Coalgebras and Modal Logic for Parameterised Endofunctors

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ABSTRACT

We study categories of coalgebras for endofunctors, which additionally depend on a parameter category. The corresponding category of coalgebras then naturally appears as cofibred over the parameters. We give examples of constructions in the cofibred framework and study the overall structure of such cofibrations. Moreover, the dependency of (modal) logics for coalgebras on a parameter category is investigated and shown to give rise to the dual of an institution.

1998 ACM Computing Classification System: F.3.2
Keywords and Phrases: Coalgebras, Modal Logic, Fibrations
Note: During the research reported here the first author was an ERCIM fellow at SEN3, CWI, the second author was a member of the PhD programme 'Logic in Computer Science' in Munich.

Introduction

When working with coalgebras for an endofunctor $\Omega : \mathbb{C} \to \mathbb{C}$, the definition of Ω generally involves parameters. Prominent examples include $\Omega X = A \times X$, giving rise to infinite lists over A and $\Omega X = \mathcal{P}(X) \times \mathcal{P}(P)$, modelling Kripke-Structures with a set P of propositional variables. Here, we consider functors $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{L}$, depending *explicitly* on a parameter category $\mathbb{L}$. Given a 'parameterised endofunctor' Ω, we obtain a category $\mathbb{C}_{\Omega(L,\cdot)}$ of coalgebras for every object $L \in \mathbb{L}$ of the parameter category. The objective of the present paper is to study the relationship between coalgebras (together with their logics) for the endofunctors $\Omega(L, \cdot)$ with different parameter objects $L \in \mathbb{L}$.

This relationship is studied within a single 'total category' $\mathbb{E}$, which encompasses all of the categories $\mathbb{C}_{\Omega(L,\cdot)}$, but also admitting morphisms between coalgebras corresponding to different parameters, which can be thought of as 'functional bisimulations up to parameter transformation'. Considering the functor which maps a coalgebra $C \to \Omega(L, C)$ to the parameter object $L \in \mathbb{L}$, $\mathbb{E}$ naturally appears as cofibred over the category $\mathbb{L}$ of parameters.

Making use of the cofibred structure, we investigate functional bisimulations up to parameter transformations. It turns out, that relabelling and restriction, as known from process algebra, can be understood 'co-reindexing', wheras 'reindexing' can be used to enforce certain safety properties on transition systems.

We then investigate the categorical structure of cofibrations of coalgebras. In particular, we prove sufficient conditions, under which all reindexing functors exist and the cofibration under consideration is actually a bifibration. Another issue is the existence of colimits and limits in the 'total' category $\mathbb{E}$.

Given a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ and a collection $(\mathcal{L}_L)_{L \in \mathbb{L}}$ of modal logics for each parameter $L \in \mathbb{L}$ we investigate the effect of parameter transformations on the logics and the associated satisfaction relation. As it turns out, the situation is completely dual to the one known from algebras and gives rise to 'co-institutions'.

Making parameters in the signatures explicit, we give a new explanation for the fact that coalgebras for multiplicative functors (algebras for hidden signatures) can be understood as algebras (coalgebras).

1. Cofibrations of Coalgebras

As mentioned in the introduction, signature functors for coalgebras are generally defined using parameters. Making the parameters explicit, this leads to the notion of "parameterised signature" as a functor $\mathbb{L} \times \mathbb{C} \to \mathbb{C}$, where we think of $\mathbb{L}$ as a category of parameters. This section demonstrates, that a parameterised signature gives rise to a cofibration $p : \mathbb{E} \to \mathbb{L}$. We also discuss basic properties of the cofibrations thus obtained and give some examples motivating both the naturality and applicability of the cofibrational approach.

1.1 Parameterised Signatures

When working with coalgebras for an endofunctor $\Omega : \mathbb{C} \to \mathbb{C}$, one generally thinks of Ω as the signature of the corresponding category $\mathbb{C}_\Omega$ of Ω-coalgebras. This leads us to call a functor, which additionally depends on a parameter category $\mathbb{L}$ a *parameterised signature.* Since this notion is fundamental for the theory developed in the subsequent sections, we formally introduce it in

Definition 1.1. Suppose $\mathbb{L}$ and $\mathbb{C}$ are categories. A *parameterised signature* is a functor $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$.

We call $\mathbb{L}$ the *parameter category* and write Ω_L for the functor $X \mapsto \Omega(L, X)$ for a (fixed) object $L \in \mathbb{L}$. Examples of parameterised signatures arise by making parameters used in (ordinary) coalgebraic signatures explicit:

Example 1.2.

(i) Input/Output automata with variable sets of I inputs and O of outputs are modelled by the parameterised signature $\Omega(I, O, X) = (O \times X)^I$. Note that $\Omega : (\mathrm{Set}^{\mathrm{op}} \times \mathrm{Set}) \times \mathrm{Set} \to \mathrm{Set}$, that is, Ω acts contravariantly on inputs whereas it acts covariantly on outputs.

(*ii*) Labelled transition systems, regarded as coalgebras for the functor $\Omega X = \mathcal{P}(L \times X)$, can also be considered as parameterised over the set L of labels. This gives rise to a parameterised signature $\Omega : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$, defined by $\Omega(L, X) = \mathcal{P}(L \times X)$.

Using the isomorphism $\mathcal{P}(L \times X) \cong \mathcal{P}(X)^L$, where L now appears in a contravariant position on the right, we can also model the dependency on the set of labels contravariantly by means of the parameterised signature $\Omega'(L, X) = \mathcal{P}(X)^L$, where $\Omega' : \mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}$. The differences between Ω and Ω' correspond to viewing labels $l \in L$ as outputs and inputs, respectively, and will be discussed in section 2.

(*iii*) Suppose $\mathbb{D}$ is a category of domains with least element $\perp$. Infinite streams with a possibly non-terminating successor function can be modelled as coalgebras for the functor $\Omega_L(X) = L_\perp \otimes X_\perp$, where L is a set. This gives rise to the parameterised signature $\Omega(L, X) = L_\perp \otimes X_\perp$, mapping $\mathrm{Set} \times \mathbb{D} \to \mathbb{D}$.

In the examples above, the endofunctors Ω_L corresponding to a parameter object $L \in \mathbb{L}$ were structurally identical. By taking $\mathbb{L}$ as (a subcategory of) the functor category $[\mathbb{C}, \mathbb{C}]$, the concept of parameterised signatures can be seen also to incorporate *structurally different* endofunctors.

Example 1.3. Suppose $\mathbb{L} \hookrightarrow [\mathbb{C}, \mathbb{C}]$ is a (possibly non-full) subcategory of the category $[\mathbb{C}, \mathbb{C}]$ of endofunctors on $\mathbb{C}$. Then $\Omega(F, X) = F(X)$ defines a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$. If F and G are endofunctors on $\mathbb{C}$, a natural transformation $\eta : F \to G$ then allows us to view every F-coalgebra $\gamma : C \to FC$ also as a G-coalgebra $\eta(C) \circ \gamma : C \to GC$.

The preceding example can be seen as an instance of the (slightly) more general fact that, given a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$, every morphism $\lambda : L \to L' \in \mathbb{L}$ gives rise to a natural transformation $\hat{\lambda} : \Omega_L \to \Omega_{L'}$.

Proposition 1.4. *Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature. Then the operation $\hat{\lambda}(C) = \Omega(\lambda, \mathrm{id}_C) : \Omega_L C \to \Omega_{L'} C$ defines a natural transformation $\hat{\lambda} : \Omega_L \to \Omega_{L'}$.*

Since every natural transformation $\eta : F \to G$ between endofunctors on a category $\mathbb{C}$ defines a functor $\eta^\dagger : \mathbb{C}_F \to \mathbb{C}_G$, a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{L}$ gives rise to a functor $\mathbb{L} \to \mathrm{Cat}$, taking values in the category of categories. We note this as

Proposition 1.5. *Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature. For a morphism $\lambda : L \to L' \in \mathbb{L}$ let $\lambda^\dagger$ denote the functor given by*

$$\lambda^\dagger(C, \gamma) = (C, \Omega(\lambda, \mathrm{id}_C) \circ \gamma).$$

Then the assignment

$$L \mapsto \mathbb{C}_{\Omega_L}, \quad (L \xrightarrow{\lambda} L') \mapsto \lambda^\dagger$$

defines a functor $\mathcal{I}(\Omega) : \mathbb{L} \to \mathrm{Cat}$.

In this way, every parameterised signature Ω can be seen to define a split co-indexed category $\mathcal{I}(\Omega)$, a concept originally introduced by Paré and Schumacher in [17]. Instead of working with co-indexed categories, it is technically more convenient (and aesthetically more pleasing) to describe the phenomenon of "variation over a parameter category" in terms of (co-)fibrations, which is the programme of the next section.

1.2 Coalgebras, Cofibred

A cofibration over a category $\mathbb{L}$ of parameters is given by a category $\mathbb{E}$ (the *total* category of the cofibration) and a functor $p : \mathbb{E} \to \mathbb{L}$, which maps "structures" to "parameters", subject to a universal property (described later). One advantage of the cofibred approach over the co-indexed view is that it provides us with a *single* category $\mathbb{E}$, which contains structures corresponding to *different* parameter objects and thus allows us to relate structures corresponding to different parameters within just one category.

In the case of coalgebras for a parameterised signature, morphisms in the total category are not only the functional bisimulations, but can be thought of as "functional bisimulations up to parameter transformation", and provide us with additional categorical structure, see Sections 2 and 3.

We finally remark that we can recover the structures corresponding to a fixed parameter $L \in \mathbb{L}$ as the (non-full) subcategory of those structures, which are mapped to L by the projection functor p. Thus $\mathbb{E}$ incorporates the concept of varying parameters (via morphisms between structures corresponding to different parameters) as well that of fixed parameters (via the fibres).

We begin by describing the cofibration arising through a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ before recalling some basic fibred terminology, providing the reader not familiar with the theory of (co-)fibrations with a concrete instance of this concept.

Definition 1.6. Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature. The *cofibration* $p : \mathbb{E} \to \mathbb{L}$ *induced by* Ω is given by the following data:

(*i*) Objects of $\mathbb{E}$ are pairs $(L, (C, \gamma))$ with $(C, \gamma) \in \mathbb{C}_{\Omega_L}$, that is, $\gamma : C \to \Omega(L, C)$ is a coalgebra structure for C.

(*ii*) Morphisms from $(L, (C, \gamma))$ to $(M, (D, \delta))$ in $\mathbb{E}$ are pairs of morphisms $(\lambda : L \to M, f : C \to D) \in \mathbb{L} \times \mathbb{C}$ making the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ \Omega(L, C) \arrow[r, "\Omega(\lambda, f)"] & \Omega(M, D) \end{tikzcd}$$

commute.

(*iii*) The functor $p : \mathbb{E} \to \mathbb{L}$ is first projection.

When the parameter L of an object $(L, (C, \gamma))$ is clear from the context we simply write (C, γ).

We recall some standard terminology. $\mathbb{E}$ is called the *total category* and $\mathbb{L}$ the *base category* of the cofibration. An object $A \in \mathbb{E}$ with $pA = L$ and a morphism $f : A \to B \in \mathbb{E}$ with $pf = \lambda$ are called *over* L and *over* λ, respectively. The subcategory of objects over L and morphism over id_L is called the *fibre over* L and is denoted by $\mathbb{E}_L$. In cofibrations of coalgebras, the fibre over L is isomorphic to the category of coalgebras for Ω_L.

The structure of the total category $\mathbb{E}$ is determined by the fibres and the cocartesian morphisms: A morphism $f : A \to B \in \mathbb{E}$ is called *cocartesian*, if for all morphisms $g : A \to C \in \mathbb{E}$ and all morphisms $\lambda : pB \to pC \in \mathbb{L}$ with $\lambda \circ pf = pg$ there exists a unique morphism $h : B \to C$ over λ with $h \circ f = g$. The defining property of a cofibration is now that for all $\lambda : pA \to L \in \mathbb{L}$ there exists a cocartesian morphism $f : A \to B \in \mathbb{E}$ such that $pf = \lambda$. Such a morphism is called a *cocartesian lifting* of λ. If $\dagger(A, \lambda) : A \to \lambda^\dagger(A)$ is a particular choice of cocartesian liftings of $\lambda : L \to M \in \mathbb{L}$ for every A over L, the assignment $A \mapsto \lambda^\dagger(A)$ extends to a functor $\lambda^\dagger : \mathbb{E}_L \to \mathbb{E}_M$. A functor obtained in this way is called a *co-reindexing*. We will use the fibred terminology freely and refer to [4, 9] regarding further reading on this subject.

That the cofibration associated to a parameterised signature is indeed a cofibration follows from (λ, id_C) being a cocartesian lifting of $\lambda : L \to M \in \mathbb{L}$ for every C over L. We conclude this section with some properties of cofibrations of coalgebras.

Proposition 1.7 (Characterisation of Cocartesian Morphisms). *Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature and $p : \mathbb{E} \to \mathbb{L}$ the induced cofibration. Then a morphism $(\lambda, f) \in \mathbb{E}$ is cocartesian iff f is an isomorphism in $\mathbb{C}$.*

Proposition 1.8 (Co-reindexing Preserves Colimits). *Let $G : I \to \mathbb{E}_L$ be a diagram and $\lambda : L \to M \in \mathbb{L}$. Then $\lambda^\dagger(colimG) \cong colim(\lambda^\dagger G)$. Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature with induced cofibration $p : \mathbb{E} \to \mathbb{C}$. If $\lambda : L \to M \in \mathbb{L}$, then $\lambda^\dagger : \mathbb{E}_L \to \mathbb{E}_M$ preserves all colimits which exist in $\mathbb{E}_L$.*

Proof. Follows from proposition 1.7 and the fact that colimits of coalgebras are calculated as colimits in the category $\mathbb{C}$ of carriers: Let $((C, \gamma), c_i)$ be a colimit for G and $((D, \delta), d_i) = (\lambda^\dagger(C, \gamma), \lambda^\dagger(c_i))$. Consider the forgetful functors $U_L : \Omega_{\mathbb{C}_L} \to \mathbb{C}$ and $U_M : \Omega_{\mathbb{C}_M} \to \mathbb{C}$. By proposition 1.7 there is an iso $f : C \to D$ and a natural iso $\eta : U_L G \to U_M \lambda^\dagger G$ such that $f \circ U_L(c_i) = U_M(d_i) \circ \eta_i$. It follows from $(C, U_L(c_i))$ being a colimit of $U_L G$ that $(D, U_M(d_i))$ is a colimit of $U_M \lambda^\dagger G$. Since U_M creates colimits, $((D, \delta), d_i)$ is a colimit of $\lambda^\dagger G$. □

Generalising the definition of Aczel and Mendler [1] to arbitrary categories by taking a bisimulation between two coalgebras (C, γ) and (D, δ) in the same fibre $\mathbb{E}_L$ to be a monic span $(\pi_C : R \to C, \pi_D : R \to D)$ in $\mathbb{C}$ which can be equipped with a transition structure $\rho : R \to \Omega_L(R)$ turning π_C and π_D into coalgebra-morphisms, it is easy to see that co-reindexing preserves bisimulations in the following sense:

Proposition 1.9 (Co-reindexing Preserves Bisimulation). *If $(R, \pi_C : R \to C, \pi_D : R \to D)$ is a bisimulation between two coalgebras (C, γ) and $(D, \delta) \in \mathbb{E}_L$, and $f : (C, \gamma) \to \lambda^\dagger(C, \gamma)$, $g : (D, \delta) \to \lambda^\dagger(D, \delta)$ are cocartesian over $\lambda : L \to M$, then $(R, f \circ \pi_C, g \circ \pi_D)$ is a bisimulation between $\lambda^\dagger(C, \gamma)$ and $\lambda^\dagger(D, \delta) \in \mathbb{E}_M$.*

Proof. The span $(R, f \circ \pi_C, g \circ \pi_D)$ is monic in $\mathbb{C}$ since f and g are cocartesian and hence isomorphisms between the carriers by 1.7. A transition structure $\hat{\rho} : R \to \Omega_M(R)$ can be obtained by transporting an appropriate transition structure $\rho : R \to \Omega_L(R)$ along λ. □

The above result can be seen as generalisation of the corresponding result of [23], section 15.

The notion of morphism in the total category allows morphisms between coalgebras of different signature functors. It is therefore surprising, that we can recover this category as a category of coalgebras of an endofunctor, dispensing with the fibrational structure. The resulting description is sometimes technically easier to work with and will be used in section 5.

Proposition 1.10. *Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature and $\mathbb{L}$ has a terminal object 1. If $\hat{\Omega}$ is defined by*

$$\hat{\Omega} : \mathbb{L} \times \mathbb{C} \to \mathbb{L} \times \mathbb{C}, \quad (L, C) \mapsto (1, \Omega(L, C))$$

then the category $(\mathbb{L} \times \mathbb{C})_{\hat{\Omega}}$ of $\hat{\Omega}$-coalgebras is isomorphic to the total category $\mathbb{E}$ of the cofibration induced by Ω.

2. Coalgebras and Parameter Transformations

This section studies the effect on coalgebras of cocartesian and cartesian liftings of input and output parameter transformations. We consider fibred automata and labelled transition systems (example 1.2). The corresponding signatures are parameterised over $\mathrm{Set}^{\mathrm{op}}$ or Set. We call parameters from $\mathrm{Set}^{\mathrm{op}}$ *input parameters* and parameters from Set *output parameters*. In particular, we will study the following picture where (C, γ) is a coalgebra over outputs O and inputs I, respectively, and o_1, o_2 and

i_1, i_2 are corresponding parameter transformations.

$$\begin{tikzcd} o_1^*(C,\gamma) \arrow[r] & (C,\gamma) \arrow[r] \arrow[d, mapsto, "p"'] & o_2^\dagger(C,\gamma) \\ O_1 \arrow[r, "o_1"] & O \arrow[r, "o_2"] & O_2 \end{tikzcd} \qquad \begin{tikzcd} (i_1^{\mathrm{op}})^*(C,\gamma) \arrow[r] & (C,\gamma) \arrow[r] \arrow[d, mapsto, "p"'] & (i_2^{\mathrm{op}})^\dagger(C,\gamma) \\ I_1 & I \arrow[l, "i_1"'] & I_2 \arrow[l, "i_2"'] \end{tikzcd}$$

These parameter transformation give rise to transformations of the coalgebra (C, γ) where we use $(-)^*$ and $(-)^\dagger$ to indicate the coalgebra that arises from a cartesian lifting (ie. via reindexing)[1] and, respectively, from a cocartesian lifting (ie. via co-reindexing).

Co-reindexing wrt. output is relabelling. Given a signature $\Omega : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$, a coalgebra $C \xrightarrow{\gamma} \Omega(O, C)$, and a parameter transformation $o_2 : O \to O_2$, the cocartesian lifting of o_2 gives rise to a coalgebra $o_2^\dagger(C, \gamma) =$

$$C \xrightarrow{\gamma} \Omega(O, C) \xrightarrow{\Omega(o_2, \mathrm{id}_C)} \Omega(O_2, C)$$

where, intuitively, composition with $\Omega(o_2, \mathrm{id}_C)$ just relabels the outputs of γ in O according to o_2. It is not difficult to check that in the examples $\Omega(O, X) = (O \times X)^I$ and $\Omega(O, X) = \mathcal{P}(O \times X)$ one obtains indeed the usual notion of relabelling.

Co-reindexing wrt. input inclusion is restriction. Given a signature $\Omega : \mathrm{Set}^{\mathrm{op}} \times \mathrm{Set} \to \mathrm{Set}$, a coalgebra $C \xrightarrow{\gamma} \Omega(I, C)$, and a parameter transformation $i_2 : I_2 \to I$, the cocartesian lifting of i_2 gives rise to a coalgebra $(i_2^{\mathrm{op}})^\dagger(C, \gamma)$. Formally, $(i_2^{\mathrm{op}})^\dagger(C, \gamma)$ is as in the output case $(C, \Omega(i_2^{\mathrm{op}}, C) \circ \gamma)$ but due to the contravariance dependence on the input transformation the effect is now quite different. Intuitively, $(i_2^{\mathrm{op}})^\dagger(C, \gamma)$ takes an input in I_2, translates it via i_2 to I, and then runs the machine (C, γ). In case of the examples $\Omega(I, X) = (O \times X)^I$ and $\Omega(I, X) = \mathcal{P}(X)^I$ consider an inclusion $i_2 : I_2 \hookrightarrow I$. Then $(i_2^{\mathrm{op}})^\dagger(C, \gamma)$ is the transition system resulting from (C, γ) by deleting all transitions with labels in $I - I_2$.

Let us also consider the case where $i_2 : I_2 \to I$ in the preceding paragraph is not a mono but an epi. Then $(i_2^{\mathrm{op}})^\dagger(C, \gamma)$ is the transition system in which each transition labelled with $l \in I$ is replaced by $|i_2^{-1}(l)|$ copies, each labelled with a distinct element from $i_2^{-1}(l)$.

Next, we will discuss the effect of cartesian liftings on coalgebras. In this section, only cartesian liftings of monos are considered, the more general case is treated in section 3. The basic result is:

Theorem 2.1. *Let $\Omega : \mathbb{L} \times \mathrm{Set} \to \mathrm{Set}$ be a parameterised signature that preserves monos and let p be the induced cofibration. Then all monos $L \to pB \in \mathbb{L}$ have cartesian liftings $\bullet \to B$.*

Proof. Let (C, γ) be a coalgebra over L and $\lambda : L_1 \to L$ a mono in $\mathbb{L}$. We construct the cartesian lifting $(D, \delta) \to (C, \gamma)$ of λ as a composition

$$\begin{tikzcd} D \arrow[r, "\mathrm{id}_D"] \arrow[d, "\delta"'] & D \arrow[r, hook, "in"] \arrow[d, "\delta'"'] & C \arrow[d, "\gamma"'] \\ \Omega(L_1, D) \arrow[r, "\Omega(\lambda, \mathrm{id}_D)"'] & \Omega(L, D) \arrow[r, "\Omega(\mathrm{id}_L, in)"'] & \Omega(L, C) \end{tikzcd}$$

where (D, δ') is the largest subcoalgebra of (C, γ) such that δ' factors through $\Omega(\lambda, \mathrm{id}_D)$. Note that the left-hand square is a cocartesian morphism.

[1] Cartesian liftings and reindexing is defined dually to cocartesian liftings and co-reindexing, see Section 1.2.

Consider all subcoalgebras $in_i : (C_i, \gamma_i) \hookrightarrow (C, \gamma)$ over L such that γ_i factors through $\Omega(\lambda, \mathrm{id}_{C_i})$ as $\gamma_i = \Omega(\lambda, \mathrm{id}_{C_i}) \circ \gamma_i'$ for some γ_i'. Let (D, δ') be the union of all these subcoalgebras, that is, there are injections $e_i : (C_i, \gamma_i) \hookrightarrow (D, \delta')$ and $in : (D, \delta') \hookrightarrow (C, \gamma)$ s.t. $in \circ e_i = in_i$. It follows $\delta' \circ e_i = \Omega(\lambda, \mathrm{id}_D) \circ \Omega(\mathrm{id}_{L_1}, e_i) \circ \gamma_i'$. This implies, due to the e_i being jointly epi and $\Omega(\lambda, \mathrm{id}_D)$ being mono, that there is a unique 'diagonal' $\delta : D \to \Omega(L_1, D)$ such that $\Omega(\lambda, \mathrm{id}_D) \circ \delta = \delta'$. This completes the definition of the diagram above. We leave the verification that the morphism $(D, \delta) \to (C, \gamma)$ is indeed cartesian to the reader. □

Remark 2.2. (*i*) The condition that Ω preserves monos is rather weak and e.g. satisfied in all our examples.

(*ii*) Theorem and proof generalise to coalgebras over arbitrary categories admitting a factorisation system which allows to form unions of subcoalgebras. Such categories have been investigated in [14].

(*iii*) The proof of the theorem shows how to actually calculate cartesian liftings of monos as certain subcoalgebras. This allows to determine cartesian liftings in our examples.

Reindexing wrt. output inclusion gives the largest subsystem not producing 'bad ouputs'. Given a signature $\Omega : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$, a coalgebra $C \xrightarrow{\gamma} \Omega(O, C)$, and a parameter transformation $o_1 : O_1 \hookrightarrow O$, the cartesian lifting of o_1 gives rise to a coalgebra $(D, \delta) = o_1^*(C, \gamma)$ which is the largest subcoalgebra of (C, γ) such that no state $d \in D$ can output a label in $O - O_1$.

Reindexing wrt. epi input transformations. Let $C \xrightarrow{\gamma} \Omega(I, C)$ be a coalgebra for one of the signatures $\Omega(I, X) = (O \times X)^I$ or $\Omega(I, X) = \mathcal{P}(X)^I$ and $i_1 : I \to I_1$ an epi in Set (which is a mono in $\mathrm{Set}^{\mathrm{op}}$). According to the proof of theorem 2.1, the cartesian lifting of i_1 is given by a composition $(D, \delta) \to (D, \delta') \to (C, \gamma)$ where the left-hand morphism is cocartesian. First, (D, δ') is the largest subcoalgebra of (C, γ) s.t. for all $c \in D$ and all $l, l' \in I$

$$i_1(l) = i_1(l') \implies \gamma(c)(l) = \gamma(c)(l').$$

Second, (D, δ) now arises from (D, δ') by relabelling input labels in I according to i_1.

3. Categorical Structure of Cofibrations for Coalgebras

Whereas the previous section has given examples of the effect of (co)reindexing on coalgebras, this section investigates the structure present in cofibrations from a categorical point of view. In particular, we give conditions under which the cofibration under consideration is actually a bifibration. Other topic treated are the existence of fibred limits and colimits.

Let us start by recalling some standard terminology. If $\mathbb{C}$ is any category and $\mathbb{I}$ is small, we say that $\mathbb{C}$ *has (co)limits of type* $\mathbb{I}$, if every diagram $D : \mathbb{I} \to \mathbb{C}$ has a (co)limit. A (co)fibration is said to have (co)fibred (co)limits iff every fibre has (co)limits and these are preserved by (co)reindexing. (Co)limits in the total category are obtained from (co)fibred (co)limits as follows:

Lemma 3.1. *Let $p : \mathbb{E} \to \mathbb{L}$ be a cofibration and $\mathbb{L}$ have colimits of type $\mathbb{I}$. Then $\mathbb{E}$ has these colimits and p preserves them iff each fibre of p has these colimits and they are preserved by co-reindexing.*

Using proposition 1.8 and that forgetful functors for coalgeabras create colimits we can summarise:

Proposition 3.2. *Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature with associated cofibration $p : \mathbb{E} \to \mathbb{L}$. If $\mathbb{C}$ has colimits of type $\mathbb{I}$, then p has cofibred colimits of type $\mathbb{I}$. If, moreover, $\mathbb{L}$ has colimits of type $\mathbb{I}$ then $\mathbb{E}$ has colimits of type $\mathbb{I}$ and they are preserved by p.*

In case that cofree coalgebras exist, cofibrations of coalgebras turn out to have a much richer structure. We call (p, q, U, F) as in the left-hand diagram below an **adjunction of bifibrations**

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, bend left, "U"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathbb{E}' \arrow[ll, bend left, "F"] \arrow[dl, "q"] \\
& \mathbb{L} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{E} \arrow[rr, bend left, "U"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathbb{L} \times \mathbb{C} \arrow[ll, bend left, "F"] \arrow[dl, "\pi_1"] \\
& \mathbb{L} &
\end{tikzcd}$$

whenever (1) $U \dashv F$ and (co)units vertical, (2) U cofibred and F fibred,[2] (3) p, q bifibrations.[3]

In particular, we are interested in the case where in the right-hand diagram above p is a cofibration of coalgebras, $U : \mathbb{E} \to \mathbb{L} \times \mathbb{C}$ the corresponding (global) forgetful functor, F the (global) cofree construction, and $\pi_1 : \mathbb{L} \times \mathbb{C} \to \mathbb{L}$ is the first projection:

Theorem 3.3. *Let $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ be a functor, $p : \mathbb{E} \to \mathbb{L}$ the induced cofibration, and $\pi_1 : \mathbb{L} \times \mathbb{C} \to \mathbb{L}$ the first projection. If each fibre $\mathbb{E}_L$ has equalisers and each forgetful functor $U_L : \mathbb{E}_L \to \mathbb{C}$ has a right adjoint F_L, then there is an adjunction of bifibrations (p, π_1, U, F). Moreover, p has then fibred limits if $\mathbb{C}$ is complete.*

Proof. For the first statement we have to show (1) $U \dashv F$ and (co)units vertical, (2) U cofibred and F fibred, (3) p, π_1 bifibrations.

(1): Define $F(L, C) = F_L(C)$, $\varepsilon(L, C) = \varepsilon_L(C)$ (ε_L being the counit of $U_L \dashv F_L$). Consider $A \in \mathbb{E}$ and $(\lambda, f) : UA \to (L, C)$. Let $(\lambda, i) : A \to \lambda^\dagger(A)$ be cocartesian. There is a unique $g : \lambda^\dagger(A) \to F(L, C)$ such that $\varepsilon(L, C) \circ Ug = (\mathrm{id}_{U(\lambda^\dagger(A))}, f \circ i^{-1})$. Hence, $g \circ (\lambda, i) : A \to F(L, C)$ is the unique morphism s.t. $\varepsilon(L, C) \circ g \circ (\lambda, i) = (\lambda, f)$. This shows that the F_L have a unique extension F to all of $\mathbb{L} \times \mathbb{C}$ satisfying (1).

(2): We show that F preserves cartesian morphism. Let $f : X \to Y \in \mathbb{L} \times \mathbb{C}$ be cartesian and consider $Ff : FX \to FY$. Given $g : A \to FX$ and $\lambda \in \mathbb{L}$ such that $pg = pFf \circ \lambda$, adjunction yields a $g' : UX \to B$ with $\pi_1 g' = \pi_1 f \circ \lambda$ (counit is vertical). That f is cartesian provides us with a unique $h' : UA \to X$ with $g' = f \circ h'$ and by adjunction we obtain the required $h : A \to FX$.

(3): π_1 is a bifibration and p a cofibration. For p being a bifibration, it is sufficient to show that co-reindexing functors $\lambda^\dagger$ have right adjoints λ^* (see [9], 9.1.2). Consider the diagram below.

$$\begin{tikzcd}
\mathbb{E}_L \arrow[rr, "\lambda^\dagger"] \arrow[dr, "U_L"'] & & \mathbb{E}_M \arrow[dl, "U_M"] \\
& \mathbb{C} &
\end{tikzcd}$$

Note that U_L, U_M are comonadic and recall that $\mathbb{E}_L$ has equalisers. It follows from the adjoint lifting theorem (see [4], vol.2, corollary 4.5.7) that $\lambda^\dagger$ has a right adjoint λ^*.

Second, that each fibre $\mathbb{E}_L$ has limits, follows again from U_L being comonadic and $\mathbb{E}_L$ having equalisers (see [4], vol.2, proposition 4.3.4). That reindexing preserves limits follows from reindexing being right adjoint, see (3). $\square$

Let us comment on this theorem. First, the proof of this theorem does not exhibit how limits and cartesian liftings can actually be calculated. As shown in [12], both can be obtained by factoring

[2] F is fibred (dualise for 'U cofibred') iff $pF = q$ and F preserves cartesian liftings. Note that this definition makes sense even when p fails to be a fibration.

[3] Note that this need neither be a fibred nor a cofibred adjunction. Indeed, below U will generally not be fibred and F will not be cofibred.

appropriate sinks with cofree codomain thus giving a possibility to calculate limits and cartesian liftings in concrete examples.

Second, (1) and (2) hold whenever p is a cofibration, π_1 is a fibration, and we have adjunctions $U_L \dashv F_L$. In particular, (1) and (2) do not depend on the condition that fibres have equalisers. On the other hand, this condition is required to show both that limits and reindexing exist. Fortunately, fibres have indeed equalisers in many common situations: (i) In case that $\mathbb{C} = \mathrm{Set}$ (because Ω_L preserves (split) equalisers, hence U_L creates them); (ii) In case that fibres have factorisation structures for sinks (E, M) with sinks in E being epi (see [2], 15.7); (iii) In case that $\mathbb{C}$ is locally presentable and the Ω_L are accessible (then the fibres are locally presentable and hence complete, see [19]).

In the remainder of this section we investigate when one can dispense with the assumption that fibres have equaliseres. The crucial observation is that, for the use of the adjoint lifting theorem, it is enough that fibres have equalisers of **coreflexive pairs**, ie. equalisers of parallel pairs f, g with a common retract r, $r \circ f = r \circ g = \mathrm{id}$. The following lemmas will help us to exploit this fact.

Lemma 3.4. *Suppose $\mathbb{C}$ has equalisers of coreflexive pairs and $T : \mathbb{C} \to \mathbb{C}$ is an endofunctor which preserves monic arrows and weakly preserves pullbacks. Then $\mathbb{C}_T$ has equalisers of coreflexive pairs.*

Proof. Suppose $f, g : (C, \gamma) \to (D, \delta)$ have a common retract $r : (D, \delta) \to (C, \gamma)$. Then r is also a common retract of the parallel pair $f, g : C \rightrightarrows D$ in $\mathbb{C}$. If $e : E \to C$ is an equaliser of f and g in $\mathbb{C}$, the existence of the common retract implies that (e, e) is a pullback of (f, g) in $\mathbb{C}$. Since T weakly preserves pullbacks, this implies the existence of $\epsilon : E \to TE$ such that $e : (E, \epsilon) \to (C, \gamma)$ is a morphism in $\mathbb{C}_T$. That e has indeed the required universal property follows from $T(e)$ being mono. $\square$

Lemma 3.5. *Let $T : \mathbb{C} \to \mathbb{C}$ be a functor. If $\mathbb{I}$ is a small category then the functor category $[\mathbb{I}, \mathbb{C}_T]$ is itself a category of coalgebras. Moreover, if $\mathbb{C}_T$ has cofree coalgebras then also $[\mathbb{I}, \mathbb{C}_T]$.*

Proof. Define a functor $T^{\mathbb{I}} : [\mathbb{I}, \mathbb{C}] \to [\mathbb{I}, \mathbb{C}]$ via $H \mapsto TH$. It follows $[\mathbb{I}, \mathbb{C}_T] \cong [\mathbb{I}, \mathbb{C}]_{T^{\mathbb{I}}}$. For the second statement, let $U : \mathbb{C}_T \to \mathbb{C}$, $U^{\mathbb{I}} : [\mathbb{I}, \mathbb{C}]_{T^{\mathbb{I}}} \to [\mathbb{I}, \mathbb{C}]$ be the forgetful functors and $U \dashv F$. Define $F^{\mathbb{I}} : [\mathbb{I}, \mathbb{C}] \to [\mathbb{I}, \mathbb{C}]_{T^{\mathbb{I}}}$ via $H \mapsto FH$. Then $U^{\mathbb{I}} \dashv F^{\mathbb{I}}$. $\square$

Note that the lemma implies that U and $U^{\mathbb{I}}$ are comonadic. We now use the adjoint lifting theorem to prove a theorem about the existence of limits in categories of coalgebras.

Theorem 3.6. *Let $\mathbb{C}$ have equalisers (of coreflexive pairs), let $\Omega : \mathbb{C} \to \mathbb{C}$ be a functor that preserves monos and weakly preserves pullbacks and assume that $\mathbb{C}_\Omega$ has cofree coalgebras. Then $\mathbb{C}_\Omega$ has every type of limit that $\mathbb{C}$ has.*

Proof. Using lemma 3.4, it is sufficient to show that if $\mathbb{C}_\Omega$ has equalisers of coreflexive pairs and cofree coalgebras then $\mathbb{C}_\Omega$ has every type of limit that $\mathbb{C}$ has. Consider

$$\begin{tikzcd}
\mathbb{C}_T \arrow[r, "\Delta_{\mathbb{C}_T}"] \arrow[d, "U"'] & {[\mathbb{I}, \mathbb{C}_T] \cong [\mathbb{I}, \mathbb{C}]_{T^{\mathbb{I}}}} \arrow[d, "U^{\mathbb{I}}"] \\
\mathbb{C} \arrow[r, "\Delta_{\mathbb{C}}"'] & {[\mathbb{I}, \mathbb{C}]}
\end{tikzcd}$$

where $\Delta_{\mathbb{C}}, \Delta_{\mathbb{C}_T}$ map objects to the corresponding constant functors. Using lemma 3.5 and the adjoint lifting theorem (see [4], vol.2, theorem 4.5.6, exercise 4.8.6) it follows that $\Delta_{\mathbb{C}_T}$ has a right adjoint if $\Delta_{\mathbb{C}}$ has. $\square$

As a corollary to all of the above, we now obtain a version of theorem 3.3 not needing the requirement that fibres have equalisers.

Corollary 3.7. *If $\mathbb{C}$ has equalisers (of coreflexive pairs) and Ω_L preserves monos and weakly preserves pullbacks for each $L \in \mathbb{L}$, then the assumption that fibres have equalisers may be dropped from Theorem 3.3.*

4. Modal Logics and Parameter Transformations

Having studied the overall structure present in a cofibration of coalgebras, we turn to the interplay between logics for coalgebras and parameter transformations. Since coalgebras for an endofunctor are a natural generalisation of transition systems, where often properties of interest are bisimulation invariant and described using modal logic, we use the term "modal logic" loosely to refer to logics where formulas are interpreted as bisimulation invariant predicates on (state spaces of) coalgebras.

Suppose that we are given a category $\mathbb{S}$, which we think of as a category of signatures. Relating Σ-structures to Σ-formulas for a particular signature $\Sigma \in \mathbb{S}$ amounts to defining a parameterised satisfaction relation $\models_\Sigma$ between Σ-Structures and Σ-formulas.

The requirement, that satisfaction interacts with signature morphisms is usually expressed by the so-called satisfaction condition in the framework of institutions.

This section shows, that the dual of the satisfaction condition holds, when one considers coalgebras for a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{L}$ and a (coherent) family of logics $(\mathcal{L}_L)_{L \in \mathbb{L}}$ indexed by the objects $L \in \mathbb{L}$ of the parameter category. In order to establish this duality formally, we begin with translating the satisfaction condition, as known from institutions, to (co-)fibrational setting.

4.1 Institutions, Fibrationally

Institutions were introduced in [6] in order to describe the effect of signature morphisms wrt. logics for different signatures. In an institution $\mathcal{I} = (\mathbb{S}, \mathrm{Mod}, \mathrm{Sen}, (\models_\Sigma)_{\Sigma \in \mathbb{S}})$, one associates a contravariant functor $\mathrm{Mod}(\sigma) : \mathrm{Mod}(\Sigma') \to \mathrm{Mod}(\Sigma)$ between Σ'- and Σ-structures to every signature morphism $\sigma : \Sigma \to \Sigma'$. On the level of syntax, σ induces a (covariant) translation $\mathrm{Sen}(\Sigma) \to \mathrm{Sen}(\Sigma')$ between the sets $\mathrm{Sen}(\Sigma)$ and $\mathrm{Sen}(\Sigma')$ of Σ- and Σ'-formulas. The quadruple $\mathcal{I}$ is an institution, if the *satisfaction condition*

$$\mathrm{Mod}(\sigma)(M') \models_\Sigma \phi \quad \Leftrightarrow \quad M' \models_{\Sigma'} \mathrm{Sen}(\sigma)(\phi) \tag{1}$$

holds for all $\sigma : \Sigma \to \Sigma'$, $M' \in \mathrm{Mod}(\Sigma')$ and $\phi \in \mathrm{Sen}(\Sigma)$.

From a (co-)fibrational viewpoint, Mod and Sen give rise to a (split) fibration $m : \mathbb{M} \to \mathbb{S}$ and a (split) cofibration $l : \mathcal{L} \to \mathbb{S}$ over the same base category $\mathbb{S}$ of signatures, where cofibration $\mathcal{L} \to \mathbb{S}$ arises via considering every set $\mathrm{Sen}(\Sigma)$ as a discrete category.

Denoting the (co-)reindexing functors associated to a signature morphism $\sigma : \Sigma \to \Sigma'$ by $\sigma^* : \mathbb{M}_{\Sigma'} \to \mathbb{M}_\Sigma$ and $\sigma^\dagger : \mathcal{L}_\Sigma \to \mathcal{L}_{\Sigma'}$, respectively, we can reformulate (1) as

$$\sigma^*(M') \models_\Sigma \phi \quad \Leftrightarrow \quad M' \models_{\Sigma'} \sigma^\dagger(\phi) \tag{2}$$

for all models $M \in \mathbb{M}_{\Sigma'}$ and all formulas $\phi \in \mathcal{L}_\Sigma$.

The fibrational formulation of the satisfaction condition has the advantage that it generalises smoothly to arbitrary (co-)fibrations, in particular, we can accommodate syntactical deduction into the logics $\mathcal{L}_\Sigma$, where the logical consequence relation $\phi \vdash \psi$ corresponds to a morphism $\phi \to \psi$ in the fibre $\mathbb{L}_\Sigma$ over Σ.

However, a little bit of extra care is needed when working with condition (2), since it depends on a choice of cleavages for m and l. Requiring (2) to hold for *every* choice of cleavages for m and l amounts to requiring that isomorphic models have the same theory and that logically equivalent formulas are satisfied by the same class of models. This is not present in the original formulation (1), which corresponds to considering only the cleavages induced by the (co-)indexed structures.

The next two sections establish the dual of condition (2) wrt. two different conceptions of logic for coalgebras.

4.2 Predicate Liftings and (Modal) Logic

Viewing coalgebras on the category of sets as transition systems, several authors [10, 11, 15, 22, 21] have developed modal logics, where formulas are interpreted as bisimulation invariant predicates on the state space. The logics we will be concerned with here are interpreted (and assumed to be given) by a set of predicate liftings for the signature functor.

If Ω is an endofunctor on Set, then a *predicate lifting for* Ω is a natural transformation $\mu(X) : 2^X \to 2^{\Omega X}$, where $2^{(\,\cdot\,)}$ is the contravariant powerset functor.

The relationship between predicate liftings and logics for coalgebra has (to the authors' knowledge) first been made explicit in [10, 11], where one associates a set of predicate liftings to a signature functor according to its syntactical structure. Closer investigation reveals that predicate liftings also occur implicitly in the logics discussed in [15, 22, 21]. Also, the concept of natural relation in [18] can be seen as a special instance. This leads us to consider logics interpreted by means of a set $\mathcal{S}$ of (arbitrary) predicate liftings for the signature functor $\Omega : \mathrm{Set} \to \mathrm{Set}$.

Note that every predicate lifting μ for Ω defines a fibred endofunctor on the fibration SubSet $\to$ Set and vice versa, which allows for a generalisation of the approach presented to coalgebras on a category which is equipped with a (preorder) fibration. Since the general theory of coalgebraic (modal) logic in this setting is not well developed yet, we restrict our attention to the Set-based case.

The notion of predicate liftings allows us to treat both modal operators and atomic propositions in a uniform setting. Given a predicate lifting μ for Ω and a Ω-coalgebra $(\gamma : C \to \Omega C)$, we obtain an operator $\heartsuit_\mu = \gamma^{-1} \circ \mu : \mathcal{P}(C) \to \mathcal{P}(C)$ by composing with inverse image $\gamma^{-1} : 2^{\Omega C} \to 2^C$ under γ. These operators interpret both modalities and atomic propositions, as exemplified by

Example 4.1. Suppose $\Omega X = \mathcal{P}(X) \times \mathcal{P}(P)$, modelling Kripke models with a fixed set P of propositional constants. The function $[P](X) : 2^X \to 2^{\mathcal{P}(X) \times \mathcal{P}(P)}$ defined by $[P](X)(\mathfrak{x} \subseteq X) = \{(\mathfrak{y}, \mathfrak{p}) \in \mathcal{P}(X) \times \mathcal{P}(P) \mid \mathfrak{y} \subseteq \mathfrak{x}\}$ is a predicate lifting for Ω.

Given a Ω-coalgebra $\gamma : C \to \mathcal{P}(C) \times \mathcal{P}(P)$, the associated operator $\gamma^{-1} \circ [P]$ is the $\Box$-operator of modal logic: Indeed, given $\mathfrak{c} \subseteq C$, we obtain $\heartsuit_{[P]}(\mathfrak{c}) = \gamma^{-1} \circ [P](\mathfrak{c}) = \{c \in C \mid \forall c' \in C.c \to c' \Rightarrow c' \in \mathfrak{c}\}$, where we have written $c \to c'$ for $\gamma(c) = (\mathfrak{c}, \mathfrak{p}) \wedge c' \in \mathfrak{c}$.

Also, given a propositional constant $p \in P$, the constant function $[p \in P]$ defined by $[p \in P](X)(\mathfrak{x} \subseteq X) = \{(\mathfrak{c}, \mathfrak{p} \in \mathcal{P}(X) \times \mathcal{P}(C) \mid p \in \mathfrak{p})$ is a predicate lifting. Given $\gamma : C \to \mathcal{P}(C) \times \mathcal{P}(P)$, then the (constant) operator $\gamma^{-1} \circ [p \in P]$ associated to $[p \in P]$ gives rise to the set of states, which validate p. More precisely, given any subset $\mathfrak{c} \subseteq C$, we have $\heartsuit_{[p \in P]}(\mathfrak{c}) = \gamma^{-1} \circ [p \in P](\mathfrak{c}) = \{c \in C \mid c \models p\}$, where $c \models p$ is a shorthand for $\gamma(c) = (\mathfrak{c}, \mathfrak{p}) \wedge p \in \mathfrak{p}$.

Given a set $\mathcal{S}$ of predicate liftings for $\Omega : \mathrm{Set} \to \mathrm{Set}$, we consider the language $\mathcal{L}(\mathcal{S})$, we obtain a language $\mathcal{L}(\mathcal{S})$ as laid out in

Definition 4.2 (Syntax and Semantics of $\mathcal{L}(\mathcal{S})$). Suppose $\Omega : \mathrm{Set} \to \mathrm{Set}$ and $\mathcal{S}$ is a set of predicate liftings for Ω.

The language $\mathcal{L}(\mathcal{S})$ is the least set containing the (propositional) constants $\mathtt{tt}$ and $\mathtt{ff}$ and closed under conjunction $\wedge$ and negation $\neg$ and containing the formula $\heartsuit_\mu \phi$ for every $\mu \in \mathcal{S}$ and every $\phi \in \mathcal{L}(\mathcal{S})$.

The interpretation $[\![\phi]\!]_\gamma \subseteq C$ of $\phi \in \mathcal{L}(\mathcal{S})$ wrt. a coalgebra (C, γ) is defined by induction on the structure of ϕ, where $[\![\heartsuit_\mu \phi]\!]_\gamma = \gamma^{-1} \circ \mu(C)([\![\phi]\!]_\gamma)$.

The approach taken here assumes a given set $\mathcal{S}$ of prediacte liftings. In the cases considered in [10, 11, 15, 22, 21], where the class of signature functors under consideration is generated inductively, the underlying inductive definition also gives rise to a set of predicate liftings for the signature functor.

For a Ω-coalgebra (C, γ) and $c \in C$, we also write $c \models_\gamma \phi$ if $c \in [\![\phi]\!]_\gamma$ and $(C, \gamma) \models \phi$ if $[\![\phi]\!]_\gamma = C$.

We briefly remark that the interpretation of a language based on predicate liftings co-operates with coalgebra morphisms and is bisimulation invariant. The notion of bisimulation referred to in the next proposition it that of is that of Aczel and Mendler [1], see also Rutten [23].

Proposition 4.3. *Suppose Ω is an endofunctor on* Set *and $\mathcal{S}$ is a set of predicate liftings for Ω. If $f : (C, \gamma) \to (D, \delta)$ is a morphism of coalgebras and $\phi \in \mathcal{L}(\mathcal{S})$, then $[\![\phi]\!]_\gamma = f^{-1}[\![\phi]\!]_\delta$.*

Proof. By induction on the structure of ϕ, where the modality case $\phi = \heartsuit_\mu \psi$ follows from the naturality of predicate liftings. $\square$

Proposition 4.3 immediately implies that bisimilar points satisfy the same sets of formulas. The formulation of coalgebraic modal logic in terms of predicate liftings is also well suited to study the effect of a natural transformation $\eta : \Omega \to \Omega'$ on a language described by means of predicate liftings for Ω':

Proposition 4.4. *Suppose $\eta : \Omega \to \Omega'$ is a natural transformation and $\mu(X) : 2^X \to 2^{\Omega' X}$ is a predicate lifting for Ω'. Then the operation $\eta * \mu(X) : 2^X \to 2^{\Omega X}$, defined by*

$$(\eta * \mu)(X)(\mathfrak{x} \subseteq X) = \eta(X)^{-1} \circ \mu(X)$$

is a predicate lifting for Ω.

Proof. Follows immediately from the naturality of η and μ. $\square$

Example 4.5. We re-examine example 4.1 and consider Kripke models parameterised over the set of atomic propositions. These are modelled by the parameterised signature $\Omega(L, X) = \mathcal{P}(X) \times 2^L$. Note that the category of parameters is $\mathrm{Set}^{\mathrm{op}}$.

A morphism $\lambda : Q \to P$ in the parameter category $\mathrm{Set}^{\mathrm{op}}$ then corresponds to a function $\lambda : P \to Q$ in Set. As in Proposition 1.4, $\hat{\lambda} = \Omega(\lambda, \cdot) : \Omega_Q \to \Omega_P$ denotes the natural transformation determined by λ.

An easy calculation shows, that the predicate liftings $[P]$ and $[p \in P]$ from example 4.1 translate along a morphism along $\lambda : Q \to P \in \mathrm{Set}^{\mathrm{op}}$ in the expected way, that is, $\hat{\lambda} * [Q] = [P]$ and $\hat{\lambda} * [p \in P] = [\lambda(p) \in Q]$.

Considering a natural transformation $\eta : \Omega \to \Omega'$ as a signature morphism, the above proposition can be seen as an example of the fact, that signature morphisms act contravariantly on syntax, which is a first duality to the situation one has in an institution.

We proceed to demonstrate the relationship between logics (which we assume as given by a set of predicate liftings for the purpose of this section) corresponding to different signature functors and prove a satisfaction condition in the context of coalgebraic modal logic. In order to be able to translate languages along a natural transformation, we have to impose a coherence condition regarding the sets of predicate liftings for the respective signature functors. We remind the reader of the fact, that every morphism $\lambda \in \mathbb{L}$ induces a natural transformation $\hat{\lambda}$, as described in Proposition 1.7.

Definition 4.6. Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{L}$ is a parameterised signature and for each $L \in \mathbb{L}$, $\mathcal{S}_L$ is a set of predicate liftings for Ω_L. We call the family $(\mathcal{S}_L)_{L \in \mathbb{L}}$ *coherent*, if $\hat{\lambda} * \mu \in \mathcal{S}_L$ for every $\lambda : L \to M \in \mathbb{L}$ and every $\mu \in \mathcal{S}_M$.

Given a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ and a coherent family of predicate liftings $(\mathcal{S}_L)_{L \in \mathbb{L}}$, every morphism $\lambda : L \to L'$ induces a translation $\lambda^* : \mathcal{L}(\mathcal{S}_{L'}) \to \mathcal{L}(\mathcal{S}_L)$ by replacing every modality $\heartsuit_\mu$ by its translation $\heartsuit_{\hat{\lambda} * \mu}$ along λ.

Considering $\mathcal{L}(\mathcal{S}_L)$ as a discrete category, this induces a fibration $l : \mathcal{L} \to \mathbb{L}$ by the Grothendieck construction. We are now in the position to establish the relationship regarding satisfaction of formulas in the different fibres $\mathbb{E}_L$ of the total category.

Theorem 4.7. *Suppose $\Omega : \mathbb{L} \times \mathrm{Set} \to \mathrm{Set}$ is a parameterised signature with associated cofibration $p : \mathbb{E} \to \mathbb{L}$ and $(\mathcal{S}_L)_{L \in \mathbb{L}}$ is a coherent family of predicate liftings for Ω, giving rise to the logical fibration $\mathcal{L} \to \mathbb{L}$. Then*

$$(C, \gamma) \models \lambda^* \phi \quad \Leftrightarrow \quad \lambda^\dagger(C, \gamma) \models \phi$$

for all $(C, \gamma) \in \mathbb{E}_L$ and all $\phi \in \mathcal{L}_{L'}$, where $\lambda^\dagger$ denotes co-reindexing wrt. an arbitrary choice of cocartesian liftings.

Proof. Suppose $\lambda : L \to M \in \mathbb{L}$. If $(\lambda, f) : (C,\gamma) \to (C',\gamma')$ is cocartesian, we have to show that $(C,\gamma) \models \lambda^*\phi$ if and only if $(C',\gamma') \models \phi$ for all $\phi \in \mathcal{L}(\mathcal{S}_L)$. Lemma 1.7 provides us with an isomorphism $i : C' \to C \in \mathrm{Set}$. By induction on the structure of ϕ, one now shows that $[\![\lambda^\dagger\phi]\!]_{\gamma'} = i^{-1}([\![\phi]\!]_\gamma)$ using Proposition 4.3. The case $\phi = \heartsuit_\mu\psi$ follows from the naturality of μ. ☐

Comparing this to the discussion of institutions in Section 4.1, we see that the situation is completely dual to the algebraic case.

4.3 A Semantical View on Logics for Coalgebras

This section discusses a semantical approach to logics for coalgebras based on work of the first author's thesis [14]. Viewing coalgebras as a generalisation of Kripke frames, a modal formula, describing a property of Ω-coalgebras is seen to be a regular subobject of an appropriate cofree coalgebra. This interpretation establishes both a duality with the algebraic case (where a formula containing variables $x \in X$ describes a quotient of the free algebra over X) and generalises the interpretation of modal formulas wrt. Kripke frames to coalgebras for arbitrary signature functors.

Example 4.8. Following [13], we show how Kripke frames are generalised in a coalgebraic setting. Consider $\Omega X = \mathcal{P}_\kappa X$ where κ is a cardinal and $\mathcal{P}_k X = \{Y \subset X : |Y| < \kappa\}$.[4] Ω-coalgebras are Kripke frames, the notion *frame* referring to our intention to interpret the propositions in modal formulas as propositional *variables*: Given a set of propositions P, a Kripke model $\langle\gamma, v\rangle : C \to \Omega(C) \times \mathcal{P}P$ is a Ω-coalgebra (C,γ) together with a valuation $v : C \to \mathcal{P}P$. One then says that a frame (or a coalgebra) (C,γ) satisfies a modal formula ϕ in variables P iff for all valuations $v : C \to \mathcal{P}P$ the models $(C, \langle\gamma, v\rangle)$ satisfy ϕ in the usual sense. This can be rephrased in categorical terms—identifying ϕ with the largest subcoalgebra $M_\phi \hookrightarrow F(\mathcal{P}P)$ satisfying ϕ—as follows: $(C,\gamma) \models \phi$ iff all coalgebra morphisms $w : (C,\gamma) \to F(\mathcal{P}P)$ factor through $M_\phi \hookrightarrow F(\mathcal{P}P)$.

Given an endofunctor $\Omega : \mathbb{C} \to \mathbb{C}$, this leads us to the abstract notion of formula as regular mono $\phi : M \to FC$ (in $\mathbb{C}_\Omega$), which can be thought of as formula in context (or in colours) C. In order to allow change of context (ie. renaming of propositional variables), one additionally has to require that the projection functor $r : \mathrm{RegMono}(\mathbb{C}_\Omega) \to \mathbb{C}_\Omega$ is a fibration, that is, $\mathbb{C}_\Omega$ has pullbacks of regular monos. We refer to [14] for an elaboration of this issue and to Section 3 for sufficient conditions.

Under these conditions, the construction of the *full language* $\mathcal{F}_\Omega$ associated to Ω can be described by the change of base situation

$$\begin{tikzcd} \mathcal{F}_\Omega \arrow[r] \arrow[d, "l"] & \mathrm{RegMono}(\mathbb{C}_\Omega) \arrow[d, "r"] \\ \mathbb{C} \arrow[r, "F"] & \mathbb{C}_\Omega, \end{tikzcd}$$

that is, $\mathcal{F}_\Omega$ arises as the pullback in Cat of r along F. In this context, we call a category $\mathcal{L}$ together with a functor $[\![\cdot]\!] : \mathcal{L} \to \mathcal{F}_\Omega$ an *abstract language* for Ω.

The relation $(C,\gamma) \models \phi$ is now given as in the example above. Suppose (C,γ) is a Ω-coalgebra and $\phi \in \mathcal{F}_\Omega$ is a Ω-formula (that is, a regular mono $\phi : M \to FX$ for some $X \in \mathbb{C}$). Then $(C,\gamma) \models \phi$, if for all morphisms $c : (C,\gamma) \to FX$ (which correspond to colourings $C \to X$ under the adjunction $U \dashv F$) there is a coalgebra morphism $h : M \to FX$ such that $\phi \circ h = c$, ie.

$$\begin{tikzcd} (C,\gamma) \arrow[r, dashed, "h"] \arrow[dr, "c"'] & M \arrow[d, "\phi"] \\ & FX \end{tikzcd}$$

[4]This cardinality restriction ensures the existence of cofree coalgebras. The ordinary powerset can be recovered when we take κ to be an inaccessible cardinal (admitting existence).

commutes.

The next example shows, that a language based on predicate liftings can be seen as a concrete representation of an abstract modal language.

Example 4.9. Suppose $\Omega : \mathrm{Set} \to \mathrm{Set}$ is an endofunctor preserving weak pullbacks such that the forgetful functor $U : \mathrm{Set}_\Omega \to \mathrm{Set}$ has a right adjoint F. Denote the terminal Ω-coalgebra $F1$ by (Z, ζ). Given a set of predicate liftings $\mathcal{S}$, the language $\mathcal{L}(\mathcal{S})$ can be seen as a concrete representation of an abstract language in the sense of the present section.

For a formula $\phi \in \mathcal{L}(\mathcal{S})$, denote the largest subcoalgebra of (Z, ζ), which is contained in $[\![\phi]\!]_{(Z,\zeta)}$ by $(Z, \zeta)|\phi$ and let $i(\phi)$ denote the inclusion $(Z, \zeta)|\phi \hookrightarrow (Z, \zeta)$ (regarding the existence of $(Z, \zeta)|\phi$ we refer to [23]).

Then, for any Ω-coalgebra (C, γ) we have that $(C, \gamma) \models \phi$ wrt. $\mathcal{L}(\mathcal{S})$ iff $(C, \gamma) \models i(\phi)$ wrt. the definition of satisfaction in this section. This follows form the fact that the image $![C] \subseteq Z$ of C under the unique coalgebra-morphism $! : (C, \gamma) \to (Z, \zeta)$ gives rise to a subcoalgebra of (Z, ζ), see [23]. Thus the assignment $\phi \mapsto i(\phi)$ allows us to view $\mathcal{L}(\mathcal{S})$ as abstract language for Ω.

Note that in contrast to languages based on predicate liftings, as discussed in the previous section, we also have morphisms between formulas, since the category $\mathcal{F}_\Omega$ is in general not discrete. The meticulous reader is invited to check that for all morphisms $\phi \to \psi$, which can be thought of as syntactical consequence $\phi \vdash \psi$ and all Ω-coalgebras (C, γ) with $(C, \gamma) \models \phi$, we also have $(C, \gamma) \models \psi$.

We now broaden the perspective by moving to a parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$. The assumptions on Ω needed to express abstract modal logic in a cofibrational setting are summarised in

Definition 4.10. Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ is a parameterised signature with associated cofibration $p : \mathbb{E} \to \mathbb{C}$. We say that Ω *supports abstract modal logic*, if

- There is an adjunction $U \dashv F$ of bifibrations, where $U : \mathbb{E} \to \mathbb{L} \times \mathbb{C}$ is the global forgetful functor (cf. Section 3)
- Every fibre $\mathbb{E}_L$ has pullbacks.

The first condition provides us with a cofree construction in every fibre $\mathbb{C}_{\Omega_L}$ of p, which is needed to formulate the notion of formula as regular mono with cofree codomain. The second assumption ensures that the formulas in the fibres are well-behaved with respect to change of context.

In order to present abstract languages as fibred over the parameter category, we need an easy

Lemma 4.11. *Suppose $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$ supports abstract modal logic and $p : \mathbb{E} \to \mathbb{L}$ is the associated bifibration. Denote the reindexing functor associated to $\lambda : L \to L' \in \mathbb{L}$ by λ^*.*

(i) If $\phi \in \mathbb{E}_{L'}$ is a regular mono, then so is $\lambda^* \phi \in \mathbb{E}_L$.

(ii) If $\phi \in \mathcal{F}_{\Omega_{L'}}$, then $\lambda^* \phi \in \mathcal{F}_{\Omega_L}$.

Proof. The first statement follows form the adjunction $\lambda^\dagger \dashv \lambda^*$. For the second, note that reindexing functors preserve cofree coalgebras as demonstrated in the proof of Theorem 3.3. □

The preceding lemma says that assigning the full language $\mathcal{F}_{\Omega_L}$ to every parameter $L \in \mathbb{L}$ gives rise to a pseudofunctor $\mathbb{L} \to \mathrm{Cat}$, or, equivalently, to a fibration $l : \mathcal{F}_\Omega \to \mathbb{L}$. We call this fibration the *full language* associated to the parameterised signature $\Omega : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$. An *abstract modal language* for Ω is now a fibred category $l : \mathcal{L} \to \mathbb{L}$, equipped with a fibred functor $[\![\cdot]\!] : \mathcal{L} \to \mathcal{F}_\Omega$. Note that the coherence condition, needed in the case of languages given by a family of predicate liftings, is incorporated into the requirement that $\mathcal{L} \hookrightarrow \mathcal{F}_\Omega$ is a subfibration.

Having the language (which appears as fibred over $\mathbb{L}$) and the definition of satisfaction in the different fibres at hand, we now establish the main result of this section, relating the fibrewise defined satisfaction relations to the (co-)reindexing functors.

Theorem 4.12. *Suppose Ω is a parameterised signature supporting abstract modal logic, and $p : \mathbb{E} \to \mathbb{L}$ is the associated bifibration. If $l : \mathcal{L} \to \mathbb{L}$ is an abstract language for Ω and $\lambda : L \to L' \in \mathbb{E}$, then*

$$(C, \gamma) \models \lambda^* \phi \quad \Leftrightarrow \quad \lambda^\dagger (C, \gamma) \models \phi$$

for all coalgebras $(C, \gamma) \in \mathbb{E}$ and formulas $\phi \in \mathcal{L}_{L'}$.

Proof. We only show one implication, since the other can be proved essentially along the same lines. So suppose $(C, \gamma) \models \lambda^* \phi$, where $\phi : M \to F(L', X)$ is a regular mono in $\mathbb{C}_{\Omega'_L}$. In order to show that $\lambda^\dagger (C, \gamma) \models \phi$, pick any coalgebra morphism $v : \lambda^\dagger (C, \gamma) \to F(L', X)$.

Pick a cocartesian morphism $c : (C, \gamma) \to \lambda^\dagger (C, \gamma)$ and factor the composite $v \circ c$ as $v \circ c = c' \circ v'$, where $v' : (C, \gamma) \to F(L, X)$ is vertical and and $c' = F(\lambda, \mathrm{id}_C)$ is cartesian (compare the proof of 3.3).

By the construction of reindexing functors, this gives rise to a map $\hat{h} : (C, \gamma) \to M$, where M is the codomain of ϕ. Now $\hat{h}$ factors as $c \circ d$, where $c : (C, \gamma) \to \lambda^\dagger (C, \gamma)$ as above and $d : \lambda^\dagger (C, \gamma) \to M$. An easy diagram chase shows that $\phi \circ d = v$, ie. $\lambda^\dagger (C, \gamma) \models \phi$. □

We conclude the section on logics by re-examining Example 4.9 and show, under which circumstances Theorem 4.7 can be seen as a special instance of Theorem 4.12.

In order to be able to relate the two results, we have to make some assumptions about the signature Ω, ensuring the applicability of both. So suppose $\Omega : \mathbb{L} \times \mathrm{Set} \to \mathrm{Set}$ is a parameterised signature which supports abstract moral logic and Ω_L preserves weak pullbacks for every object $L \in \mathbb{L}$ of the parameter category. Given a coherent family $(\mathcal{S}_L)_{L \in \mathbb{L}}$ of predicate liftings for Ω and a morphism $\lambda : L \to M \in \mathbb{L}$, denote the translation operation by $(\cdot)^\lambda : \mathcal{L}(\mathcal{S}_M) \to \mathcal{L}(\mathcal{S}_L)$.

The following lemma uses the notation from Example 4.9.

Lemma 4.13. *Under the above conditions, we have an isomorphism*

$$i(\phi^\lambda) \cong \lambda^*(i(\phi))$$

in $\mathbb{C}_{\Omega_L}/Z_L$ for all formulas $\phi \in \mathcal{L}(\mathcal{S})_M$ and all $\lambda : L \to M \in \mathbb{L}$, where $Z_L \in \mathbb{C}_{\Omega_L}$ is the final coalgebra.

Proof. Denote the final coalgebras over L and M by Z_L and Z_M, respectively and note that $\lambda^* Z_M \cong Z_L$ by Theorem 3.3. By induction on the structure of ϕ one obtains a pullback with cartesian horizontal arrows as in the left diagram below.

$$\begin{tikzcd} Z_L|\phi^\lambda \arrow[r] \arrow[d, "i(\phi^\lambda)"'] & Z_M|\phi \arrow[d, "i(\phi)"] \\ Z_L \arrow[r] & Z_M \end{tikzcd} \qquad \begin{tikzcd} \lambda^*(Z_M|\phi) \arrow[r] \arrow[d, "\lambda^*(i(\phi))"'] & Z_M|\phi \arrow[d, "i(\phi)"] \\ Z_L \arrow[r] & (Z_M) \end{tikzcd}$$

Since also the right diagram is a pullback (with cartesian horizontal arrows) in $\mathbb{E}$, the result follows. □

This lemma allows us to conclude Theorem 4.7 from Theorem 4.12. Suppose $\lambda : L \to M \in \mathbb{L}$, and we have a formula $\phi \in \mathcal{L}(\mathcal{S}_M)$ and $C \in \mathbb{E}_L$. Then $\lambda^\dagger C \models \phi \Leftrightarrow \lambda^\dagger \models i(\phi) \Leftrightarrow C \models \lambda^*(i(\phi)) \Leftrightarrow C \models i(\phi^\lambda)$, where the first equivalence of this chain was established in Example 4.9, the second is Theorem 4.12 and the last is an application of Lemma 4.13.

Finally, let us emphasise the differences of the approaches presented in Sections 4.2 and 4.3. Definition 4.2 used the notion of predicate liftings to cover the existing logics for coalgebras whose languages and semantics are inductively defined. In particular, the proof of Theorem 4.7 showed the effect of paramter transformations on (inductively given) modal languages. On the other hand, the semantical approach covers any logic where formulas are bisimulation invariant predicates on carriers, independently of how the logic is given. An important example is Moss' coalgebraic logic [16] (which fits in the semantical but not in the precicate lifting approach).

5. Hidden and Multiplicative Signatures

In specification formalisms using algebras and/or coalgebras one often restricts attention to special signature functors, namely, hidden signatures [7] in case of algebras and multiplicative functors in case of coalgebras. We show here that we can characterise, roughly speaking, hidden signatures as functors on Set^n having a right adjoint (and these right adjoints are multiplicative) and multiplicative functors as functors on Set^n having a left adjoint (and these left adjoints are hidden signatures). This yields a new explanation for the well-known fact that categories of hidden algebras are isomorphic to categories of coalgebras for multiplicative functors.

These results are an elaboration of the following simple observation:

Proposition 5.1. *Let $\Xi : \mathbb{C} \to \mathbb{C}$ be a functor and Σ a left adjoint of Ξ. Then the category $\mathbb{C}_\Xi$ of Ξ-coalgebras is isomorphic to the category $\mathbb{C}^\Sigma$ of Σ-algebras.*

Proof. For $X, Y \in \mathbb{C}$, let $\phi_{X,Y} : \mathbb{C}(X, \Xi Y) \to \mathbb{C}(\Sigma X, Y)$ be the natural isomorphism given by the adjunction. The required isomorphism between the category of Ξ-coalgebras and Σ-algebras is then given on objects by $(X, \xi : X \to \Xi X) \mapsto (X, \phi_{X,X}(\xi) : \Sigma X \to X)$ and on morphisms by the identity (naturality of ϕ guarantees that coalgebra morphisms are indeed algebra morphisms). $\square$

At first sight, this observation seems to be of limited interest because the only functors Ξ on Set that have a left adjoint are of the form $\Xi X = X^A$ for some $A \in \mathrm{Set}$. But this changes when we make parameters explicit, motivating

Definition 5.2 (ASC and CSA). We call an endofunctor Ω on a category $\mathbb{C}$ with terminal element 1 (inital element 0) an algebraic signature for coalgebras or ASC (a coalgebraic signature for algebras or CSA) iff there is $n \in \mathbb{N}$ and $L \in \mathbb{C}^n$ such that (1) there is $\Omega_e : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ with $\Omega_e(L, -)$ naturally isomorphic to Ω and (2) $\hat{\Omega}_e : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$ defined via $\hat{\Omega}_e(L, X) = (1, \Omega_e(L, X))$ is a right adjoint (defined via $\hat{\Omega}_e(L, X) = (0, \Omega_e(L, X))$ is a left adjoint).

The terms 'algebraic signature for coalgebras' and 'coalgebraic signature for algebras' are justified by the following proposition which follows from propositions 1.10 and 5.1.

Proposition 5.3. *Let Ω be an ASC (CSA) over $\mathbb{C}$ and $p : \mathbb{E} \to \mathbb{L}$ the cofibration (fibration[5]) induced by Ω_e. Then $\mathbb{C}_\Omega$ ($\mathbb{C}^\Omega$) is isomorphic to a fibre of p and the total category $\mathbb{E}$ is isomorphic to a category of algebras (coalgebras) for an endofunctor.*

The main example for ASC are the *multiplicative functors*, that is, functors which are built from identity, constants, products and exponentiation with constants.

Proposition 5.4. *Let $\mathbb{C}$ be a bicartesian closed category and Ω a multiplicative signature over $\mathbb{C}$. Then Ω is ASC.*

Proof. Every multiplicative functor can be written as $\Omega X = \prod_{i=1}^m X^{A_i} \times \prod_{j=1}^n B_j$. Making the parameters B_j explicit this gives rise to a functor $\Omega_e : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$. Now, $\hat{\Omega}_e$ has a left adjoint Σ:

$$\Sigma \begin{pmatrix} X \\ B_1 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m A_i \times X \\ X \\ \vdots \\ X \end{pmatrix}$$

$\square$

The main examples of CSA are hidden signatures [7].

[5] The results of this paper dualise to algebras for an endofunctor. In particular, a parameterised signature induces a fibration of algebras.

Proposition 5.5. *Let $\mathbb{C}$ be a bicartesian closed category. Then hidden signatures are CSA.*

Proof. A hidden signature can be written as a functor $\Omega X = \sum_{j=1}^{n} C_j + \sum_{i=1}^{m} A_i \times X$. Making the parameters C_j explicit this can be written as a functor $\Omega_e : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, giving rise to a functor $\hat{\Omega}_e$ having a right adjoint Ξ:

$$\Xi \begin{pmatrix} X \\ C_1 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^{m} X^{A_i} \\ X \\ \vdots \\ X \end{pmatrix}$$

□

We have seen that a mulitplicative signature can be extended to a functor having a left adjoint and that a hidden signature can be extended to a functor having a right adjoint. We now show that in the case of signatures over Set, multiplcative and hidden signatures are even characterised by this property. The key to this result is the following lemma which generalises theorem 5.7 in Arbib and Manes [3] from Set to Set^n.[6]

Lemma 5.6. *Let Σ be a functor on* Set^n*. Then the following are equivalent.*

(i) Σ *has a right adjoint.*

(ii) Σ *preserves coproducts.*

(iii) There is a $(n \times n)$-matrix M over Set *such that $\Sigma X = MX$, $X \in \mathrm{Set}^n$.*[7]

Proof. "(1) $\Rightarrow$ (2)" is a standard result on adjoints. "(2) $\Rightarrow$ (3)": Let $1 \le i \le n$. Write X_i for the i-th component of X and E^i for the vector in Set^n that has 0 everywhere but 1 in the i-th component. Then $\Sigma X = \Sigma(\sum_{1\le i\le n} X_i \times E^i) = \sum_{1\le i\le n} \Sigma(X_i \times E^i) = \sum_{1\le i\le n} \Sigma(\sum_{|X_i|} E^i) = \sum_{1\le i\le n} \sum_{|X_i|} \Sigma E^i = \sum_{1\le i\le n} X_i \times \Sigma E^i$, using that Σ preserves coproducts. Now define the components of M by letting M_{ij} be the j-th component of ΣE^i. "(3) $\Rightarrow$ (1)": Let $\Sigma X = MX$ for some $(n \times n)$-matrix M over Set. Then define a right adjoint Ξ of Σ by $X_i \mapsto \prod_{1\le j\le n} X_j^{M_{ij}}$. □

As corollaries we obtain converses to the propositions above.

Corollary 5.7. *Let $\Omega : \mathrm{Set}^n \to \mathrm{Set}^n$ be ASC. Then Ω is a multiplicative functor.*

Corollary 5.8. *Let $\Omega : \mathrm{Set}^n \to \mathrm{Set}^n$ be CSA. Then Ω is a hidden signature.*

These results shed a new light on hidden algebra and on logics for coalgebras. We now see that hidden algebras are given by precisely those signatures which give rise to coalgebras via proposition 5.1. Conversely, the use of equational logic for coalgebras (as opposed to modal logic) in e.g. [5, 20, 8] implicitly relies on multiplicative signatures giving rise to categories of algebras via proposition 5.1.

Acknowledgments

The authors would like to thank Horst Reichel for the organisation of the workshops "Categorical and Coalgebraic Models of Concurrency", Dresden, October 1999, and CMCS 2000, Berlin, where earlier versions of this paper were presented. We gained much from comments by Bart Jacobs, Ataru Nakagawa and Jan Rutten on the versions presented there. Moreover, the comments of the anonymous referees were very helpful. Diagrams have been drawn with Paul Taylor's and with Kristoffer Rose's macro packages.

[6] We would like to thank Bart Jacobs for pointing out this theorem.

[7] MX is matrix multiplication, thinking of X as a vector and using the operations $+, \times$ on sets as addition and multiplication.

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