Definable Model Classes in Polynomial Coalgebraic Logic

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Victoria University of Wellington

Workshop on Coalgebraic Logic, Oxford, August 2007

Theme:

Lift ideas and results from

propositional modal logic

to

• polynomial coalgebraic logic.

Issue:

how to handle infinite sets of "observables" ?

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References

- Observational ultraproducts of polynomial coalgebras. Annals of Pure and Applied Logic, 123:235–290, 2003.
- Enlargements of polynomial coalgebras. In Rod Downey et al., editors, *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 152–192. World Scientific, 2003.
- Duality for some categories of coalgebras. *Algebra Universalis*, 46(3):389–416, 2001.

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Theoretical Computer Science, 360:1–22, 2006.

see also www.mcs.vuw.ac.nz/~rob

T-Coalgebras

 $T: \mathbf{Set} \to \mathbf{Set}$ is a functor on the category \mathbf{Set} of sets and functions

Definition A *T*-coalgebra (A, α) is given by a function of the form

$$A \xrightarrow{\alpha} TA$$

- A is the state set
- α is the transition structure.

Morphism of *T*-Coalgebras

$$(A, \alpha) \xrightarrow{f} (B, \beta)$$

given by a function f for which



$$\beta \circ f = Tf \circ \alpha$$

Polynomial functors $T : \mathbf{Set} \to \mathbf{Set}$

constructed from

- the identity functor $Id: A \mapsto A$ and/or
- constant functors $\overline{D}: A \mapsto D$, by forming
- products $T_1 \times T_2 : A \mapsto T_1 A \times T_2 A$,
- coproducts (disjoint unions) $T_1 + T_2 : A \mapsto T_1A + T_2A$,
- exponential functors $T^D: A \mapsto (TA)^D$ with constant exponent D.

Definition

Polynomial coalgebras $A \xrightarrow{\alpha} TA$ have polynomial T.

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Syntax for polynomial T

Notation:

M:S

means M is a term of type S, with S a component functor of T.

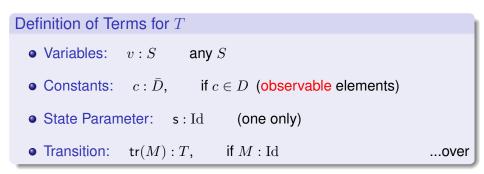
Definition of Terms for T• Variables: v: S any S• Constants: $c: \overline{D}$, if $c \in D$ (observable elements)• State Parameter: s: Id (one only)• Transition: tr(M): T, if M: Id ...over

Syntax for polynomial T

Notation:

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... continued

• Products: $\langle M_1, M_2 \rangle : S_1 \times S_2$, if $M_j : S_j$ $\pi_j M : S_j$, if $M : S_1 \times S_2$ • Exponentials: $\lambda v M : S^D$, if $v : \overline{D}$ and M : S M(N) : S, if $M : S^D$ and $N : \overline{D}$ • Coproducts: $\iota_j M : S_1 + S_2$, if $M : S_j$ [case N of v_1 in M_1 or v_2 in M_2] : S if $N : S_1 + S_2$, $v_j : S_j$, $M_j : S$

Semantics of Terms

In a *T*-coalgebra (A, α) ,

the denotation/ interpretation of a term M : S with free variables $v_1 : S_1, \ldots, v_n : S_n$, is a function

$$\llbracket M \rrbracket_{\alpha} : A \times S_1 A \times \dots \times S_n A \to S A.$$

Definition

ground term: has no free variables

 $\llbracket M \rrbracket_{\alpha} : A \to SA.$

Example

$$\llbracket \operatorname{tr}(\mathsf{s}) \rrbracket_{\alpha} \quad \text{is} \quad A \xrightarrow{\alpha} TA.$$

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Ground observable (GO) term:

a ground term of "observable" type \overline{D} , some D.

Ground equation:

 $M_1 \approx M_2$

with M_1, M_2 ground terms of same type.

Truth-sets of ground equations:

 $\|M_1 \approx M_2\|^{\alpha}$

is the set

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\{x \in A : [\![M_1]\!]_{\alpha}(x) = [\![M_2]\!]_{\alpha}(x)\}
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of all states in coalgebra (A, α) at which the equation $M_1 \approx M_2$ is true.

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Ground formula:

built from ground equations by logical connectives $\neg,$ $\wedge.$

$$\|\neg\varphi\|^{\alpha} = A - \|\varphi\|^{\alpha}$$

$$\|\varphi_1 \wedge \varphi_2\|^{\alpha} = \|\varphi_1\|^{\alpha} \cap \|\varphi_2\|^{\alpha}.$$

Truth/satisfaction relation:

•
$$\alpha, x \models \varphi$$
 means $x \in \|\varphi\|^{\alpha}$.

•
$$\alpha \models \varphi$$
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Ground observable (GO) formula:

built from equations between GO terms.

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Ground observable (GO) formula:

built from equations between GO terms.

GO formulas	equations
polynomial coalgebras	abstract algebras

The GO terms and formulas provide a natural language for

- specifying properties of polynomial coalgebras.
- characterizing morphisms in terms of term-value preservation.
- characterizing the *bisimilarity* relation of *observational indistinguishability* of states by satisfaction of the same formulas (Hennessy-Milner property).

Modally Definable Classes of Kripke Frames

Theorem

Let *K* be a class of Kripke frames that is closed under ultrapowers. Then *K* is modally definable iff it is

closed under subframes, p-morphic images and disjoint unions;

and

• its complement is closed under ultrafilter extensions (i.e. it reflects ultrafilter extensions).

Note:

can weaken

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• K is closed under ultrafilter extensions,

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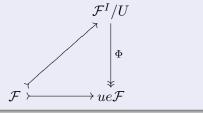
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• K is closed under ultrafilter extensions,

because ...

the ultrafilter extension $ue\mathcal{F}$ of frame \mathcal{F} is a p-morphic image of a suitably saturated ultrapower of \mathcal{F} :



$$\Phi: f^U \mapsto \{X \subseteq \mathcal{F} : f \in_U X\}$$

Venema's analogue for Kripke models

Theorem

A class of Kripke models is modally definable iff it is

- closed under images of bisimulation relations and disjoint unions, and
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Let U be an ultrafilter on a set I.

Given *T*-coalgebra $A \xrightarrow{\alpha} TA$, we construct $\alpha^+ : A^+ \to T(A^+)$, an "observational ultrapower" of α with respect to *U*.

The state set of α^+ is a sub-quotient of the *I*-th power A^I of *A*.

Key Requirements:

• Every GO formula valid in α is valid in α^+ .

 If every GO formula valid in α is valid also in coalgebra β, then β is a bisimilarity image of α⁺, i.e. each state of β is bisimilar to a state of α⁺.

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Theorem

Let T be polynomial with an observable component \bar{D} having $|D| \geq 2$.

For any class K of T-coalgebras, the following are equivalent.

• K is GO-definable,

i.e. is the class of all models of some set of GO formulas.

- K is closed under coproducts (disjoint unions), images of bisimulations, and observational ultrapowers.
- K is closed under coproducts, domains and images of T-morphisms, and observational ultrapowers.

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can replace closure under observational ultrapowers here by closure under certain (definable) ultrafilter extensions.

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Standard Ultrapowers:

• A^U is the quotient $A^I/=_U$, where

$$f =_U g \text{ iff } \{i \in I : f(i) = g(i)\} \in U.$$

 $f =_U g$ means that f and g agree "almost everywhere".

• $f^U :=$ the equivalence class of f.

•
$$A^U = \{ f^U : f \in A^I \}.$$

• A natural embedding $e_A : A \rightarrow A^U$ allows us to assume $A \subseteq A^U$.

Liftings:

Any map $A_1 \times \cdots \times A_n \xrightarrow{\theta} B$ has a *U*-lifting

$$A_1^U \times \cdots \times A_n^U \xrightarrow{\theta^U} B^U$$

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Problem:

A T-coalgebra

$$A \xrightarrow{\alpha} TA$$

has the U-lifting

$$A^U \xrightarrow{\alpha^U} (TA)^U,$$

but a T-coalgebra based on A^U should look like

 $A^U \to T(A^U).$

Solution:

Restrict α^U to the subset $A^+ \subseteq A^U$ of elements f^U that are "observable".

Idea:

For a GO term $M : \overline{D}$, the α -denotation

 $\llbracket M \rrbracket_{\alpha} : A \to D$

has the U-lifting $\llbracket M \rrbracket^U_{\alpha} : A^U \to D^U.$

Since $D \subseteq D^U$, we ask does $\llbracket M \rrbracket^U_{\alpha}(f^U) \in D$? If YES for all M, put f^U in A^+ .

Example

every member of A is observable, so $A \subseteq A^+$.

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Defining the transition structure $\alpha^+ : A^+ \to TA^+$:

• an intricate analysis of the components involved in the inductive formation of functor *T*.

• A path

$$T \xrightarrow{p} S$$

from T to a component functor S is a finite list of symbols expressing the way T is formed from S.

• A path induces a partial function

$$p_A: TA \longrightarrow SA$$

for each set A.

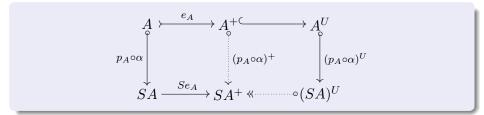
Definition

• $T \xrightarrow{\langle \rangle} T$ is the empty path.

• from $T_j \xrightarrow{p} S$ form $T_1 \times T_2 \xrightarrow{\pi_j \cdot p} S$ for j = 1, 2.

• from
$$T_j \xrightarrow{p} S$$
 form $T_1 + T_2 \xrightarrow{\varepsilon_j \cdot p} S$ for $j = 1, 2$.

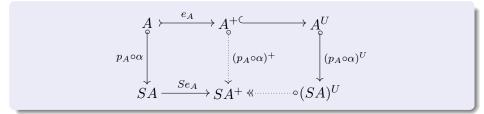
• from $T \xrightarrow{p} S$ form $T^D \xrightarrow{ev_d \cdot p} S$ for all $d \in D$.



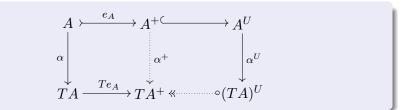
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Łoś-type Theorem

$\alpha^+, f^U \models \varphi \quad \text{ if, and only if, } \quad \{i \in I : \alpha, f(i) \models \varphi\} \in U.$

Proof method:

an analysis for each term M:S of the relationship between the U-lifting

$$\llbracket M \rrbracket^U_\alpha : A^U \to \ (SA)^U$$

of the α -denotation

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Corollary

 $\alpha \models \varphi$ if, and only if, $\alpha^+ \models \varphi$.

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Coalgebraic Logic, Oxford '07

Observational Ultraproducts

Given *T*-coalgebras $\{(A_i, \alpha_i) : i \in I\}$, define a *T*-coalgebra

$$\Pi_U A_i^+ \xrightarrow{\alpha^+} T(\Pi_U A_i^+)$$

whose states are "observable" members of the ultraproduct $\Pi_U A_i$

Łoś:

$$\{i \in I : \alpha_i \models \varphi\} \in U \quad \text{implies} \quad \alpha^+ \models \varphi,$$

and conversely if $\Pi_U A_i^+ = \Pi_U A_i$.

(Converse does hold for ultrapowers.)

NB: could have $\Pi_U A_i^+ = \emptyset$

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Example

 $\bullet \ T = \bar{\omega}$

•
$$(A_n, \alpha_n)$$
 is the $\bar{\omega}$ -coalgebra

$$\{n, n+1, \dots\} \hookrightarrow \omega.$$

• $\Pi_U A_n^+ = \emptyset$ whenever U non-principal.

Compactness Property:

Possible definitions

- a set of formulas has a non-empty model whenever each of its finite subsets does.
- a set of formulas is satisfiable at some state whenever each of its finite subsets is.

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Both of these fail for \{tr(s) \approx i\}
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If every observational ultraproduct of nonempty T-coalgebras is nonempty, then Compactness does hold for T.

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Ultrafilter Enlargements

Definitions

 An ultrafilter *F* on the state set of (*A*, α) is observationally rich if for each GO term *M* : *D* there exists some *c* ∈ *D* such that

$$||M \approx c||^{\alpha} = \{x \in A : \llbracket M \rrbracket_{\alpha}(x) = c\} \in F,$$

i.e. every GO term takes a constant value on an F-large set.

• The ultrafilter enlargement of (A, α) is a coalgebra

$$A^* \xrightarrow{\alpha^*} TA^*,$$

whose state set A^* is the set of all rich ultrafilters on A.

The definition of α^* involves path functions, similarly to α^+ .

Example

Every principal ultrafilter on A is rich, giving an embedding

 $\eta_A: (A, \alpha) \rightarrowtail (A^*, \alpha^*)$

that is a coalgebraic morphism (contra the modal case!)

Truth Lemma

For each GO formula φ ,

 $\alpha^*, F \models \varphi \quad \text{if, and only if,} \quad \|\varphi\|^\alpha \in F$

i.e. φ is true at state F in A^* iff true in an "F-large" set of states in A.

Corollary

$$\alpha \models \varphi$$
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Corollary

$$\alpha \models \varphi$$
 if, and only if, $\alpha^* \models \varphi$.

 (A^*, α^*) as a quotient of some (A^+, α^+)

- $\Phi_U : A^+ \to A^*$ acts by $f^U \mapsto \{X \subseteq A : f \in_U X\}.$
- Φ_U is a coalgebraic morphism $(A^+, \alpha^+) \to (A^*, \alpha^*)$.
- Φ_U is surjective if A^U enlarges A:

every collection of subsets of A with the finite intersection property has non-empty intersection in A^U .

Definable Ultrafilter Enlargements

- Def^α = { ||φ||^α : φ is GO } is the Boolean algebra of definable subsets of A.
- The definable enlargement $A^{\delta} \xrightarrow{\alpha^{\delta}} TA^{\delta}$ of (A, α) has A^{δ} = the set of all rich ultrafilters in Def^{α}

•
$$\alpha^{\delta}, F \models \varphi$$
 if, and only if, $\varphi^{\alpha} \in F$.

•
$$\alpha \models \varphi$$
 if, and only if, $\alpha^{\delta} \models \varphi$.

There is an epimorphism α^{*} → α^δ.
α^δ is isomorphic to the bisimilarity quotient of α^{*}.

Infinitary Proof Theory

Path formulas

• Halting formulas:

for any path $T \xrightarrow{p} S$ there is a formula $p \downarrow$ with

 $\alpha, x \models p \downarrow \quad \text{iff} \quad \alpha(x) \in \text{Dom} \, p_A$

• Observation formulas: for any path $T \xrightarrow{p} \overline{D}$ and $c \in D$ there is a formula (p)c with $\alpha, x \models (p)c$ iff $\alpha(x) \in \text{Dom } p_A$ and $p_A(\alpha(x)) = c$.

Modalities:

for any path $T \xrightarrow{p} \operatorname{Id}$ and formula φ there is a formula $[p]\varphi$ with $\alpha, x \models [p]\varphi$ iff $\alpha(x) \in \operatorname{Dom} p_A$ implies $\alpha, p_A(\alpha(x)) \models \varphi$.

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Deducibility Relation $\Gamma \vdash_T \varphi$

defined syntactically using axioms and infinitary rules, e.g.

$$\frac{\neg(p)c \quad \text{for all} \quad c \in D}{\neg(p\downarrow)} \qquad \qquad \frac{\psi \to [q] \neg(p)c \quad \text{for all} \quad c \in D}{\psi \to [q] \neg(p\downarrow)}$$

- Γ is *T*-consistent if $\Gamma \nvDash_T \bot$.
- Γ is *T*-maximal if it is consistent, negation complete, and closed under certain infinitary rules.

Example For any α, x , $\{\varphi : \alpha, x \models \varphi\}$ is *T*-maximal.

 Completeness (i.e. "consistent implies satisfiable") depends on some cardinality constraint, as with ω-logic.

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Lindenbaum Functors

Definition

T is Lindenbaum if every T-consistent set of formulas can be extended to a T-maximal set.

Lemma

T is Lindenbaum if any of the following hold:

- **1** T has no constant component \overline{D} , or
- 2 Every constant component \overline{D} has D finite, or
- Solution \mathbf{S}^{D} of T has countable exponent D.

Example

Let $T = \bar{\omega}^{\mathbb{R}}$.

There is a set Σ of formulas such that

- Σ is *T*-consistent.
- Σ is not satisfiable at any state of any *T*-coalgebra.
 - hence Completeness fails.
- every countable subset of Σ is satisfiable.

Theorem

If T is Lindenbaum, then every T-consistent set of formulas is satisfiable in a T-coalgebra.

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If T is Lindenbaum, then every T-consistent set of formulas is satisfiable in a T-coalgebra.

Canonical *T*-coalgebra

$$A_T \xrightarrow{\alpha_T} T(A_T)$$

- A_T is the set of all *T*-maximal sets.
- Truth Lemma: $\alpha_T, x \models \varphi$ iff $\varphi \in x$.

• If *T* is Lindenbaum, then every *T*-consistent set is satisfiable.

 (A_T, α_T) is a final coalgebra.

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is the unique morphism from any (A, α) to (A_T, α_T) .

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Questions

- Can this theory be extended to finitary Kripke polynomial functors, involving \mathcal{P}_{ω} ?
- is there a universal property characterising observational ultraproducts ?