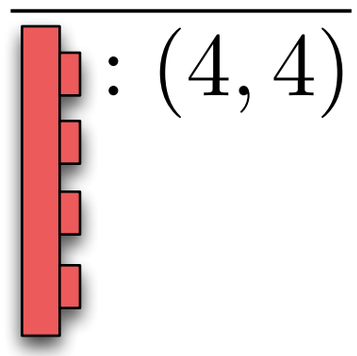


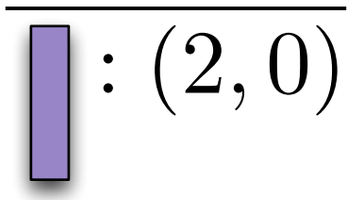
Plan

- **algebra of magic Lego**
- categories
- monoidal categories
- string diagrams
- symmetric monoidal categories

A simple type system





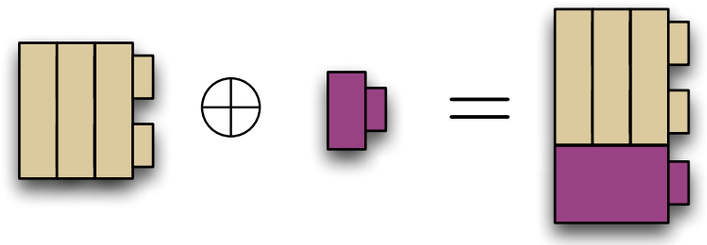


$$\frac{X : (k, l) \qquad Y : (m, n)}{X \oplus Y : (k+m, l+n)}$$

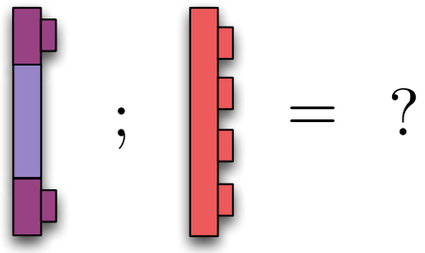
$$\frac{X : (k, l) \qquad Y : (l, m)}{X ; Y : (k, m)}$$

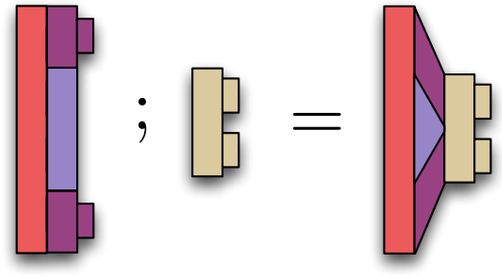
The magic in magic Lego

$$\frac{X : (k, l) \qquad Y : (m, n)}{X \oplus Y : (k+m, l+n)}$$



$$\frac{X : (k, l) \qquad Y : (l, m)}{X ; Y : (k, m)}$$





Pieces stretch and shrink as needed — the only important thing is the type (number of holes, number of studs)

The mathematics of magic Lego

- "stacking things on top of each other" and "plugging things together" are very common operations in science and mathematics

- the mathematical structure that's behind these operations is called a (strict) **monoidal category**

- let's start with categories…

Plan

- algebra of magic Lego
- **categories**
- monoidal categories
- string diagrams
- symmetric monoidal categories

Categories

- A *category* **C**, seen as a mathematical structure, consists of two sets: objects and arrows

- think of objects as **types**

- think of arrows as **operations**

  - every arrow C has two associated objects k, m, called **domain** and **codomain**, written $C : k \to m$

- Every object has an **identity arrow** (think of this as a trivial “do-nothing” operation)

$$\frac{}{I_k : k \to k}$$

Operations on arrows

- There is a partial operator on arrows ; called **composition**

$$\frac{C : k \to l \qquad D : l \to m}{C \,;\, D : k \to m}$$

— like the plugging in magic Lego

Associativity

- Composition is associative, i.e.

$$\frac{C : k \to l \qquad D : l \to m \qquad E : m \to n}{(C \,;\, D) \,;\, E = C \,;\, (D \,;\, E) : k \to n}$$

Composing identities

- Composing with identity arrows on both sides does not change the arrow, *i.e.*

$$\frac{C : m \to n}{I_m \,;\, C = C = C \,;\, I_n}$$

Examples of (small) categories

- **0** - the empty category
- **1** - the category with one object * and one arrow
  - *Exercise*: check that everything works
- **2** - the category with objects $\{0, 1\}$, one non-identity arrow $0 \to 1$.
  - *Exercise*: check that everything works
- Any preorder is a category with
  - objects the elements of X, and at most one arrow between any 2 objects: $x \to y$ iff $x \leq y$.
  - conversely, any category with at most one arrow from any object to another is a preorder.

Examples of (big) categories

- **Set**
  - *Objects*: sets
  - *Arrows*: functions
- **$\mathbf{Set}_f$**
  - *Objects*: finite sets
  - *Arrows*: functions
- **Rel**
  - *Objects*: sets
  - *Arrows*: relations
- **Mon**
  - *Objects*: monoids
  - *Arrows*: homomorphisms
- **Top**
  - *Objects*: topological spaces
  - *Arrows*: continuous functions
- **Cat**
  - *Objects*: categories
  - *Arrows*: functors

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In the spirit of HOTT

- A 2-category has
  - objects “0-cells”
  - arrows “1-cells”
    - 2-cells (arrows between arrows)
- so that (1 and 2) composition is associative and has identities
- e.g. Topological space: objects = points, arrows = paths, 2-cells = homotopies
- two different ways of composing 2-cells!
- **A (strict) monoidal category is a 2-category with one object**

Monoidal categories

- A (strict) *monoidal category* **C**, seen as a mathematical structure, consists of two sets: objects and arrows
  - think of objects as (products of) **types**
  - think of arrows as **operations**
    - every arrow C has two associated objects k, m, called **domain** and **codomain**, written $C : k \to m$

- there is an associative operation on objects $\oplus$ called **monoidal product**

$$(k \oplus m) \oplus n = k \oplus (m \oplus n)$$

- Every object has an **identity arrow** (think of this as a trivial “do-nothing” operation)

$$\frac{}{I_k : k \to k}$$

- There is an **identity object** I for monoidal product (think of this as a unit type). This makes the set of object a **monoid**.

$$\frac{}{k \oplus I = I \oplus k = k}$$

Operations on arrows

- There is a partial operation or arrows ; called **composition**

$$\frac{C : k \to l \qquad D : l \to m}{C ; D : k \to m}$$

— like the plugging in magic Lego

- There is a total operation on arrows $\oplus$ (abusing notation) also called **monoidal product**.

$$\frac{C : k \to l \qquad D : m \to n}{C \oplus D : k \oplus m \to l \oplus n}$$

— like the stacking in magic Lego

Associativity

- Composition is associative, i.e.

$$\frac{C : k \to l \quad D : l \to m \quad E : m \to n}{(C ; D) ; E = C ; (D ; E) : k \to n}$$

- Monoidal product is associative, also on arrows, i.e.

$$\frac{C : m \to n \quad D : m' \to n' \quad E : m'' \to n''}{(C \oplus D) \oplus E = C \oplus (D \oplus E) : m \oplus m' \oplus m'' \to n \oplus n' \oplus n''}$$

Monoidal product and functoriality

- The monoidal product is actually a **functor**, a morphism of categories

$$F: \mathbf{X} \to \mathbf{Y}$$

- Functors consist of two functions, one each for objects and arrows, and preserve **domains** and **codomains** (*c.f.* morphism of digraphs)

$$\frac{C : m \to n}{F(C) : F(m) \to F(n)}$$

- as well as **composition** and **identities**

$$F(C \mathbin{;} D) = F(C) \mathbin{;} F(D)$$

$$F(I_k) = I_{F(k)}$$

How is monoidal product a functor?

- A *monoid* is a set M equipped with a **function**

$$\cdot : M \times M \to M$$

(M×M is the cartesian product of sets)

- … with a unit element, satisfying associativity

- Similarly, a *strict monoidal category* is a category C equipped with a **functor**

$$\oplus : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

(**C**×**C** is the cartesian product of categories)

- … with a unit object, satisfying associativity

Exercise

- Suppose that **C** and **D** are categories. How do we define of the cartesian product **C**×**D** of **C** and **D**?
  - *What are the objects?*
  - *What are the arrows?*
  - *How does composition work?*
  - *What are the identity arrows?*

Monoidal product and composition

- Suppose that A can be composed with C, and B can be composed with D. Then since $\oplus$, being a functor, preserves composition, we have:

$$\oplus(A;B,\ C;D) = \ \oplus(\ (A,C)\ ;\ (B,D)\ ) = \oplus(A,C)\ ;\ \oplus(B,D)$$

i.e.

$$(A\ ;\ B) \oplus (C\ ;\ D) = (A \oplus C)\ ;\ (B \oplus D)$$

This equation is sometimes called “middle-four interchange”

Exercise

- Suppose that A: $p \rightarrow q$ and B: $r \rightarrow s$. Show that

$$(A \oplus I_r) ; (I_q \oplus B) = A \oplus B = (I_p \oplus B) ; (A \oplus I_s)$$

Composing identities

- Composing with identity arrows on both sides does not change the arrow, *i.e.*

$$C\colon m \to n$$

$$I_m \,;\, C = C = C \,;\, I_n$$

- Since $\oplus$ must preserve identities, we also require

$$\oplus(I_{(m,n)}) = \oplus(I_m,\ I_n) = I_{m \oplus n} \qquad \text{i.e.}$$

$$I_m \oplus I_n = I_{m \oplus n}$$

Examples of monoidal categories

- **Set**$_f$ with × (cartesian product) or with + (disjoint union) as monoidal product
  - it's not quite strict… e.g. monoidal product is not associative on the nose
  - it is a (non-strict) monoidal category, where associativity is up to coherent isomorphism
- **F** - the strict monoidal version of **Set**$_f$
  - *Objects*: finite ordinals $\underline{m} := \{1,2,\ldots,m\}$
  - *Arrows*: functions
  - Monoidal product 1: on objects $\underline{m} + \underline{n} := \underline{m + n}$
    - what is the monoidal identity object?
    - how to define monoidal product on morphisms?
  - Monoidal product 2: on objects $\underline{m} \times \underline{n} := \underline{m \times n}$
    - what is the monoidal identity object?
    - how to define monoidal product on morphisms?

Magic Lego

- Is a monoidal category with
  - *Objects*: natural numbers (keeping track of numbers of holes and studs)
  - *Arrows*: lego constructions

Plan

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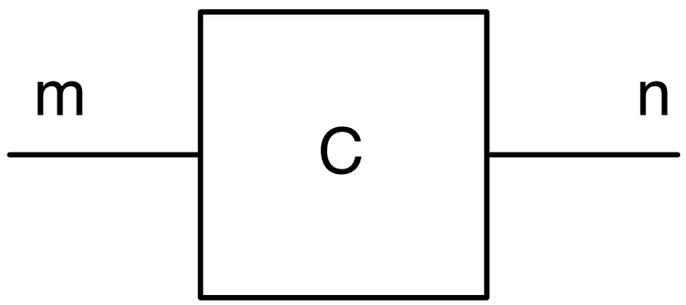
- **string diagrams**
- symmetric monoidal categories

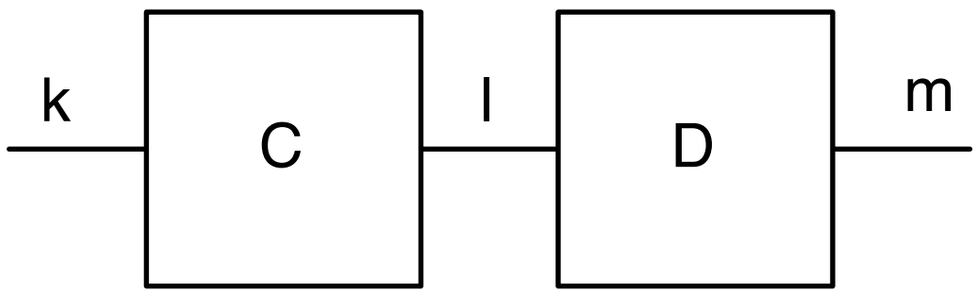
String diagrams

- A graphical notation for the arrows of monoidal categories
  - We have been writing $C\colon m \to n$
  - We will now **draw**



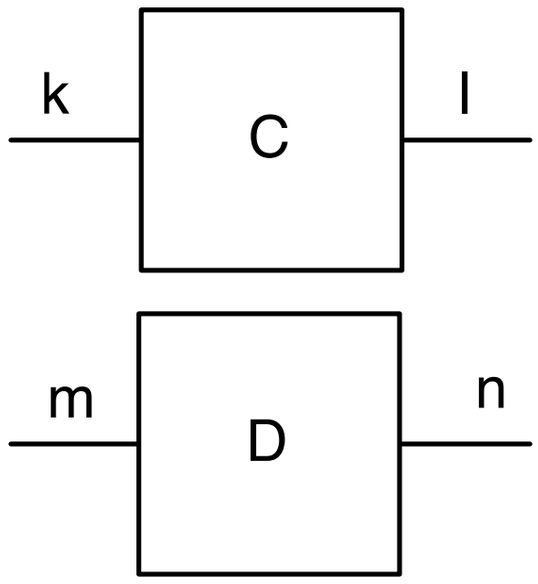
Composition

$$\frac{C : k \rightarrow l \qquad D : l \rightarrow m}{C ; D : k \rightarrow m}$$



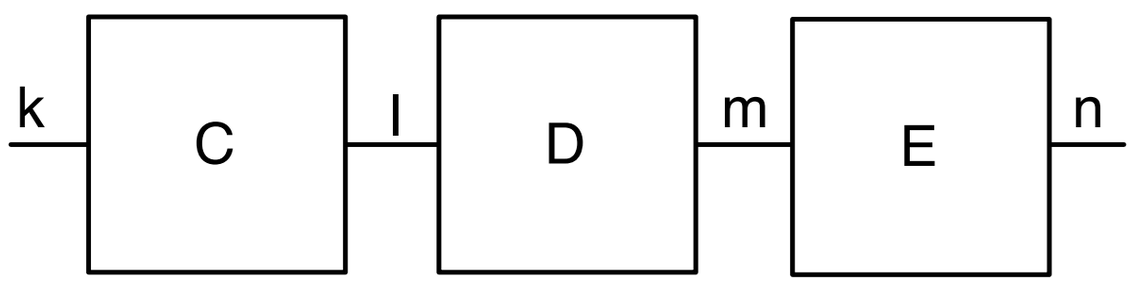
Monoidal product

$$\frac{C : k \to l \qquad D : m \to n}{C \oplus D : k \oplus m \to l \oplus n}$$



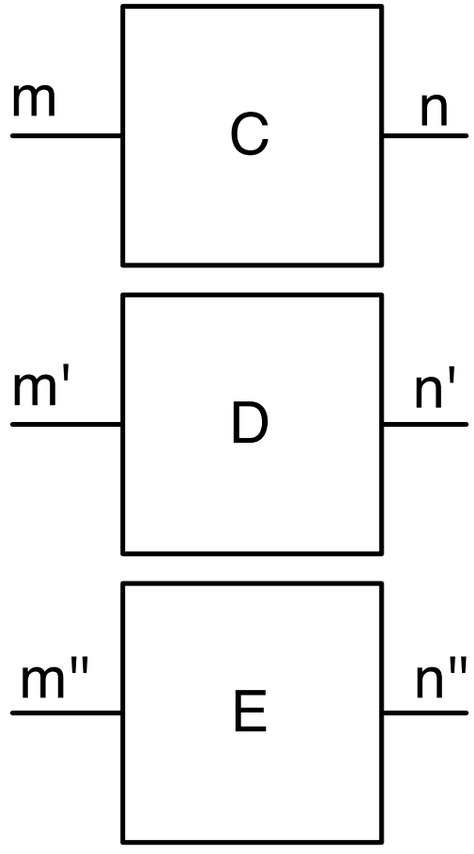
Perks of the notation I

$$\frac{C : k \to l \qquad D : l \to m \qquad E : m \to n}{(C \,;\, D) \,;\, E = C \,;\, (D \,;\, E) : k \to n}$$



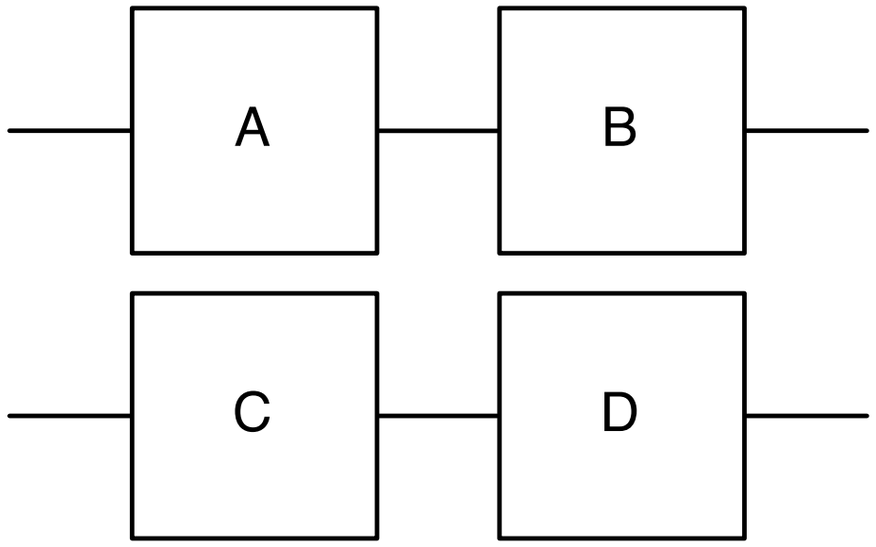
Perks of the notation II

$$\frac{C : m \to n \quad D : m' \to n' \quad E : m'' \to n''}{(C \oplus D) \oplus E = C \oplus (D \oplus E) : m \oplus m' \oplus m'' \to n \oplus n' \oplus n''}$$



Perks of the notation III

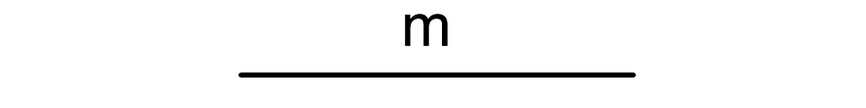
$$(A \,;\, B) \oplus (C \,;\, D) = (A \oplus C) \,;\, (B \oplus D)$$



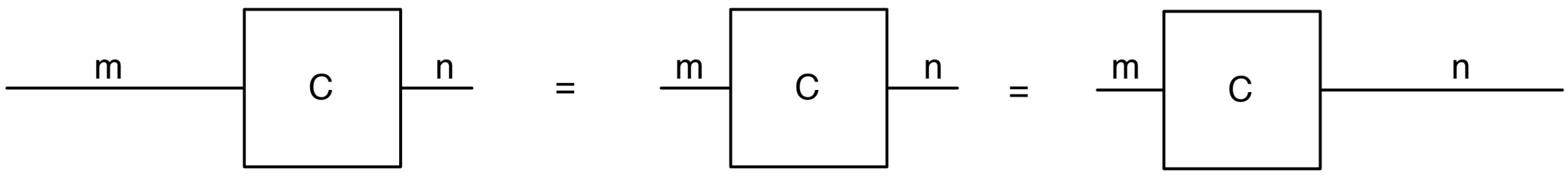
Diagrammatic reasoning I

Stretching

- Identity on *m* is simply drawn as a wire



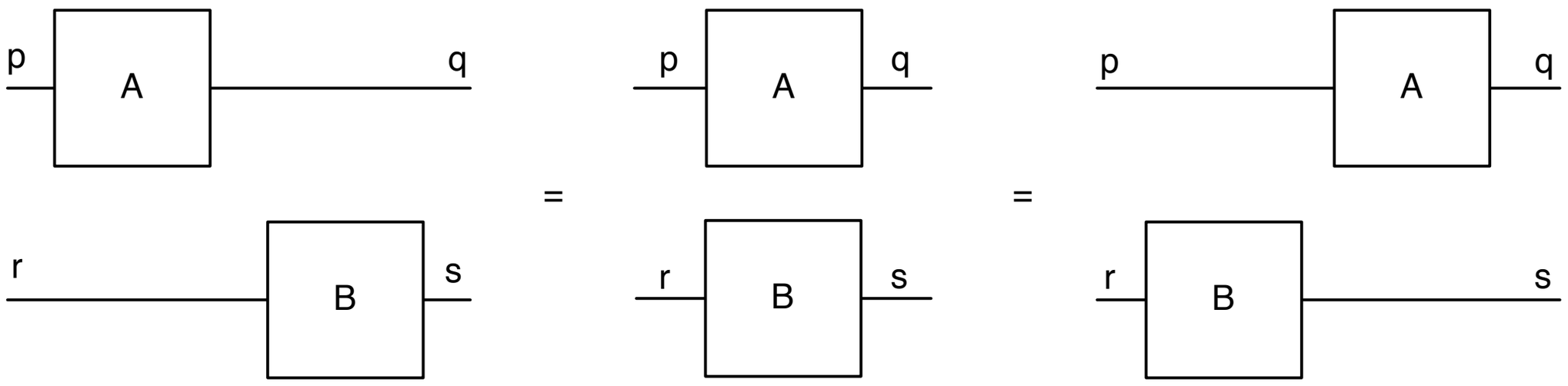
$$\frac{C : m \to n}{I_m \,;\, C = C = C \,;\, I_n}$$



Diagrammatic reasoning II

Sliding

$$(A \oplus I_r) ; (I_q \oplus B) = A \oplus B = (I_p \oplus B) ; (A \oplus I_s)$$



General story

- A strict monoidal category is the same thing as a 2-category with one object, a particularly simple kind of *higher category*
    - string diagrams are a kind of graph theoretical dual, i.e.
        - zero dimensional things (objects) become two dimensional things
        - one dimensional things (arrows in a 2-cat = objects in a strict monoidal cat) stay as one dimensional things — wires
        - two dimensional things (2-cells in a 2-cat = arrows in a strict monoidal cat) become zero dimensional things (points, or boxes as we have been drawing)
- See Globular (Vicary, Kissinger, Bar): http://globular.science

Plan

- algebra of magic Lego
- categories
- monoidal categories
- string diagrams
- **symmetric monoidal categories**

Symmetric monoidal categories

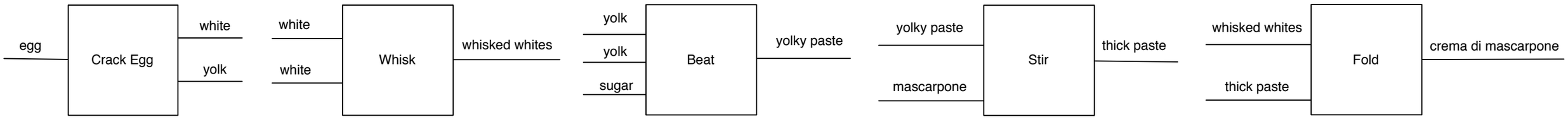
- When wiring things up using the algebra of connecting and stacking, we often want to permute the wires

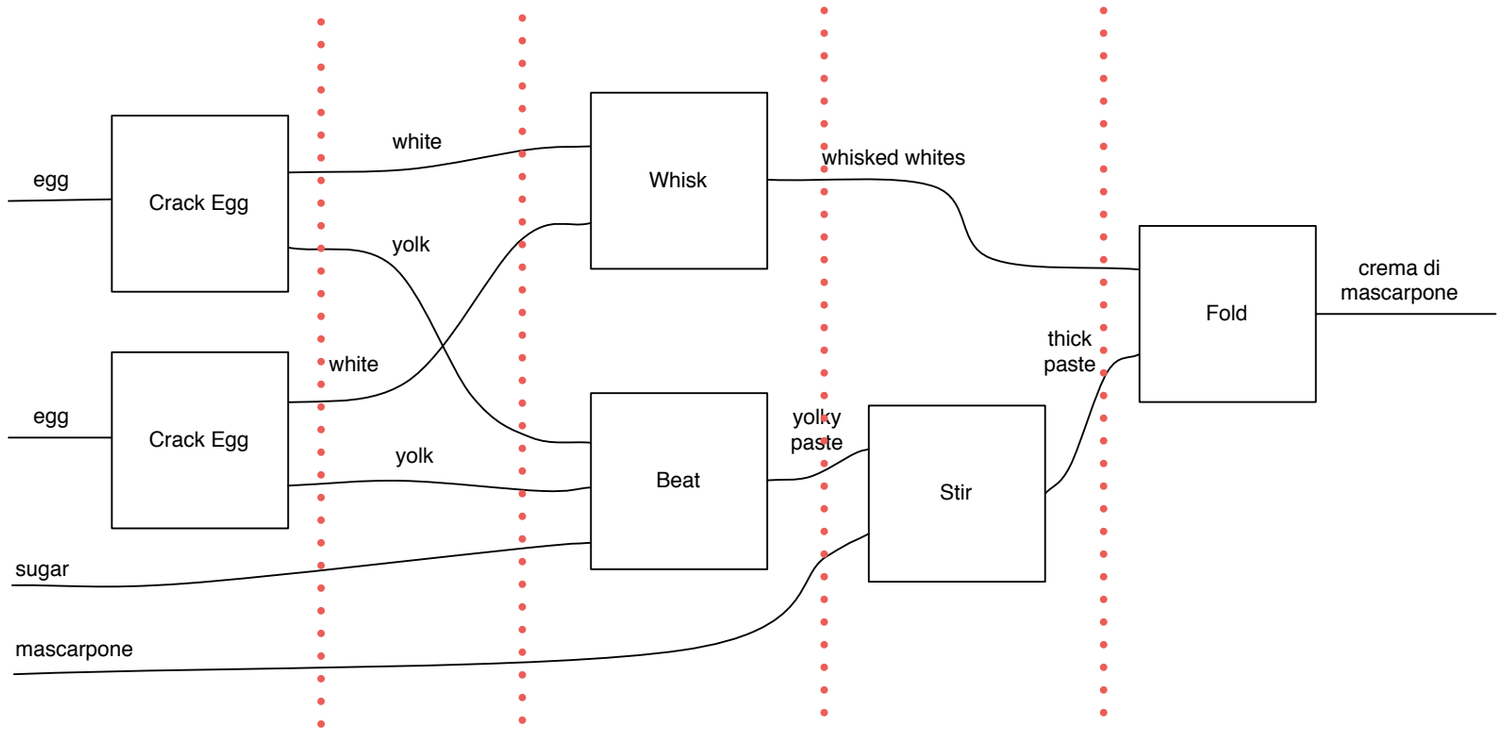
- Mathematically, this means moving from monoidal categories to *symmetric* monoidal categories

- In a symmetric monoidal category, for any two objects m, n, there is a *symmetry*, or *twist*

$$\mathrm{tw}_{m,n} \colon m \oplus n \to n \oplus m$$

Example - Crema di Mascarpone





$$(C \oplus C \oplus id_2) \; ; \; (id \oplus tw \oplus id_3) \; ; \; (W \oplus B \oplus id) \; ; \; (id \oplus S) \; ; \; F$$

Natural transformations

- We have seen categories, and functors: morphisms between categories
- A natural transformation is a *morphism between functors*

$$F \Rightarrow G: \mathbf{X} \rightarrow \mathbf{Y}$$

- A natural transformation $\alpha : F \Rightarrow G$ is a collection of arrows of **Y**, one for each object in $m \in \mathbf{X}$

$$\alpha_m : Fm \rightarrow Gm$$

- These must satisfy a condition with respect to the arrows of **X**, namely for each arrow C: $m \rightarrow n$

$$\begin{array}{ccc} Fm & \xrightarrow{\alpha_m} & Gm \\ {\scriptstyle FC}\downarrow & & \downarrow{\scriptstyle GC} \\ Fn & \xrightarrow{\alpha_n} & Gn \end{array}$$

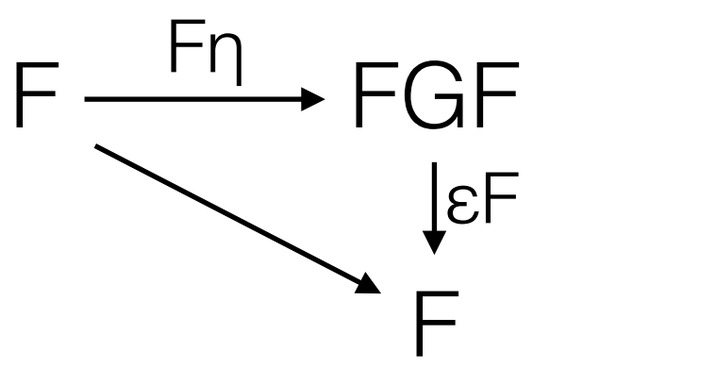
commutes.

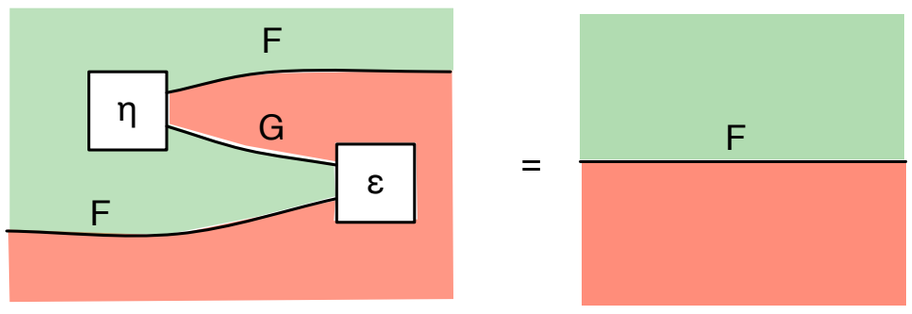
These are sometimes called naturality squares

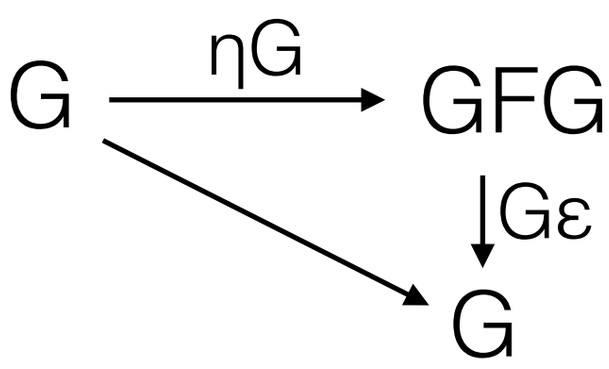
Aside — string diagrams for **Cat**

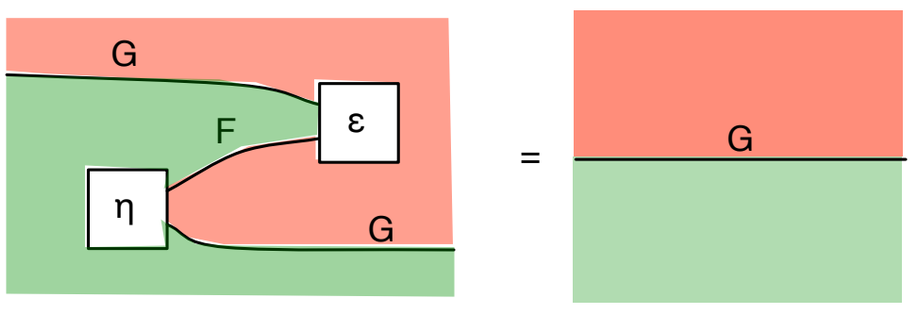
- **Cat** is a 2-category
  - *Objects (0-morphisms)*: categories
  - *Arrows (1-morphisms)*: functors
  - *2-cells (2-morphisms)*: natural transformations

- Suppose that F: **C**→**D** has right adjoint G: **D**→**C**, then the triangle equations can be drawn as follows, using string diagrams:









Symmetry as natural transformation

- For a monoidal category **C**, there are actually two functors $\mathbf{C}\times\mathbf{C} \to \mathbf{C}$ given by $\oplus$

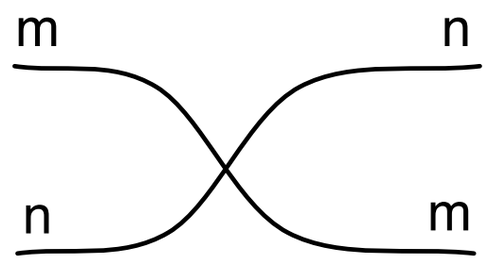
  $\oplus = -_1 \oplus -_2 : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ – stack the first on the second

  $\oplus' = -_2 \oplus -_1 : \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ – stack the second on the first

- *tw* is a natural transformation from the first to the second, with components $tw_{m,n} : m \oplus n \to n \oplus m$

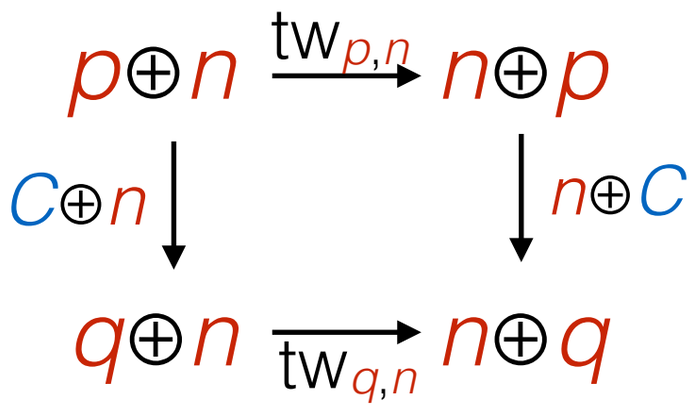
Drawing twists

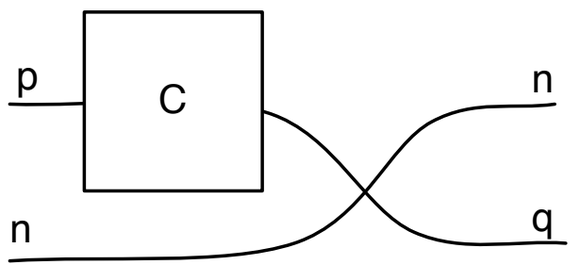
$\mathrm{tw}_{m,n}$: $m \oplus n \to n \oplus m$



Diagrammatic reasoning III

naturality = sliding across twists





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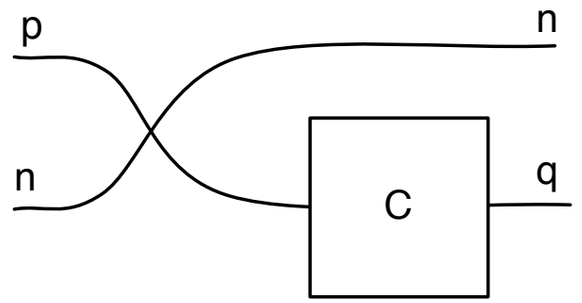

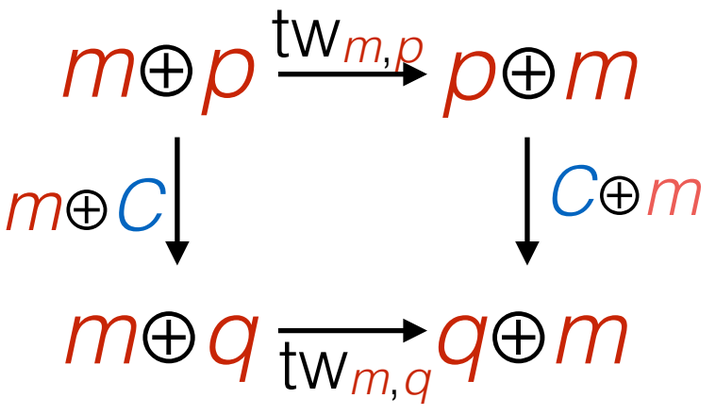

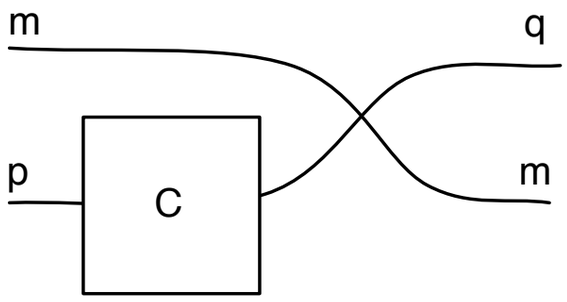

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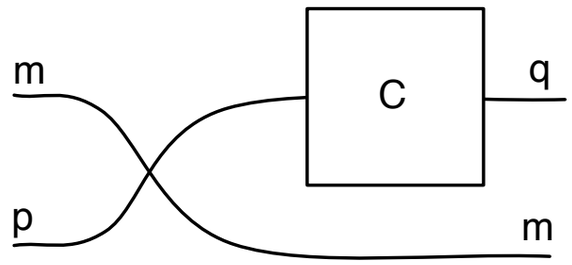
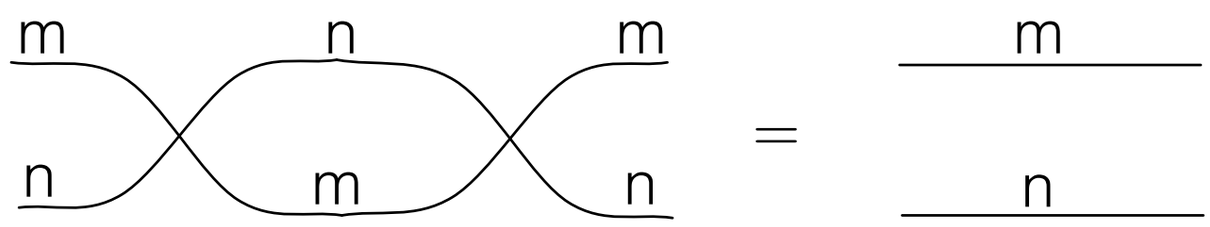


Diagrammatic reasoning IV

tightening (without wires tangling)

- In any symmetric monoidal category, the twist is **invertible**, and has itself as inverse, in the following sense

$$tw_{m,n} ; tw_{n,m} = I_{m \oplus n} : m \oplus n \to m \oplus n$$



In a braided monoidal category, the twist is invertible, but it is not, in general, it's own inverse

Symmetric monoidal categories

- A *strict symmetric monoidal* category **C**, is a strict monoidal category with a natural family of arrows $tw_{m,n} : m \oplus n \rightarrow n \oplus m$, indexed by pairs of objects of **C**, such that

$$tw_{m,n} ; tw_{n,m} = I_{m \oplus n} : m \oplus n \rightarrow m \oplus n$$

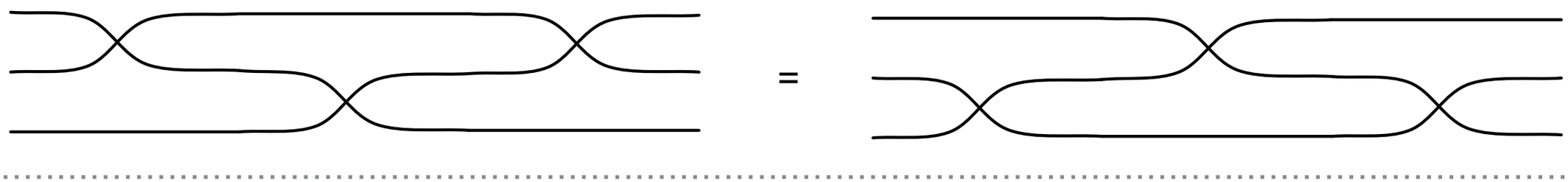
$$tw_{p,\, m \oplus n} = (tw_{p,m} \oplus I_n); (I_m \oplus tw_{p,n}): p \oplus m \oplus n \rightarrow m \oplus n \oplus p$$

$$tw_{m \oplus n,\, p} = (I_m \oplus tw_{n,p}) ; (tw_{p,m} \oplus I_n): m \oplus n \oplus p \rightarrow p \oplus m \oplus n$$

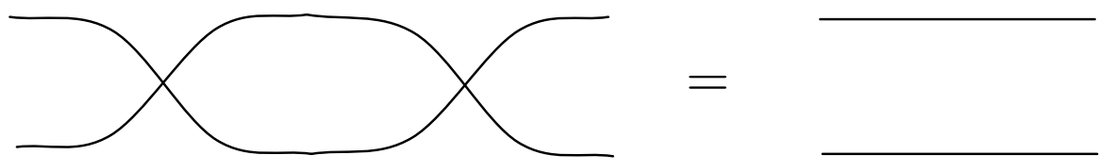
$$tw_{I,m} = I_m = tw_{m,I}$$

Yang-Baxter

- *Exercise*: The famous Yang-Baxter equation is an instance of naturality of the twist



- together with



we get that “pure” wiring diagrams $t^{\oplus m} \to t^{\oplus m}$ are in 1-1 correspondence with permutations of the m-element set

The category of permutations

- *Objects*: finite ordinals $\underline{m} = \{1,\ldots,m\}$
- *Arrows*: no arrows from m to n if $m \neq n$, otherwise the permutations

- Strict symmetric monoidal, with $\underline{m} \oplus \underline{n} := \underline{m+n}$