Definable Model Classes in Polynomial Coalgebraic Logic

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Theme:

Lift ideas and results from

- propositional modal logic

to

- polynomial coalgebraic logic.

Issue:

how to handle infinite sets of “observables” ?

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References

- Observational ultraproducts of polynomial coalgebras.
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- Enlargements of polynomial coalgebras.
  In Rod Downey et al., editors, *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 152–192. World Scientific, 2003.

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  Theoretical Computer Science, 360:1–22, 2006.

see also `www.mcs.vuw.ac.nz/~rob`

T-Coalgebras

$T : \mathbf{Set} \to \mathbf{Set}$ is a functor on the category $\mathbf{Set}$ of sets and functions

Definition

A T-coalgebra (A, α) is given by a function of the form

$$A \xrightarrow{\alpha} TA$$

- A is the state set
- α is the transition structure.

Morphism of T-Coalgebras

$$(A, \alpha) \xrightarrow{f} (B, \beta)$$

given by a function f for which

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ TA & \xrightarrow{Tf} & TB \end{array}$$

$$\beta \circ f = Tf \circ \alpha$$

Polynomial functors $T : \mathbf{Set} \to \mathbf{Set}$

constructed from

- the identity functor $\mathrm{Id} : A \mapsto A$ and/or
- constant functors $\bar{D} : A \mapsto D$, by forming
- products $T_1 \times T_2 : A \mapsto T_1 A \times T_2 A$,
- coproducts (disjoint unions) $T_1 + T_2 : A \mapsto T_1 A + T_2 A$,
- exponential functors $T^D : A \mapsto (TA)^D$ with constant exponent D.

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Polynomial coalgebras $A \xrightarrow{\alpha} TA$ have polynomial T.

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Polynomial coalgebras $A \xrightarrow{\alpha} TA$ have polynomial T.

Syntax for polynomial T

Notation:

$$M : S$$

means M is a term of type S, with S a component functor of T.

Definition of Terms for T

- Variables: $v : S$ any S
- Constants: $c : \bar{D}$, if $c \in D$ (observable elements)
- State Parameter: $\mathsf{s} : \mathrm{Id}$ (one only)
- Transition: $\mathrm{tr}(M) : T$, if $M : \mathrm{Id}$

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... continued

- Products: $\langle M_1, M_2 \rangle : S_1 \times S_2$, if $M_j : S_j$

  $\pi_j M : S_j$, if $M : S_1 \times S_2$

- Exponentials: $\lambda v M : S^D$, if $v : \bar{D}$ and $M : S$

  $M(N) : S$, if $M : S^D$ and $N : \bar{D}$

- Coproducts: $\iota_j M : S_1 + S_2$, if $M : S_j$

  [case N of v_1 in M_1 or v_2 in M_2] $: S$ if $N : S_1 + S_2$, $v_j : S_j$, $M_j : S$

Semantics of Terms

In a T-coalgebra (A, α),

the denotation/ interpretation of a term $M : S$ with free variables $v_1 : S_1, \ldots, v_n : S_n$, is a function

$$[\![M]\!]_\alpha : A \times S_1 A \times \cdots \times S_n A \to SA.$$

Definition

ground term: has no free variables

$$[\![M]\!]_\alpha : A \to SA.$$

Example

$[\![\mathsf{tr(s)}]\!]_\alpha$ is $A \xrightarrow{\alpha} TA.$

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Ground observable (GO) term:

a ground term of "observable" type $\bar{D}$, some D.

Ground equation:

$$M_1 \approx M_2$$

with M_1, M_2 ground terms of same type.

Truth-sets of ground equations:

$$\|M_1 \approx M_2\|^\alpha$$

is the set

$$\{x \in A : [\![M_1]\!]_\alpha(x) = [\![M_2]\!]_\alpha(x)\}$$

of all states in coalgebra (A, α) at which the equation $M_1 \approx M_2$ is true.

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of all states in coalgebra (A, α) at which the equation $M_1 \approx M_2$ is true.

Ground formula:

built from ground equations by logical connectives $\neg$, $\wedge$.

$$\|\neg\varphi\|^\alpha = A - \|\varphi\|^\alpha$$

$$\|\varphi_1 \wedge \varphi_2\|^\alpha = \|\varphi_1\|^\alpha \cap \|\varphi_2\|^\alpha.$$

Truth/satisfaction relation:

- $\alpha, x \models \varphi$ means $x \in \|\varphi\|^\alpha$.
- $\alpha \models \varphi$ means $\|\varphi\|^\alpha = A$.

Ground observable (GO) formula:

built from equations between GO terms.

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$$\frac{\text{GO formulas}}{\text{polynomial coalgebras}} = \frac{\text{equations}}{\text{abstract algebras}}$$

The GO terms and formulas provide a natural language for

- specifying properties of polynomial coalgebras.
- characterizing morphisms in terms of term-value preservation.
- characterizing the *bisimilarity* relation of *observational indistinguishability* of states by satisfaction of the same formulas (Hennessy-Milner property).

Modally Definable Classes of Kripke Frames

Theorem

Let K be a class of Kripke frames that is closed under ultrapowers. Then K is modally definable iff it is

- *closed under subframes, p-morphic images and disjoint unions;*

and

- *its complement is closed under ultrafilter extensions (i.e. it reflects ultrafilter extensions).*

Note:

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the ultrafilter extension $ue\mathcal{F}$ of frame $\mathcal{F}$ is a p-morphic image of a suitably saturated ultrapower of $\mathcal{F}$:

$$\begin{tikzcd} & \mathcal{F}^I/U \arrow[d, two heads, "\Phi"] \\ \mathcal{F} \arrow[ur, tail] \arrow[r, tail] & ue\mathcal{F} \end{tikzcd}$$

$$\Phi : f^U \mapsto \{X \subseteq \mathcal{F} : f \in_U X\}$$

Venema's analogue for Kripke models

Theorem

A class of Kripke models is modally definable iff it is

- *closed under images of bisimulation relations and disjoint unions, and*
- *is invariant under ultrafilter extensions.*

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Observational Ultrapowers

Let U be an ultrafilter on a set I.

Given T-coalgebra $A \xrightarrow{\alpha} TA$, we construct $\alpha^+ : A^+ \to T(A^+)$, an "observational ultrapower" of α with respect to U.

The state set of α^+ is a sub-quotient of the I-th power A^I of A.

Key Requirements:

- Every GO formula valid in α is valid in α^+.
- If every GO formula valid in α is valid also in coalgebra β, then β is a bisimilarity image of α^+, i.e. each state of β is bisimilar to a state of α^+.

For suitably chosen U, α^+ is sufficiently "saturated" to make this work, and leads to the following co-Birkhoff Theorem for polynomial coalgebras.

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Theorem

Let T be polynomial with an observable component $\bar{D}$ having $|D| \geq 2$.

For any class K of T-coalgebras, the following are equivalent.

1. *K is GO-definable,*
   i.e. is the class of all models of some set of GO formulas.
2. *K is closed under coproducts (disjoint unions), images of bisimulations, and observational ultrapowers.*
3. *K is closed under coproducts, domains and images of T-morphisms, and observational ultrapowers.*

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Standard Ultrapowers:

- A^U is the quotient $A^I/{=_U}$, where

$$f =_U g \text{ iff } \{i \in I : f(i) = g(i)\} \in U.$$

$f =_U g$ means that f and g agree "almost everywhere".

- $f^U :=$ the equivalence class of f.
- $A^U = \{f^U : f \in A^I\}$.
- A natural embedding $e_A : A \rightarrowtail A^U$ allows us to assume $A \subseteq A^U$.

Liftings:

Any map $A_1 \times \cdots \times A_n \xrightarrow{\theta} B$ has a U-lifting

$$A_1^U \times \cdots \times A_n^U \xrightarrow{\theta^U} B^U.$$

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Problem:

A T-coalgebra

$$A \xrightarrow{\alpha} TA$$

has the U-lifting

$$A^U \xrightarrow{\alpha^U} (TA)^U,$$

but a T-coalgebra based on A^U should look like

$$A^U \to T(A^U).$$

Solution:

Restrict α^U to the subset $A^+ \subseteq A^U$ of elements f^U that are "observable".

Idea:

For a GO term $M : \bar{D}$, the α-denotation

$$[\![M]\!]_\alpha : A \to D$$

has the U-lifting $\quad [\![M]\!]^U_\alpha : A^U \to D^U.$

Since $D \subseteq D^U$, we ask $\quad$ does $[\![\boldsymbol{M}]\!]^{\boldsymbol{U}}_{\boldsymbol{\alpha}}(\boldsymbol{f}^{\boldsymbol{U}}) \in \boldsymbol{D}$?

If YES for all M, put f^U in A^+.

Example

every member of A is observable, so $A \subseteq A^+$.

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Defining the transition structure $\alpha^+ : A^+ \to TA^+$:

- an intricate analysis of the components involved in the inductive formation of functor T.
- A path

$$T \overset{p}{\rightsquigarrow} S$$

from T to a component functor S is a finite list of symbols expressing the way T is formed from S.

- A path induces a partial function

$$p_A : \ TA \circ\!\!\longrightarrow SA$$

for each set A.

Definition

- $T \overset{\langle\rangle}{\rightsquigarrow} T$ is the empty path.
- from $T_j \overset{p}{\rightsquigarrow} S$ form $T_1 \times T_2 \overset{\pi_j.p}{\rightsquigarrow} S$ for $j = 1, 2$.
- from $T_j \overset{p}{\rightsquigarrow} S$ form $T_1 + T_2 \overset{\varepsilon_j.p}{\rightsquigarrow} S$ for $j = 1, 2$.
- from $T \overset{p}{\rightsquigarrow} S$ form $T^D \overset{ev_d.p}{\rightsquigarrow} S$ for all $d \in D$.

$$
\begin{tikzcd}
A \arrow[r, tail, "e_A"] \arrow[d, "p_A \circ \alpha"'] & A^+ \arrow[r, hook] \arrow[d, dotted, "(p_A \circ \alpha)^+"] & A^U \arrow[d, "(p_A \circ \alpha)^U"] \\
SA \arrow[r, "Se_A"] & SA^+ & (SA)^U \arrow[l, two heads, dotted]
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When $p =$ the empty path $T \rightsquigarrow T$, get

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Łoś-type Theorem

$\alpha^+, f^U \models \varphi$ if, and only if, $\{i \in I : \alpha, f(i) \models \varphi\} \in U$.

Proof method:

an analysis for each term $M : S$ of the relationship between the U-lifting

$$[\![M]\!]^U_\alpha : A^U \to (SA)^U$$

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and its α^+-denotation

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Corollary

$\alpha \models \varphi$ *if, and only if,* $\alpha^+ \models \varphi$.

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Observational Ultraproducts

Given T-coalgebras $\{(A_i, \alpha_i) : i \in I\}$, define a T-coalgebra

$$\Pi_U A_i^+ \xrightarrow{\alpha^+} T(\Pi_U A_i^+)$$

whose states are "observable" members of the ultraproduct $\Pi_U A_i$

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$$\{i \in I : \alpha_i \models \varphi\} \in U \quad \text{implies} \quad \alpha^+ \models \varphi,$$

and conversely if $\Pi_U A_i^+ = \Pi_U A_i$.

(Converse does hold for ultrapowers.)

NB: could have $\Pi_U A_i^+ = \emptyset$

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Example

- $T = \bar{\omega}$
- (A_n, α_n) is the $\bar{\omega}$-coalgebra

$$\{n, n+1, \dots\} \hookrightarrow \omega.$$

- $\Pi_U A_n^+ = \emptyset$ whenever U non-principal.

Compactness Property:

Possible definitions

- a set of formulas has a non-empty model whenever each of its finite subsets does.
- a set of formulas is satisfiable at some state whenever each of its finite subsets is.

Both of these fail for

$$\{\mathsf{tr}(\mathsf{s}) \not\approx n : n \in \omega\}$$

with $T = \bar{\omega}$.

Theorem

If every observational ultraproduct of nonempty T-coalgebras is nonempty, then Compactness does hold for T.

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Ultrafilter Enlargements

Definitions

- An ultrafilter F on the state set of (A, α) is observationally rich if for each GO term $M : \bar{D}$ there exists some $c \in D$ such that

$$\|M \approx c\|^\alpha = \{x \in A : [\![M]\!]_\alpha(x) = c\} \in F,$$

  i.e. every GO term takes a constant value on an F-large set.

- The ultrafilter enlargement of (A, α) is a coalgebra

$$A^* \xrightarrow{\alpha^*} TA^*,$$

  whose state set A^* is the set of all rich ultrafilters on A.

  The definition of α^* involves path functions, similarly to α^+.

Example

Every principal ultrafilter on A is rich, giving an embedding

$$\eta_A : (A, \alpha) \rightarrowtail (A^*, \alpha^*)$$

that is a coalgebraic morphism (contra the modal case!)

Truth Lemma

For each GO formula φ,

$$\alpha^*, F \models \varphi \quad \text{if, and only if,} \quad \|\varphi\|^\alpha \in F$$

i.e. φ is true at state F in A^* iff true in an "F-large" set of states in A.

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(A^*, α^*) as a quotient of some (A^+, α^+)

- $\Phi_U : A^+ \to A^*$ acts by $f^U \mapsto \{X \subseteq A : f \in_U X\}$.
- Φ_U is a coalgebraic morphism $(A^+, \alpha^+) \to (A^*, \alpha^*)$.
- Φ_U is surjective if A^U enlarges A:

  every collection of subsets of A with the finite intersection property has non-empty intersection in A^U.

Definable Ultrafilter Enlargements

- $\boldsymbol{Def}^{\alpha} = \{\|\varphi\|^{\alpha} : \varphi \text{ is GO}\}$
  is the Boolean algebra of definable subsets of A.
- The definable enlargement $A^{\delta} \xrightarrow{\alpha^{\delta}} TA^{\delta}$ of (A, α) has
  $A^{\delta} =$ the set of all rich ultrafilters $\boxed{\text{in } \boldsymbol{Def}^{\alpha}}$
- $\alpha^{\delta}, F \models \varphi$ if, and only if, $\varphi^{\alpha} \in F$.
- $\alpha \models \varphi$ if, and only if, $\alpha^{\delta} \models \varphi$.
- There is an epimorphism $\alpha^{*} \twoheadrightarrow \alpha^{\delta}$.
  α^{δ} is isomorphic to the bisimilarity quotient of α^{*}.

Infinitary Proof Theory

Path formulas

- Halting formulas:

  for any path $T \overset{p}{\rightsquigarrow} S$ there is a formula $p{\downarrow}$ with

  $\alpha, x \models p{\downarrow} \quad \text{iff} \quad \alpha(x) \in \mathrm{Dom}\, p_A$

- Observation formulas:

  for any path $T \overset{p}{\rightsquigarrow} \bar{D}$ and $c \in D$ there is a formula $(p)c$ with

  $\alpha, x \models (p)c \quad \text{iff} \quad \alpha(x) \in \mathrm{Dom}\, p_A$ and $p_A(\alpha(x)) = c$.

- Modalities:

  for any path $T \overset{p}{\rightsquigarrow} \mathrm{Id}$ and formula φ there is a formula $[p]\varphi$ with

  $\alpha, x \models [p]\varphi \quad \text{iff} \quad \alpha(x) \in \mathrm{Dom}\, p_A$ implies $\alpha, p_A(\alpha(x)) \models \varphi$.

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Deducibility Relation $\Gamma \vdash_T \varphi$

- defined syntactically using axioms and infinitary rules, e.g.

$$\frac{\neg(p)c \quad \textit{for all} \quad c \in D}{\neg(p\downarrow)} \qquad \frac{\psi \to [q]\neg(p)c \quad \textit{for all} \quad c \in D}{\psi \to [q]\neg(p\downarrow)}$$

- Γ is T-consistent if $\Gamma \nvdash_T \bot$.
- Γ is T-maximal if it is consistent, negation complete, and closed under certain infinitary rules.

Example

For any α, x,

$\{\varphi : \alpha, x \models \varphi\}$ is T-maximal.

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Lindenbaum Functors

Definition

T is Lindenbaum if every T-consistent set of formulas can be extended to a T-maximal set.

Lemma

T is Lindenbaum if any of the following hold:

1. *T has no constant component $\bar{D}$, or*
2. *Every constant component $\bar{D}$ has D finite, or*
3. *Any exponential component S^D of T has countable exponent D.*

Example

Let $T = \bar{\omega}^{\mathbb{R}}$.

There is a set Σ of formulas such that

- Σ is T-consistent.
- Σ is not satisfiable at any state of any T-coalgebra.

  – hence Completeness fails.
- every countable subset of Σ is satisfiable.

Theorem

If T is Lindenbaum, then every T-consistent set of formulas is satisfiable in a T-coalgebra.

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Canonical T-coalgebra

$$A_T \xrightarrow{\alpha_T} T(A_T)$$

- A_T is the set of all T-maximal sets.
- Truth Lemma: $\alpha_T, x \models \varphi$ iff $\varphi \in x$.
- If T is Lindenbaum, then every T-consistent set is satisfiable.

(A_T, α_T) is a final coalgebra.

$$x \longmapsto \{\varphi : \alpha, x \models \varphi\}$$

is the unique morphism from any (A, α) to (A_T, α_T).

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Questions

- Can this theory be extended to finitary Kripke polynomial functors, involving $\mathcal{P}_\omega$?
- is there a universal property characterising observational ultraproducts ?