Lecture 3

Interacting Hopf monoids and graphical linear algebra

Plan

- **relational intuitions**
- Frobenius monoids
- the equations of interacting Hopf monoids
- linear relations
- rational numbers, diagrammatically

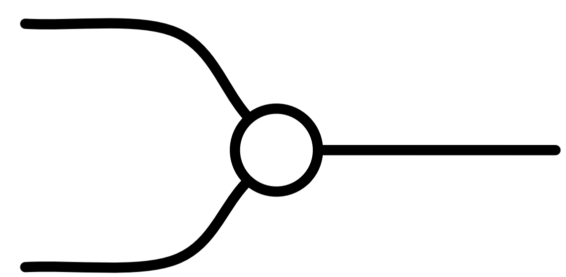
Relational intuitions

- We have been saying that numbers go from left to right in diagrams
  - this is a **functional**, input/output interpretation

> The input/output framework is totally inappropriate for dealing with all but the most special system interconnections. [The input/output representation] often needlessly complicates matters, mathematically and conceptually. A good theory of systems takes the behavior as the basic notion.
>
> J.C. Willems, *Linear systems in discrete time,* 2009

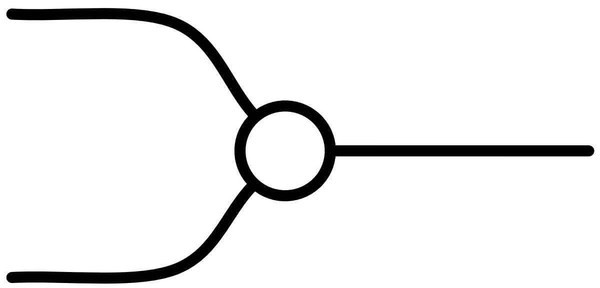
- From now on, we will take a **relational** point of view, a diagram is a contract that allows certain numbers to appear on the left and on the right

Intuition upgrade



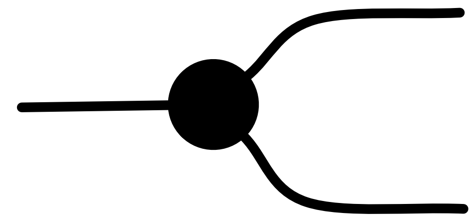
- Intuition so far is this as a function +: $D \times D \to D$
- From now it will be as a relation of type $D \times D \to D$
- Composition is relational composition

Mirror images

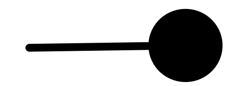


$\begin{matrix}x\\y\end{matrix}\ ,\ x+y$

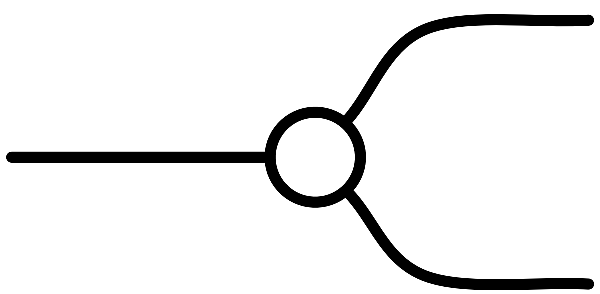
$()\ ,\ 0$



$x\ ,\ \begin{matrix}x\\x\end{matrix}$

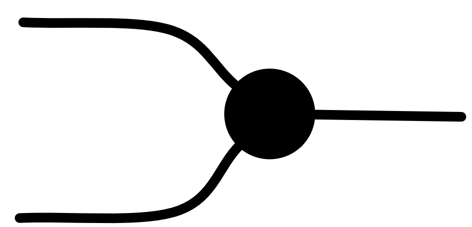


$x\ ,\ ()$

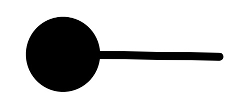


$x+y\ ,\ \begin{matrix}x\\y\end{matrix}$

$0\ ,\ ()$

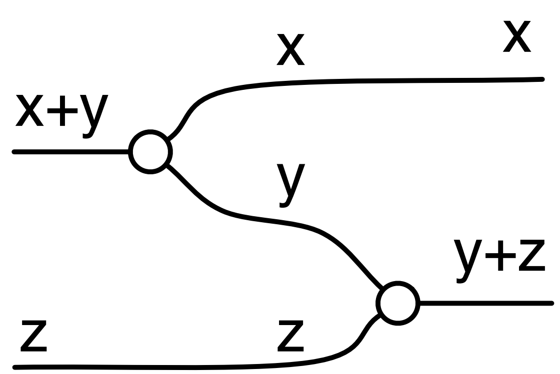


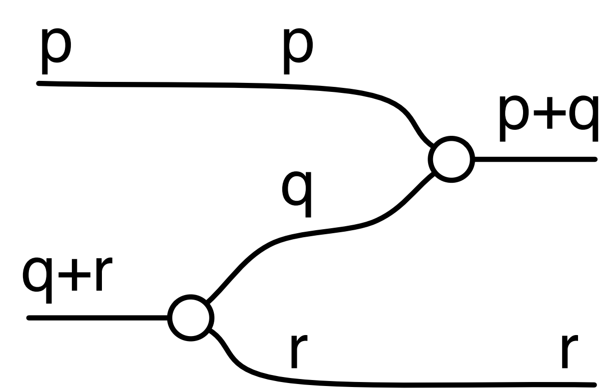
$\begin{matrix}x\\x\end{matrix}\ ,\ x$



$()\ ,\ x$

Adding meets adding





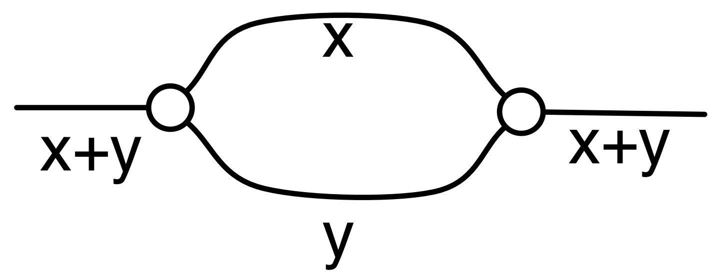
$x = p+q \qquad p = x+y$

$z = q+r \qquad r = y+z$

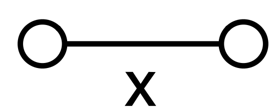
$\Rightarrow \quad y = -q$

Provided addition yields abelian group (i.e. there are additive inverses), the two are **the same** relation

More adding meets adding

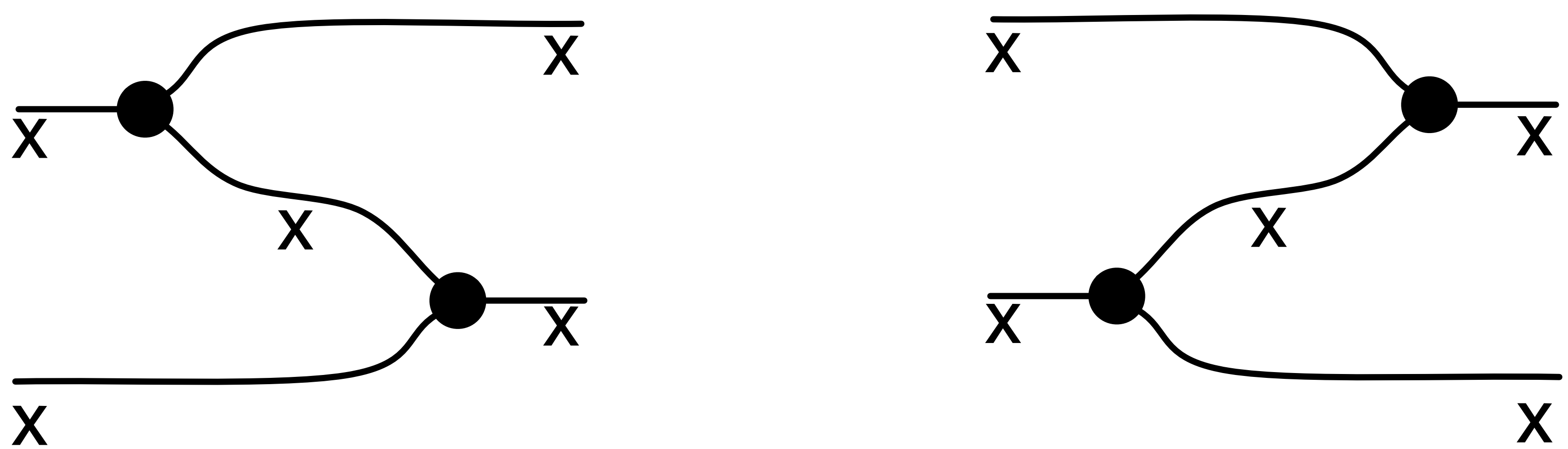


since x and y are free, this is the identity relation

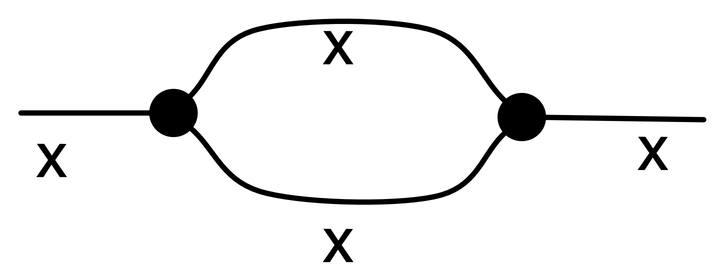


empty relation

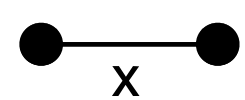
Copying meets copying



clearly both give the same relation



identity relation



empty relation

Two Frobenius structures

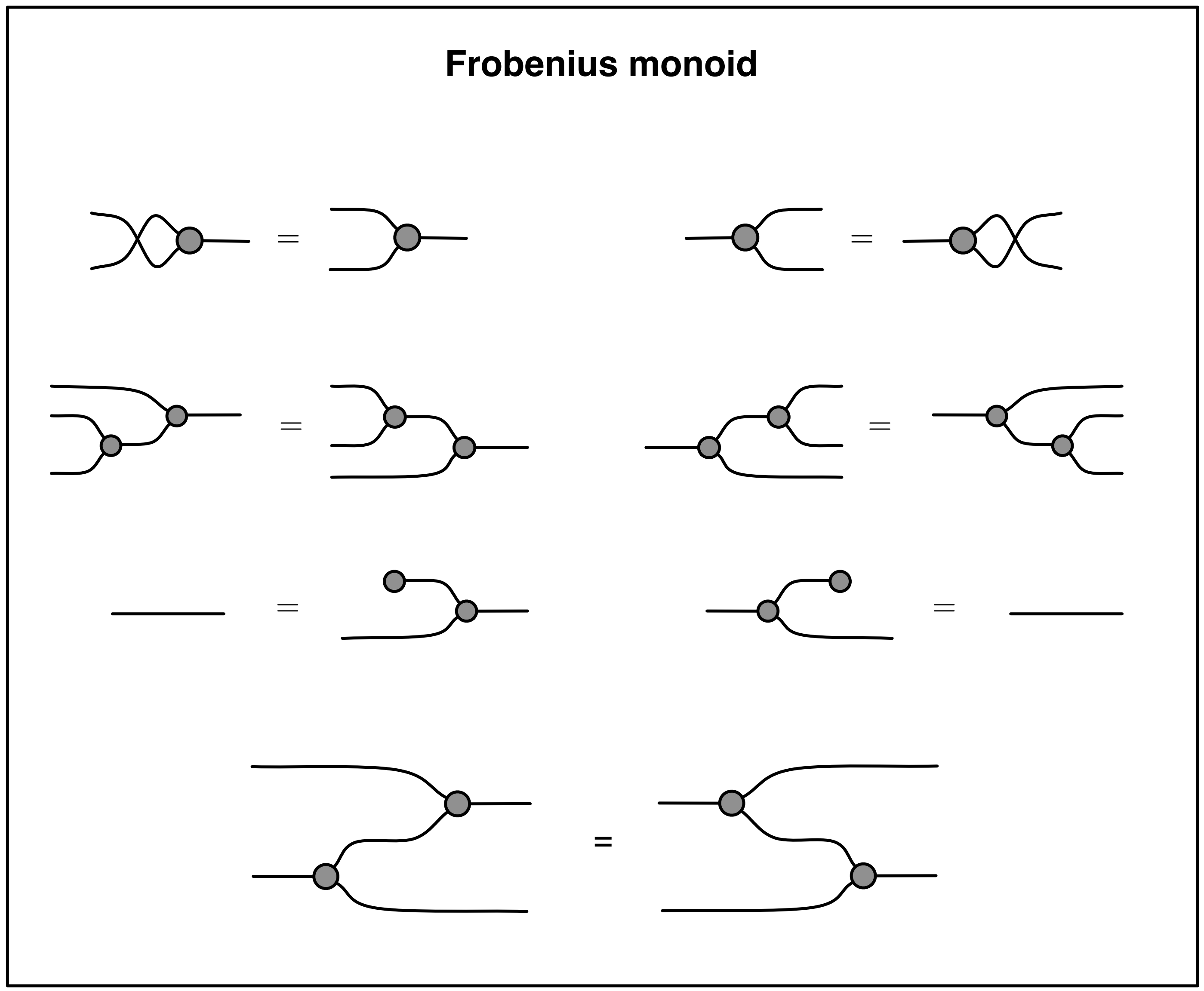
+ special / strongly separable equations

+ “bone” equations

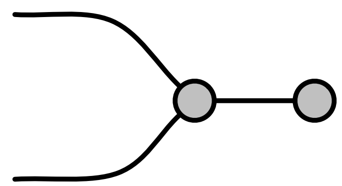
Plan

- relational intuitions
- **Frobenius monoids**
- the equations of interacting Hopf monoids
- rational numbers and linear relations
- graphical linear algebra

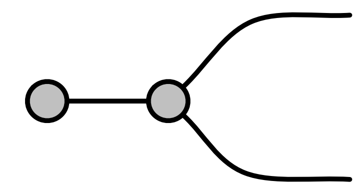
Frobenius monoids



Snakes



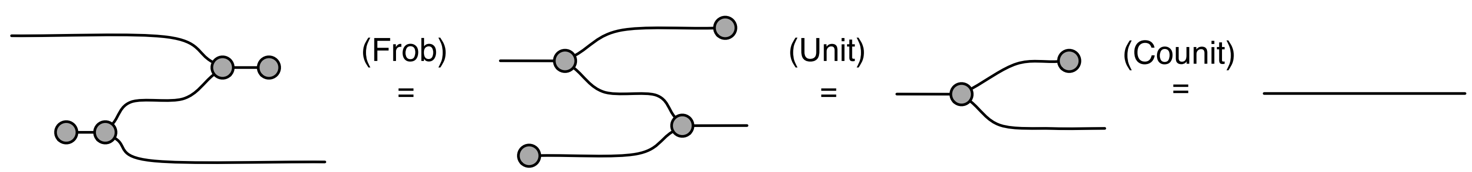
“cup”



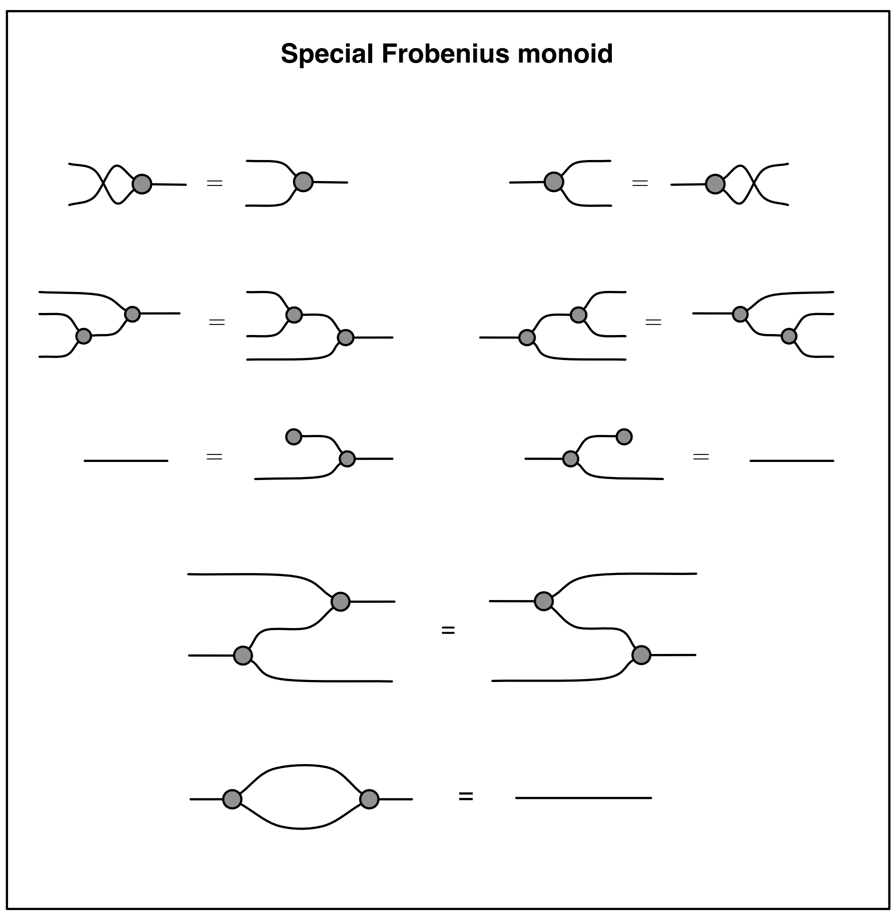
“cap”

Snake lemma

Proof:



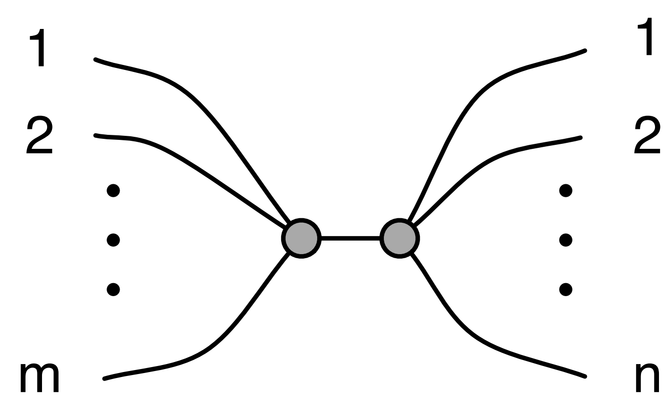
Normal forms



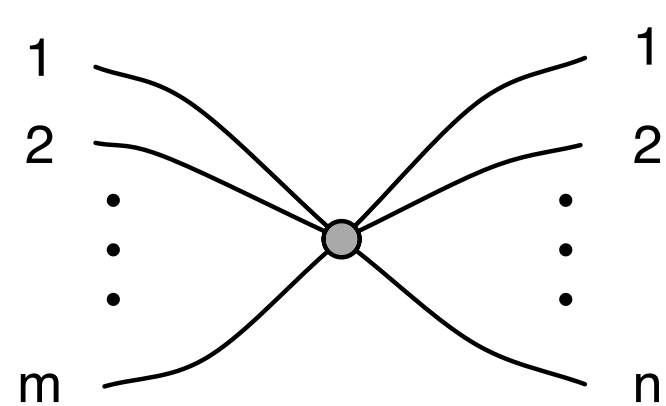
- In **B**, we saw that every diagram can be factorised into comonoid structure ; monoid structure, this gave us **centipedes**
- In **Frob**, every diagram can be factored into monoid structure ; comonoid structure, these are often referred to as **spiders**

Spiders in special Frobenius monoids 1

- In a special Frobenius monoid every connected diagram is equal to one of the form

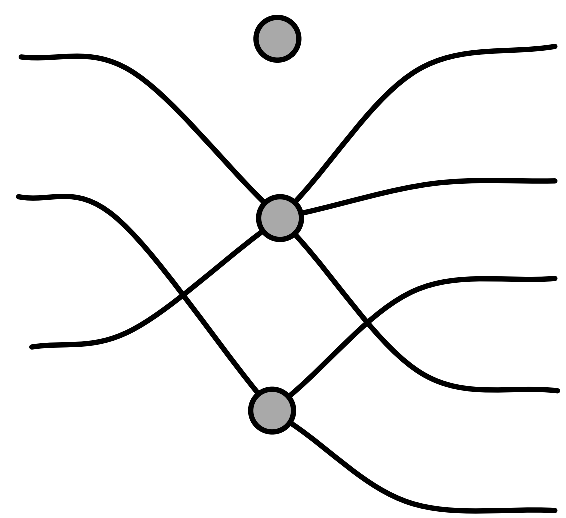


- which suggests the "spider notation"



Spiders in special Frobenius monoids 2

- In general, diagrams are collections of spiders



- when two spiders connect, they fuse into one
    - i.e. any *connected* diagram of type m→n is equal

Plan

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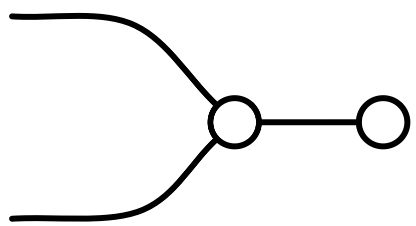
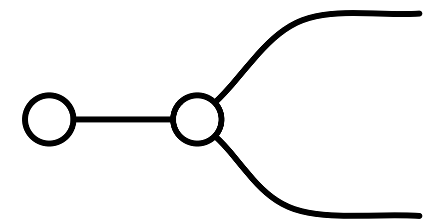
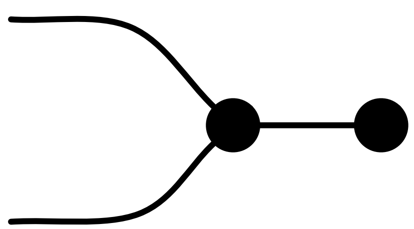
- Frobenius monoids
- **the equations of interacting Hopf monoids**
- rational numbers and linear relations
- graphical linear algebra

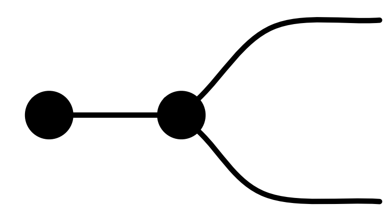
Black and white cups and caps



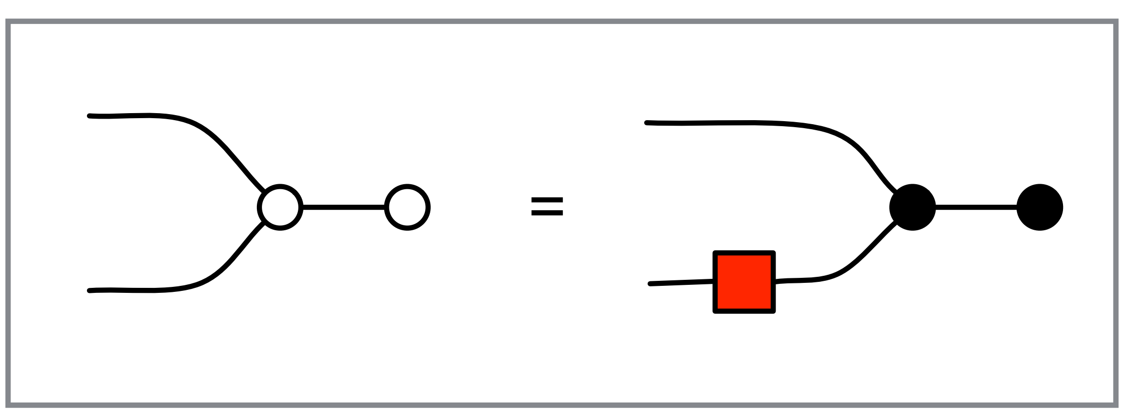


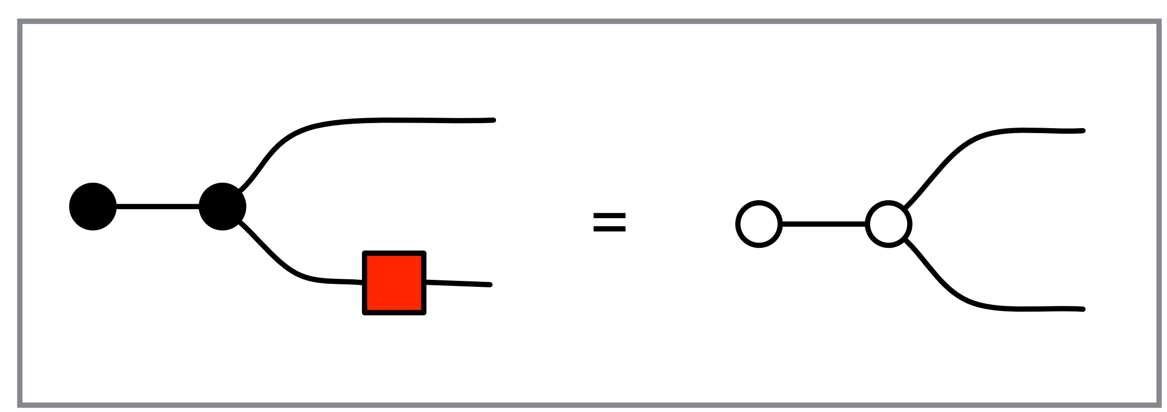
$$\{(\begin{pmatrix} x \\ y \end{pmatrix}, ()) \mid x + y = 0\}$$





$$\{(\begin{pmatrix} x \\ x \end{pmatrix}, ())\}$$





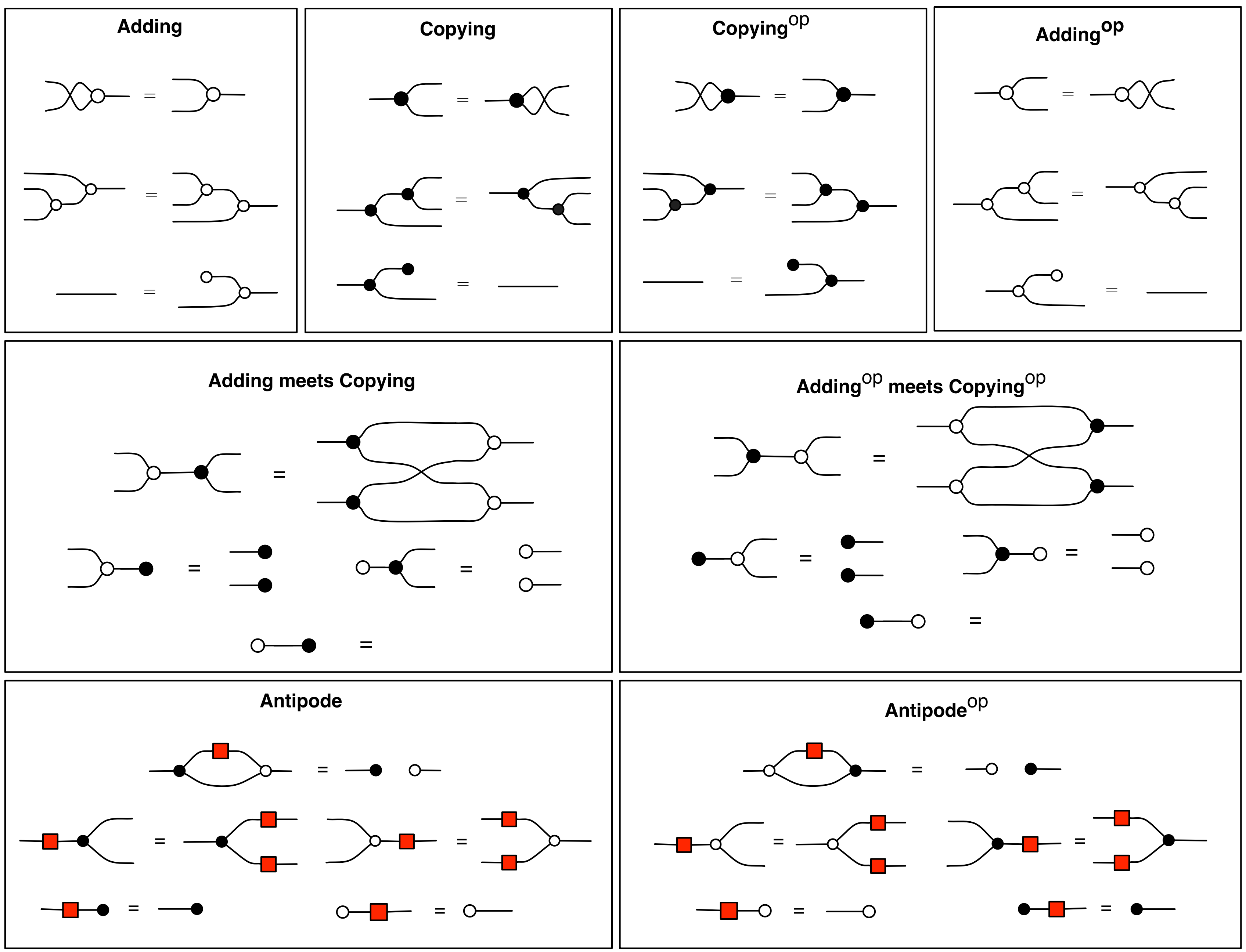
Scalars meet scalars

if multiplication on the left by p is injective
(e.g. if p ≠ 0 in a field)

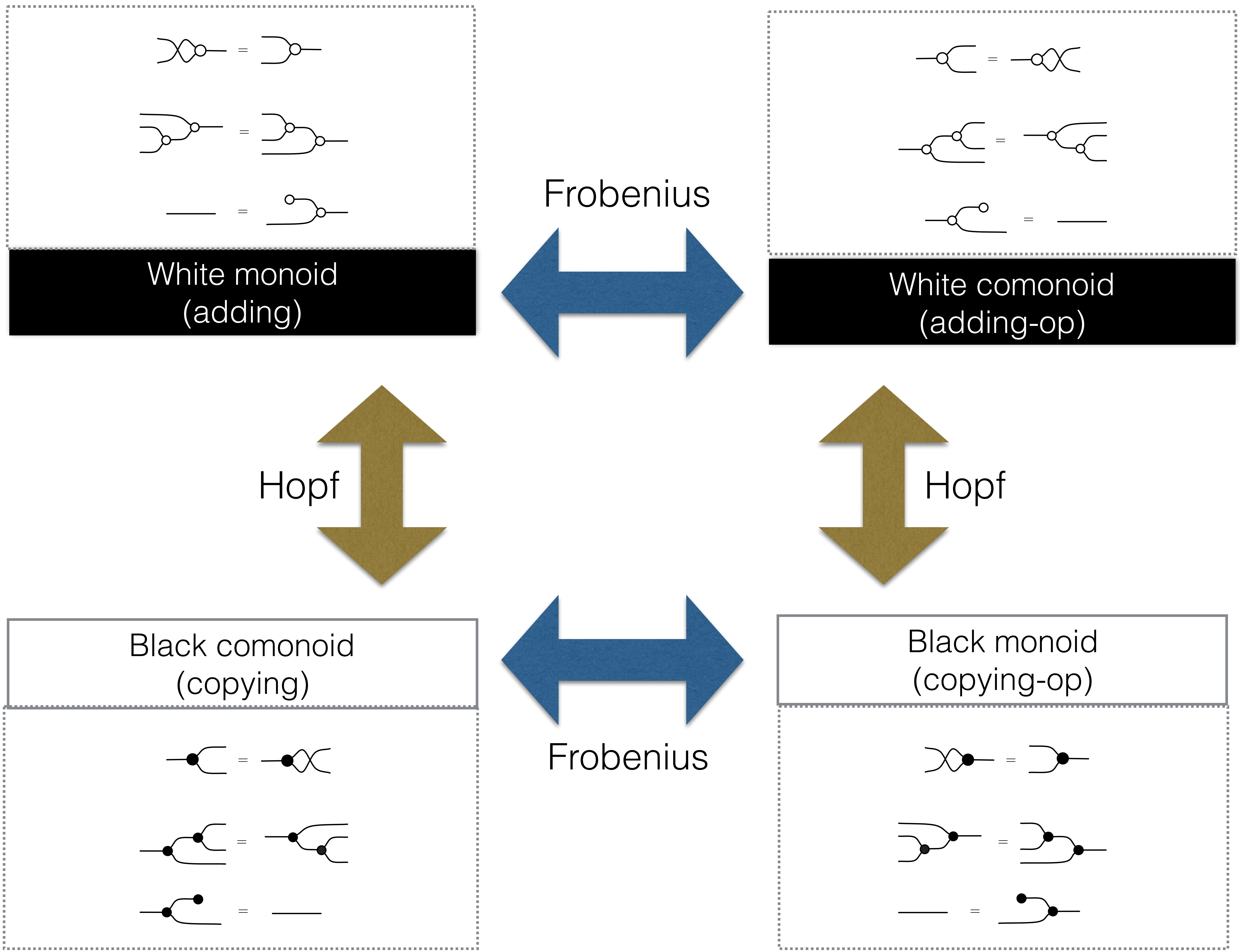
if multiplication on the left by p is surjective
(e.g. if p ≠ 0 in a field)

Interacting Hopf Monoids

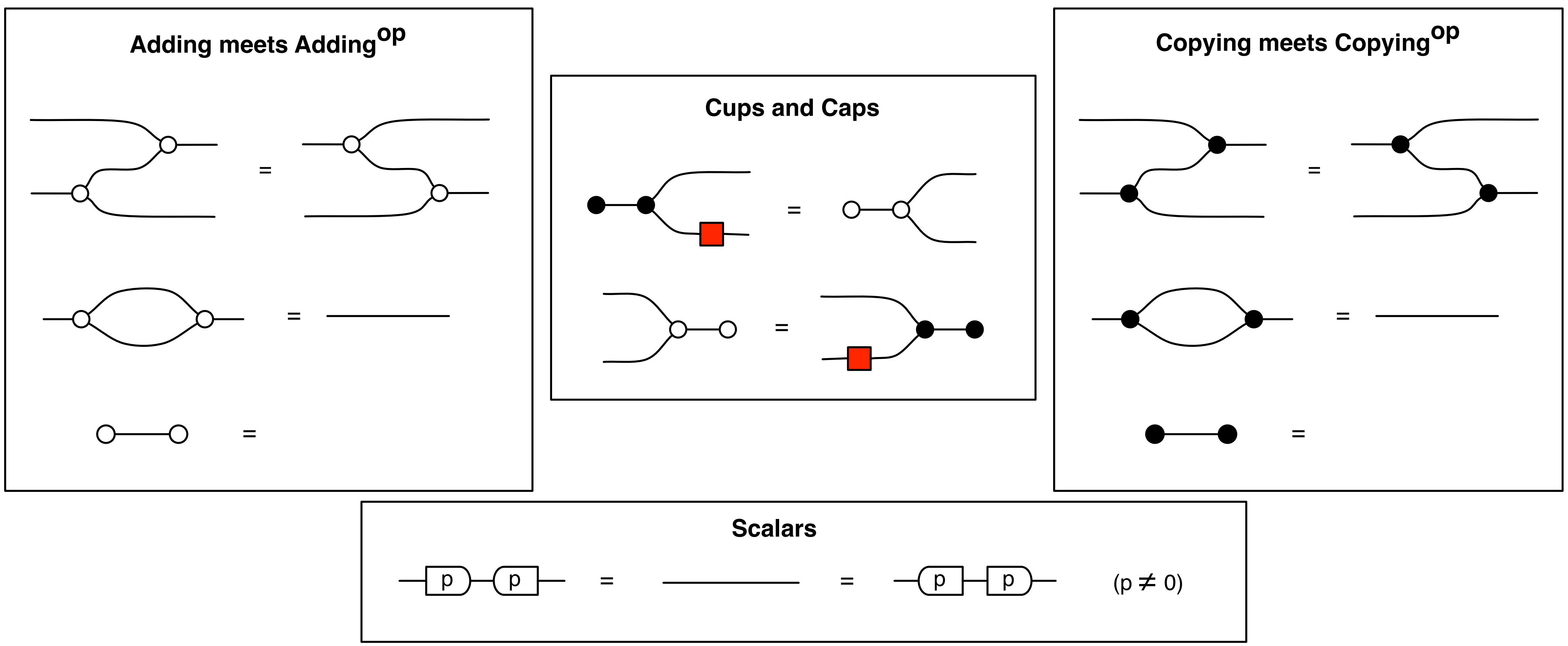
(Bonchi, S., Zanasi, '13, '14)



H $\mathbf{H}^{op}$

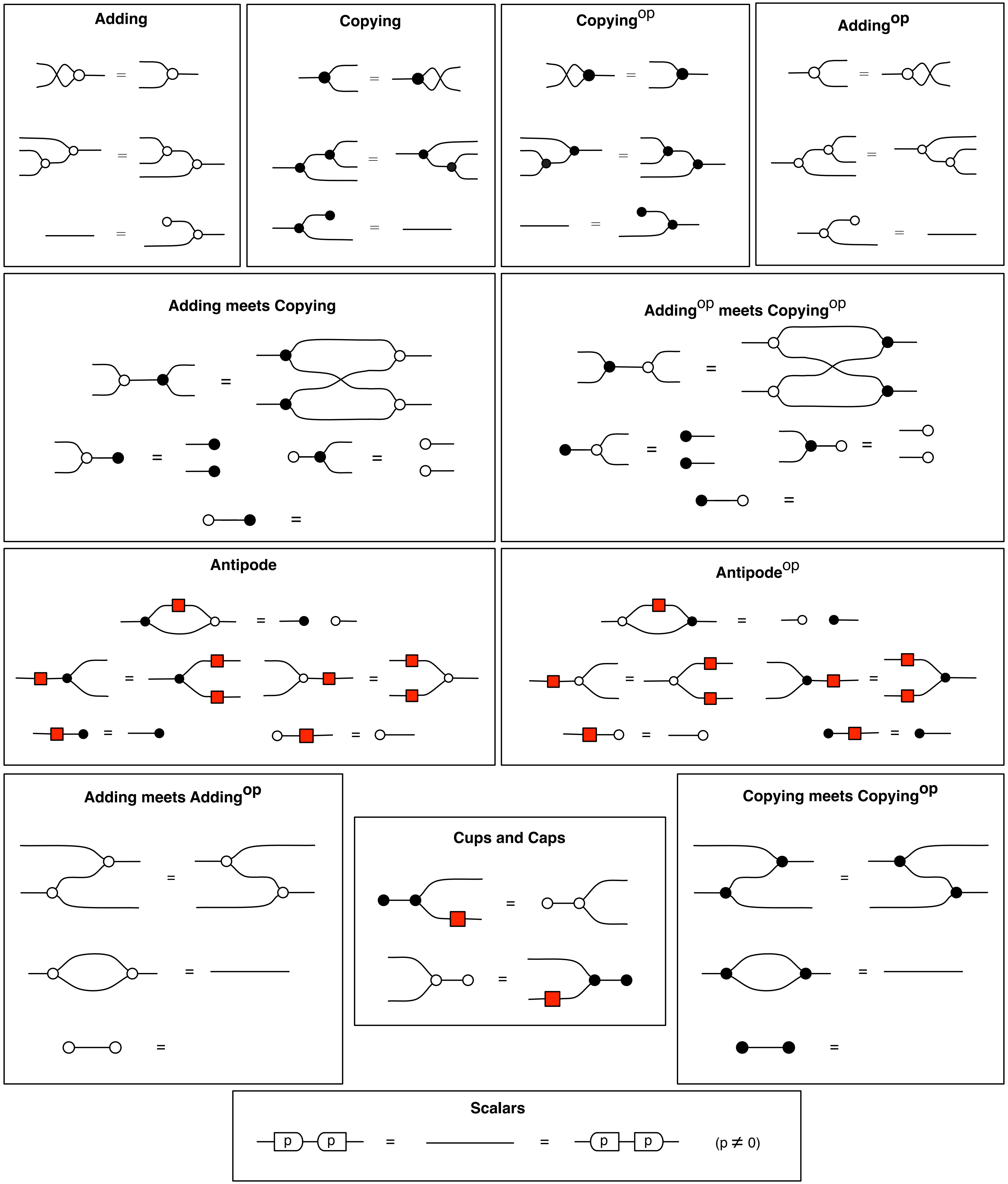

White monoid
(adding)
White comonoid
(adding-op)
Frobenius
Hopf
Hopf
Black comonoid
(copying)
Black monoid
(copying-op)
Frobenius

Interacting Hopf Monoids



cf. ZX-calculus (Coecke, Duncan)

IH



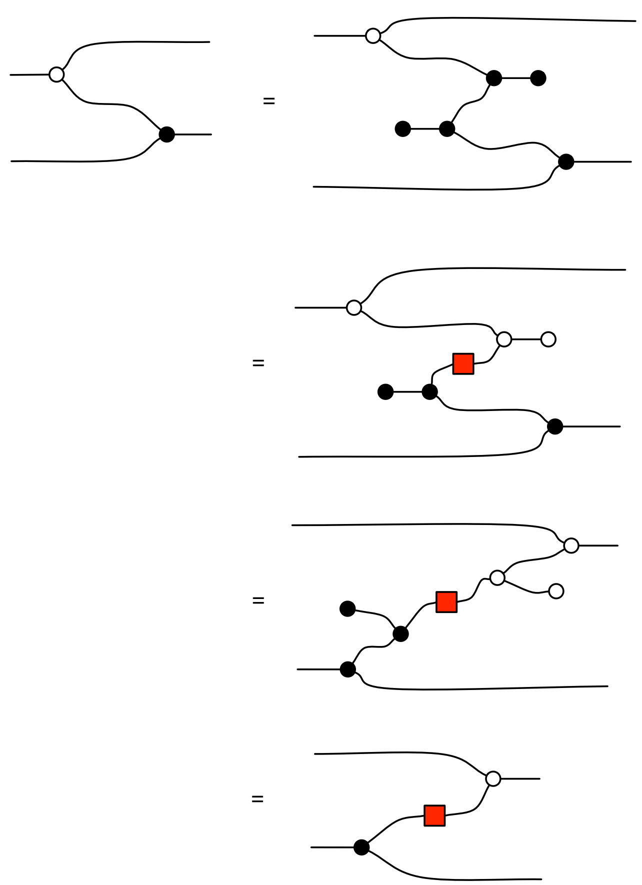
Symmetry 1 - colour inversion

Symmetry 2 - mirror image

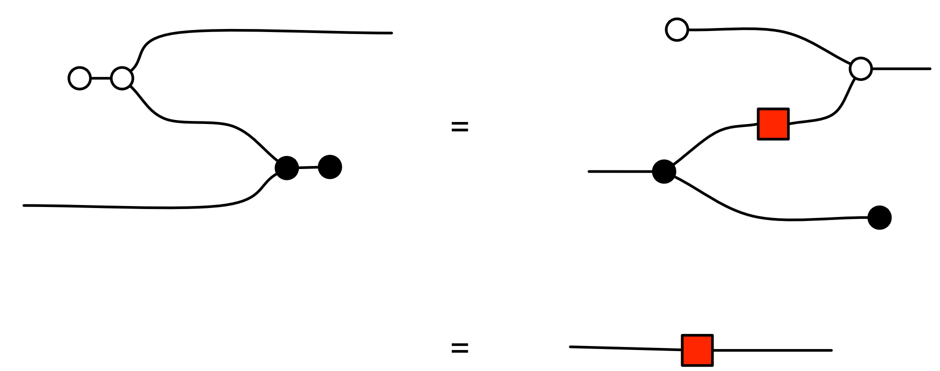
Redundancy

- Generators are expressible in terms of other generators, e.g.

Lemma



so



Plan

- relational intuitions
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- **rational numbers and linear relations**
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Linear subspaces

- Suppose that V is a vector space over field k
  - A *linear subspace* $U \subseteq V$ is a subset that
    - contains the zero vector, $\boldsymbol{0} \in V$
    - closed under addition, if $u, u' \in U$ then $u + u' \in U$
    - closed under scalar multiplication, if $u \in U$ and $p \in k$ then $p \cdot u \in U$
  - *e.g.* $\mathbf{R}^2$ is an $\mathbf{R}$-vector space. What are the linear subspaces?

Exercise

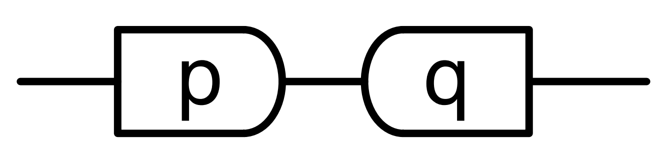
- Suppose that U, V, W are k vector spaces,
  - $R \subseteq U \times V$ is a subspace and
  - $S \subseteq V \times W$ is a subspace
- Show that the relational composition $R;S \subseteq U \times W$ is a subspace

LinRel

- PROP of linear relations over the rationals
  - arrows m to n are subspaces of $\mathbf{Q}^m \times \mathbf{Q}^n$
  - composed **as relations**
  - monoidal product is direct sum
- **IH** is isomorphic to **LinRel**

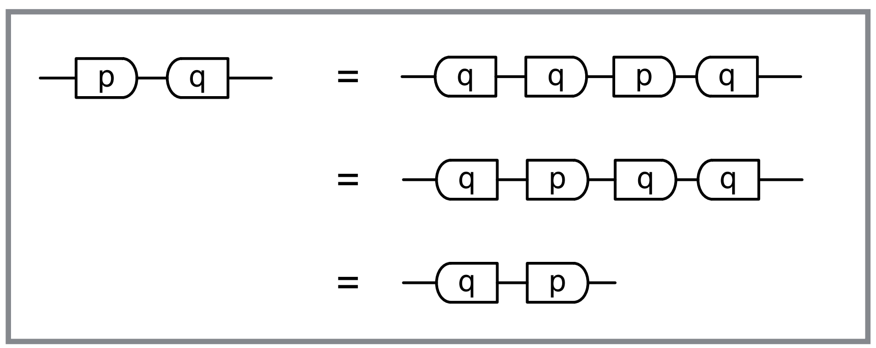
Where did the rationals come from?

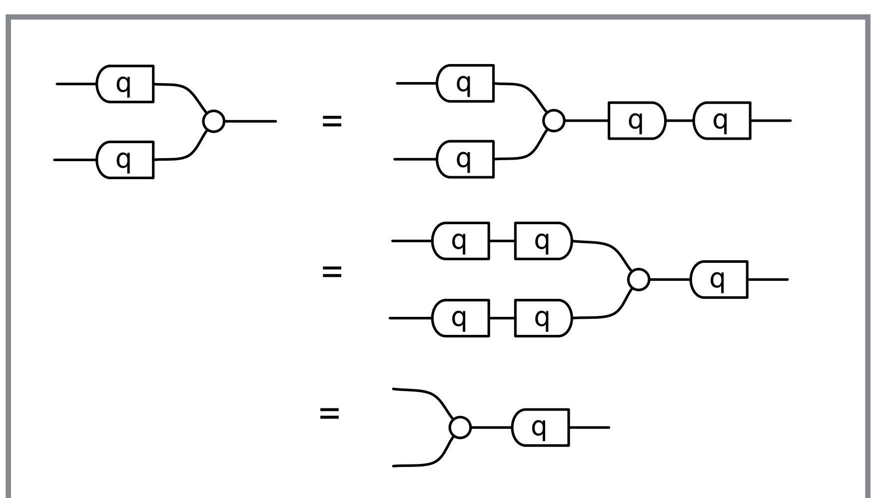
- Recall
  - in **B**, the (1,1) diagrams were the natural numbers
  - in **H**, the (1,1) diagrams were the integers
  - In **IH**, the (1,1) diagrams include the rationals p/q



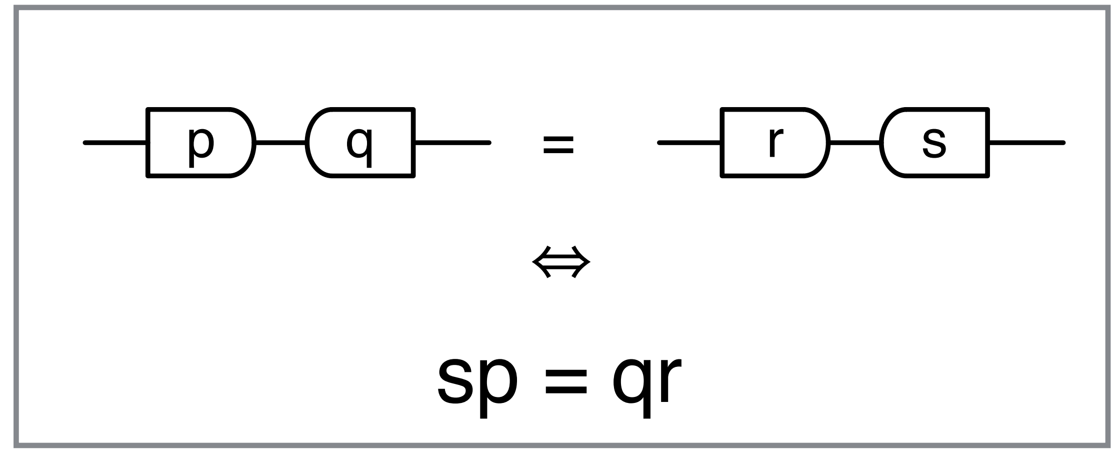
Some Lemmas

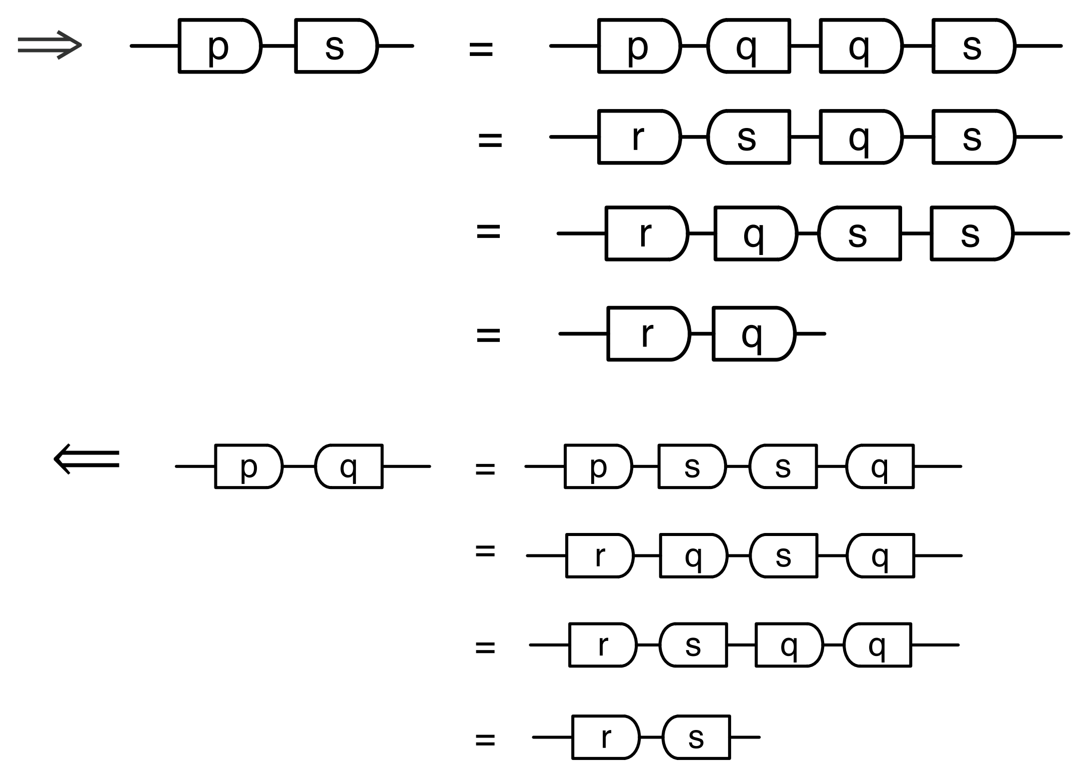
if $q \neq 0$:





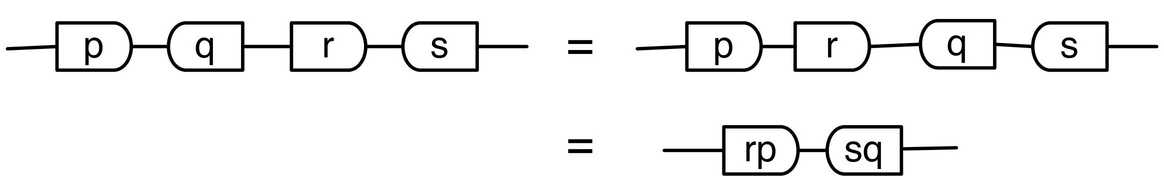
suppose $q,s \neq 0$:

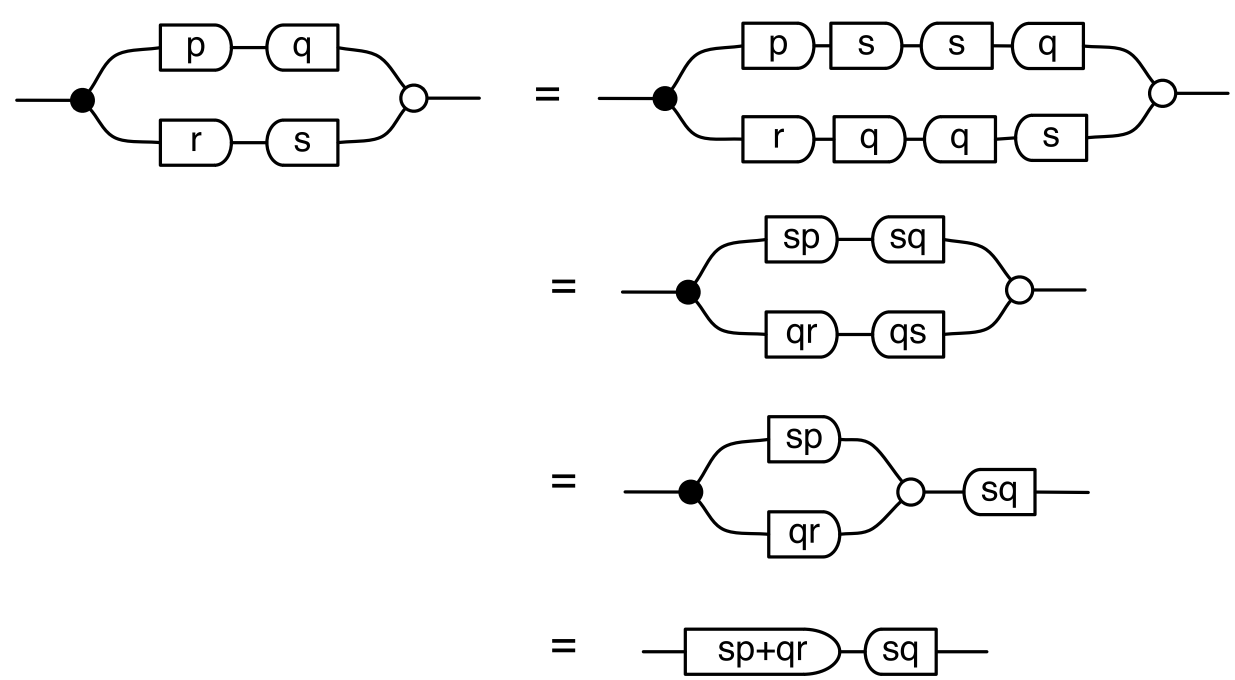




Rational arithmetic

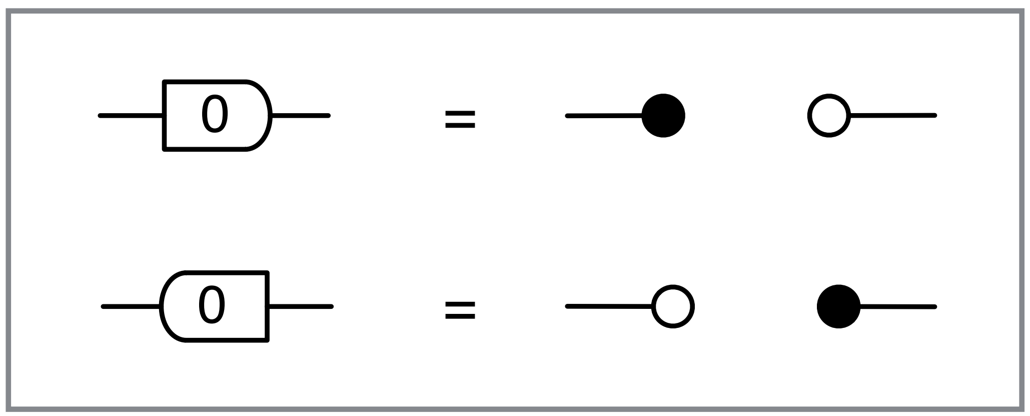
$(q,s \neq 0)$





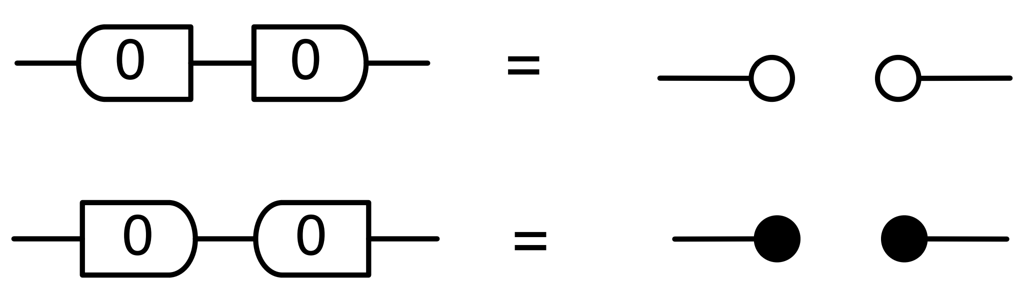
Keep calm and divide by zero

- it’s ok, nothing blows up



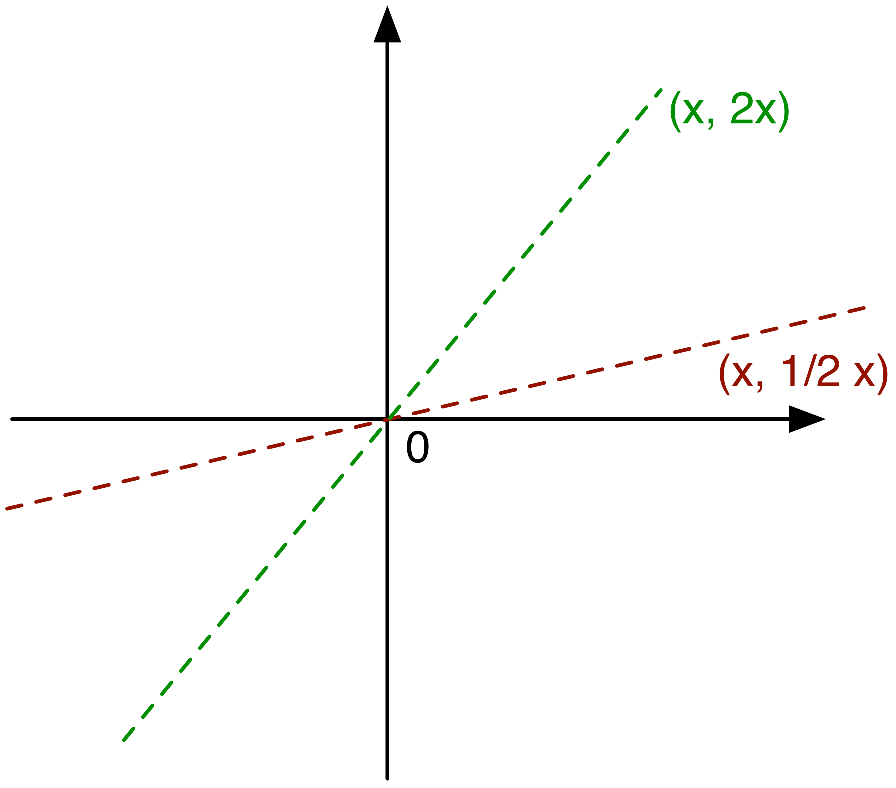
- of course, arithmetic with 1/0 is not quite as nice as with proper rationals.

- two ways of interpreting 0/0 (0 · /0 or /0 · 0)



Projective arithmetic++

- Projective arithmetic identifies numbers with one-dimensional spaces (lines) of $\mathbf{Q}^2$
    - one for each rational p : $\{ (x,px) \mid x \in \mathbf{Q} \}$
    - and "infinity" : $\{ (0, x) \mid x \in \mathbf{Q} \}$
- The extended system includes all the subspaces of $\mathbf{Q}^2$, in particular:
    - the unique zero dimensional space $\{ (0, 0) \}$
    - the unique two dimensional space $\{ (x,y) \mid x,y \in \mathbf{Q} \}$



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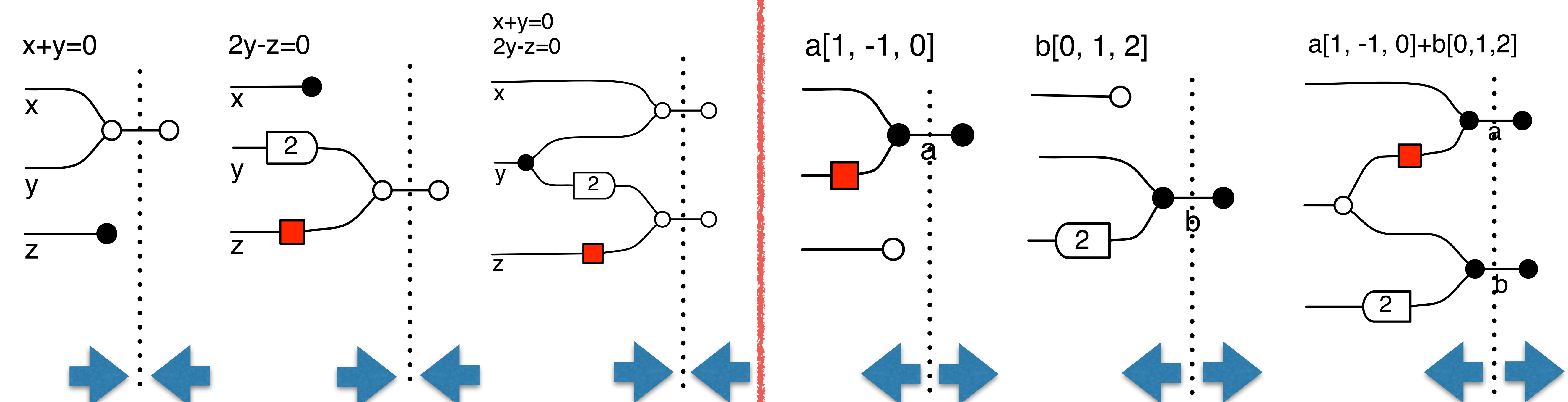
Factorisations

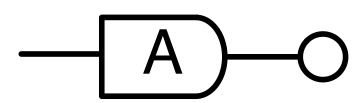
- Every diagram can be factorised as a span or a cospan of matrices
- This gives us the two different ways one can think of spaces

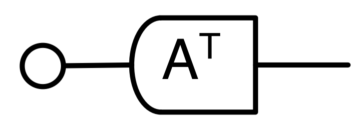
solutions of a list of homogeneous equations	linear combinations of basis vectors

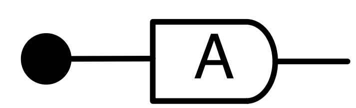


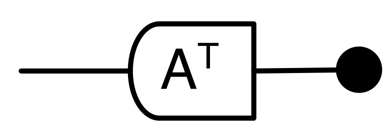
Cospans	Spans

Image and kernel

- **Definition**
  - The **kernel** of A is 

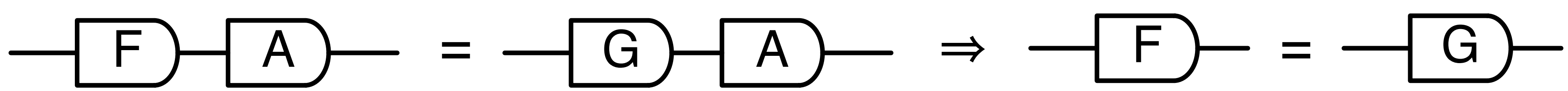
  - The **cokernel** of A is 

  - The **image** of A is 

  - The **coimage** of A is 

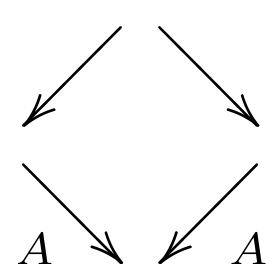
Injectivity

Injective matrices are the monos in $\mathbf{Mat}_{\mathbb{Z}}$



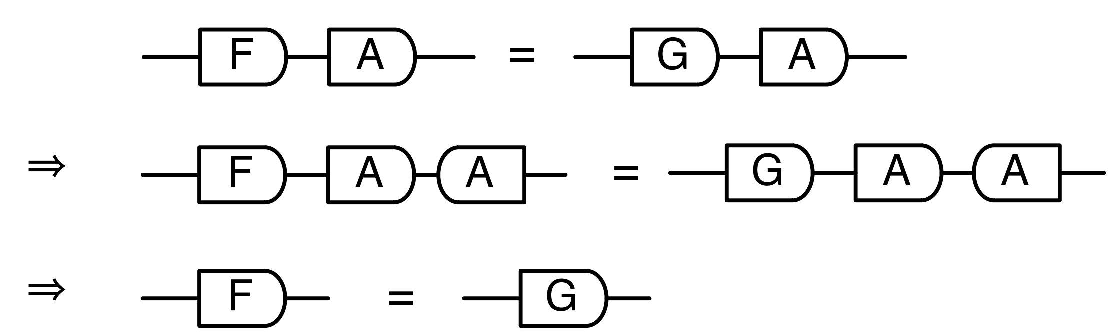
Theorem. A is injective iff

$\Rightarrow$



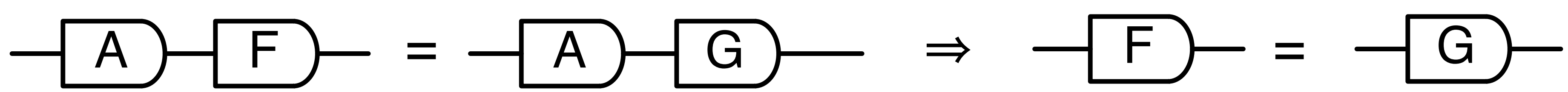
is pullback in $\mathbf{Mat}_{\mathbb{Z}}$

$\Leftarrow$

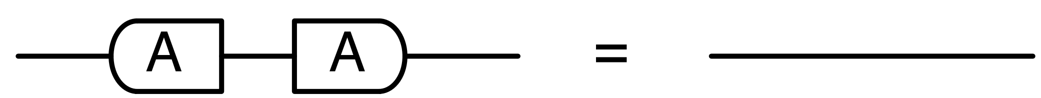


Surjectivity

- Surjective matrices are the epis in $\mathbf{Mat}_{\mathbf{Z}}$, i.e.



- **Theorem**. A is surjective iff

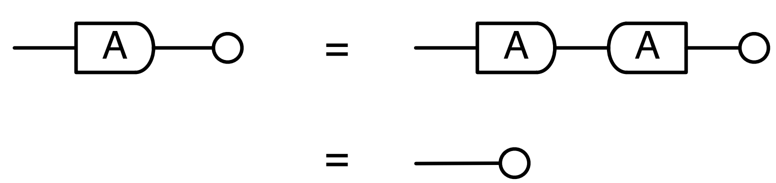


Proof: Bizarro of last slide

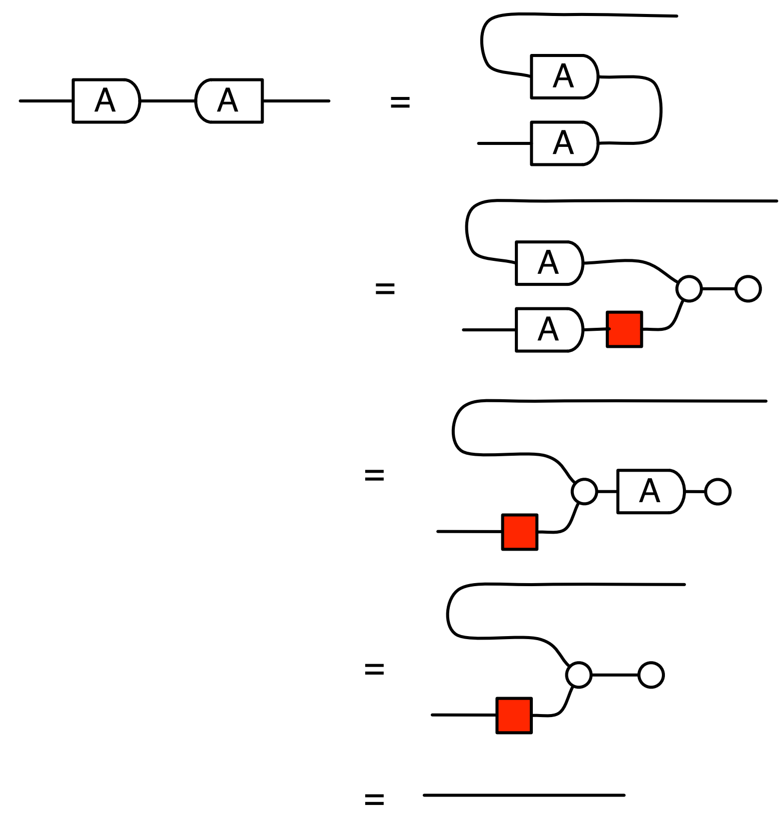
Injectivity and kernel

- **Theorem**. A is injective iff ker A = 0

⇒

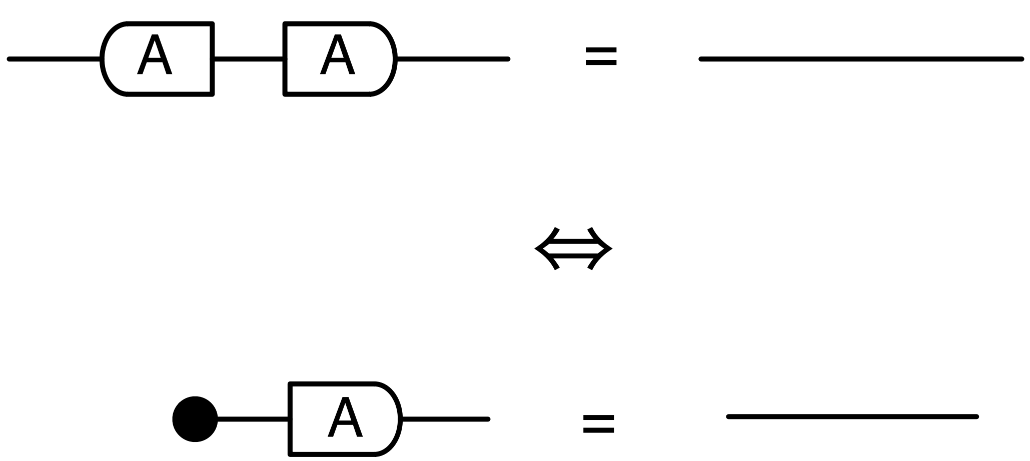


⇐



Surjectivity and image

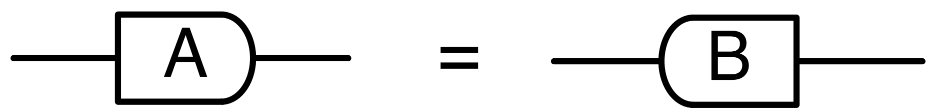
- **Theorem**. A is surjective iff im(A)=codomain



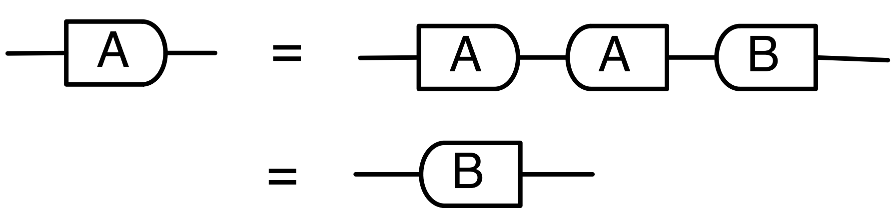
Proof: bizarro of last slide

Invertible matrices

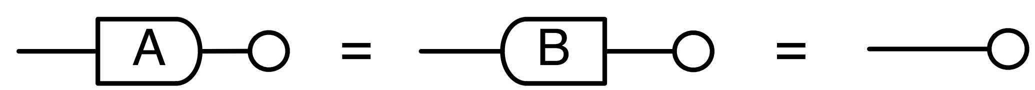
- **Theorem**: A is invertible with inverse B iff



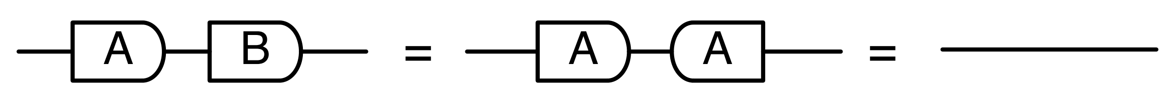
$\Rightarrow$



$\Leftarrow$



so A is injective



bizarro argument yields other half

Summary

- We have done a bit of linear algebra without mentioning
  - vectors, vector spaces and bases
  - linear dependence/independence, spans of a vector list
  - dimensions
- Similar stories can be told for other parts of linear algebra: decompositions, eigenvalues/eigenspaces, determinants
  - much of this is work in progress: check out the blog! :)