# Plan

- **Lecture 1** - String diagrams and symmetric monoidal categories
- **Lecture 2 - Resource-sensitive algebraic theories**
- **Lecture 3** - Interacting Hopf monoids and graphical linear algebra
- **Lecture 4** - Signal Flow Graphs and recurrence relations

# Lecture 2

Resource sensitive algebraic theories

# Plan

- **algebraic theories**
- symmetric monoidal theories (resource sensitive algebraic theories)
- props
- bimonoids and matrices of natural numbers
- Hopf monoids and matrices of integers

# Algebraic theories

## Universal Algebra

- A (presentation of) *algebraic theory* is a pair $(\Sigma, E)$ where
  - $\Sigma$ is a set of *generators* (or *operations*), each with an *arity,* a natural number
  - $E$ is a set of *equations* (or *relations)*, between $\Sigma$-*terms* built up from generators and *variables*

### Example 1 - monoids

$$\Sigma_M = \{ \cdot{:}2,\ e{:}0 \}$$

$$E_M = \{ \cdot(\cdot(x, y), z) = \cdot(x, \cdot(y, z)),\ \cdot(x, e) = x,\ \cdot(e, x) = x \}$$

### Example 2 - abelian groups

$$\Sigma_G = \Sigma_M \cup \{ i{:}1 \}$$

$$E_G = E_M \cup \{ \cdot(x, y) = \cdot(y, x),\ \cdot(x, i(x)) = e \}$$

# $\Sigma$ - terms (cartesian)

$$\frac{x \in Var}{x} \qquad \frac{t_1\ t_2\ \dots\ t_m \quad \sigma \in \Sigma \quad ar(\sigma) = m}{\sigma(t_1, t_2, \dots, t_m)}$$

i.e. terms a *trees* with internal nodes labelled by the *generators* and the leaves labelled by *variables* and *constants (generators with arity 0)*

# Models - classically

- To give a model of an algebraic theory $(\Sigma,E)$, choose a set $X$
  - for each operation $\sigma : k$ in $\Sigma$, choose a function $[[\sigma]] : X^k \to X$
  - now for each term $t$, given an assignment of variables $\alpha$, we can recursively compute the element of $[[t]]_\alpha \in X$ which is the "meaning" of $t$
  - need to ensure that for every assignment of variables $\alpha$, and every equation $t_1 = t_2$ in $E$, we have $[[t_1]]_\alpha = [[t_2]]_\alpha$ as elements of $X$
- Example 1: to give a model of the algebraic theory of monoids is to give a monoid
- Example 2: to give a model of the theory of abelian groups is to give an abelian group

# Algebraic theories, categorically

- There is a nice way to think of algebraic theories categorically, due to Lawvere in the 1960s
  - get rid of “countably infinite set of variables”, “variable assignments” etc.
  - generalise - models don’t need to be sets (e.g. topological groups)
  - relies on the notion of *categorical product*

# Categorical product

- Suppose that $X$, $Y$ are objects in a category **C**. Then $X$ and $Y$ have a product if ∃ object $X{\times}Y$ and arrows $\pi_1$: $X{\times}Y \to X$, $\pi_2$: $X{\times}Y \to Y$ so that the following universal property holds

$$
\begin{tikzcd}
X & X{\times}Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y \\
& Z \arrow[ul, "f"] \arrow[u, dotted, "h"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

for any object $Z$ and arrows
$f$: $Z \to X$, $g$: $Z \to Y$,
∃ unique $h$: $Z \to X{\times}Y$ s.t.
$h \,;\, \pi_1 = f$ and $h \,;\, \pi_2 = g$

- *Example*: in the category **Set** of sets and functions, the cartesian product satisfies the universal property

- Any category with (binary) categorical products is monoidal, with the categorical product as monoidal product

# Exercise

- If $X$ is a preorder, considered as a category, what does it mean if $X$ has (binary) categorical products?
- In **Set**, the categorical product is the cartesian product
    - What is the product in the category of categories and functors?
    - What is the product in the category of monoids and homomorphisms?

# Lawvere categories

- Suppose that ($\Sigma$, $E$) is an algebraic theory
- Define a category $\mathbf{L}_{(\Sigma,E)}$ with
  - *Objects*: natural numbers
  - *Arrows* from m to n: n tuples of $\Sigma$-terms, each using possibly m variables $x_1$, $x_2$, …, $x_m$, modulo the equations of $E$
- Composition is *substitution*

## Examples in the theory of monoids

$$2 \xrightarrow{(x_1 \cdot x_2)} 1 \qquad 2 \xrightarrow{(x_2 \cdot x_1)} 1$$

$$1 \xrightarrow{(x_1 \cdot e)} 1 \quad = \quad 1 \xrightarrow{(x_1)} 1$$

It is also possible (and elegant) to view $\mathbf{L}_{(\Sigma,E)}$ as the *free category with products* on the data specified in ($\Sigma$,E)

# Exercise

- Lawvere categories have (binary) categorial products: $m \times n := m+n$.

    *Q1*. What are the projections?

- In any category with binary products there is a canonical arrow $\Delta: X \to X \times X$ called the *diagonal.*

    *Q2.* How is it defined?

*Q3*. What is $L_{(\varnothing,\varnothing)}$? Can you find a simple way of describing it?

# Models categorically (Functorial semantics)

- A functor F: C → D is product-preserving if

$$F(X \times Y) = F(X) \times F(Y)$$

- **Theorem.** To give a model of $(\Sigma, E)$ is to give a product-preserving functor F: $\mathbf{L}_{(\Sigma,E)} \to \mathbf{Set}$

*Proof idea*: since m = 1+1+…+1 (m times), to give a product preserving functor F from $\mathbf{L}_{(\Sigma,E)}$ it is enough to say what F(1) is.

- By changing **Set** to other categories, we obtain a nice generalisation of classical universal algebra, with examples such as topological groups, etc.

# Limitations of algebraic theories

- Copying and discarding built in

$$2 \xrightarrow{(x_1)} 1 \qquad 2 \xrightarrow{(x_2)} 1 \qquad 1 \xrightarrow{(x_1,\, x_1)} 2$$

- But in computer science (and elsewhere), we often need to be more careful with resources

- Consequently, there are also no bona fide operations with *coarities* other than one

$$1 \xrightarrow{c} 2 \qquad = \qquad 1 \xrightarrow{(c_1, c_2)} 2$$

# Plan

- algebraic theories
- **symmetric monoidal theories (resource sensitive algebraic theories)**
- props
- bimonoids and matrices of natural numbers
- Hopf monoids and matrices of integers

# Symmetric monoidal theories

- *symmetric monoidal theories* (*SMTs*) give rise to special kinds of symmetric monoidal categories called *props*
- Symmetric monoidal theories generalise *algebraic theories*, a classical concept of universal algebra, but
  - no built in copying and discarding
  - can consider operations with coarities other than 1

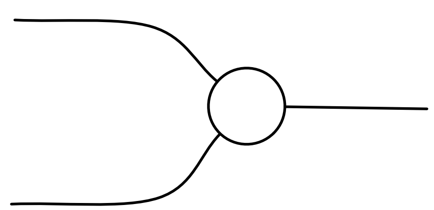
# Symmetric monoidal theories

- A *symmetric monoidal theory* is a pair $(\Sigma, E)$ where
  - $\Sigma$ is a set of *generators* (or *operations*), each with an *arity,* and *coarity*, both natural numbers
  - $E$ is a set of *equations* (or *relations)*, between compatible $\Sigma$-*terms*

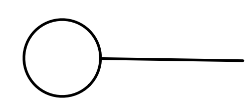
- Since generators can have coarities, and since we need to be careful with resources, we can't use the standard notion of term (tree).
- Instead, terms are arrows in a certain symmetric monoidal category, which we will construct a la magic Lego

# Generators and terms

Running example: the SMT of commutative monoids



$: (2, 1)$

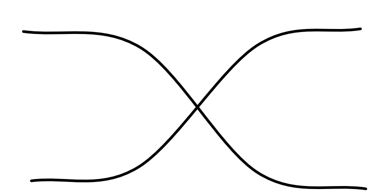


$: (0, 1)$

we always have the following "basic tiles" around



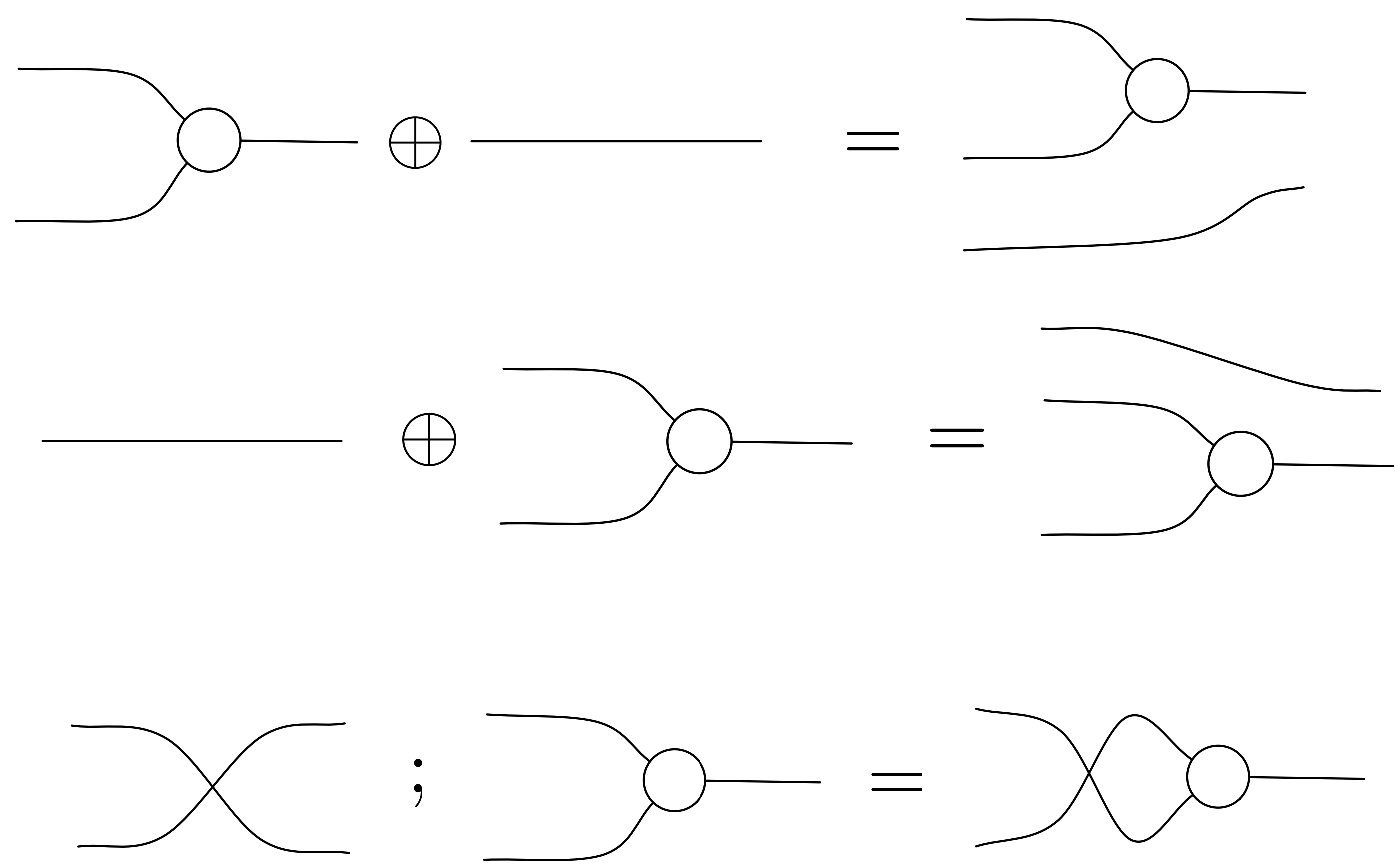
$: (1, 1)$



$: (2, 2)$

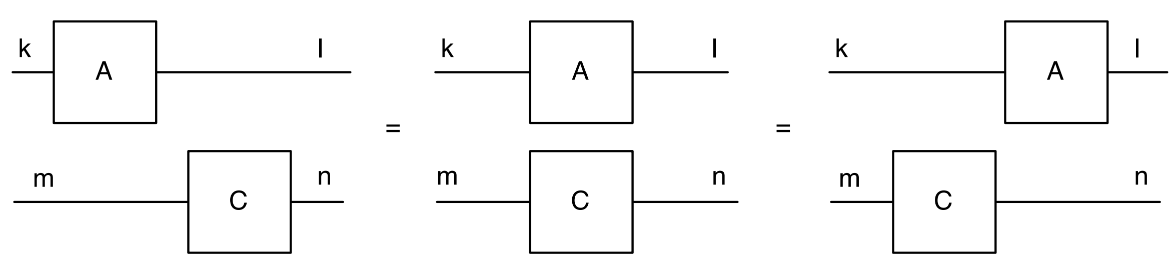
# Some string diagrams

- String diagrams: constructions built up from the generators and basic tiles, with the two operations of magic Lego

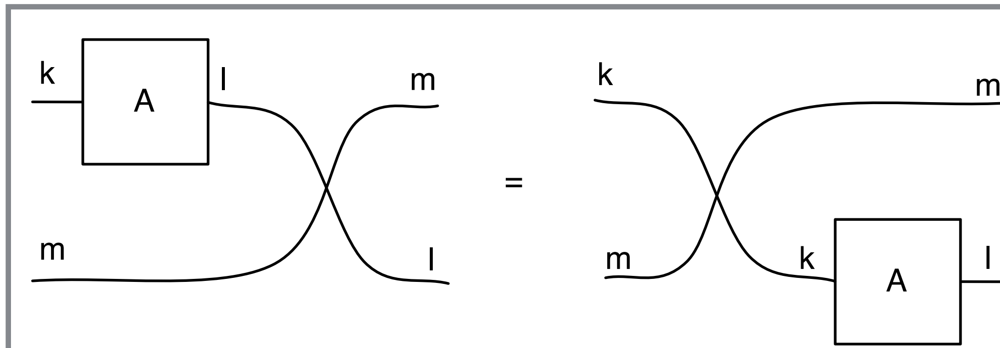


# Recall: diagrammatic reasoning

- diagrams can slide along wires

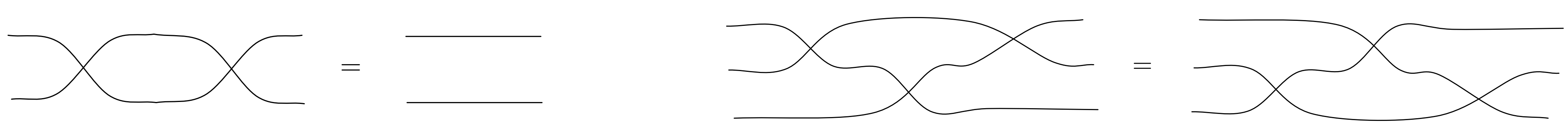



functoriality




naturality

- wires don’t tangle, i.e.



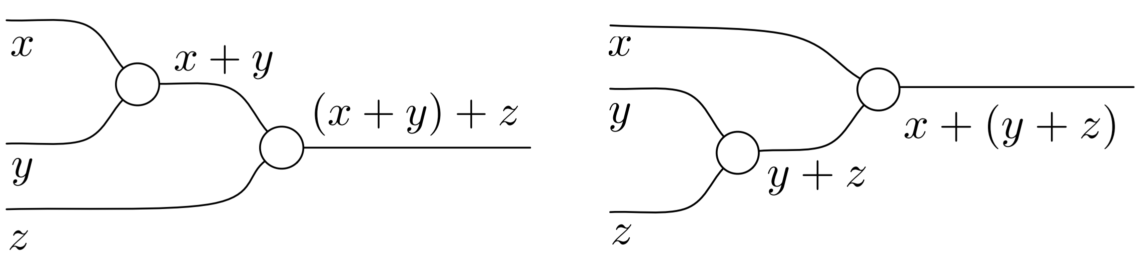

i.e. pure wiring obeys the same equations as permutations

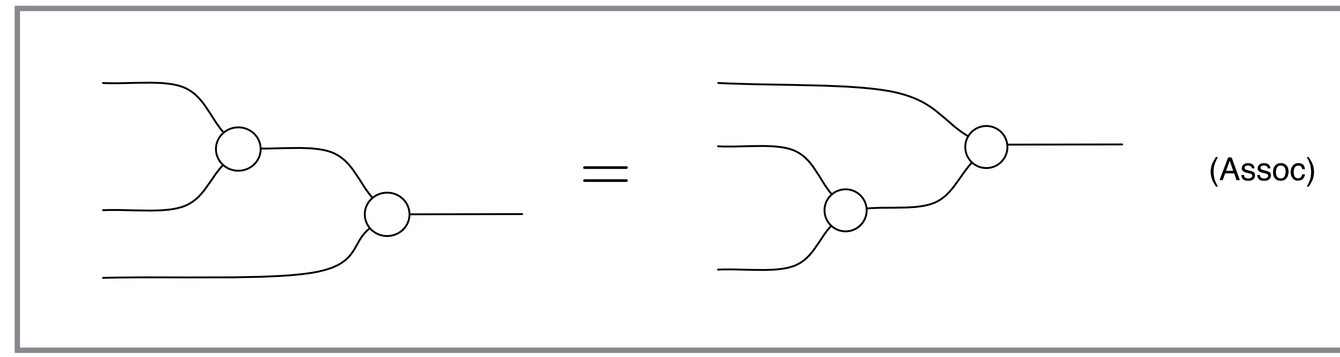
- sub-diagrams can be replaced with equal diagrams (compositionality)

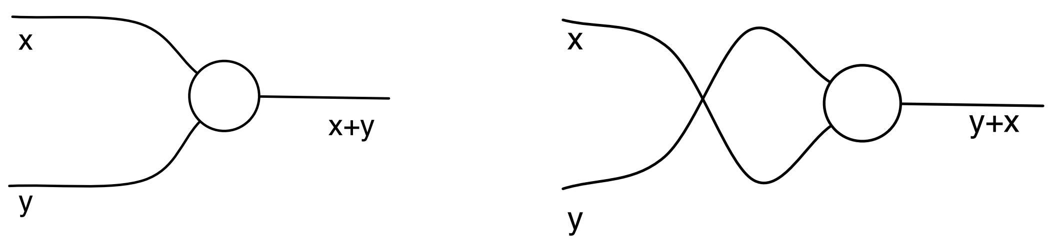
# $\Sigma$ - Terms (monoidal)

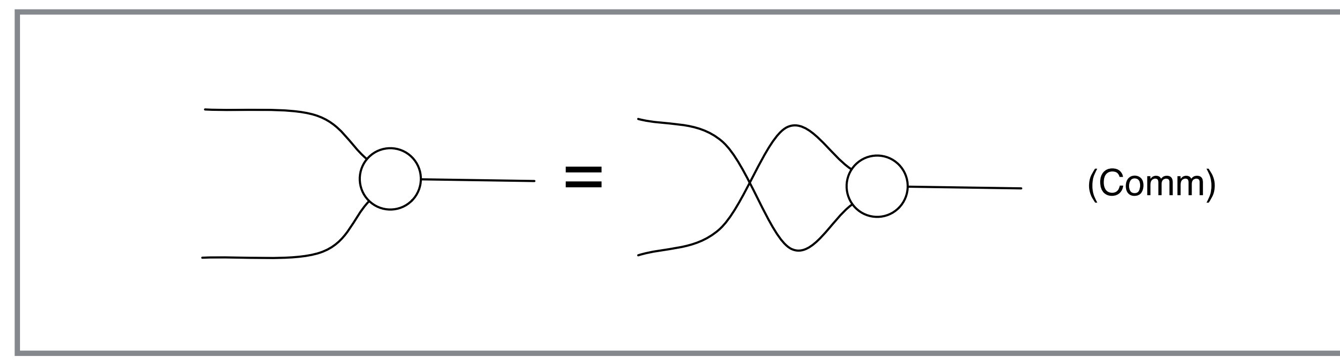
- Are thus the arrows of the free symmetric monoidal category $\mathbf{S}_{\Sigma}$ on $\Sigma$
- *Objects*: natural numbers
- *Arrows from m to n*: string diagrams constructed from generators, identity and twist, modulo diagrammatic reasoning
- Monoidal product, on objects: $m \oplus n := m + n$

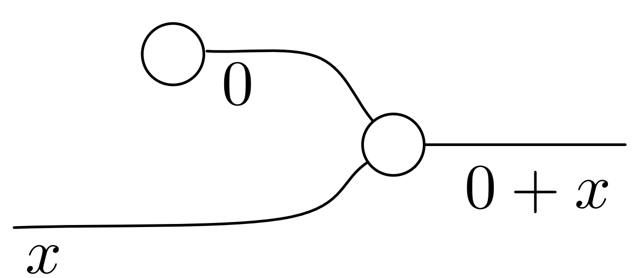
# Equations

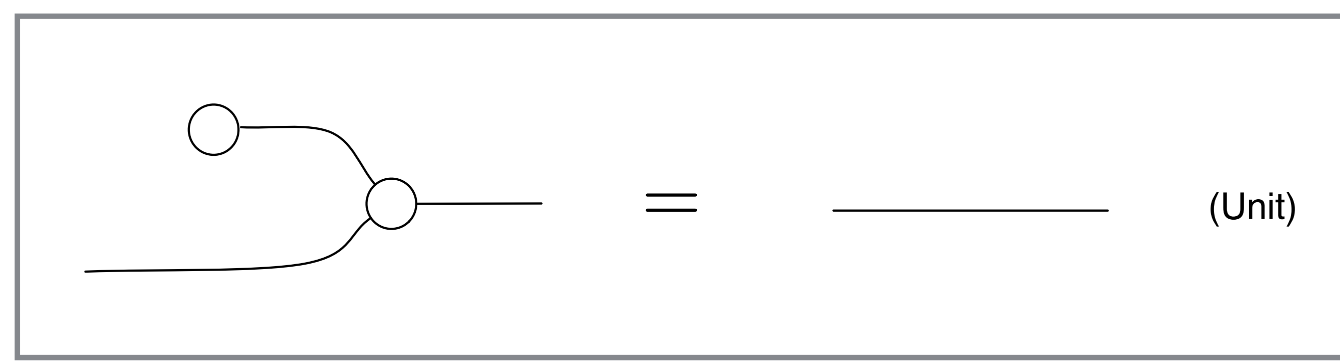


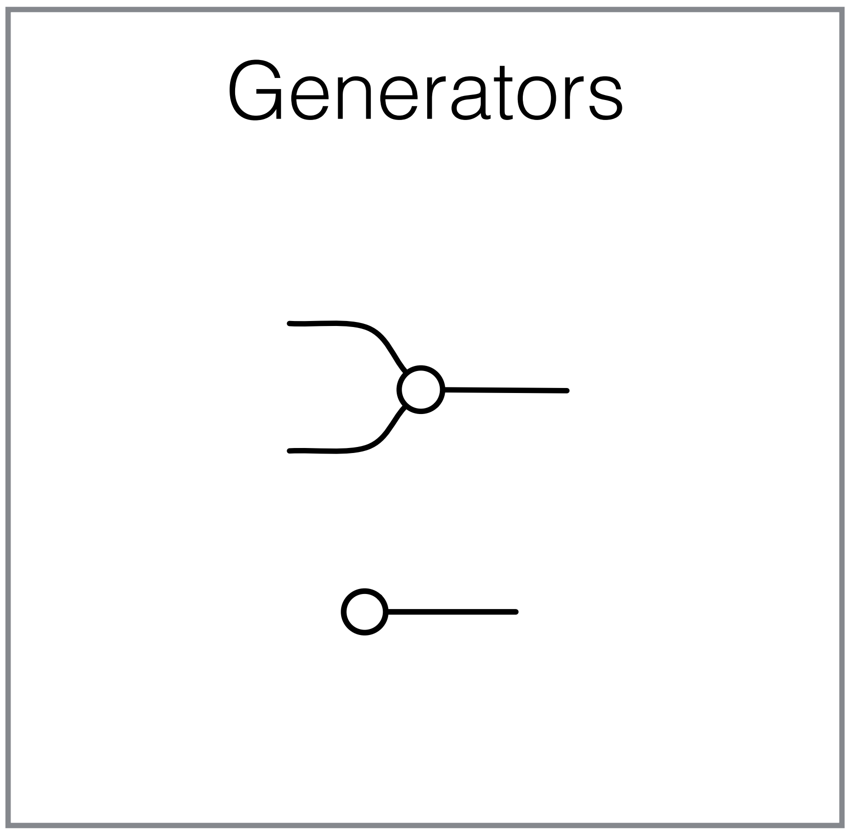



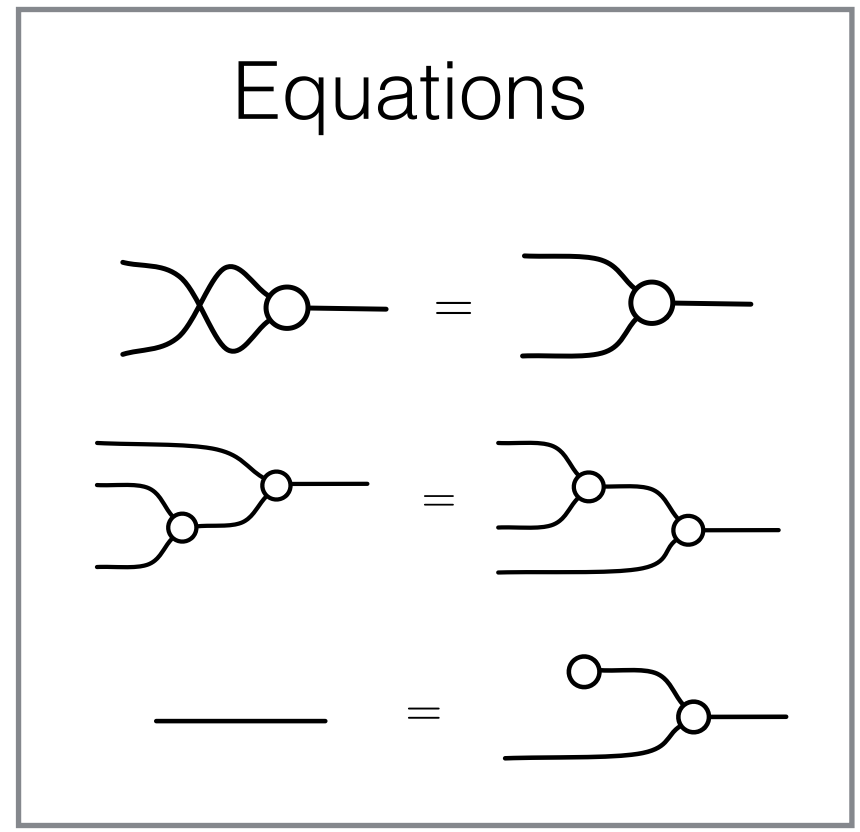




Note that all equations are of the form $t_1 = t_2$ : (m, n), that is, $t_1$ and $t_2$ must agree on domain and codomain

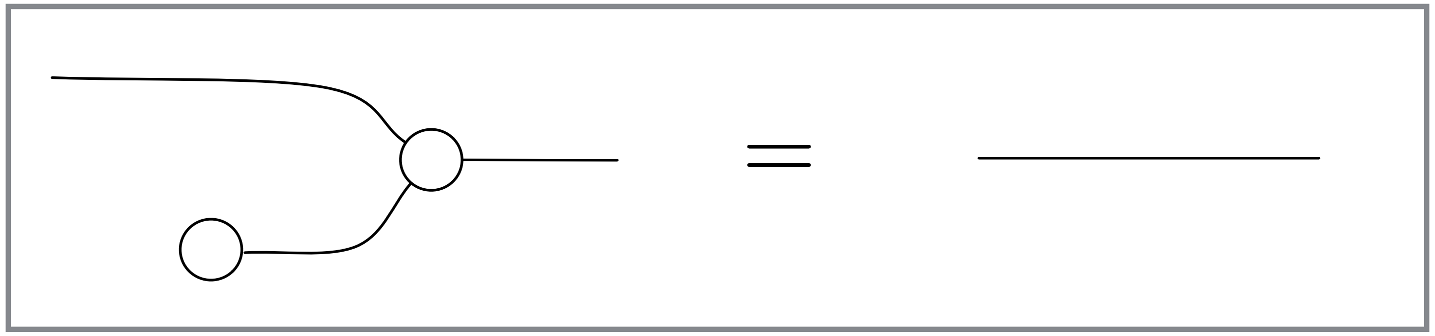
# The SMT of commutative monoids

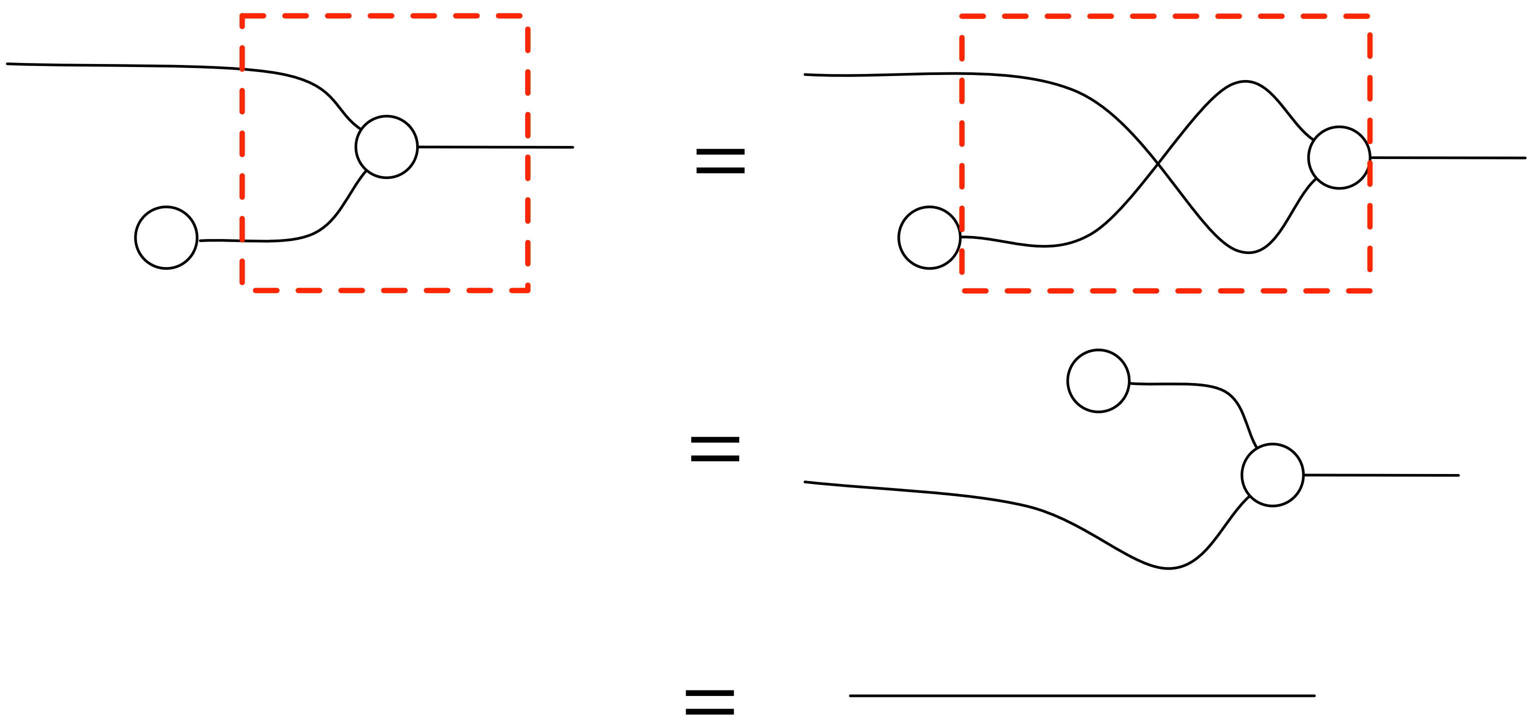



Let's call this SMT **M**, for monoid

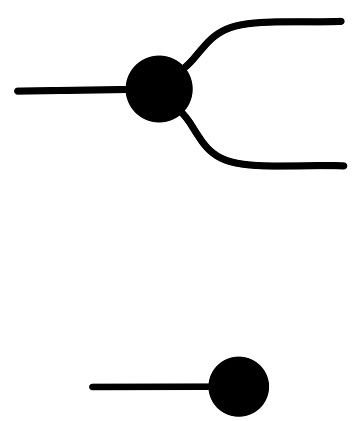
# Diagrammatic reasoning example



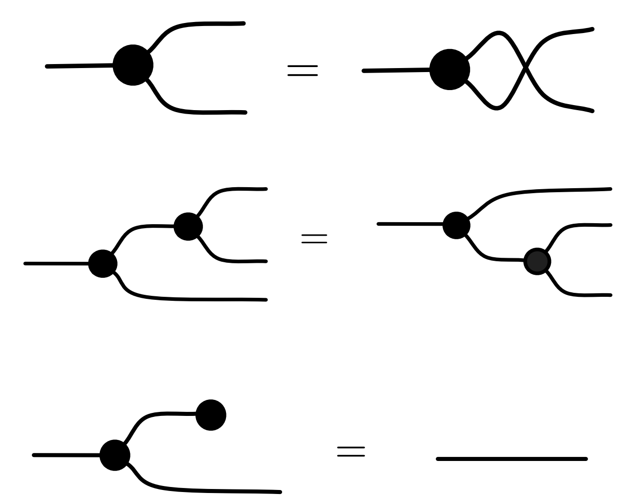


# Another SMT: commutative comonoids

Generators



Equations



# Plan

- algebraic theories
- symmetric monoidal theories (resource sensitive algebraic theories)
- **props**
- bimonoids and matrices of natural numbers
- Hopf monoids and matrices of integers

# From SMTs to symmetric monoidal categories

- Every symmetric monoidal theory $(\Sigma,E)$ yields a free strict symmetric monoidal category $\mathbf{S}_{(\Sigma,E)}$
  - Object: natural numbers
  - Arrows: monoidal $\Sigma$-terms, taken modulo equations in $E$
- Such categories are an instance of *props* (product and permutation categories)

# props

- A **prop** (product and permutation category) is
  - strict symmetric monoidal
  - objects = natural numbers
  - monoidal product on objects = addition
    - i.e. $m \oplus n = m+n$

# Examples

1. Any symmetric monoidal theory gives us a prop

2. The strict symmetric monoidal category **F**

   - arrows from m to n are all functions from the *m* element set {0, …, m-1} to the n element set {0, … , n-1}

3. The free strict symmetric monoidal category on one object, the category **P** of permutations

4. The category **I** with precisely one arrow from any m to n is a prop

# Morphisms of props

- A morphism of props F: $\mathbf{X} \to \mathbf{Y}$ is an identity on objects symmetric monoidal functor
    - identity-on-objects: $F(m) = m$
    - strict: $F(C \oplus D) = F(C) \oplus F(D)$
    - symmetric monoidal: $F(\mathrm{tw}_{m,n}) = \mathrm{tw}_{m,n}$
    - functor $F(I_m)=I_m$, $F(C\ ;\ D) = F(C)\ ;\ F(D)$
- In other words, all the structure is simply preserved on the nose — easy peasy

# Models

- Recall: models of algebraic theories are finite product preserving functors, often to **Set**
- We can define models of an SMT to be symmetric monoidal functors, a generalisation of the notion of finite product preserving
- Some computer science intuitions:
    - SMTs, like **M**, are a *syntax*
    - props like **F** are a *semantics*
    - homomorphisms map syntax to semantics
    - when the map is an isomorphisms, we have an equational characterisation, and a sound and fully complete proof system to reason about things in **F**

# Example

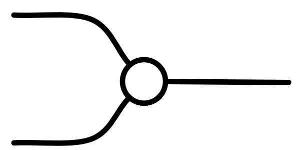
$\boxed{\text{As props, } \mathbf{M} \text{ is } \textit{isomorphic} \text{ to } \mathbf{F}}$

- So **M** is an equational characterisation of **F**
- or the “commutative monoids is the theory of functions”

# Morphisms from (props obtained from) SMTs

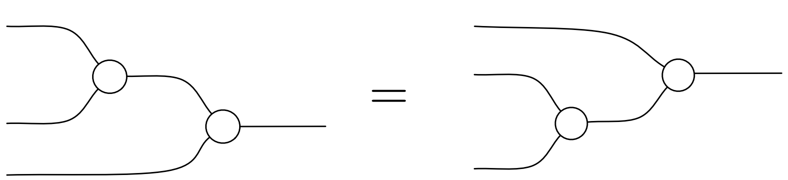
- Let us define a morphism [[-]] : **M** → **F**
  - M is obtained from a symmetric monoidal theory (Σ, E), thus its arrows are constructed inductively
  - To define [[-]] it thus suffices to
    - say where the generators in Σ are mapped
    - check that the equations in hold in **F**
  - This is a general pattern when defining morphisms from a prop obtained from an SMT

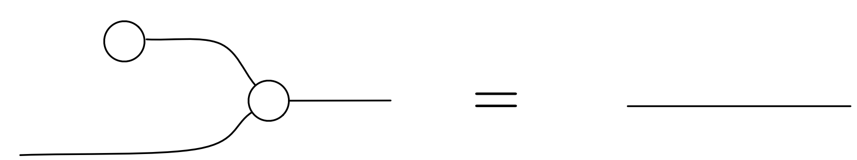
# $[[-]] : \mathbf{M} \to \mathbf{F}$

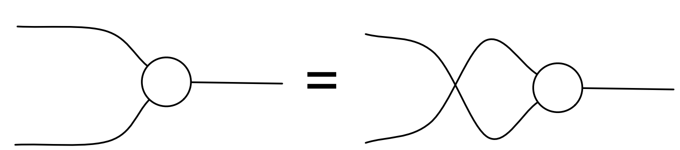
 $\longmapsto$ $\{1,2\} \to \{1\}$

 $\longmapsto$ $\{\} \to \{1\}$

**Simple exercise: check the following hold in F**

 (Assoc)

 (Unit)

 (Comm)

# Soundness

- Simple observation: the fact that we have a homomorphism $[\![-]\!] : \mathbf{M} \to \mathbf{F}$ means that diagrammatic reasoning in **M** is sound for **F**

*Q1*. What property of $[\![-]\!]$ do we need to ensure completeness?

*Q2.* If we have soundness and completeness, is this enough for $[\![-]\!]$ to be an *isomorphism*? (i.e. invertible)

# Full and faithful

- To show that a morphism of props F: **X**→**Y** is an isomorphism it suffices to show that it is full and faithful

  - **full**: for every arrow *g* of **Y** there exists an arrow *f* of **X** such that F(*f*) = *g*

  - **faithful**: given arrows *f*, *f'* in X, if F(*f*)=F(*f'*) then *f* = *f'*

So full and faithful functor from a (free PROP on an) SMT = sound and fully complete equational charaterisation

$$[\![-]\!] : \mathbf{M} \to \mathbf{F}$$

- **full:** every function between finite sets can be constructed from the two basic building blocks together with permutations

- **faithful**: every diagram in **M** can be written as multiplications followed by units, which corresponds to a factorisation of a function as an surjection followed by an injection. This factorisation is unique “up-to-permutation”.

# Free things

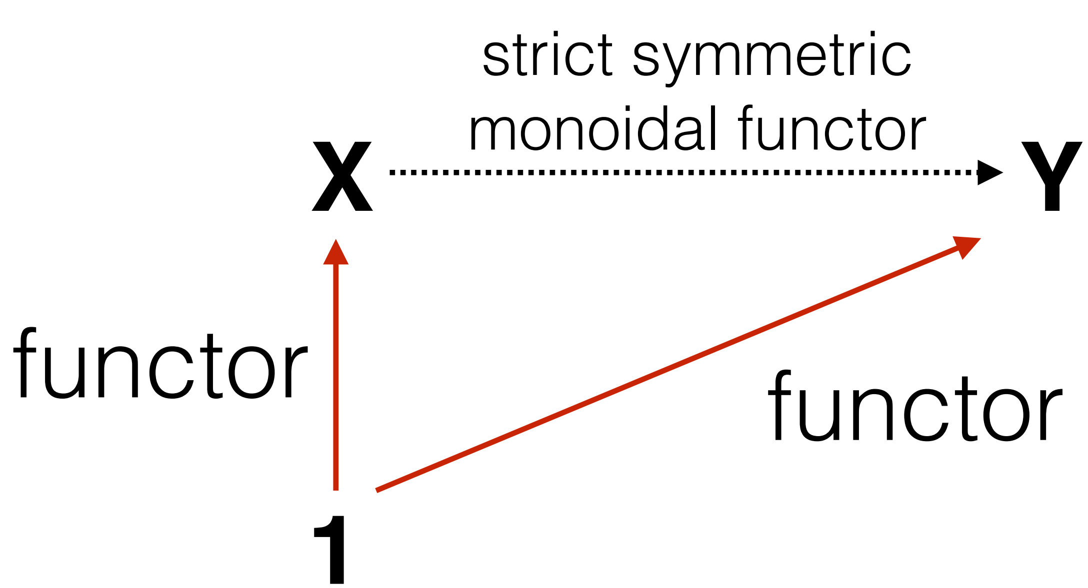
- A free "something on X" is one that satisfies a universal property — it's the "smallest" thing that contains X which satisfies the properties of "something"

$$\begin{tikzcd}
F \arrow[r, dotted] & G \\
X \arrow[u] \arrow[ur] &
\end{tikzcd}$$

- e.g. free "monoid on a set $\Sigma$" is the set of finite words $\Sigma^*$

# Free strict symmetric monoidal category on one object

- Any ideas?
    - Recall: there is a category **1** with one object and one arrow
    - Let **X** be the free symmetric monoidal category on **1**
    - There should be a functor from **1** to **X**
    - For any functor to a strict symmetric monoidal category **Y**, there should be a strict symmetric monoidal functor **X** to **Y** such that the diagram below commutes



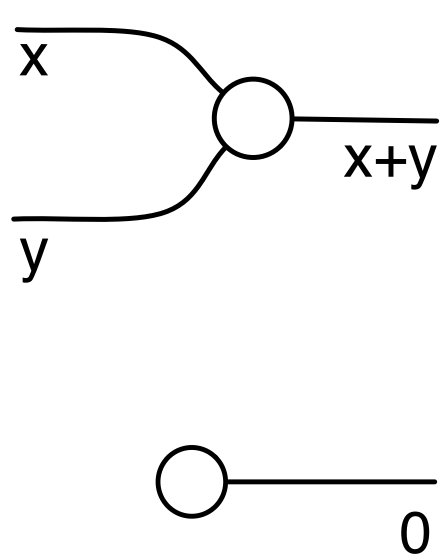
# Plan

- algebraic theories
- symmetric monoidal theories (resource sensitive algebraic theories)
- props
- **bimonoids and matrices of natural numbers**
- Hopf monoids and matrices of integers

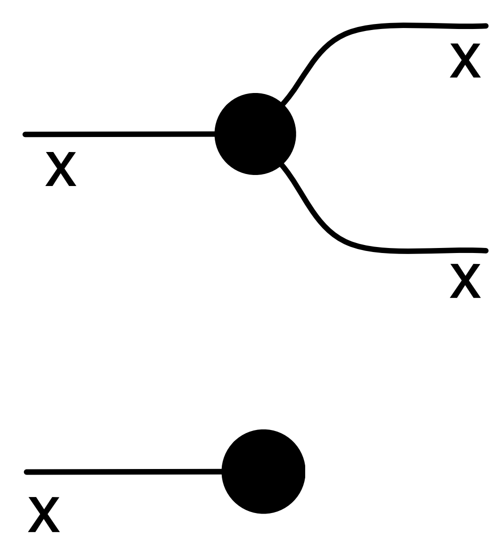
# The SMT of bimonoids

- Combines generators and equations of the SMTs of monoids and comonoids
- *Intuition*: “numbers” travel on wires from left to right

The monoid structure acts as addition/zero

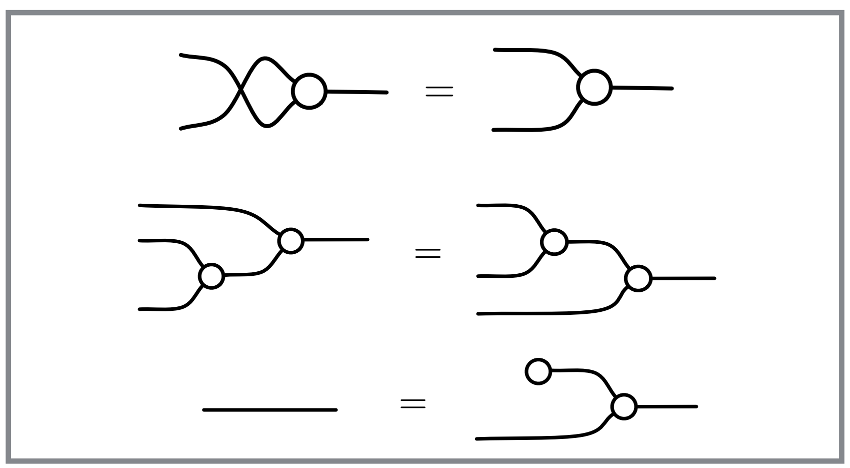


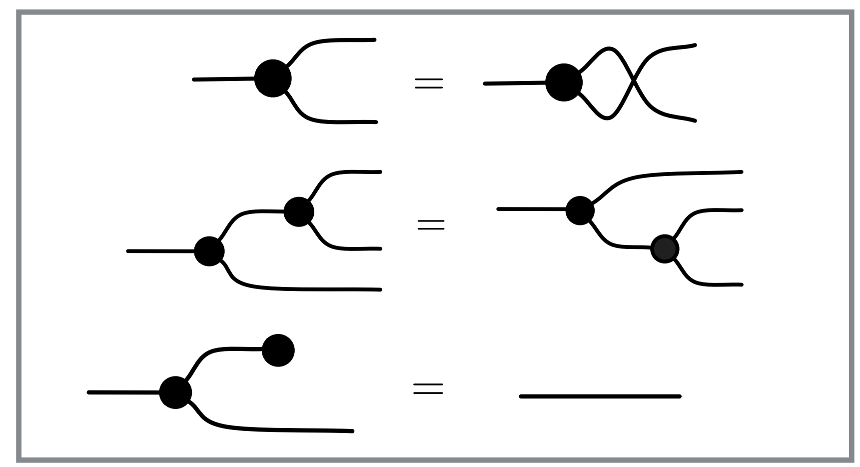

The comonoid structure acts as copying/discarding



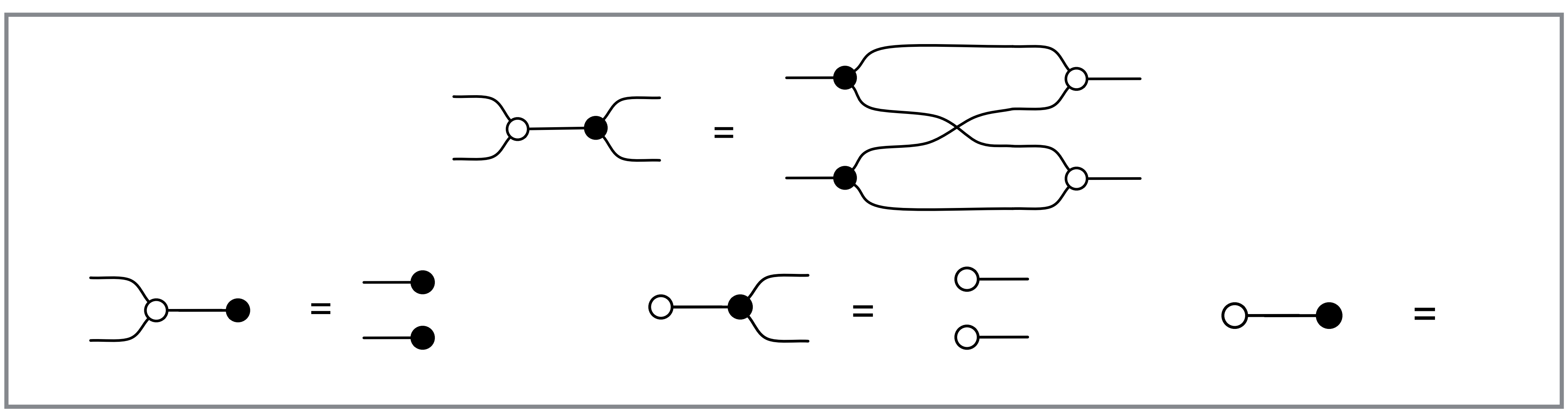
# The SMT of bimonoids

- all the generators we have seen so far
- monoid and comonoid equations





- “adding meets copying” - equations compatible with intuition



# Mat

- A PROP where arrows m to n are n×m matrices of natural numbers
  - e.g. $\begin{pmatrix} 0 & 5 \end{pmatrix} : 2 \to 1$ $\begin{pmatrix} 3 \\ 15 \end{pmatrix} : 1 \to 2$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} : 2 \to 2$
- Composition is matrix multiplication
- Monoidal product is direct sum

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

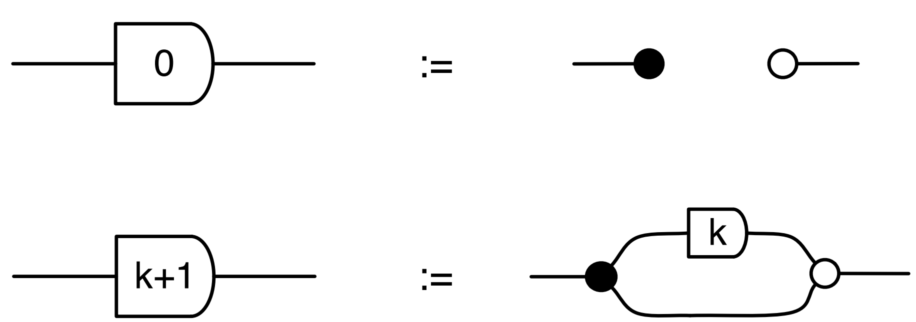
- Symmetries are *permutation matrices*

# **B** and **Mat**

- *Theorem*. **B** is isomorphic to the **Mat**
    - ie. bimonoids is the theory of natural number matrices

- natural numbers themselves can be seen as certain (1,1) diagrams, with the recursive definition below

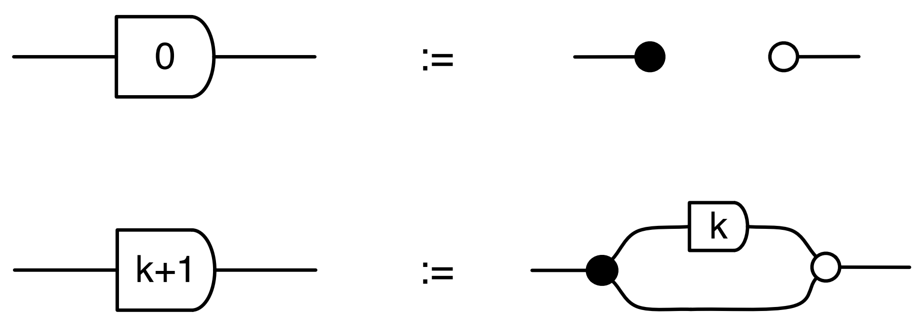
- as we will see, the algebra (rig) of natural numbers follows




+1 is “add one path”

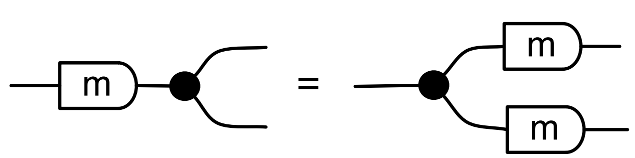
# Exercise

Given




, prove

1.

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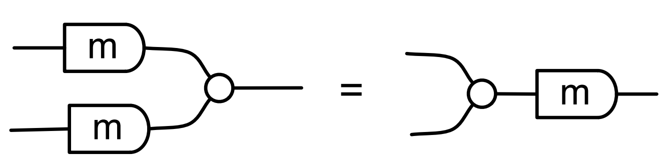
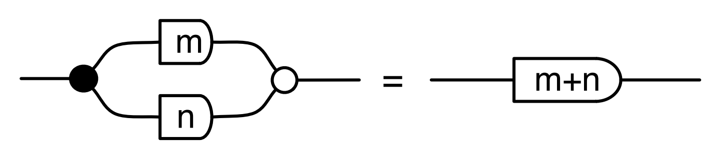



3.

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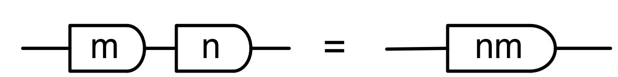
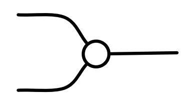
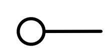
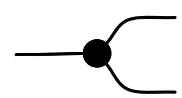


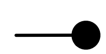
# Proof $\mathbf{B} \cong \mathbf{Mat}$

*Recall*: Since **B** is an SMT, suffices to say where generators go
(and check that equations hold in the codomain)

 $\mapsto \begin{pmatrix} 1 & 1 \end{pmatrix} : 2 \to 1$

 $\mapsto () : 0 \to 1$

 $\mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 1 \to 2$

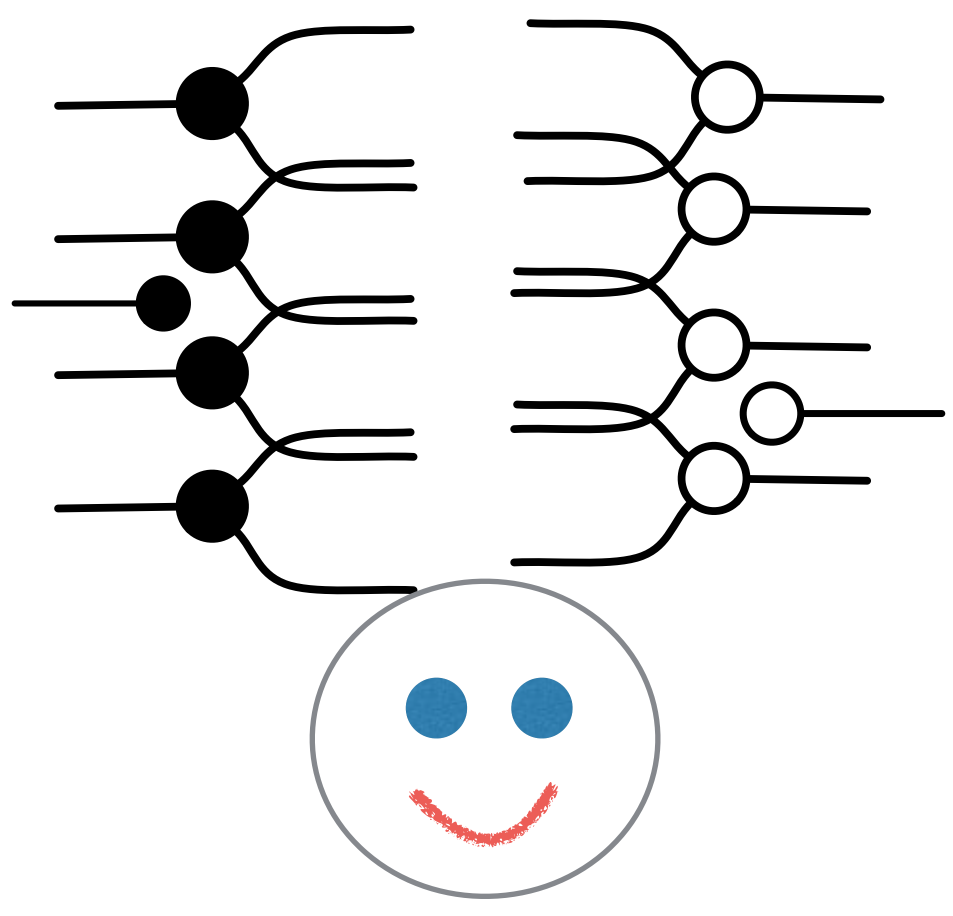
 $\mapsto () : 1 \to 0$

**Full** - easy!
Recursively define a syntactic sugar for matrices

**Faithful** - harder
Use the fact that equations are a presentation of a *distributive law*, obtain factorisation of diagrams as comonoid structure followed by monoid structure - **normal form**

# Normal form for **B**

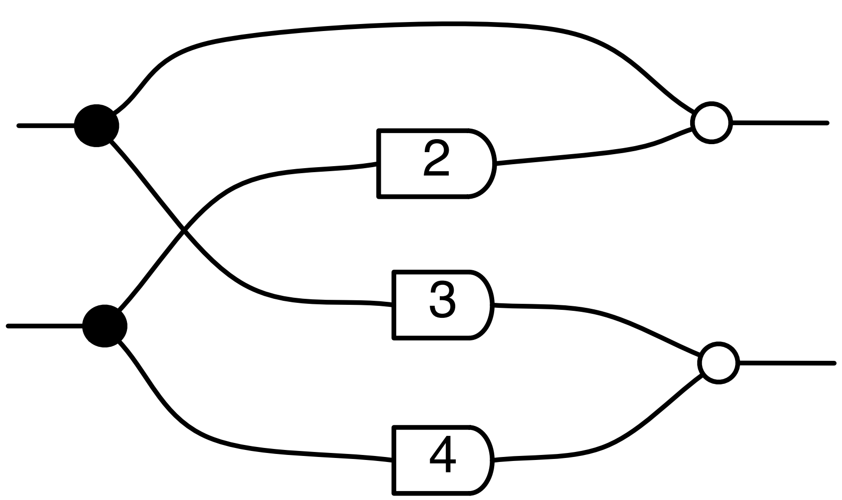
- Every diagram can be put in the form
  - **comonoid** ; **monoid**
- Centipedes



# Matrices

- To get the ijth entry in the matrix, count the paths from the j*th* port on the left to the ith port on the right

- Example:




$$\Longleftrightarrow \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

# Exercise

*Q1*. Show that the monoidal product in **B**≅**Mat** is the categorical product

*Q2*. The categorical coproduct of X, Y, if it exists satisfies the following universal property

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "f"'] & X+Y \arrow[d, dashed, "h"] & Y \arrow[l, "i_2"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

for any object $Z$ and arrows
$f\colon X \to Z$, $g\colon Y \to Z$,
$\exists$ unique $h\colon X+Y \to Z$ s.t.
$i_1 \,;\, h = f$ and $i_2 \,;\, h = g$

show that the monoidal product in B≅Mat is the categorical coproduct.

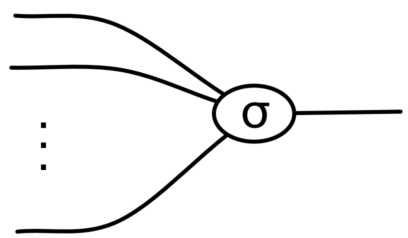
When a monoidal product satisfies both the universal properties of products and coproducts, we say that it is a *biproduct*.

In fact **B**≅**Mat** is the free category with biproducts on one object.

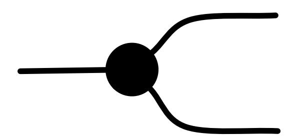
*Q3 (challenging)*. Given a category **C**, describe the free category with biproducts on **C**.

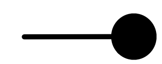
# Lawvere categories with string diagrams

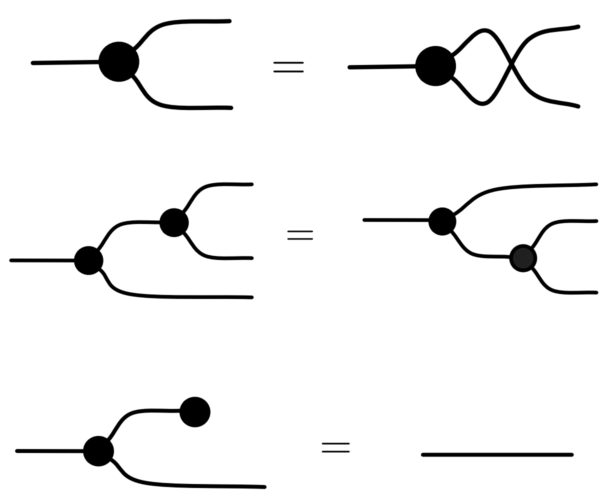
(i.e. how ordinary syntax looks, with string diagrams)



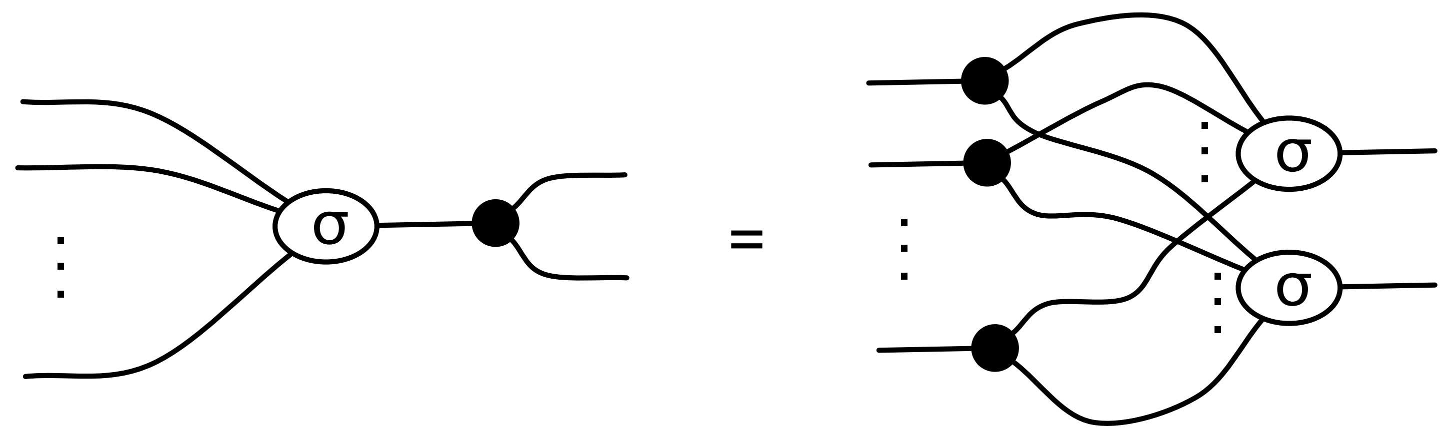

(σ ∈ Σ)

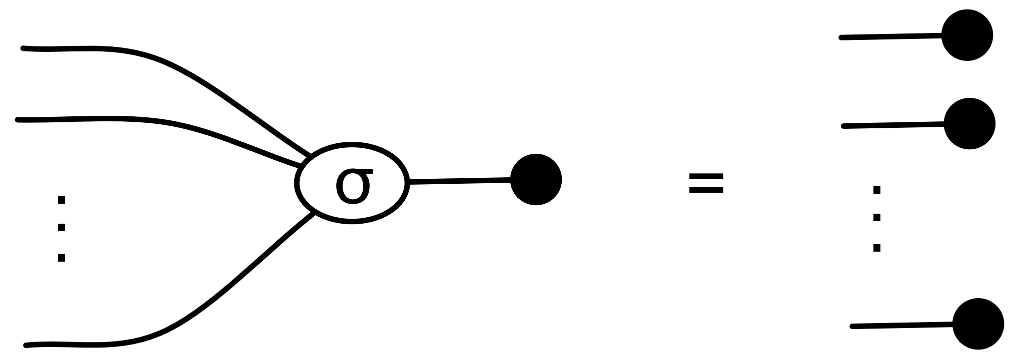






and what else?




***Exercise***: show that the monoidal product now becomes a *categorical* product

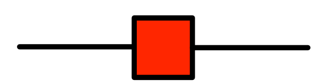
In particular, notice that B is isomorphic (as a symmetric monoidal category) to the Lawvere category of commutative monoids!

# Plan

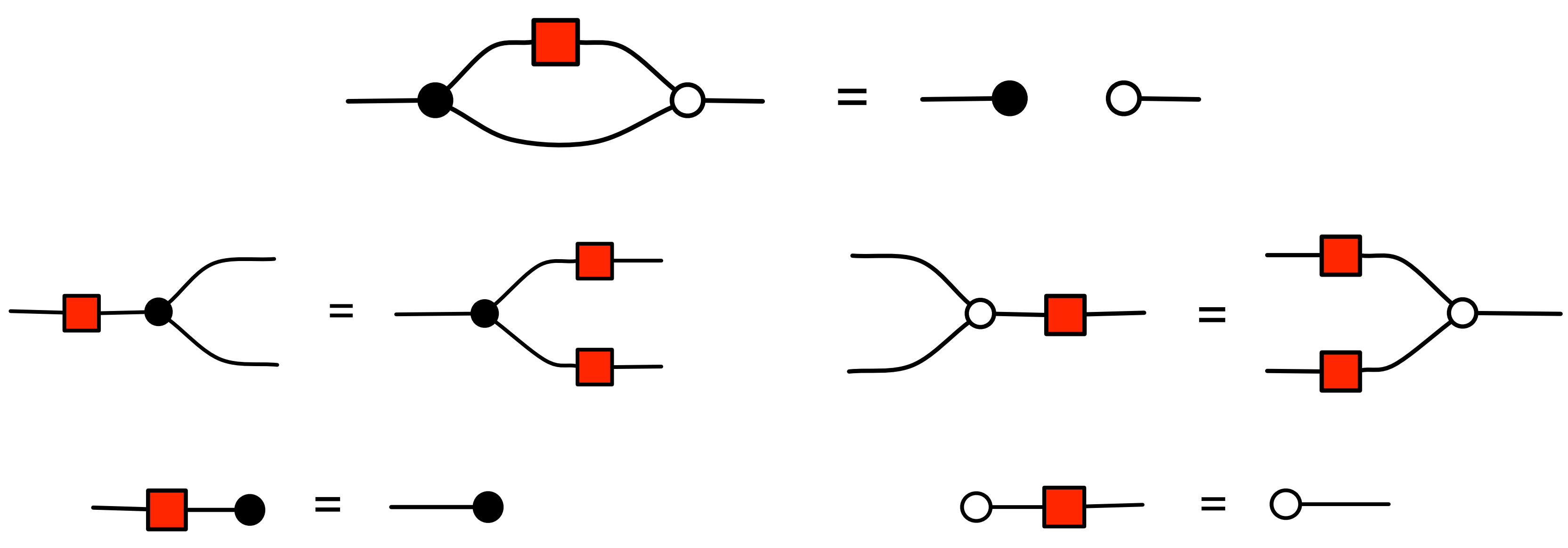
- algebraic theories
- symmetric monoidal theories (resource sensitive algebraic theories)
- props
- bimonoids and matrices of natural numbers
- **Hopf monoids and matrices of integers**

# Putting the n in ring: Hopf monoids

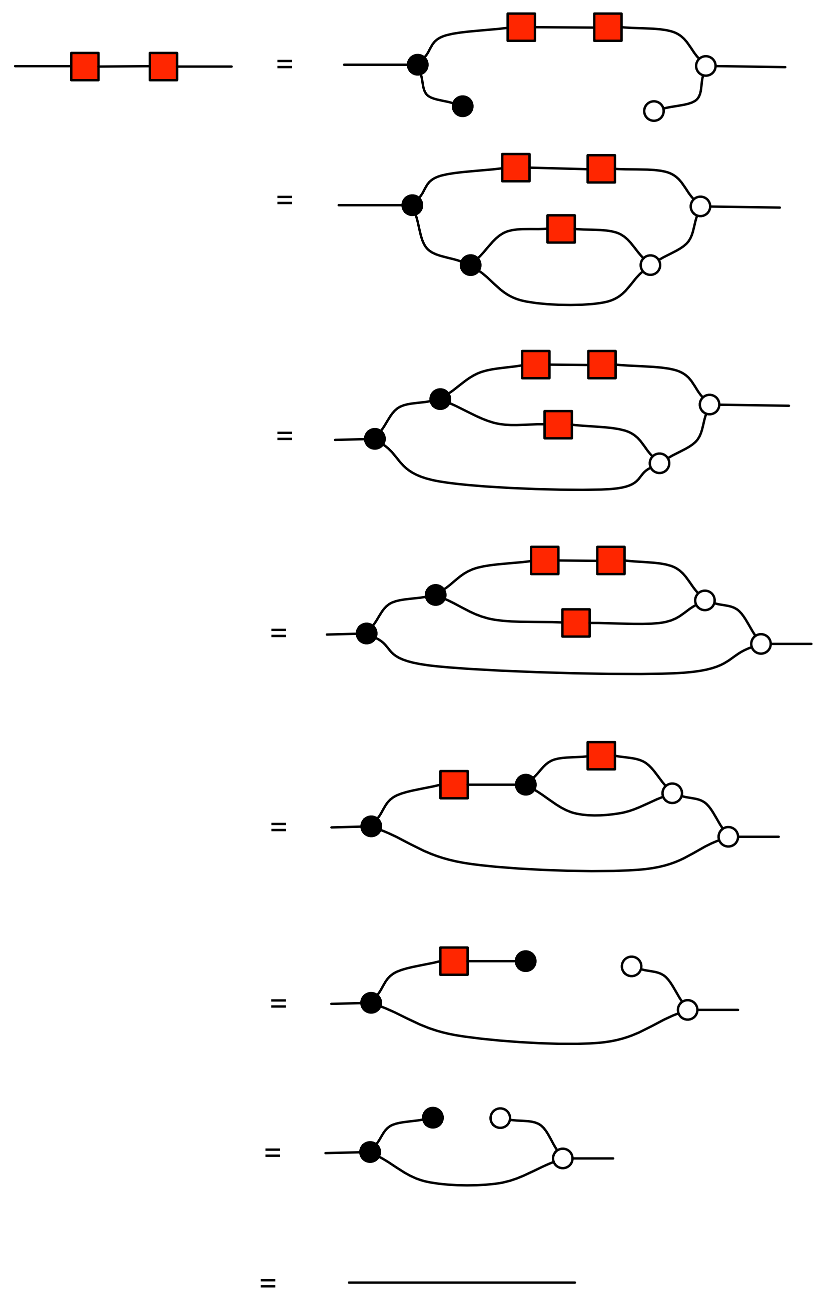
- generators of bimonoids + **antipode**
- think of this as acting as -1



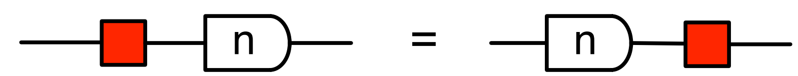
- equations of bimonoids and the following



$$-1 \cdot -1 = 1$$

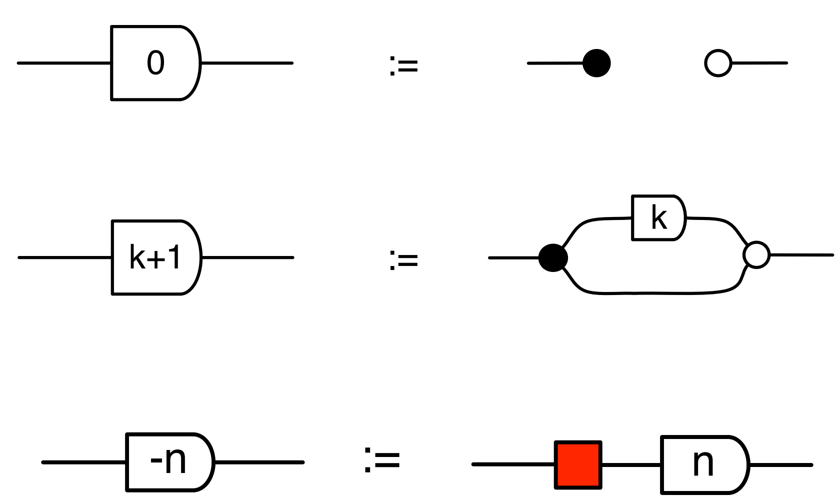


# The ring of integers

- Simple induction: 

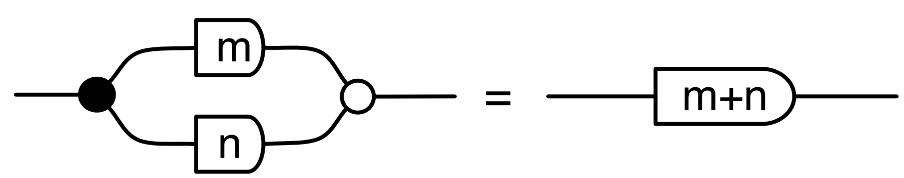

- Recall: in **B**, the arrows 1→1 were in one-to-one correspondence with natural numbers

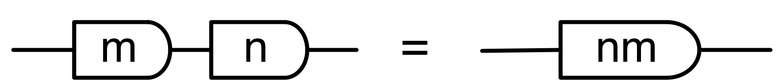
- In **H**, the arrows 1→1 are in one-to-one correspondence with the integers



# Exercise

- Verify that, in **H**, for all integers *m*, *n* we have



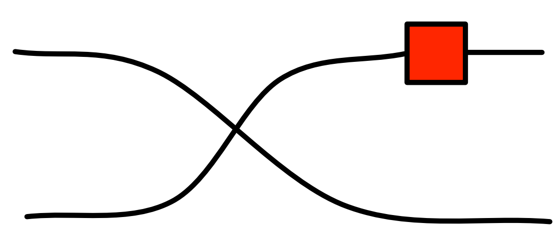



# $\mathbf{Mat_Z}$

- Arrows m to n are n×m matrices of integers
  - composition is matrix multiplication
  - monoidal product is direct sum
- $\mathbf{Mat_Z}$ is equivalent to the category of finite dimensional free **Z**-modules
- SMT **H** is isomorphic to the PROP $\mathbf{Mat_Z}$

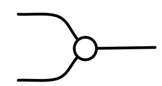
# Path counting in MatZ

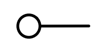
- To get the ij*th* entry in the matrix, count the
  - positive paths from the j*th* port on the left to the i*th* port on the right (where antipode appears an even number of times)
  - negative paths between these two ports (where antipode appears an odd number of times)
  - subtract the negative paths from the positive paths
- Example:

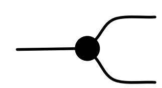


$$\Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

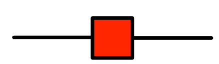
# Proof $\mathbf{H} \cong \mathbf{Mat_Z}$

 $\mapsto \begin{pmatrix} 1 & 1 \end{pmatrix} : 2 \to 1$

 $\mapsto () : 0 \to 1$

 $\mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} : 1 \to 2$

 $\mapsto () : 1 \to 0$

 $\mapsto (-1) : 1 \to 1$

- Fullness easy
- Faithfulness more challenging: put diagrams in the form

copying ; antipode ; adding

# Exercise

- We saw that **B** is the isomorphic, as a symmetric monoidal category, to the Lawvere category of *commutative monoids*.
- Which Lawvere category is **H** isomorphic to?